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Dedicated to the memory of Academician Mitrofan M. Cioban (1942-2021)

Dynamics of a cubic differential advertising model

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Abstract. In this paper a cubic differential advertising model with a singular point $O(0,0)$ is studied. The dynamics of the system is investigated and the global phase portraits are obtained. A small amplitude limit cycle is emitted by the Hopf supercritical bifurcation.

2010 Mathematics Subject Classification: 34C05.

Keywords: cubic differential system, advertising model, Hopf bifurcation, phase portrait.

Dinamica unui model diferenţial cubic de publicitate

Rezumat. În lucrare este studiat un model diferenţial cubic de publicitate cu un punct singular $O(0,0)$. Se studiază dinamica sistemului şi se obţin portretele globale de fază. Un ciclu limită de amplitudine mică este obţinut la bifurcaţia Hopf supercritică.

Cuvinte cheie: sistem diferenţial cubic, model de publicitate, bifurcaţie Hopf, portret de fază.

1. Introduction

Numerous mathematical models have been developed in recent decades to analyse ecological, physical, chemical and economical phenomena and processes. Some of these are optimization models, static or dynamic; others describe the evolution of these phenomena. Evolution models, in which time-dependent variables verify a system of autonomous differential equations, are considered in this work. This system depends on several parameters and changing the values of a parameter causes a change in the behaviour of the system and can produce bifurcations.

The famous Lotka-Volterra equations play a fundamental role in the mathematical modelling of various ecological, physical and chemical systems. The dynamics of this model described by quadratic differential systems was investigated in works [1], [2], [3], [4]. The dynamics of Lotka-Volterra models (Kolmogorov systems) described by cubic differential systems was studied in works [5], [6]. A new modification of these equations was suggested in [7] to model the structure of marine phage populations, which are the most abundant biological entities in the biosphere.

Evolution models used in the practice of economic-mathematical modelling are important tools for studying economic cyclicity, equilibriums as well as imbalance in the economy. Of the many mathematical models of the dynamics of various economic variables, models of economic growth, models of the dynamics of the business cycle or models of inflation, unemployment and advertising are of particular interest [8], [9]. We will consider an advertising model inspired by the mathematical theory of epidemics, advertising being seen as a spreading virus. Potential buyers "get infected" with these viruses through advertising and get in touch with users of a product [10], [11], [12]. The model is described by a system of two differential equations, whose variables represent the number of potential buyers and the number of users of a product. Advertising has become an essential marketing activity in the modern era of large scale production and serve as competition in the market.

In this paper we study the dynamics of a cubic differential advertising model. We show that this model can be brought to the Kukles cubic system with a small amplitude limit cycle bifurcating from a weak focus.

2. Advertising model description

We will consider an advertising model described by a system of two differential equations [10], [12]

$$\begin{cases} \dot{x} = k - \gamma xy + \beta y, \\ \dot{y} = \gamma xy - \delta y, \end{cases} \tag{1}$$

where $x(t)$ represents the number of potential buyers at time t and $y(t)$ is the number of users of the brand at time t. System (1) contains several parameters: k - the rate at which new potential buyers enter the market; γ - the advertising contact rate at time t; β - the migration rate to the rival brand and $\delta = \beta + \varepsilon$, where ε is the rate of migration or death of potential buyers.

In our study we'll assume that γ is not constant, but varies in time:

$$\gamma(t) = \alpha y(t). \tag{2}$$

In this case, system (1) becomes a cubic differential system of the form

$$\begin{cases} \dot{x} = k - \gamma xy + \beta y, \\ \dot{y} = \alpha xy^2 - \delta y. \end{cases} \tag{3}$$

