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Dedicated to the memory of Academician Mitrofan M. Cioban (1942-2021)

Lyapunov's stability of the unperturbed motion governed by the $s^3(1,3)$ differential system of Darboux type

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Abstract. There were obtained the conditions of stability after Lyapunov of the unperturbed motion for the system $s^3(1,3)$ in the non-critical case. It was constructed the Lyapunov series for the ternary differential system $s^3(1,3)$ of Darboux type in the critical case and determined the conditions of stability of the unperturbed motion governed by this system.

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Stabilitatea după Lyapunov a mişcării neperturbate guvernate de sistemul diferenţial $s^3(1,3)$ de tip Darboux

Rezumat. Au fost obţinute condiţiile de stabilitate după Lyapunov a mişcării neperturbate pentru sistemul $s^3(1,3)$ în cazul necritic. A fost construită seria Lyapunov pentru sistemul diferenţial ternar de tip Darboux $s^3(1,3)$ în cazul critic şi determinate condiţiile de stabilitate a mişcării neperturbate guvernate de acest sistem.

Cuvinte cheie: sistem diferenţial, stabilitatea mişcării neperturbate, comitant şi invariant centro-afin.

1. Introduction

Problems which required a general formulation of stability, both of equilibrium and motion, arose in science and technics in the middle of XIX-th century.

Lyapunov (1857-1918) published his PhD thesis concerning the stability of motion in 1882. Later it was translated into French and published in France in 1907. This work was reprinted in Russian, according to the French version, with some additions, in his collection of works [1] in 1956. The mentioned work contains many fruitful ideas and results of great importance. So that all the history related to the theory on stability of motion is considered to be divided into periods before and after Lyapunov.

First of all, A. M. Lyapunov gave a strict definition of the stability of motion, which was so successful that all scientists took it as a fundamental one for their researches.

A lot of papers were written in the field of stability of motion. The universal scientific literature on the stability of motion contains thousands of papers, including hundreds of monographs and textbooks of many authors. This literature is rich in the theory development, as well as in its application in practice.

It should be noted that many problems on stability treated in these works are governed by bi-dimensional (or multi-dimensional) autonomous polynomial differential systems. For such systems methods of the theory of invariants were elaborated in the school of differential equations from Chişinău. Moreover, it was developed the theory of Lie algebras and Sibirsky graded algebras with applications in the qualitative theory of these equations [2-7].

The stability of unperturbed motions using the theory of algebras, of invariants and of Lie algebras was studied for the first time in [8]. In this paper, the similar investigations are done for ternary differential systems with polynomial nonlinearities.

2. Lyapunov form of the ternary critical differential system $s^3(1,3)$ with cubic nonlinearities

We examine the ternary differential system with cubic nonlinearities $s^3(1,3)$ of the form

$$\frac{dx^j}{dt} = a^j_\alpha x^\alpha + a^j_{\alpha\beta\gamma} x^\alpha x^\beta x^\gamma \quad (j, \alpha, \beta, \gamma = \overline{1,3}), \tag{1}$$

where $a^j_{\alpha\beta\gamma}$ is a symmetric tensor in lower indices in which the total convolution is done.

Definition 2.1. According to I. G. Malkin [9], we will say that system (1) is critical if the characteristic equation of this system has one zero root, and all other roots of this equation have negative real parts.

Lemma 2.1. *System* (1) *is critical if and only if the center-affine invariant conditions hold*

$$L_{1,3} > 0, \quad L_{2,3} > 0, \quad L_{3,3} = 0, \tag{2}$$

where

$$L_{1,3} = -\theta_1, \quad L_{2,3} = \frac{1}{2}(\theta_2 - \theta_1^2), \quad L_{3,3} = \frac{1}{6}(-\theta_1^3 + 3\theta_1\theta_2 - 2\theta_3), \tag{3}$$

and

$$\theta_1 = a^\alpha_\alpha, \quad \theta_2 = a^\alpha_\beta a^\beta_\alpha, \quad \theta_3 = a^\alpha_\gamma a^\beta_\alpha a^\gamma_\beta. \tag{4}$$

By means of the Lyapunov theorems on stability or instability of unperturbed motion [1] and the Hurwitz theorem we obtain the following theorems:

Theorem 2.1. *Assume that the center-affine invariants* (3) *of system* (1) *satisfy the inequalities*

$$L_{1,3} > 0, \quad L_{2,3} > 0, \quad L_{1,3}L_{2,3} - L_{3,3} > 0, \tag{5}$$

then the unperturbed motion $x^1 = x^2 = x^3 = 0$ of this system is asymptotically stable.

