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Dedicated to the memory of Academician Mitrofan M. Cioban (1942-2021)

Singular integral operators in the case of a piecewise Lyapunov contour

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Abstract. The article attempts to present the results obtained by the author in recent years (in a number of cases with some additions) on the theory of singular integral equations and Riemann boundary value problems in the case of a piecewise Lyapunov contour. It has a survey character of problems and results related to the influence of corner points of integration contour on various properties of singular operators. At the same time, much attention is paid to the research of other authors related to the scientific interests of the author of this work.

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Keywords: singular integral operators, noetherian operators, piecewise Lyapunov contour.

Operatori integrali singulari. Cazul conturului de tip Lyapunov pe porţiuni

Rezumat. În lucrarea de faţă se încearcă o expunere a rezultatelor obţinute de către autor în ultimii ani (în mai multe cazuri, cu unele completări) în teoria ecuaţiilor integrale singulare şi a problemelor la frontieră de tip Riemann în cazul conturului Lyapunov pe porţiunii. Lucrarea are un caracter de sinteză a problemelor şi rezultatelor ce ţin de influenţa punctelor unghiulare ale conturului de integrare asupra diferitor proprietăţi ale operatorilor singulari. Totodată, o atenţie deosebită este acordată cercetărilor altor autori legate de interesele ştiinţifice ale autorului acestei lucrări.

Cuvinte cheie: operatori integrali singulari, operatori noetherieni, contur Lyapunov pe porţiuni.

1. Introduction

The classical theory of singular integral operators of the form

$$A = aP + bQ, \tag{1}$$

where $P = (I + S)/2$, $Q = I - P$ and S is a singular operator with Cauchy kernel, is closely connected with boundary problems of the theory of analytic functions, mainly with the problem of linear conjugation,

$$\Phi^{+}(t) = G(t)\,\Phi^{-}(t) + g(t)\,. \tag{2}$$

For the first time this connection was discovered by Carleman, and he outlined the idea how to construct the explicit solution of problem (2) with the help of integrals of Cauchy type and Sohotsky-Plemely formulas. Investigations by Carleman played an important role because, as it was found out afterwards, his works contained ideas which had far-reaching generalizations and received fast and complete developments. This scientific direction was summarized by well-known monographs of N. Muskhelishvili [1], F. Gakhov [2], I. Vekua [3], F. Gakhov and Iu. Chersky [4], containing also large list of literature.

Under the assumption that the coefficients a, b are piesewise Holder and nondegenerated in book [1] a complete investigation of operators (1) in Muskhelishvili classes is given. In this case the equation is $A\phi = f$ and the corresponding boundary problem of linear conjugation (2) permits an effective solution in explicit form, which allows to obtain a full picture of the solvability of the equation in H_{μ}. These results were extended by I. Vekua to the matrix case under the assumption that at each node of a curve not more than two are met. Here also the case is considered where the operator

$$B = PaI + QbI$$

is taken as the operator A.

Retaining the condition that the coefficients a, b are piecewise Holder, the above mentioned theory of operators along Lyapunov curves has been transferred by B. Khvedelidze [5] to the case of space L_p with weight

$$\rho(t) = \prod_{k=1}^{n} |t - t_k|^{\beta_k} \quad (-1 < \beta_k < p - 1,\ t_k \in \Gamma)\,.$$

The case, where the coefficients a, b are only piecewise continuous required, a special approach, because the application of methods of the theory of functions encountered serious difficulties. First results in this direction belong to S. Mikhlin [6], I. Gohberg [7] etc. These methods are based, on one hand, on the idea of the factorization of elements of Banach algebra, originating in M. Krein work [8] about Wiener-Hopf equations on a semi-axis, on the other hand, on the use of so-called local principle. This principle was introduced by I. Simonenko [9]. I. Gohberg and N. Krupnik [10] obtained criteria of Noetherian property and a formula for the index of elements of minimal algebra generated by operators (1) in the space $L_p(\Gamma, \rho)$. Using the same method the analogous results were obtained by R. Duduchava [11] for the space H_{μ} with weight. Under various assumptions problem (2) was studied by F. Gahov, D. Kveselava, I. Vekua, G. Mandjavidze and B. Khvedelidze, I. Simonenko, L. Chibrikova (see [2], [5] and references below).

The first works, where the study of equation (1) and problem (2) with measurable bounded coefficients along piecewise Lyapunov curves and curves with self-intersection curves have begun, seem to be [12]–[18]. Note that we have found rather large classes of coefficients which ensure the Noetherian property of operator (1). These classes are given in Section 3 of the present work. In a series of works by V. Paatashvili and V. Kokilashvili the equation of admissible curves for which the Noether theory remains applicable were also considered. Vast literature is devoted to the equations of boundedness of operator (1) in various spaces (see [5], [11], [20]–[22]).

There is a large number of works (see [23]–[26]) and references that devoted to the investigation of normal solvability and to the evaluation of index of different classes of equations of contraction type, singular integral and others – containing terms with a reflection or a shift, with complex conjugation etc. For singular integral equations, in the case when some iteration of shifts is the identity transformation, the general theory was constructed by G. Litvinchuk and V. Kravchenko [23]. It is based on the idea of the elimination of terms with shift by means of transition to a system of equations. The equations which has been investigated has the form

$$\left(A_0 + VA_1 + \cdots + V^{n-1}A_{n-1}\right)\phi = f, \tag{3}$$

where V is a generalized involuntary operator and operators A_j belong to certain class of operators, theory of which is already well known. Some classes of such equations were investigated in works by Z. Khalilov, Yu. Chersky, S. Samko, N. Karapetiants and S. Samko [23]. However, in these works the investigation is, in fact, restricted to the frameworks of the theory of singular equations without shift, embracing only the cases where the coefficients A_j in (3) are "invariant" with respect to V:

$$A_jV = VA_j + T_j,$$

where T_j are completely continuous. But in the case of equations with shift and a piecewise Lyapunov contour it is, in general, not true [19]. As was shown in [20]–[22], in the case of a piecewise Lyapunov contour the corresponding singular operator and operator with shift (3) are not simultaneously Noetherian. Namely, the Noetherian property of corresponding operator implies the Noetherian property of initial operator, but the converse is already not true. The violation of simultaneous Noetherian property of the given and corresponding operator is a result of discontinuity of the derivative of shift function $\alpha(t)$ at corner points of the contour.

The further stage in development of the theory was the study of operators $AP + BQ$ in the case, when A and B are generated by shifts of non-Carlemanian type or by any finite group of shifts. Corresponding results were obtained independently by Yu. Karlovich, V. Kravchenko, A. Myasnikov, L. azonov, A. Antonevich, A. Nechaev, A. Soldatov (see [23] and bibliography there). In connection with the mentioned above works we note that in some of them the problem of construction of the Noether theory is studied in a very general statement, but then the infinite smoothness of contour and coefficients are assumed, and the index of operator is expressed via characteristics, the practical calculation of which is rather difficult.

A large number of works are devoted to the theory of singular integral equations with discontinuous coefficients. We shall dwell mainly on the monographs [1]–[5], [10], [11], [13], [19], [22], [26] which contain history of the question and detailed bibliography. As to operators (1) that do not satisfy the condition of Hausdorff normal solvability, see the book by Z. Presdorf [20].