This system has a cubic term, four non-zero real coefficients and the singular point $M\left(\dfrac{\delta\varepsilon}{\alpha k}, \dfrac{k}{\varepsilon}\right)$. Via the linear transformation $u = \dfrac{\alpha k}{\delta\varepsilon}x - 1$, $v = \dfrac{\varepsilon}{k}y - 1$, we can bring this singular point to the origin of coordinates and the system changes to the form

$$\begin{cases} \dot{u} = -\frac{\alpha k^2}{\delta \varepsilon^2} \left(\delta u + (\delta + \varepsilon) v + 2\delta uv + \delta v^2 + \delta uv^2\right), \\ \dot{v} = \delta \left(u + v + 2uv + v^2 + uv^2\right). \end{cases} \tag{4}$$

Let us denote $a = \dfrac{\alpha k^2}{\delta \varepsilon^2}$, $b = 2 - \dfrac{\beta}{\delta}$, $u = x$, $v = y$. Then, changing the time $\tau = \delta t$, system (4) takes the form

$$\begin{cases} \dot{x} = -a \left(x + by + 2xy + y^2 + xy^2\right), \\ \dot{y} = x + y + 2xy + y^2 + xy^2, \end{cases} \tag{5}$$

where a and b are real parameters. Since $\delta > \beta$, it follows that $b > 1$. We are interested to use this system to model an advertising campaign, therefore we set $a > 0$. System (5) has only one singular point $O(0,0)$ in the finite part of the phase plane. A straight line $y + 1 = 0$ is an invariant straight line for system (5) with the cofactor $K(x, y) = x + y + xy$.

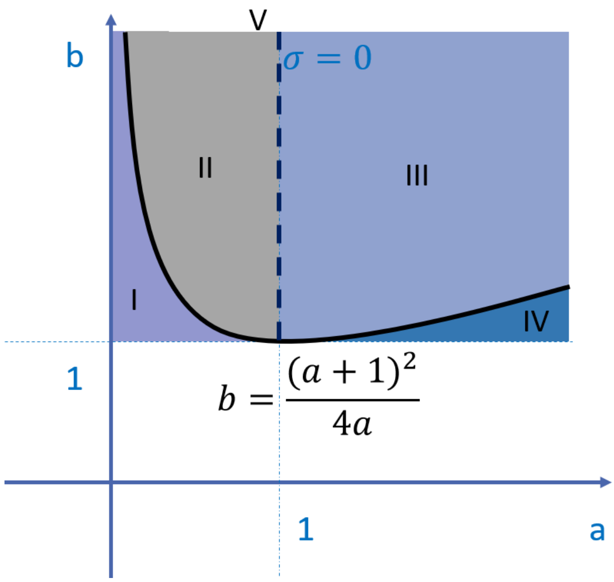


Figure 1. Bifurcation sectors in the space of coefficients.

3. Singular points and Hopf bifurcation

To study the singular point $O(0,0)$, we linearize system (5):

$$\begin{cases} \dot{x} = -a\,(x + by), \\ \dot{y} = x + y. \end{cases} \tag{6}$$

Denote

$$A = \begin{pmatrix} -a & -ab \\ 1 & 1 \end{pmatrix}, \ \Delta = \det A = a(b-1), \ \sigma = \operatorname{tr} A = 1 - a.$$

Then the characteristic equation of A is $\lambda^2 - \sigma\lambda + \Delta = 0$. Since $a > 0$ and $b > 1$, we have that $\Delta > 0$ and the discriminant of the characteristic equation is $D = (a+1)^2 - 4ab$.

The curve defined by equation $D = 0$ and the straight line $\sigma = 0$ divide the plane of coefficients (a, b) in 5 sectors (Figure 1). Depending on the values of parameters a and b, the singular point $O(0,0)$ of (6) belongs to one of the five types given in Table 1.

Table 1. The type of singular point $O(0,0)$.