Theorem 2.2. *If at least one of the center-affine invariant expressions* (3) *of system* (1) *is negative, then the unperturbed motion $x^1 = x^2 = x^3 = 0$ of this system is unstable.*

Lemma 2.2. *In case of conditions* (2)*, by a center-affine transformation, system* (1)*, can be brought to the critical Lyapunov form*

$$\begin{aligned} \frac{dx^1}{dt} &= a^1_{\alpha\beta\gamma}x^\alpha x^\beta x^\gamma, \\ \frac{dx^j}{dt} &= a^j_\alpha x^\alpha + a^j_{\alpha\beta\gamma}x^\alpha x^\beta x^\gamma \quad (j = 2,3; \alpha,\beta,\gamma = \overline{1,3}), \end{aligned} \tag{6}$$

where the first equation from (6) *is called the **critical equation** and the second one – the **non-critical equation**.*

Proof. We will show that the system of the first approximation

$$\frac{dx^j}{dt} = a^j_\alpha x^\alpha \quad (j,\alpha = \overline{1,3}), \tag{7}$$

for system (1), under conditions (2), admits a linear first integral (this was shown in [1], for any multidimensional differential system with analytical nonlinearities).

We will look for this integral in the form

$$Ax^1 + Bx^2 + Cx^3 = C_1 \quad (A^2 + B^2 + C^2 \neq 0), \tag{8}$$

where A, B, C are unknown constants and C_1 is an arbitrary constant. Then we have

$$a^1_\alpha Ax^\alpha + a^2_\alpha Bx^\alpha + a^3_\alpha Cx^\alpha = 0.$$

Identifying the coefficients of x^1, x^2 and x^3, we obtain

$$\begin{aligned} a^1_1A + a^2_1B + a^3_1C &= 0, \\ a^1_2A + a^2_2B + a^3_2C &= 0, \\ a^1_3A + a^2_3B + a^3_3C &= 0. \end{aligned} \tag{9}$$

For this system to admit a non-trivial solution $A^2 + B^2 + C^2 \neq 0$, it is necessary and sufficient that

$$\begin{vmatrix} a^1_1 & a^2_1 & a^3_1 \\ a^1_2 & a^2_2 & a^3_2 \\ a^1_3 & a^2_3 & a^3_3 \end{vmatrix} \equiv L_{3,3} = 0,$$

where $L_{3,3}$ is from (3). This condition is contained in (2). Therefore, in this situation, an integral of the form (8) always exists for system (1).

We assume that in (8) the condition $A \neq 0$ holds. Then, considering the center-affine substitution

$$\bar{x}^1 = Ax^1 + Bx^2 + Cx^3, \quad \bar{x}^2 = x^2, \quad \bar{x}^3 = x^3, \tag{10}$$

we have

$$\frac{d\bar{x}^1}{dt} = A\frac{dx^1}{dt} + B\frac{dx^2}{dt} + C\frac{dx^3}{dt}, \quad \frac{d\bar{x}^2}{dt} = \frac{dx^2}{dt}, \quad \frac{d\bar{x}^3}{dt} = \frac{dx^3}{dt}$$

or by virtue of system (7), for the first equality we obtain

$$\frac{d\bar{x}^1}{dt} = (a_1^1A + a_1^2B + a_1^3C)x^1 + (a_2^1A + a_2^2B + a_2^3C)x^2 + (a_3^1A + a_3^2B + a_3^3C)x^3.$$

Taking into account (9), we find $\frac{d\bar{x}^1}{dt} = 0$, while the other equations in system (7) retain their form. Performing substitution (10) with conditions (2) in (1), similarly we obtain (6), because this substitution does not change the form of the cubic parts of system (1).