2. On the essential norm of singular operators

Let Γ be a piecewise Lyapunov contour with a finite number of self-intersection points. In 1927 M. Riesz proved the boundedness of the operator

$$(S_\Gamma\phi)(t) = \frac{1}{\pi i}\int\limits_\Gamma \frac{\phi(\tau)}{\tau - t}d\tau \qquad (t \in \Gamma) \tag{4}$$

in the space $L_p(\Gamma_0)$ $(\Gamma_0 = \{z \in C : |z| = 1\})$. Then G. Hardy and J. Littlewood and K. Babanko transferred this result to the spaces $L_p(\mathrm{R}, \rho)$ with weight $\rho(x) = |x|^\alpha$ $(1 < p < \infty,\ -1 < \alpha < p - 1)$. In work [5] B. Khvedelidze proved the boundedness of operator S in the space $L_p(\Gamma, \rho)$ for an arbitrary Lyapunov contour Γ and the weight

$$\rho(t) = \prod_{k=1}^{n} |t - t_k|^{\beta_k} \quad (t_k \in \Gamma, 1 < p < \infty, -1 < \beta_k < p - 1). \tag{5}$$

E. Gordadze [27] transferred this result to an arbitrary piecewise Lyapunov contour without cusps. Using this result one can prove the boundedness of the operator S in the case of a composite contour with a finite number of self-intersection points in the space L_p with weight (5). The condition $-1 < \beta_k < p - 1$ is necessary for the boundedness of the operator S in the space $L_p(\Gamma, \rho)$. It is confirmed by the following

Lemma 2.1. *Let S be bounded in $L_p(\Gamma, \rho)$, then $\rho^{1/p} \in L_p(\Gamma)$ and $\rho^{-1/p} \in L_q(\Gamma)$* $\left(p^{-1} + q^{-1} = 1\right)$.

Proof. The boundedness of the operator S in the space $L_p(\Gamma,\rho)$ implies the boundedness in $L_p(\Gamma)$ of the operator $R = \pi i \rho^{1/p}(RS - SR)\rho^{-1/p}I$, where $(R\phi)(t) = \frac{1}{t-z_0}\phi(t)$. And $z_0 \notin \overline{\Gamma}$. But

$$(R\phi)(t) = \rho^{\frac{1}{p}}(t)\frac{1}{t-z_0}\int_\Gamma \frac{\rho^{\frac{-1}{p}}(\tau)\phi(\tau)}{\tau - z_0}d\tau.$$

Therefore $\rho^{\frac{1}{p}} \in L_p(\Gamma)$ and $\rho^{\frac{-1}{p}} \in L_q(\Gamma)$. □

Corollary 2.1. *If the operator S is bounded in the space* $L_p(\Gamma,\rho)$, $(\rho(t) = \prod_{k=1}^n |t-t_k|^{\beta_k})$, *then from the above relation* $\rho^{1/p} \in L_p(\Gamma)$, *and* $\rho^{-1/p} \in L_q(\Gamma)$, *it follows that the numbers* β_k *verify the inequalities*

$$-1 < \beta_k < p-1,\ k = 1,2,\dots,n.$$

Remark 2.1. If the contour of integration Γ is unbounded, then operator S is continuous at $L_p(\Gamma,\rho)$ if and only if $-1 < \beta_k < p-1$ and $-1 < \sum_{k=1}^n \beta_k < p-1$.

In [10] it was shown that the norm $\|S_0\|_p$ in the space $L_p(\Gamma_0)$ $(\Gamma_0 = \{t : |t| = 1\})$ for $p = 2^n$ and $p = 2^n(2^n-1)^{-1}$ is equal to $\nu(p)$, where

$$\nu(p) = \begin{cases} ctg\pi/2p & if \quad 2 \le p \le \infty, \\ tg\pi/2p & if \quad 1 < p \le 2. \end{cases}$$

After this result had been obtained, a series of works appeared where the norms of singular operators in various spaces were evaluated and calculated. In [28] the norm of the operator

$$(C\phi)(t) = \frac{1}{2\pi}\int_0^{2\pi}\phi(y)ctg\frac{y-t}{2}dy.$$

was calculated. It turned out that $\|C\|_p = \nu(p)$ (in the space $L_p(0,2\pi)$. This permitted to prove that $\|S_0\|_p = \nu(p)$ for any $p \in (1,\infty)$.

In monograph [10] the estimation of the essential norm $|S_\Gamma|_{P,\rho} = \inf_{T\in I}\|S_\Gamma + T\|_{L_p(\Gamma,\rho)}$ in the space $L_p(\Gamma,\rho)$ in the case of a Lyapunov contour was obtained. In the book of F. Zigmund it is proved that $\|S_\mathrm{R}\|_p \le \|C\|_p$, therefore, $\|S_\mathrm{R}\|_p = \nu(p)$. Taking also in consideration the equality $\|S_\mathrm{R}\|_p = \|S_0\|_{p,|t-t_0|^{p-2}}$ (see [29]–[31]) and M. Riesz interpolation theorem, we obtain that if $min(0,p-2) \le \beta \le max(0,p-2)$, then $\|S_0\|_{p,|t-t_0|^\beta} = \|S_0\|_p$.

Further it was found out that the norms of operator S in the spaces $L_p(\Gamma)$ and $L_p(\Gamma,\rho)$ play an important role in different problems (see, for example, [10], [30]–[32]).

In works [13], [30] exact constants for the factor-norms of singular operators S_α, P_α are established. Suppose that Γ_α has one corner point with angle $\pi\alpha$ $(0 < \alpha \le 1)$, then

$|S_\alpha|_2 = ctg\theta(\alpha)/2, |P_\alpha|_2 = |Q_\alpha|_2 = (\sin\theta(\alpha))^{-1}$, where $S_\alpha = S_{\Gamma_\alpha}$ and

$$ctg\theta(\alpha) = \frac{1}{2}\max_{-1\le x\le 1}\left|(1+x)\left(\frac{1-x}{1+x}\right)^{\alpha/2} - (1-x)\left(\frac{1+x}{1-x}\right)^{\alpha/2}\right|. \tag{6}$$

In particular, $\left|S_{1/3}\right|_2 = \frac{1+\sqrt{5}}{2}$ and $\left|S_{1/2}\right|_2 = \sqrt{2}$.

Theorem 2.1. *Let Γ be a piecewise Lyapunov contour with corner points $t_1,\ldots,t_n$ and $\rho(t) = \prod_{k=1}^n |t-t_k|^{\beta_k}$ $(-1<\beta_k<1)$, then $|S_\Gamma|_{L_2(\Gamma,\rho)} = \max\limits_{1\le k\le n}\left|S_{\alpha_k}\right|_{L_2(\Gamma_{\alpha_k},|t|^{\beta_k})}$. Let $\min\limits_{1\le k\le n}(\alpha_1,\ldots,\alpha_n) = \alpha_{k_0}$. If $\alpha_{k_0} = 1$, then $|S_\Gamma|_{L_2(\Gamma,\rho)} = \max\limits_{1\le k\le n} ctg\pi\frac{1-|\beta_k|}{4}$. If $\rho(t)\equiv 1$, then $|S_\Gamma|_{L_2(\Gamma)} = ctg\frac{\theta(\alpha_{k_0})}{2}$. For the operators P_Γ and Q_Γ the equalities*

$$|P_\Gamma| = |Q_\Gamma| = (|S_\Gamma|^2+1)/2\,|S_\Gamma|.$$

hold. In the space $L_p(\Gamma)$ the estimates

$$|S_\Gamma|_p \leqslant \begin{cases} \operatorname{ctg}\dfrac{\theta\left(\alpha_{k_0}\right)}{p}, & \text{if } p = 2^n,\\ \operatorname{ctg}^t\dfrac{\theta\left(\alpha_{k_0}\right)}{2^n}\cdot\operatorname{ctg}^{1-t}\dfrac{\theta\left(\alpha_{k_0}\right)}{2^{n+1}}, & \text{if } 2^n < p < 2^{n+1},\end{cases}$$

where $t = (2^{n+1}-p)/p$ are valid.

Remark 2.2. Equality $|P_\Gamma| = |Q_\Gamma| = (|S_\Gamma|^2+1)/2\,|S_\Gamma|$ confirms the following hypothesis of the mathematician S. Marcus: let B be some Banach space and L_1, L_2 subspaces from B such that $L_1\cap L_2 = \emptyset$ and $B = L_1 + L_2$, then equality

$$|P| = |Q| = \frac{|S|^2+1}{2\,|S|}$$

takes place, where P and Q are projectors projecting the space B onto L_1, respectively, on L_2 and $S = P - Q$.