Conditions	$\sigma > 0, D > 0$	$\sigma > 0, D < 0$	$\sigma < 0, D < 0$	$\sigma < 0, D > 0$	$\sigma = 0$
$\lambda_{1,2}$	$\lambda_{1,2} > 0$	$\lambda_{1,2} = \alpha \pm i\beta$ $\alpha > 0, \beta \neq 0$	$\lambda_{1,2} = \alpha \pm i\beta$ $\alpha < 0, \beta \neq 0$	$\lambda_{1,2} > 0$	$\lambda_{1,2} = \pm i\sqrt{b-1}$
Type of Singularity	Unstable Node	Unstable Focus	Stable Focus	Stable Node	Weak Focus

On the half line $\sigma = 0$, i.e. in the set $H = \{(a,b) | \, a = 1, b > 1\}$, the Hopf bifurcation conditions can be fulfilled. A Hopf bifurcation is a local bifurcation in which the singular point of a dynamic system looses its stability as the eigenvalues of its linearized system around the fixed point traverse the imaginary axis of the complex plane. In this type of bifurcation, a small amplitude limit cycle will appear from the singular point.

Let $D < 0$, then the singular point $O(0,0)$ is:

- an unstable focus, if $a < 1$;
- a weak focus, if $a = 1$;
- a stable focus, if $a < 1$.

Consider the set $H = \{(a,b) | \, a = 1, b > 1\}$. By change of variables $X = x + y$, $Y = \sqrt{b-1}y$, $dT = -\sqrt{b-1}d\tau$, system (5) can be brought to the Kukles differential system

$$\begin{cases} \dot{X} = Y, \\ \dot{Y} = -X - \frac{2\sqrt{b-1}}{b-1}XY + \frac{1}{b-1}Y^2 - \frac{1}{b-1}XY^2 + \frac{\sqrt{b-1}}{(b-1)^2}Y^3, \end{cases} \tag{7}$$

with a weak focus at $O(0,0)$. Necessary and sufficient conditions for the Kukles system to have a center at a weak focus $O(0,0)$ were obtained in [13], [14]. The first Lyapunov quantity calculated for system (7) is

$$L_1 = -\frac{\sqrt{b-1}}{(b-1)^2} < 0$$

and this means that a non-degenerate supercritical Hopf bifurcation can occur [15].

The limit cycle is attractive and it exists for parameter values close to the bifurcation value for $a < 1$, i.e. in the region where the singular point is an unstable focus [15]. In Figure 2 it is represented a limit cycle for $a = 0.891$ and $b = 2$.

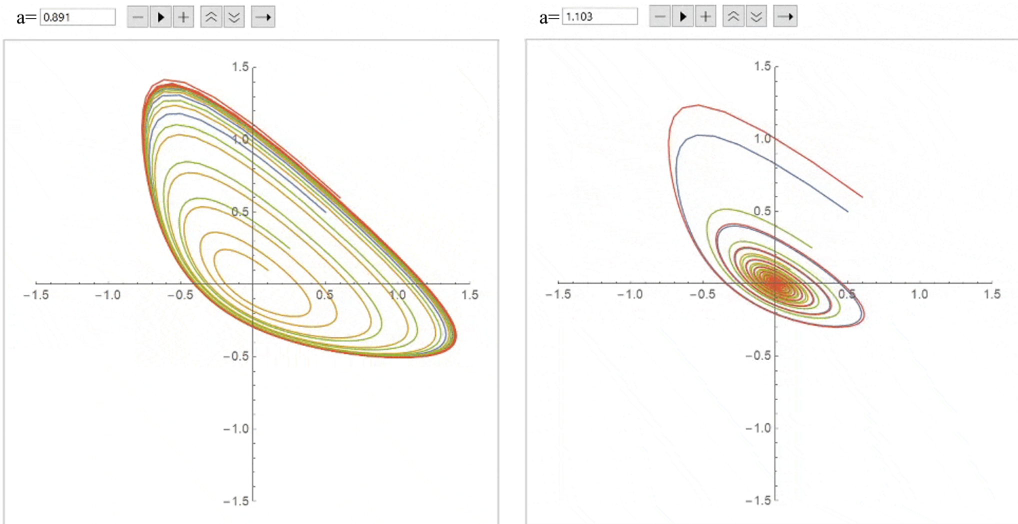


Figure 2. Hopf bifurcation.