It can easily be verified that in case $B \neq 0$ the substitution

$$\bar{x}^1 = Ax^1 + Bx^2 + Cx^3, \quad \bar{x}^2 = x^1, \quad \bar{x}^3 = x^3,$$

in system (1), brings it to the form (6), and for $C \neq 0$ the substitution

$$\bar{x}^1 = Ax^1 + Bx^2 + Cx^3, \quad \bar{x}^2 = x^2, \quad \bar{x}^3 = x^1$$

in system (1) has the same effect. Lemma 2.2 is proved. □

3. Stability conditions of unperturbed motion governed by critical three-dimensional differential system $s^3(1,3)$ of Darboux type

By a center-affine transformation, system (1) can be brought to the critical Lyapunov form [1] and considering the center-affine condition

$$\eta_1 = a^{\alpha}_{\beta,\gamma,\delta}x^{\beta}x^{\gamma}x^{\delta}x^{\mu}y^{\nu}\varepsilon_{\alpha\mu\nu} \equiv 0, \tag{11}$$

from [2], system (1) becomes a critical one of Darboux type, and has the following form

$$\begin{aligned} \frac{dx}{dt} &= 3x(ax^2 + by^2 + cz^2 + 2dxy + 2exz + 2fyz), \\ \frac{dy}{dt} &= px + qy + rz + 3y(ax^2 + by^2 + cz^2 + 2dxy + 2exz + 2fyz), \\ \frac{dz}{dt} &= sx + my + nz + 3z(ax^2 + by^2 + cz^2 + 2dxy + 2exz + 2fyz), \end{aligned} \tag{12}$$

where $a, b, c, d, e, f, m, n, p, q, r, s$ are real arbitrary coefficients.

We analyze the noncritical equations

$$\begin{aligned} &px + qy + rz + 3y(ax^2 + by^2 + cz^2 + 2dxy + 2exz + 2fyz) = 0, \\ &sx + my + nz + 3z(ax^2 + by^2 + cz^2 + 2dxy + 2exz + 2fyz) = 0. \end{aligned} \tag{13}$$

Due to $L_{2,3} = nq - mr > 0$ in system (12), according to conditions (2), we can assume, without losing generality, that $nq \neq 0$.

Then from the first equation of (13) we express y, while from the second equation of (13) we express z:

$$\begin{aligned} &y = -\frac{p}{q}x - \frac{r}{q}z - \frac{2}{q}y(ax^2 + by^2 + cz^2 + 2dxy + 2exz + 2fyz), \\ &z = -\frac{s}{n}x - \frac{m}{n}y - \frac{2}{n}z(ax^2 + by^2 + cz^2 + 2dxy + 2exz + 2fyz). \end{aligned} \tag{14}$$

We seek y and z as holomorphic functions of x. Then we can write

$$\begin{aligned} y(x) &= A_1x + A_2x^2 + A_3x^3 + A_4x^4 + A_5x^5 + \ldots, \\ z(x) &= B_1x + B_2x^2 + B_3x^3 + B_4x^4 + B_5x^5 + \ldots \end{aligned} \tag{15}$$

Substituting (15) into (14) we have

$$\begin{gathered} A_1x + A_2x^2 + A_3x^3 + A_4x^4 + A_5x^5 + \ldots = -\frac{p}{q}x - \frac{r}{q}(B_1x + B_2x^2 + B_3x^3 + \\ +B_4x^4 + B_5x^5 + \ldots) - \frac{3}{q}(A_1x + A_2x^2 + A_3x^3 + A_4x^4 + A_5x^5 + \ldots)[(ax^2 + \\ +b(A_1x + A_2x^2 + A_3x^3 + A_4x^4 + A_5x^5 + \ldots)^2 + c(B_1x + B_2x^2 + B_3x^3 + B_4x^4 + \\ +B_5x^5 + \ldots)^2 + 2dx(A_1x + A_2x^2 + A_3x^3 + A_4x^4 + A_5x^5 + \ldots) + 2ex(B_1x + B_2x^2 + \\ +B_3x^3 + B_4x^4 + B_5x^5 + \ldots) + 2f(A_1x + A_2x^2 + A_3x^3 + A_4x^4 + A_5x^5 + \ldots)(B_1x + \\ +B_2x^2 + B_3x^3 + B_4x^4 + B_5x^5 + \ldots))], \\ B_1x + B_2x^2 + B_3x^3 + B_4x^4 + B_5x^5 + \ldots = -\frac{s}{n}x - \frac{m}{n}(A_1x + A_2x^2 + A_3x^3 + \\ +A_4x^4 + A_5x^5 + \ldots) - \frac{3}{n}(B_1x + B_2x^2 + B_3x^3 + B_4x^4 + B_5x^5 + \ldots)[(ax^2 + \\ +b(A_1x + A_2x^2 + A_3x^3 + A_4x^4 + A_5x^5 + \ldots)^2 + c(B_1x + B_2x^2 + B_3x^3 + B_4x^4 + \\ +B_5x^5 + \ldots)^2 + 2dx(A_1x + A_2x^2 + A_3x^3 + A_4x^4 + A_5x^5 + \ldots) + 2ex(B_1x + B_2x^2 + \\ +B_3x^3 + B_4x^4 + B_5x^5 + \ldots) + 2f(A_1x + A_2x^2 + A_3x^3 + A_4x^4 + A_5x^5 + \ldots)(B_1x + \\ +B_2x^2 + B_3x^3 + B_4x^4 + B_5x^5 + \ldots))]. \end{gathered}$$