Consider the case when Γ has selfintersection points. To formulate one result we introduce some notations, which will be also used further. Let Γ be a composite contour consisting of m simple piecewise Lyapunov closed curves $\gamma_1,\ldots,\gamma_m$, which have a point t_0 in common,

$$h_k = p(1+\beta_k)^{-1}\ \ (k = 1,2,\ldots,n),\ \ h_{n+1} = p,$$
$$\overline{h}_k = max(h_k, h_k(h_k-1)^{-1})\ \ (k = 1,2,\ldots,n+1)\ \text{ and }\ h = max(\overline{h}_1,\overline{h}_2,\ldots,\overline{h}_{n+1}).$$

Theorem 2.2. *For the essential norm of operators P_Γ, Q_Γ and S_Γ in the space $L_p(\Gamma,\rho)$ the following estimates are true:*

$$|P_\Gamma|_{p\rho},\ |Q_\Gamma|_{p\rho} \ge max((\sin\pi/h)^{-1}, (\sin\pi/m\overline{h}_0)^{-1}),$$

$$|S_\Gamma|_{p\rho} \geq max(ctg\pi/2h, ctg\pi/2m\overline{h}_0). \tag{7}$$

These estimates are in concordance with the corresponding results from [26], [35]–[38] and embrace all the cases of boundedness of operator S_Γ in $L_p(\Gamma, \rho)$. Remark that for one class of contours the above estimates are exact. So, if the tangents to Γ at selfintersection points are perpendicular and $\rho\,(t) \equiv 1$, then (7) becomes an equality.

Let Γ be a simple closed piecewise Lyapunov contour which bounds a domain G_Γ^+, ω be Riemann function mapping G_Γ^+ into $G_\Gamma^+ = \{z : |z| < 1\}$ and $t_1, \ldots, t_n$ be all the corner points of contour Γ with angles $\alpha_k\pi$ $(0 < \alpha_k \leq 1)$.

Theorem 2.3. *Operator*

$$(K\phi)\,(t) = \frac{1}{\pi i}\int_\Gamma \left(\frac{\omega'\,(\tau)}{\omega\,(\tau) - \omega\,(t)} - \frac{1}{\tau - t}\right)\phi\,(\tau)\,d\tau$$

is compact (see [30]) in the space $L_p(\Gamma, \rho)$, if and only if $\sum_{k=1}^{l} \alpha_k = l$.

Theorem 2.4. *The operator S_Γ^* acting in the space $L_q\left(\Gamma, \rho^{1-q}\right)$ has the form (see [10])*

$$S_\Gamma^* = -VhSVhI,$$

where $(V\phi)\,(t) = \overline{\phi\,(t)}$ and h is a piecewise Hölder function on Γ.

Theorem 2.5. *The operator $S_\Gamma^* - S_\Gamma$ is compact in the space $L_2\,(\Gamma)$ if and only if (see [30])*

$$\sum_{k=1}^{l} \alpha_k = l.$$

3. Factorization. Noether theorem

In the theory of singular integral equations with measurable bounded coefficients an important role is played by the assertion that if a function g can be represented in the form $g = ha$, where $h^\pm \in L_\infty\,(\Gamma)$, and the range of function a lies inside an angle with vertex zero and value less than $\frac{\pi}{2r}$ $(2 \leq r \leq \infty)$, then [8] the singular operator $gP + Q$ is inevasible in the spaces $L_p\,(\Gamma)$ for all $p \in \left[r\,(r-1)^{-1}, r\right]$. This assertion was generalized in different directions, and each such application found application in the Noether theory of singular equations. However, the question remains to be open, whether it is possible to transfer these results to the case of a piecewise Lyapunov contour and what additional conditions one has to impose on the function a in order that the operator $gP + Q$ be inevasible in spaces with weight. These questions were considered in the author's works

[13], [15], [16], [32]–[34]. To expound some results of these works we shall introduce several notations.

Let Γ be a closed composite piecewise Lyapunov contour which bounds a domain G_Γ^+. By G_Γ^- we denote the domain which complements $G_\Gamma^+ \cup \Gamma$ to the full plane. Assume that $0 \in G_\Gamma^+$ and $\infty \in G_\Gamma^-$. Let $\mathbf{B}^{mxm}$ be the set of square matrices of order m with elements from $\mathbf{B}$; $P_\Gamma = \left\| \frac{\delta_{ij}(I+S_\Gamma)}{2} \right\|_{i,j=1}^m$, $Q_\Gamma = I - P_\Gamma$; ${}^+L_p^m(\Gamma,\rho) = P_\Gamma L_p^m(\Gamma,\rho)$; ${}^-L_p^m(\Gamma,\rho) = Q_\Gamma L_p^m(\Gamma,\rho) + \mathbb{C}$ ($\mathbb{C}$ is the set of complex numbers).

1⁰. Class $Fact_{p\rho}^m(\Gamma)$**.** The generalized factorization of a matrix a $\left(\in GL_\infty^{mxm}(\Gamma)\right)$ with respect to contour Γ in the space $L_p^m(\Gamma,\rho)$ is defined (see [10], [34], [35]). Its representation has the form

$$a = a_- D a_+, \tag{8}$$

where $D = \left\|\delta_{jk}(t-z_0)^{\kappa_j}\right\|_1^m$, $j,i \le 1$, $z_0 \in G_\Gamma^+$; κ_j are integers, $(\kappa_1 \ge \kappa_2 \ge \ldots \ge \kappa_m)$, and the factors $a_\pm$ satisfy the following conditions:

(i) $a_- \in {}^-L_p^m(\Gamma,\rho)$; $a_+ \in {}^+L_p^m\left(\Gamma,\rho^{1-q}\right)$; $a_-^{-1} \in {}^-L_p^m\left(\Gamma,\rho^{1-q}\right)$ and $a_+^{-1} \in {}^+L_p^m(\Gamma,\rho)$; $\left(p^{-1}+q^{-1}=1\right)$;

(ii) the operator $a_+^{-1} P_\Gamma a_+ I$ is bounded in the space $L_p^m(\Gamma,\rho)$.

The set of all matrix–functions a $\left(\in GL_\infty^{mxm}(\Gamma)\right)$ admitting generalized factorization with respect to contour Γ in the space $L_p^m(\Gamma,\rho)$ will be denoted by $Fact_{p\rho}^m(\Gamma)$.

2⁰. Class $Nt_{p\rho}^m(\Gamma)$**.** We regard measurable essentially bounded matrix-functions $a(t)$ as belonging to the Noether class (denoted by $Nt_{p\rho}^m(\Gamma)$), if the operator $A = aP_\Gamma + Q_\Gamma$ is Noetherian.

Theorem 3.1. $Nt_{p\rho}^m(\Gamma) = Fact_{p\rho}^m(\Gamma)$ *(see [34]).*

3⁰. The connection between $Fact_{p\rho}^m(\Gamma)$ ***and*** $Fact_p^m(\Gamma)$.

Let h be a function from $Fact_p^m(\Gamma)$ such that

$$h = h_- \cdot h_+.$$

Denote $\mathbf{B}_1 = L_p^m(\Gamma)$ and $\mathbf{B}_2 = L_p^m\left(\Gamma,\rho\,|h_+|^{-p}\right)$, $(1 < p < \infty)$.

Theorem 3.2. *There exist inversible operators* $B : \mathbf{B}_2 \to \mathbf{B}_1$ *and* $C : \mathbf{B}_1 \to \mathbf{B}_2$ *such that for any pair of matrix-functions* $a, b \in L_\infty^{mxm}(\Gamma)$ *the equality*

$$B\left(aP_\Gamma + bQ_\Gamma\right)C = haP_\Gamma + bQ_\Gamma \tag{9}$$

holds

The proof is contained in [12].