4. Global phase portraits

In order to obtain the global phase portraits of system (5) we study the system at infinity. The polynomial $yP_3(x, y) - xQ_3(x, y)$ shows that there are 3 pairs of equilibrium points at infinity $I_{1,4}\ (\pm 1, 0, 0)$, $I_{2,5}\left(\pm\frac{a}{\sqrt{a^2+1}}, \mp\frac{a}{\sqrt{a^2+1}}, 0\right)$, and $I_{3,6}\ (0, 1, 0)$. Via first Poincaré transformation $x = \frac{1}{z}$, $y = \frac{u}{z}$ we can study the singular points $I_{1,4}$ and $I_{2,5}$. Applying this transformation to system (5), we obtain the following system:

$$\begin{cases} \dot{z} = az\left(u^2 + 2zu + zu^2 + z^2 + bz^2u\right), \\ \dot{u} = z^2 + 2zu + u^2 + au^3 + (a+1)z^2u + (2a+1)zu^2 + azu^3 + abz^2u^2. \end{cases} \tag{8}$$

The singular point $O_1(0, 0)$ of system (8) corresponds to the singular point $I_{1,4}$ at infinity of system (5). Both eigenvalues are null, therefore O_1 is a multiple singular point. Using the blow-up method, we decompose this point into 4 points: $M_1\ (0, 0)$, $M_2\left(0, \frac{3\pi}{4}\right)$, $M_3(0, \pi)$, and $M_4\left(0, \frac{7\pi}{4}\right)$, where M_1, M_3 are of saddle type and M_2, M_4 are multiple.

Bringing M_2 to the origin of coordinates and applying the blow-up method once more, we obtain the following system:

$$\begin{cases} \dot{x} = x\left(\sqrt{2}ax\cos^4 y - \sqrt{2}abxcos^4 y + \dots\right)/4, \\ \dot{y} = -\cos y \sin y \left(2\cos y + 4\sqrt{2}\sin y + \sqrt{2}ax\cos^2 y + \dots\right)/4. \end{cases}$$

Solving the equation $Q^*(0, y) = 0$, i.e.

$$\cos y \sin y \left(\cos y + 2\sqrt{2}\sin y\right) = 0,$$

we obtain that the point M_2 decomposes into the following 6 points: $N_1(0,0)$, $N_2\left(0, \frac{\pi}{2}\right)$, $N_3\left(0, \pi - \operatorname{arctg}\frac{1}{2\sqrt{2}}\right)$, $N_4\,(0,\pi)$, $N_5\left(0, \frac{3\pi}{2}\right)$, $N_6\left(0, -\operatorname{arctg}\frac{1}{2\sqrt{2}}\right)$. All these points are hyperbolic, their eigenvalues and types are brought in Table 2:

Table 2. Decomposition of the singular point $M_2\left(0, \frac{3\pi}{4}\right)$.

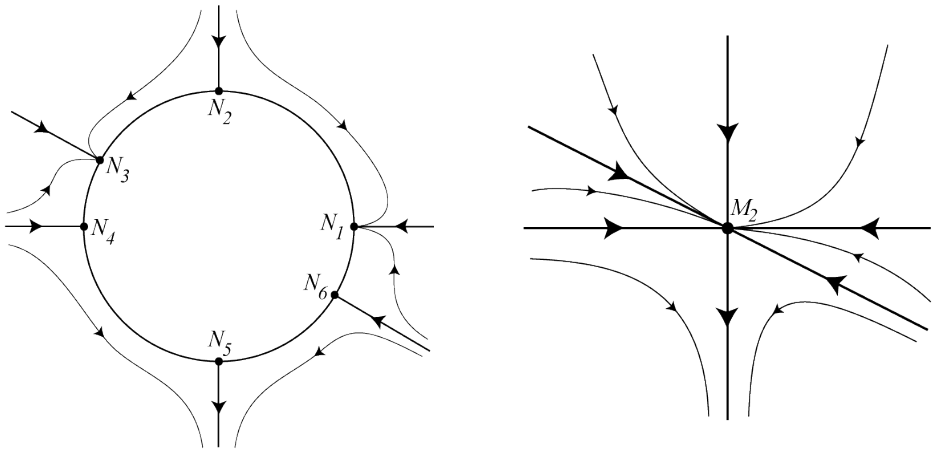
Point	N_1	N_2	N_3	N_4	N_5	N_6
λ_1, λ_2	$0, -\frac{1}{2}$	$\pm\sqrt{2}$	$0, -\frac{\sqrt{2}}{3}$	$0, \frac{1}{2}$	$\pm\sqrt{2}$	$0, \frac{\sqrt{2}}{3}$
Type	$S - N^s$	Saddle	$S - N^s$	$N^u - S$	Saddle	$N^u - S$