This implies that

$$A_1x + A_2x^2 + A_3x^3 + A_4x^4 + A_5x^5 + \ldots = -\frac{p + rB_1}{q}x - \frac{rB_2}{q}x^2 -$$

$$-\frac{3aA_1 + 3bA_1^3 + 3cA_1B_1^2 + 6dA_1^2 + 6eA_1B_1 + 6fA_1^2B_1 + rB_3}{q}x^3 - \frac{1}{q}(3aA_2 +$$

$$+9bA_1^2A_2 + 3cA_2B_1^2 + 6cA_1B_1B_2 + 12dA_1A_2 + 6eA_2B_1 + 6eA_1B_2 +$$

$$+12fA_1A_2B_1 + 6fA_1^2B_2 + rB_4)x^4 - \frac{1}{q}(3aA_3 + 9bA_1A_2^2 + 9bA_1^2A_3 + 3cA_3B_1^2 +$$

$$+6cA_2B_1B_2 + 3cA_1B_2^2 + 6cA_1B_1B_3 + 6dA_2^2 + 12dA_1A_3 + 6eA_3B_1 + 6eA_2B_2 +$$

$$+6eA_1B_3 + 6fA_2^2B_1 + 12fA_1A_3B_1 + 12fA_1A_2B_2 + 6fA_1^2B_3 + rB_5)x^5 + \ldots,$$

$$B_1x + B_2x^2 + B_3x^3 + B_4x^4 + B_5x^5 + \ldots = -\frac{mA_1 + s}{n}x - \frac{mA_2}{n}x^2 -$$

$$-\frac{3aB_1 + 3bA_1^2B_1 + 3cB_1^3 + 6dA_1B_1 + 6eB_1^2 + 6fA_1B_1^2 + mA_3)}{n}x^3 - \frac{1}{n}(6bA_1A_2B_1 +$$

$$+3aB_2 + 3bA_1^2B_2 + 9cB_1^2B_2 + 6dA_2B_1 + 6dA_1B_2 + 12eB_1B_2 + 6fA_2B_1^2 +$$

$$+12fA_1B_1B_2 + mA_4)x^4 - \frac{1}{n}(3bA_2^2B_1 + 6bA_1A_3B_1 + 6A_1A_2bB_2 + 3aB_3 + 3A_1^2bB_3 +$$

$$+9B_1B_2^2c + 9B_1^2B_3c + 6A_3B_1d + 6A_2B_2d + 6A_1B_3d + 6B_2^2e + 12B_1B_3e +$$

$$+6A_3B_1^2f + 12A_2B_1B_2f + 6A_1B_2^2f + 12A_1B_1B_3f + A_5m)x^5 + \ldots$$

From this identity we have

$$A_1 = \frac{rs - np}{nq - mr}, \quad B_1 = \frac{mp - qs}{nq - mr};$$

$$A_2 = 0, \quad B_2 = 0,$$

$$A_3 = -\frac{3}{nq - mr}(a + bA_1^2 + cB_1^2 + 2dA_1 + 2eB_1 + 2fA_1B_1)(nA_1 - rB_1),$$

$$B_3 = -\frac{3}{nq - mr}(a + bA_1^2 + cB_1^2 + 2dA_1 + 2eB_1 + 2fA_1B_1)(mA_1 - qB_1),$$

$$A_4 = 0, \quad B_4 = 0,$$

$$A_5 = -\frac{3}{nq - mr}(anA_3 + 3bnA_1^2A_3 + cnA_3B_1^2 + 2cnA_1B_1B_3 + 4dnA_1A_3 +$$