Corollary 3.1. *The operator $A = aP_\Gamma + Q_\Gamma$ $(a \in L_\infty^{mxm}(\Gamma))$ is Noetherian in the space $L_p^m(\Gamma, |h_+|^{-p})$ if and only if the operator $A_h = haP + bQ_\Gamma$ possesses the same property in the space $L_p^m(\Gamma)$. Then*

$$dimkerA|_{L_p^m(\Gamma,|h_+|^{-p})} = dimkerA_h|_{L_p^m(\Gamma)},$$

$$dimkerA^*|_{L_p^m(\Gamma,|h_+|^{-p(1-q)})} = dimkerA_h^*|_{L_q^m(\Gamma)}.$$

Corollary 3.2. $a \in Fact^m_{p\rho|h_+|^{-p}}(\Gamma) \Leftrightarrow ah \in Fact^m_{p\rho}(\Gamma)$.

Show (see [32]–[34]), that Theorem 3.2 permits to reduce the investigation of the operator $A = aP_\Gamma + bQ_\Gamma$ in the space $\rho(t) = \prod_{k=1}^{n} |t - t_k|^{\beta_k}$ to the investigation of some singular operator in the space $L_p^m(\Gamma)$ (without weight). For simplicity we assume that Γ consists of ν closed curves $\Gamma_1, ..., \Gamma_\nu$ having a point t_0 in common and

$$\rho(t) = \prod_{k=0}^{n} |t - t_k|^{\beta_k} \quad (t_k \in \Gamma, 1 < p < \infty, -1 < \beta_k < p - 1). \tag{10}$$

Denote by Γ_{i_k} $(1 \le i_k \le \nu)$ the curve containing point t_k and set

$$h_k(t) = \begin{cases} (t - z_k)^{-\frac{\beta_k}{p}} & \text{for } t \in \Gamma_{ik}, \\ 1 & \text{for } t \in \Gamma \backslash \Gamma_{ik}, \end{cases}$$

where z_k is a point of the domain $G^+_{i_k}$, bounded by the curve Γ_{i_k}, and $(t - z_k)^{\frac{-\beta_k}{p}}$ is a branch if this function continuous at any point $t \in \Gamma_{i_k}$ different from t_k.

Let $\omega_1, \ldots, \omega_\nu$ be some points belonging, respectively, to the domains $G_1^+, \ldots, G_\nu^+$; $\sigma_1, ..., \sigma_\nu$ be some real numbers and $\widetilde{h}_k(z)$ be a fixed branch of the function $(z - \omega_k)^{\sigma_k}$ defined on the complex plane $\mathbb{C}$ with a cut which joins z_k and ∞, and intersects the contour Γ in one point t_0. The functions $\widetilde{h}_k(t)$ $(k = 1, ..., \nu)$ are continuous at any point $t \in \Gamma$, perhaps, except the point t_0: $\widetilde{h}_k(t_0 \pm 0) \neq 0$ and

$$\widetilde{h}_k(t_0 - 0) / \widetilde{h}_k(t_0 + 0) = \exp(2\pi i \sigma_k), \tag{11}$$

where the numbers $\widetilde{h}_k(t_0 - 0)$ and $\widetilde{h}_k(t_0 + 0)$ are determined by equalities 3.1 of Chapter 10 of work [10] . By $h(t)$ we denote the product

$$h(t) = h_1(t) \cdot \ldots \cdot h_n(t) \widetilde{h}_1(t) \cdot \ldots \cdot \widetilde{h}_\nu(t). \tag{12}$$

Theorem 3.3. *(see [29]) Let $\sum_{k=1}^{\nu} \sigma_k = \frac{-\beta_0}{p}$. Then*

$$a \in Fact^m_{p\rho}(\Gamma) \Leftrightarrow ah \in Fact^m_p(\Gamma).$$

Corollary 3.3. *Let $A = aP_\Gamma + bQ_\Gamma$, h be a function determined by equality (12) and $A_h = ahP_\Gamma + bQ_\Gamma$. Then $A \in Nt^m_{p\rho}(\Gamma) \Leftrightarrow A_h \in Nt^m_p(\Gamma)$. Moreover $IndA = IndA_h$.*

4*[0]. *Class $M_\rho(\Gamma)$ (see [17]). In this and next items we assume that Γ consists of two curves Γ_1 and Γ_2, having one common point t_0, and besides the tangents to Γ at this point are pependicular and $\rho(t)$ is the function determined by the equality (10).

Let τ_0 be a point on Γ different from t_0. Denote by $\Lambda(\tau_0)$ the closed halfplane which does not contain the origin. By $\Lambda(\tau_0)$ we denote the angle with vertex at the origin and value of $\frac{\pi}{2}$. To the class $M_\rho(\Gamma)$ we refer essentially bounded measurable functions $a(t)$ satisfying the conditions:

(i) *essinf* $|a(t)| > 0;\ t \in \Gamma$;

(ii) for any point $\tau \in \Gamma \setminus \{t_0\}$ there exist a neighborhood $u(\tau)$ $(\subset\Gamma \setminus \{t_0\})$ of the point τ and a pair of functions $g_\tau^\pm(t)$ such that $\left(g_\tau^+(t)\right)^{\pm 1} \in L_\infty^+(\Gamma)$, $\left(g_\tau^-(t)\right)^{\pm 1} \in L_\infty^-(\Gamma)$ and the range of the function $g_\tau^+(t)\,h(t)\,a(t)\,g_\tau^-(t)$ at $t \in u(\tau)$ is contained inside $\Lambda(\tau)$;

(iii) for the point t_0 either there exist a neighborhood $u(t_0)$ and functions $g_0^\pm(t)$ such that $\left(g_0^\pm(t)\right)^{\pm 1} \in L_\infty^\pm(\Gamma)$ and the range of the function $g_0^+(t)\,h(t)\,a(t)\,g_0^-(t)$ at $t \in u(t_0)$ is contained inside $\Lambda(t_0)$, or there exist finite limits $a(t_0 \pm 0)$ and

$$h_{t_0}(t_0 - 0)\,a(t_0 - 0)\,/h_{t_0}(t_0 + 0)\,a(t_0 + 0) \notin (-\infty, 0)\,.$$

Theorem 3.4. $M_\rho(\Gamma) \subset Fact_{2\rho}(\Gamma)$.

Corollary 3.4. *Let* $a \in M_\rho(\Gamma)$. *Then the operator* $A = aP + Q$ *is Noetherian in the space* $L_2(\Gamma, \rho)$.

Corollary 3.5. $M_\rho(\Gamma) \cap PC(\Gamma) = Fact_\rho(\Gamma) \cap PC(\Gamma)$, *where* $PC(\Gamma)$ *is a set of all piecewise continuous functions on* Γ.

Note that if Γ is a simple closed Lyapunov contour and $\rho(t) \equiv 1$, then the class $M_1(\Gamma)$ coincides with the class $\widetilde{A}(2,\Gamma)$ introduced by I.B. Simonenko (see [32]). In this case, as it is known (see [32]), $aP + bQ$ is Noetherian if and only if $a \in \widetilde{A}(2,\Gamma)$ $(= M_1(\Gamma))$. From this and Theorem 3.4 it follows

Theorem 3.5. $M_\rho(\Gamma) = Nt_{2\rho}(\Gamma)$.