We denoted by $S - N^s$ a singular point of saddle-node type with stable nodal sectors, and by $N^u - S$ - a singular point of saddle-node type with unstable nodal sectors. We will distinguish the notations $S - N$ and $N - S$. In the first case, only nodal sectors matter in the qualitative portrait of a singular point M_2, and in the latter case - only saddle sectors (see Figure 3.a)).

Compressing the circle to the origin of coordinates, we obtain the qualitative behaviour of the trajectories in the neighbourhood of the singular point M_2 (see Figure 3.b)).

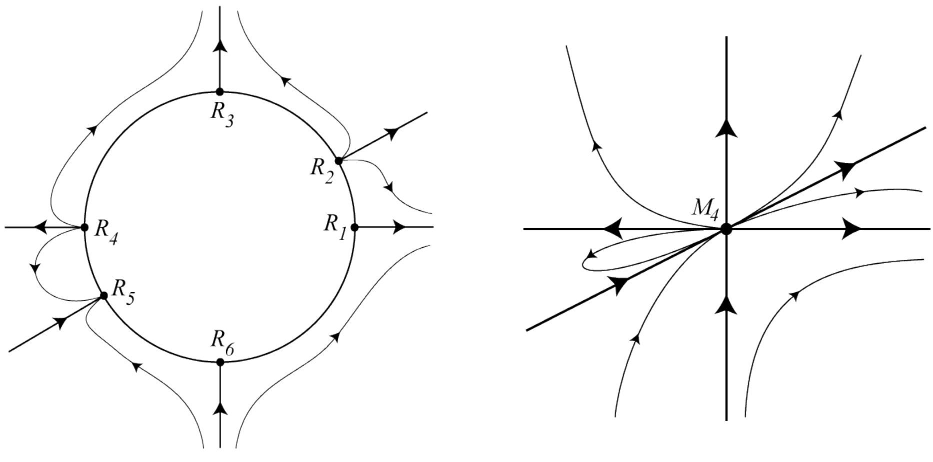
Analogous, by performing the same analysis on the singular point $M_4\left(0, \frac{7\pi}{4}\right)$, we obtain its local phase portrait (see Figure 4). Sketching the behaviour of the trajectories in the neighbourhood of singular points M_1, M_2, M_3 and M_4 on the unit circle, we get Figure 5.a) and compressing this circle to the origin, we obtain the behaviour of the trajectories near the singular points $I_{1,4}$ situated at infinity.

The singular points $I_{2,5}\left(\pm\frac{a}{\sqrt{a^2+1}}, \mp\frac{a}{\sqrt{a^2+1}}, 0\right)$ have equal eigenvalues $\lambda_1 = \lambda_2 = a$. Since $a > 0$, it follows that they are unstable nodes. It remains to study the singular point



a) b)

Figure 3. Blow-up for the point $M_2\left(0, \frac{3\pi}{4}\right)$.