$$+2enA_1B_3 + 4fnA_1A_3B_1 + 2fnA_1^2B_3 - 2brA_1A_3B_1 - arB_3 - brA_1^2B_3 -$$

$$-3crB_1^2B_3 - 2drA_3B_1 - 2drA_1B_3 - 4erB_1B_3 - 2frA_3B_1^2 - 4frA_1B_1B_3),$$

$$B_5 = -\frac{3}{nq - mr}(-amA_3 - 3bmA_1^2A_3 - cmA_3B_1^2 - 2cmA_1B_1B_3 - 4dmA_1A_3 -$$
$$-2emA_3B_1 - 2emA_1B_3 - 4fmA_1A_3B_1 - 2fmA_1^2B_3 + 2bqA_1A_3B_1 + aqB_3 + bqA_1^2B_3 +$$
$$+3cqB_1^2B_3 + 2dqA_3B_1 + 2dqA_1B_3 + 4eqB_1B_3 + 2fqA_3B_1^2 + 4fqA_1B_1B_3), \ldots \quad (16)$$

Remark 3.1. For system (12) we have

$$L_{2,3} = nq - mr,$$

which, according to condition (2), is greater than zero.

Substituting (15) into the right-hand sides of critical differential equations (12), we get the following identity

$$3x(ax^2 + by^2 + cz^2 + 2dxy + 2exz + 2fyz) = C_1x + C_2x^2 + C_3x^3 + C_4x^4 + C_5x^5 + \ldots,$$

or in detailed form

$$3x[ax^2+b(A_1x+A_2x^2+A_3x^3+A_4x^4+A_5x^5+\ldots)^2+c(B_1x+B_2x^2+B_3x^3+B_4x^4+B_5x^5+\ldots)^2+$$
$$+2dx(A_1x+A_2x^2+A_3x^3+A_4x^4+A_5x^5+\ldots)+2ex(B_1x+B_2x^2+B_3x^3+B_4x^4+B_5x^5+\ldots)+$$
$$+2f(A_1x+A_2x^2+A_3x^3+A_4x^4+A_5x^5+\ldots)(B_1x+B_2x^2+B_3x^3+B_4x^4+B_5x^5+\ldots)] =$$
$$= C_1x + C_2x^2 + C_3x^3 + C_4x^4 + C_5x^5 + \ldots$$

From here, we obtain

$$C_1 = C_2 = 0, \quad C_3 = 3(a + bA_1^2 + cB_1^2 + 2dA_1 + 2eB_1 + 2fA_1B_1),$$
$$C_4 = 6(bA_1A_2 + cB_1B_2 + dA_2 + eB_2 + fA_2B_1 + fA_1B_2),$$
$$C_5 = 3(bA_2^2 + 2bA_1A_3 + cB_2^2 + 2cB_1B_3 + 2dA_3 + 2eB_3 + 2fA_3B_1 + 2fA_2B_2 +$$
$$+2fA_1B_3), \ldots \quad (17)$$

According to Lyapunov theorem [1], we have

Theorem 3.1. *The stability of the unperturbed motion, described by Darboux type critical system* (12) *of the perturbed motion, includes all possible cases in the following three:*

$$\textbf{\textit{I.}}\ \ a + bA_1^2 + cB_1^2 + 2dA_1 + 2eB_1 + 2fA_1B_1 < 0, \quad (18)$$

then unperturbed motion is ***stable****;*

$$\textbf{\textit{II.}}\ \ a + bA_1^2 + cB_1^2 + 2dA_1 + 2eB_1 + 2fA_1B_1 > 0, \quad (19)$$

then unperturbed motion is ***unstable****;*

$$\textbf{\textit{III.}}\ \ a + bA_1^2 + cB_1^2 + 2dA_1 + 2eB_1 + 2fA_1B_1 = 0, \quad (20)$$

then unperturbed motion is ***stable****.*

In the last case, the unperturbed motion belongs to some continuous series of stabilized motions. Moreover, for sufficiently small perturbations, any perturbed motion will asymptotically approach to one of the stabilized motions of the mentioned series. The expressions A_1, B_1 *are given in* (16).

Proof. According to Lyapunov theorem [1, §32], we analyze the coefficients of the series (17). The stability or the instability of the unperturbed motion is determined by the sign of expression C_3, and we get the Cases I and II.

Therefore, if $C_3 = 0$, then all $A_i = B_i = 0$ $(\forall i)$, so we get case III of this theorem. □

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