5*[0]. *Class $M_\rho^m(\Gamma)$ (see [17]). To class $M_\rho^m(\Gamma)$ we refer matrix-functions $a(t) = \left\|a_{jk}(t)\right\|_{j,k}^m$ $(t\in\Gamma)$ of order m with elements $a_{jk} \in L_\infty(\Gamma)$ satisfying the conditions:

(i) *essinf* $|deta(t)| > 0$ $(t \in \Gamma)$;

(ii) for any point $\tau \in \Gamma$ except, perhaps, a finite number of points t_0, τ_k $(k = 1, ..., l)$, there exists a neighborhood $u(\tau)$ $(\subset \Gamma)$ of the point τ and a pair of matrix-functions $g_\tau^\pm$ such that $\left(g_\tau^+(t)\right)^{\pm 1} \in {}^+L_\infty^{m\times m}(\Gamma)$, $\left(g_\tau^-(t)\right)^{\pm 1} \in {}^-L_\infty^{m\times m}(\Gamma)$ and for any $t \in u(\tau)$

$$\operatorname{Re}\left(g_\tau^+(t)h_\tau(t)a(t)g_\tau^-(t)\right) \geq \sigma(\tau) > c(\tau)\cos\theta(\tau),$$

where $\operatorname{Re} B = (B + B^*)/2$, B^* is the matrix conjugate to B, $\theta(\tau)$ is the function from Theorem 2.1 and $c(\tau)$ is the norm of operator $h_\tau aI$ in the space $L_2^m(u(\tau))$. Note that $c(\tau)$ coincides with $\sup\limits_{t\in u(\tau)} s_1\left(h_\tau\left(t\right) a\left(t\right)\right)$, where $s_1(h_\tau a)$ is the greatest eigenvalue of the matrix $(h_\tau a, a^* h_\tau^*)^{1/2}$;

(iii) there exist finite limits $a_{2j}\,(a_{2j-1})$ of matrix-function $a\,(t)$ as t tends to t_0 along arc Γ_j $(j = 1, 2)$ directed to point t_0 (from point t_0 and the spectrum of matrix $e^{-\pi i\beta_0} a_4 a_3^{-1} a_2 a_1^{-1}$ does not intersect the negative semi-axis R^-;

(iv) at points τ_k there exist finite limits on the left and on the right $a(\tau_k - 0)$ and $a(\tau_k + 0)$ of matrix $a\,(t)$ and the spectrum of matrix $e^{-\pi i\beta(\tau_k)} a^{-1}(\tau_k + 0) a(\tau_k - 0)$ does not intersect the negative semi-axis R^-.

It is easy to see that if all the points τ_k $(k = 1, ..., l)$ at which there exist limits $a(\tau_k \pm 0)$ are ordinary (see [1], p.16), then the conditions (iv) are equivalent to conditions (ii). This can be also deduced from Lemma 3.1 of work [17].

Theorem 3.6. *Let the matrix-function a belong to the set $M_\rho^m(\Gamma)$, then the operator $A = aP + Q$ is Noetherian in the space $L_2^m(\Gamma, \rho)$.*

Remark that, for Γ being a closed Lyapunov contour and $\rho\,(t) \equiv 1$, this theorem was proved by I.B. Simonenko (see [34], Theorem 8), and the set $M_1^m(\Gamma)$ coincides with class $\widetilde{A}^m(2, \Gamma)$ introduced in [34]. For Γ, being a simple Lyapunov contour, Theorem 3.6 is contained in the work of N.Ia. Krupnik [38]. Note also that from Theorem 3.6 it follows that if $a, b \in L_\infty^{m\times m}(\Gamma)$, $essinf|det b(t)| \neq 0$ $(t \in \Gamma)$ and $b^{-1}a \in M_\rho^m(\Gamma)$, then the operator $aP + bQ$ is Noetherian in the space $L_2^m(\Gamma, \rho)$.

Corollary 3.6. $M_\rho^m\,(\Gamma) \subset Fak\,t_{2\rho}^m\,(\Gamma)$.

Theorems 3.5 and 3.6 are transferred, with corresponding changes, to the case of an unclosed contour.

6^0. Class $G_{\delta\rho}\,(\Gamma)$**.** Let Γ be a closed piecewise Lyapunov contour with corners $t_1, ..., t_n$ and $\rho\,(t)$ be the weight determined by the equality (10). By $G_{\delta\rho}\,(\Gamma)$ denote the set of all matrices a of $L_\infty^{m\times m}(\Gamma)$ satisfying the following conditions:

(i) $essinf\,|det a(t)| > 0$ $(\,t \in \Gamma)$;

(ii) for each point $\tau \in \Gamma \setminus \{t_1, ..., t_s\}$ $(s \geq n)$, there exist a neighborhood $u(\tau)(\subset \Gamma)$ of point τ and a pair of matrix-functions $g_\tau^\pm$ such that $\left(g_\tau^+(t)\right)^{\pm 1} \in {}^+L_\infty^{m\times m}\,(\Gamma)$, $\left(g_\tau^-(t)\right)^{\pm 1} \in {}^-L_\infty^{m\times m}\,(\Gamma)$, and for any $t \in u(\tau)$, the matrix $g_\tau^\pm a g_\tau^-$ is unitary and $\sigma\left(g_\tau^- a g_\tau^+\right) \subset \Lambda_\tau\,(\delta)$, where $\Lambda_\tau\,(\delta)$ $(0 < \delta < 2\pi)$ denoted angle with vertex of the point $z = 0$ and value less than δ;

(iii) for the points t_k $(k = 1, ..., n)$ there exist finite limits $a(t_k \pm 0)$ and

$$\det\left(f_k(\mu)\, a(t_k + 0) + (1 - f_k(\mu))\, a(t_k - 0)\right) \neq 0 \quad (0 \leq \mu \leq 1), \quad \text{where}$$

$$f_k(\mu) = \begin{cases} \frac{\sin\theta_k\mu \exp(i\theta_k\mu)}{\sin\theta_k \exp(i\theta_k)}, & \theta_k = \pi - 2\pi(1+\beta_k)/p, \quad if\ \theta_k \neq 0, \\ \mu, & if\ \theta_k = 0; \end{cases}$$

(iv) there exist a neighborhood $u(t_k)$ $(k = n+1, ..., s)$ and a pair of matrix-functions g_k^+, g_k^- such that $\left(g_\tau^+(t)\right)^{\pm 1} \in {}^+L_\infty^{m\times m}(\Gamma)$, $\left(g_\tau^-(t)\right)^{\pm 1} \in {}^-L_\infty^{m\times m}(\Gamma)$ for any $t \in u(t_k)$; the matrix $g_k^+ a g_k^-$ is unitary and

$$\sigma\left(g_k^- a g_k^+\right) \subset \Lambda_k(\delta) \ \left(t \in u^+(t_k)\right), \ \sigma\left(g_k^- a g_k^+ e^{\frac{-2\pi i\beta_k}{p}}\right) \subset \Lambda_k(\delta) \ (t \in u^-(t_k)),$$

where

$$u^+(t_k) = \{t \in u(t_k),\ t \succ t_k\} \text{ and } u^-(t_k) = \{t \in u(t_k), t \prec t_k\}.$$

Theorem 3.7. *(see [13]). Let $a \in G_{\delta\rho}(\Gamma)$, where*

$$tg\delta/2 = min\left(m^{\frac{2-p}{2p}} tg\frac{\pi}{2p}, m^{\frac{p-2}{2p}} ctg\frac{\pi}{2p}\right),$$

then the operator $aP + Q$ is Noetherian in the space $L_p^m(\Gamma, \rho)$.

Corollary 3.7. $G_{p\rho}(\Gamma) \subset Fact_{p\rho}^m(\Gamma)$.

4. The dependence of noetherian property of singular integral operators with shift and conjugation on the existence of corner points on contour

1⁰. Singular operators with shift. Let Γ be a closed piecewise Lyapunov contour, $\nu : \Gamma \to \Gamma$, $(V\phi)(t) = \phi(\nu(t))$. In the space $L_p(\Gamma)$ consider a singular integral operator with shift $\nu(t)$ of the form

$$A = a(t)\, I + b(t)\, S + (c(t)\, I + d(t)\, S)\, V, \tag{13}$$

where $a(t), b(t), c(t)$ and $d(t)$ are bounded measurable functions on Γ. Assume that the mapping ν satisfies the following conditions:

(i) $\nu(\nu(t)) \equiv 1$;

(ii) the derivative $\nu'(t)$ has on Γ a finite numbers of discontinuity points of the first kind and on arcs l_k, joining discontinuity points, it satisfies the Hölder conditions;

(iii) $\nu(t \pm 0) \neq 0$ $(t \in \Gamma)$.