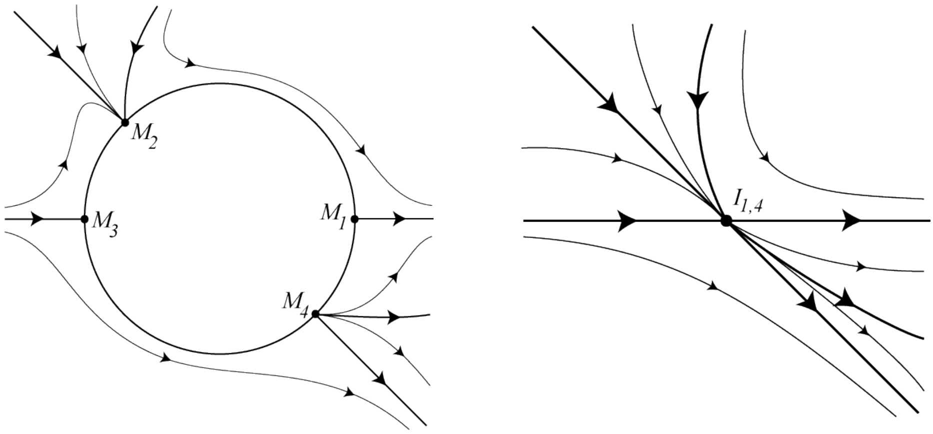


a) b)

Figure 4. Blow-up for the point $M_4\left(0, \frac{7\pi}{4}\right)$.

situated at the ends of the Oy axis. Via the second Poincaré transformation $x = \dfrac{u}{z}$, $y = \dfrac{1}{z}$, system (5) takes the following form:

$$\begin{cases} \dot{u} = -au - u^2 - az - uz - 2auz - 2u^2z - abz^2 - uz^2 - auz^2 - u^2z^2, \\ \dot{z} = -z(1+z)(u+z+uz). \end{cases} \tag{9}$$



a) b)

Figure 5. Blow-up for the point $I_{1,4}$ $(\pm 1, 0, 0)$.

The singular point $R(0,0)$ corresponds to the points $I_{3,5}$ with eigenvalues $\lambda_1 = 0, \lambda_2 = -a$. Using the transformation $u \to y, z \to x + y$, system (9) takes the following form:

$$\begin{cases} \dot{x} = x(1+x)(x^2 - y - xy), \\ \dot{y} = 2ax^2 - abx^2 + ax^3 - ay - 2axy + x^2y - ax^2y + x^3y - y^2 - 2xy^2 - x^2y^2. \end{cases} \tag{10}$$

System (10) has the form $\dot{x} = P^*(x,y)$, $\dot{y} = by + Q^*(x,y)$, where P^* and Q^* are polynomials of a degree greater than one. By solving the equation $by + Q^*(x,y) = 0$ and writing its solution as Taylor series near the point $x = 0$, we get the function:

$$y^* = (2-b)x^2 + (-3+2b)x^3 + \frac{-2+4a+3b-3ab-b^2}{a}x^4 + \ldots$$

Now we can obtain the function $\psi(x) = P^*(x, y^*)$ and express it in the Taylor series form:

$$\psi(x) = (b-1)x^3 + \frac{\left(b^2 - 3b + 2\right)x^5}{a} + \frac{x^6\left(4ab - 5a + 2b^2 - 6b + 4\right)}{a} + \ldots$$

Because the function $\psi(x)$ starts with a monomial of odd degree and $-a(b-1) < 0$, it follows that the singular points $I_{3,5}$ are topological saddles.

Using all this information, we obtain the phase portraits for system (5) when the values of the parameter a excludes the presence of limit cycles (Figure 6). Considering the limit cycle that appears near $a = 1$, we obtain Figure 7.