Together with the operator A of the form (13) consider also the operator $\tilde{A}$ determined in the space $L_p^2(\Gamma) = L_p(\Gamma) \times L_p(\Gamma)$ by the equality

$$\tilde{A} = \left\| \begin{array}{cc} aI + bS & cI + dS \\ \tilde{c}I + \varepsilon\tilde{d}S & \tilde{a}I + \varepsilon\tilde{b}S \end{array} \right\| + \left\| \begin{array}{cc} 0 & 0 \\ \tilde{d}(VsV - \varepsilon S) & \tilde{b}(VsV - \varepsilon S) \end{array} \right\| = \tilde{A}_0 + R, \qquad (14)$$

where $\tilde{f}(t) = f(\nu(t))$ and $\varepsilon = 1$ $(\varepsilon = -1)$, if the mapping ν preserves orientation on Γ. As it is known [21], if a, b, c and d are continuous functions and $\nu'(t) \in H(\Gamma)$, then the operator R is compact in $L_p^2(\Gamma)$ and the following theorem is true.

Theorem 4.1. $A \in Nt_p(\Gamma) \Leftrightarrow \tilde{A}_0 \in Nt_p^2(\Gamma)$ *by this* $IndA = \frac{1}{2}Ind\tilde{A}_0$.

Show (see [24]) that the assertion ceases to be true if Γ has corner points. In such case, usually, the derivative $\nu'(t)$ has on Γ discontinuity point, and it turns out that if the operator A is Noetherian, then the operator $\tilde{A}_0$ is also Noetherian but the converse does not hold true.

Theorem 4.2. *If operator* $A\ (a, b, c, d \in C(\Gamma))$ *is Noetherian in the space* $L_p(\Gamma)$, *then the operator* $\tilde{A}_0$ *is also Noetherian in the space* $L_p^2(\Gamma)$.

Indeed, the operator $\tilde{A}_0$ *is Noetherian if and only if*

$$\Delta_1(t) = (a(t) + b(t))\left(\tilde{a}(t) + \varepsilon\tilde{b}(t)\right) - (c(t) + d(t))\left(\tilde{c}(t) + \varepsilon\tilde{d}(t)(t)\right) \neq 0$$

and

$$\Delta_2(t) = (a(t) - b(t))\left(\tilde{a}(t) - \varepsilon\tilde{b}(t)\right) - (c(t) - d(t))\left(\tilde{c}(t) - \varepsilon\tilde{d}(t)(t)\right) \neq 0$$

for any $t \in \Gamma$.

Proof. Let the operator A be Noetherian, then the determinant of its symbol (see [26]) is not equal to zero: $detA(t_1, \xi) \neq 0$. One can check directly that

$$detA(t, -\infty) \cdot detA(t, +\infty) = \Delta_1(t) \cdot \Delta_2(t).$$

Therefore, the operator $\tilde{A}_0$ is Noetherian in $L_p^2(\Gamma)$. □

The following example shows that the converse of Theorem 4.2 is not true. Let ν reverse orientation on Γ and the corner point $t_0\ (\in \Gamma)$ with the angle θ $(0 < \theta < \pi)$ be a fixed point of the mapping $\nu : \nu(t_0) = t_0$. In this case it is easy to see that the derivative $\nu'(t)$ is discontinuous at the point t_0, and $\nu'(t_0 + 0) = \exp(i\theta + \sigma)$, $\nu'(t_0 + 0) = \exp(i\theta - \sigma)$, where σ is real number. Consider the operator

$$A = I + \delta SV,$$

where δ is a complex number. The operator $\tilde{A}$ has the form

$$\tilde{A} = \left\| \begin{matrix} I & \delta S \\ -\delta S & I \end{matrix} \right\| + \left\| \begin{matrix} 0 & 0 \\ \delta\left(VSV + S\right) & 0 \end{matrix} \right\| = \tilde{A}_0 + R.$$

If $\delta \neq \pm i$, then the operator $\tilde{A}_0$ is Noetherian. Let $A\left(t, \xi\right)$ be symbol of operator A at the point t_0. One can check directly that

$$\det A\left(t_0, \xi\right) = \delta^2 + 2\left(\alpha + \beta\right)\delta + 1,$$

where

$$\alpha = \frac{\exp\left[\left(2\pi - \theta - i\sigma\right)\left(\xi + \frac{i}{\pi}\right)\right]}{\exp\left(\xi + \frac{i}{\pi}\right) - 1} \quad \text{and} \quad \beta = \frac{\exp\left[\left(\theta + i\sigma\right)\left(\xi + \frac{i}{\pi}\right)\right]}{\exp\left(\xi + \frac{i}{\pi}\right) - 1}.$$

From this, due to Theorem 1.1 from [24], it follows that for any $\delta = -\left(\alpha + \beta\right) \pm \sqrt{\left(\alpha + \beta\right)^2 - 1}$ the operator A is not Noetherian in the space $L_p\left(\Gamma\right)$. Thus, the conditions for operator A being Noetherian depend on angle θ.

Corollary 4.1. *Let $\nu'\left(t\right) \notin H\left(\Gamma\right)$, then the operator $VSV - \varepsilon S$ is not compact in $L_p\left(\Gamma\right)$.*

Corollary 4.2. *If the operator A, determined by the equality (13) is Noetherian, then the operators $\tilde{A}$ and $\tilde{A}_0$ determined by equality (14) are also Noetherian and $IndA = Ind\tilde{A}_0$*

Corollary 4.3. *If the operator $\tilde{A}$ is Noetherian, then $\tilde{A}_0$ is also Noetherian. The converse is not true, in general.*

Note that the corresponding example of non-Noetherian operator A for which the operator $\tilde{A}_0$ is Noetherian can be given also in the case when ν preserves the orientation of contour Γ.

2^0. Singular operators with conjugation. Singular integral operators with conjugation have the form

$$A = aP + bQ + \left(cP + sQ\right)V,$$

where $a, b, c, d \in PC(\Gamma)$, $P = (I + S)/2$, $Q = I - P$, $(V\phi)(t) = \overline{\phi(t)}$ and Γ is a closed piecewise Lyapunov contour.

By constructing Nether's theory of operator A in monograph [23] an essential role was played by the fact that if at each point of contour Γ the Lyapunov condition is satisfied, then the operator $VSV + S$ is compact in the space $L_p(\Gamma, \rho)$. In this case the operator A is (see [23]) Noetherian if and only if the operator

$$A_V = \left\| \begin{matrix} a & c \\ \overline{d} & \overline{b} \end{matrix} \right\| P + \left\| \begin{matrix} b & d \\ \overline{c} & \overline{a} \end{matrix} \right\| Q$$

possesses the same property in the space $L_p^2(\Gamma, \rho) = L_p(\Gamma, \rho) \times L_p(\Gamma, \rho)$.

It is quite different if the contour Γ has corner points. It turns out that in this case the operator $VSV + S$ is not compact in $L_p(\Gamma, \rho)$ and if A is Noetherian, then A_V is also Noetherian, but the converse ceases to be true (see [39]–[40]). These are the facts which constitute significant difference between a piecewise Lyapunov contour and a Lyapunov contour.

Theorem 4.3. *The operator*

$$(VSV + S)\phi = -\frac{1}{\pi i}\int_\Gamma \frac{\phi(\tau)\overline{d\tau}}{\overline{\tau} - \overline{t}} + \frac{1}{\pi i}\int_\Gamma \frac{\phi(\tau)d\tau}{\tau - t}$$

is compact in the space $L_p(\Gamma, \rho)$ if and only if Γ is a Lyapunov contour.

The sufficient part of this assertion has been proved in [23]. Let us prove the necessity. Assume that $VSV + S$ is compact, then the operator $A_\lambda = VSV + S - \lambda I$ is Noetherian for all $\lambda \in C \setminus \{0\}$. Therefore, due to Theorem 1 from work [18], $\det A_\lambda\,(t_k, \xi) \neq 0$ for all $k + 1, ..., s$ and $-\infty < \xi < \infty$, where $t_k \;\; (k = 1, ..., s)$ are all corner points of contour Γ. From this we obtain that

$$\frac{z_k^{2\pi-\theta_k} - z_k^{\theta_k}}{z_k^{2\pi} - 1} \equiv 0 \quad \left(z_k = \exp\left(\xi + i\frac{1+\beta_k}{p}\right)\right).$$

The last is possible only for $\theta_k = \pi$. Theorem is proved.

The condition for operator A to be Noetherian, unlike singular operators not containing the operator V (i.e. $A = aP + bQ$,) depends essentially on contour. For example, the operator $A = (1 + \sqrt{2})P + (1 - \sqrt{2})Q + V$ is Noetherian in all spaces $L_p(\Gamma, \rho)$, if Γ is a Lyapunov contour and is not Noetherian in $L_2(\Gamma)$, if Γ has one corner point with angle $\pi/2$.

3^0. Generalized Riemann problem. Consider the generalized Riemann boundary value problem: find analytic functions $\Phi^+(z)$ and $\Phi^-(z)$ representable by the Cauchy integral in F_Γ^+ and F_Γ^-, whose limit values on Γ belong to the space $L_p(\Gamma, \rho)$ and satisfy the conditions

$$\Phi^+(t) = a(t)\Phi^-(t) + b(t)\overline{\Phi^-}(t) + c(t)$$

where $a(t), b(t)$ are defined on Γ continuous functions and $c(t) \in L_p(\Gamma, \rho)$.

Noether's theory of this problem, in the case of a Lyapunov contour, has been constructed in work [23]. For the investigation of this problem in the case of Lyapunov contour L. Mikhailov applied the method of I. Simonenko combining the solution of the factorization problem with the principle compressed application. L. Mikhailov, in particular, established that a necessary and sufficient condition for the problem to be Noetherian is that the inequality $|a(t)| > 0$ should be satisfied for all $t \in \Gamma$. In the case of a piecewise Lyapunov contour the following theorem is true (see [21], [39]–[41]).

Theorem 4.4. *In order that the generalized boundary Riemann problem in $L_p(\Gamma,\rho)$ be Noetherian it is necessary and sufficient that the following conditions be satisfied:*

(i) $|a(t)| > 0, \quad (t \in \Gamma)$;

(ii) $|a\,(t_k)|^2 - |b\,(t_k)|^2 \left(\frac{z_k{}^{2\pi-\theta_k} - z_k{}^{\theta_k}}{z_k{}^{2\pi}-1}\right) \neq 0$ *for all* $k = 1, \ldots n$, *where*

$$z_k = \exp\left(\xi + i\frac{1+\beta_k}{p}\right), \quad -\infty \leq \xi \leq \infty, \; \theta_k = \theta(t_k) \text{ and } \beta_k = \beta(t_k).$$

Thus, in the case of a piecewise Lyapunov contour the Noetherian property of Riemann problem depends not only on the coefficient $a(t)$, as it was in the case of a Lyapunov contour, but also on the coefficient $b(t)$.

The results of this section can be extended, without essential changes, to the case when Γ consists of a finite number of closed piecewise Lyapunov curves without common points.

5. Perturbatuon of singular integral operators

1⁰. Formulating the problem. In the monographs of N.I. Muskhelishvili and F.D. Gakhov, an operator is called complete singular integral operator if it has the form

$$(A\varphi)\,(t) = a\,(t) + \frac{1}{\pi i}\int_{\Gamma} \frac{k(\tau,t)\varphi(\tau)}{\tau - t}d\tau, \tag{15}$$

where $a(t)$ and $k(\tau,t)$ are functions satisfying the Hölder condition on Γ and $\Gamma \times \Gamma$, respectively, and the integral is understood in the sense of the principal value. The operator A, defined by equality (15), can be represented in the form $A = aI + bS + T$, where $b(t) = k(t,t)$, and T is the integral operator with kernel

$$k_0(\tau,t) = \pi i \frac{k(\tau,t) - k(t,t)}{\tau - t}. \tag{16}$$

In the case when $k(\tau,t)$ satisfies the Hölder condition on $\Gamma \times \Gamma$, the kernel (16) has a weak singularity; therefore, the operator T is completely continuous in the space $L_p(\Gamma)$. Due to this, the operator A is Noetherian in the space $L_p(\Gamma)$, if and only if the operator

$$A_0 = aI + bS$$

is Noetheiran. Operator A_0 is called the characteristic part of the operator A. In this connection, Noether's theory of singular operators was developed mainly for characteristic operators. Significant successes have been achieved in this direction: there are obtained criteria to be Noetherian for such operators with piecewise continuous coefficients, with coefficients having discontinuities of almost periodic type, and with arbitrary coefficients from $L_\infty(\Gamma)$. However, in many problems of mechanics, physics and other areas that lead to singular equations, not characteristic operators appear, but complete ones. In this regard, it becomes necessary to study the complete singular operators (15) with functions and $k(\tau, t)$ not necessarily satisfying the Hölder condition. The main difficulty here is that the operator T with kernel (16) may turn out to be not completely continuous (not compact) or (more importantly) ceases to be an Φ-admissible perturbation.

Let us show this on an example. Let Γ_0 be the unit circle, $\chi(t)$ be the characteristic function of the $\{\operatorname{Im} t > 0\} \cap \Gamma_0$; $k(\tau, t) = \chi(t) - \chi(\tau)$, $\lambda \in \mathbb{C}$,

$$(A\varphi)\,(t) = \lambda\varphi(\mathrm{t}) + \frac{1}{\pi i}\int_{\Gamma_0} \frac{k(\tau, t)\varphi(\tau)}{\tau - t} d\tau.$$

In this example, $k(\tau, t) = 0$. Therefore, the characteristic part of the operator A is a scalar operator $(A_0\varphi)\,(t) = \lambda\varphi(t)$. The operator A in this example can be represented in the form $A = \lambda I + \chi S - S\chi I$, whence it follows that it belongs to the algebra A_p, generated by singular integral operators with piecewise continuous coefficients. It was shown in (16) that on the algebra A_p one can introduce the symbol $(\gamma_{t,\mu})$ $((t, \mu) \in \Gamma_0 \times [0, 1])$, which on the generators of S and aI takes the form

$$\gamma_{t,\mu}(\alpha I) = \begin{Vmatrix} a(t+0)f_p(\mu) + a(t-0)(1 - f_p(\mu)) & (a(t+0) - a(t-0))h_p(\mu) \\ (a(t+0) - a(t-0))h_p(\mu) & a(t+0)(1 - f_p(\mu)) + a(t-0)f_p(\mu) \end{Vmatrix},$$

where

$$f_p\,(\mu) = \begin{cases} \dfrac{\sin\theta\mu}{\sin\theta} e^{i\theta(\mu-1)}, \ \left(\theta = \dfrac{\pi\,(p-2)}{2}\right), & \text{for } p \neq 2, \\ \mu, & \text{for } p = 2, \end{cases} \tag{17}$$

and $h_p(\mu)$ is some fixed continuous branch of the function $\sqrt{f_p(\mu)(1 - f_p\,(\mu))}$.

In particular, for the operator $A = \lambda I + \chi S - S\chi I$ with $p = 2$ we have: $\det\gamma_{t,\mu}(A) = \lambda^2$ for $t \neq \pm 1$ and $\det\gamma_{t,\mu}(A) = \lambda^2 + 4\mu(1 - \mu)$ for $t = \pm 1$. An operator A is Noetherian in $L_2(\Gamma)$ if and only if $\lambda^2 + 4\mu(1 - \mu) \neq 0$ for all $\mu \in [0, 1]$. This is equivalent to $\lambda \neq ti$, where $t \in [-1, 1]$.