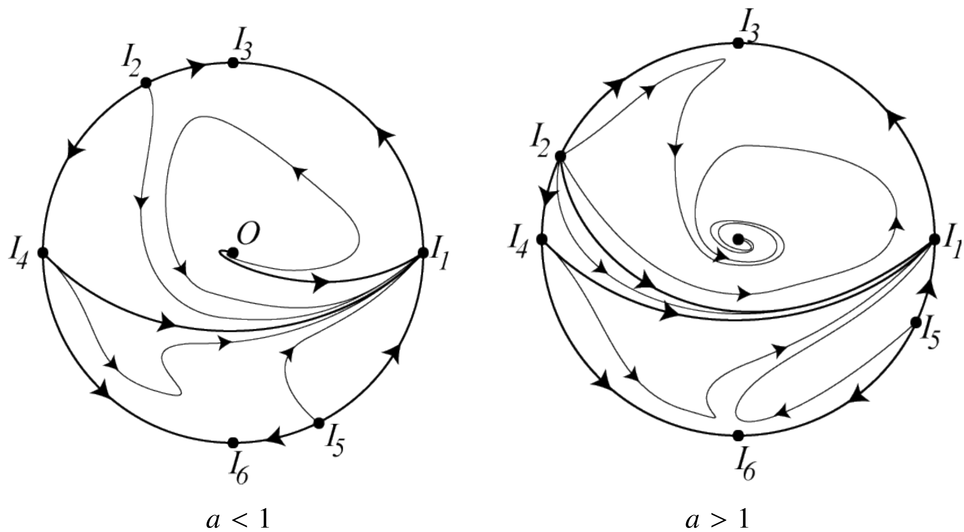


Figure 6. The phase portraits of system (5) in the absence of limit cycles.

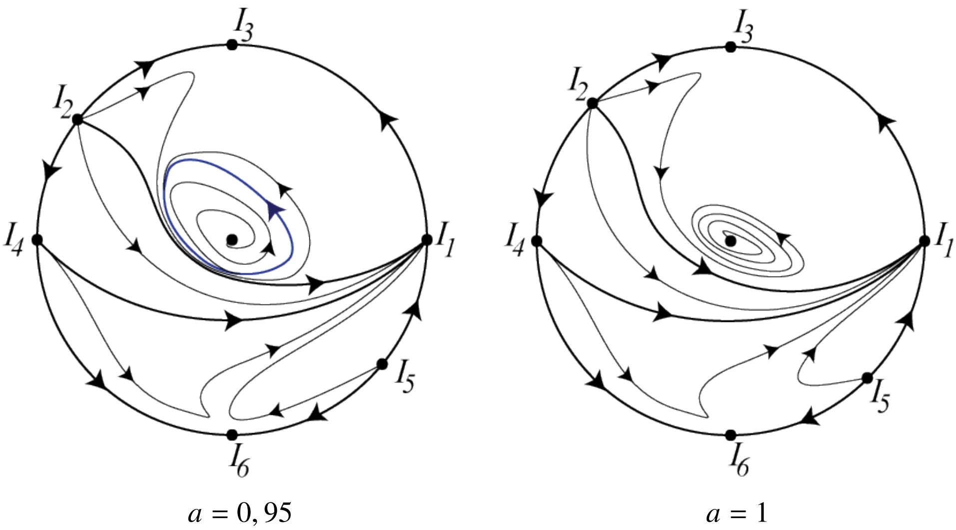


Figure 7. The phase portraits of system (5) in the neighbourhood of $a = 1$.

5. Conclusions

Depending on the variables $x = x(t)$ and $y = y(t)$, we will notice the sequence of four behavior regimes over time [9], [12]:

(1) The number of potential buyers x and the number of users y of the product increase. It is a time of prosperity when the product imposes itself and occupies a place in the market.

(2) The number of potential buyers x decreases and the number of users y of the product increases. It is a period of saturation when the product is known and the main question of the consumer is what product to buy (what product they are buying).

(3) The number of potential buyers x as well as the number of users y of the product decreases. It is a period of decline, in which the product has a degree of acceptance, its usefulness is known, but it maintains its place on the market by virtue of its past reputation.

(4) The number of potential buyers x increases and the number of users y of the product decreases. It is a period of return when the consumer is reminded that the product exists, being relaunched by a new advertising company.

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