Thus, for $\lambda = \tau i$, where $\tau \in [-1, 1] \setminus \{0\}$, the operator A is not Noetherian, but its characteristic part A_0 is Noetherian. This implies that the operator $M = A - A_0$ is not a Φ-admissible perturbation of the characteristic part of the operator A. This also implies that M is not compact.

For this operator, we managed to obtain criteria for Noetherian property due to the fact that we embedded it in the algebra A_p (see [26]). You can do the same with some other complete operators. This work will describe one class of such operators.

2^0. Perturbatuon of singular integral operators. In this section we will show that the Noetherian property of operators $aP + bQ$ is stable under perturbation by some not compact operators. Remark that analogous questions have also been studied in [41] and

[42]. For simplicity assume that $\Gamma = \{t : |t| = 1\}$ is a unit circle. Let α_k $(k = 1, ..., s)$ be some complex numbers. Introduce the following notations:

$$\Gamma_k = \{\xi : \xi = t - \alpha_k \; t \in \Gamma\} \text{ and } \widetilde{\Gamma}_k = \{\xi : \xi = t + \alpha_k, \; t \in \Gamma\}.$$

Assume that $\Gamma_j \cap \Gamma \cap \widetilde{\Gamma}_k = \emptyset$ $(j, k = 1, ..., s)$.

Theorem 5.1. *Let $a, b, c_k \in L_\infty(\Gamma)$ $(k = 1, 2, ..., s)$. In order that the operator*

$$(A\phi)(t) = a(t)\phi(t) + \frac{b(t)}{\pi i}\int_\Gamma \frac{\phi(\tau)}{\tau - t}d\tau + \sum_{k=1}^{s} c_k(t)\frac{1}{\pi i}\int_\Gamma \frac{\phi(\tau)}{\tau - t - \alpha_k}d\tau \tag{18}$$

be Noetherian in the space $L_p(\Gamma, \rho)$ it is necessary and sufficient that the operator

$$(A_0\phi)(t) = a(t)\phi(t) + \frac{b(t)}{\pi i}\int_\Gamma \frac{\phi(\tau)}{\tau - t}d\tau \tag{19}$$

possess the same property. If the operator A_0 is Noetherian, then $IndA = IndA_0$.

In particular, if $a, b \in C(\Gamma)$, the condition $a^2(t) - b^2(t) \neq 0$ $(t \in \Gamma)$ is a necessary and sufficient condition for the operator A to be Noetherian and

$$IndA = ind\frac{a(t) + b(t)}{a(t) - b(t)}.$$

In the general case $(a, b \in L_\infty(\Gamma))$ one can apply the criteria from Section 2 to the operator A. Note that the operator A_0 is (see [1]) the characteristic part of the operator A.

It turns out that operators with kernels $(\tau - t - \alpha_k)^{-1}$ are not, in general, compact in the space $L_p(\Gamma, \rho)$ (see [42]–[45]. Proof of Theorem 5.1 is based on a series of properties of operators with kernels $(\tau - t - \alpha_k)^{-1}$ and their compositions with operator S and operators aI. The conditions $\Gamma_j \cap \Gamma \cap \widetilde{\Gamma}_k = \emptyset$ is essential. For example,

$$(A\phi)(t) = \lambda\phi(t) + \frac{1}{\pi i}\int_\Gamma \frac{\phi(\tau)}{\tau - t - 1}d\tau + \frac{1}{\pi i}\int_\Gamma \frac{\phi(\tau)}{\tau - t + 1}d\tau, \tag{20}$$

$\lambda \in C$ is not Noetherian in $L_2(\Gamma)$ for $\lambda = 2i$, while $(A_0\phi)(t) = \lambda\phi(t)$ is inversible for all $\lambda \neq 0$.

Definition 5.1. The subset $(t_1, t_2), (t_2, t_3), ..., (t_m, t_1)$ of the set $\Gamma\times\Gamma$ is called [42] m-link if $t_j \neq t_k$ for $j \neq k$.

Definition 5.2. The set $M \subset \Gamma \times \Gamma$ is called admissible if there exists a neighborhood of this set which does not contain m-links for any m.

Definition 5.3. Let

$$(H\phi)(t) = \int_\Gamma h(\tau, t)\,\phi(\tau)\,d\tau \;\; (t \in \Gamma). \tag{21}$$

We say that the essential singularity of the kernel $h(\tau, t)$ is contained in the set M if the integral operators with the kernel

$$\widetilde{h}(\tau, t) = \begin{cases} 0 & \text{in a neighborhood of the set } M, \\ h(\tau, t) & \text{at other points of } \Gamma \times \Gamma, \end{cases}$$

is compact.

In [46] the following assertion is proved.

Theorem 5.2. *Let the essential singularity of the kernel $h(\tau, t)$ of the integral operator (21) be contained in an admissible set M. If $A_0 = aI + bS \in Nt_{p\rho}(\Gamma)$, then $A = aI + bS + H \in Nt_{p\rho}(\Gamma)$ and $IndA_0 = IndA$.*

Denote by M the set of pairs $(\tau, t) \in (\Gamma \times \Gamma)$ for which $\tau - t - \alpha_k = 0 \ (k = 1, ..., s)$. Assume that the numbers α_k are such that $M \neq 0$. Then the set M consists of a finite number of points $(\tau_1, t_1), ..., (\tau_N, t_N)$ and the operator

$$(K\phi)(t) = \sum_{k=1}^{S} \frac{c_k(t)}{\pi i} \int_\Gamma \frac{\phi(\tau)}{\tau - t - \alpha_k} d\tau \tag{22}$$

is not compact in the space $L_p(\Gamma, \rho)$. From Theorem 5.2 one can deduce (see [45]) the following proposition.

Theorem 5.3. *Let the set M do not contain m-links $(m = 1, ..., N)$. In order that the operator $A = aI + bS + K$ $(a, b \in PC(\Gamma))$ to be Noetherian in the space $L_p(\Gamma, \rho)$, it is necessary and sufficient that the operator $A_0 = aI + bS$ to possess the same property. If the operator A_0 is Noetherian, then $IndA = IndA_0$.*

The above example proves that the restrictions on the set of singularities are, in some sense, exact. In this connection the question of the necessity of conditions of Theorem 5.3 arises naturally. To be more exact, whether there exists an operator of the form (21) satisfying the following conditions:

(1) the set M contains an m-link;
(2) $|c_k(t)| \geq \delta > 0$ $(k = 1, ..., s)$ on the set M;
(3) the operator $A = aI + bS + K \in Nt_{p\rho}(\Gamma) \Leftrightarrow A_0 = aI + bS + K \in Nt_{p\rho}(\Gamma)$.

Such operators exist (see [13]). As operator K we take the operator acting by the rule

$$(\mathrm{K}\phi)(t) = \frac{1}{\pi i} \int_\Gamma \frac{\phi(\tau)}{\tau - t - 2} d\tau + \frac{1}{\pi i} \int_\Gamma \frac{\phi(\tau)}{\tau - t + 2} d\tau. \tag{23}$$

Note that the operator K is not (see [47-50]) compact in the space $L_p(\Gamma, \rho)$. The set M corresponding to the operator K consists of two points $(-1, 1)$ and $(1, -1)$, forming two links. Denote by **N** the set of all functions from $PC(\Gamma)$ continuous in some neighbourhood of the points $\tau = \pm 1$.

Theorem 5.4. *Let* $a, b \in \mathbf{N}$. *Then* $A = aI + bS + K \in Nt_{p\rho}(\Gamma) \Leftrightarrow A_0 = aI + bS \in Nt_{p\rho}(\Gamma)$. *If* $A_0 \in Nt_{p\rho}(\Gamma)$, *then* $IndA_0 = IndA$.

In conclusion, we remark that the results of this section can be transferred to case where Γ is an arbitrary piecewise Lyapunov contour which has no straight line parts as well as to operators of the form (18) with matrix coefficients.

Note that the case of an unbounded contour was considered in works [14], [47], [49].

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