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Dedicated to the memory of Academician Mitrofan M. Cioban (1942-2021)

On CM-groupoids with multiple identities and medial topological left loops

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Abstract. This paper studies some properties of CM-groupoids with multiple identities and medial topological left loops. The conditions for a CM-groupoid to become a CM-quasigroup were found. A new method of constructing non-associative medial topological quasigroups with left identy is given. Various examples of quasigroups with multiple identities have been constructed.

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Despre CM-grupoizi cu unități multiple și bucle de stânga mediale topologice

Rezumat. În această lucrare sunt examinate proprietăţi ale CM-groupoizilor cu unităţi multiple şi a buclelor de stânga topologice mediale. Au fost determinate condiţiile pentru care un CM-groupoid devine CM-quasigroup. Este propusă o metodă nouă de construcţie a quasigrupurilor mediale topologice cu unitate de stânga. Sunt construite diverse exemple de quasigrupuri cu unităţi multiple.

Cuvinte cheie: CM-grupoid, qasigrupuri mediale topologice, unități multiple, (n, m)-izotop omogen.

1. Introduction

Our main results can be summarized as follows. In Section 3 we prove some properties of the CM-groupoid and CM-quasigroup, using the concept of (n, m)-identities introduced by M.M. Choban and L.L. Chiriac in [1, 2]. Thus, we prove that if $(G, \cdot)$ is a CM-multiplicative groupoid, $e \in G$ and the following conditions hold: $ex = x$ for every $x \in G$; $x^2 = x \cdot x = e$ for every $x \in G$; if $xa = ya$ then $x = y$ for all $x, y, a \in G$, then $(G, \cdot)$ is a CM-quasigroup with an $(1, 2)$-identity e. In order to facilitate the study of topological quasigroups with (n, m)-identities, we expand on the notions of multiple identities and (n, m)-homogeneous isotopies. Section 4 presents some results and constructions which can be used to produce examples of medial quasigroups with left identity that are not associative. Finally, in Section 5, using the concept of the (n, m)-identities, we show some examples of the quasigroups with multiple identities and examples of the locally compact medial or paramedial quasigroup G, where exists a unique invariant Haar measure on G.

We dedicate this paper to the memory of Professor Mitrofan Choban who made many important contributions to modern mathematics.

2. Basic notions

In this section we recall some fundamental definitions and notations.

A non-empty set G is said to be a *groupoid* with respect to a binary operation denoted by $\{\cdot\}$, if for every ordered pair (a, b) of elements of G, there is a unique element $ab \in G$.

If the groupoid G is a topological space and the multiplication operation $(a, b) \to a \cdot b$ is continuous, then G is called a *topological groupoid.*

A groupoid G is called a primitive groupoid with divisions if there exist two binary operations $l : G \times G \to G$, $r : G \times G \to G$ such that $l(a, b) \cdot a = b, a \cdot r(a, b) = b$ for all $a, b \in G$. Thus, a primitive groupoid with divisions is a universal algebra with three binary operations.

A primitive groupoid G with divisions is called a quasigroup if the equations $ax = b$ and $ya = b$ have unique solutions. In a quasigroup G the divisions l, r are unique. If the multiplication operation in a quasigroup $(G, \cdot)$ with a topology is continuous, then G is called a semitopoligical quasigroup. If in a semitopological quasigroup G the divisions l and r are continuous, then G is called a topological quasigroup.

An element $e \in G$ is called an *identity* if $ex = xe = x$ every $x \in X$.

A quasigroup with an identity is called a *loop*. A groupoid G is called *medial* if it satisfies the law $xy \cdot zt = xz \cdot yt$ for all $x, y, z, t \in G$. A groupoid G is called *paramedial* if it satisfies the law $xy \cdot zt = ty \cdot zx$ for all $x, y, z, t \in G$. A groupoid G is called *bicommutative* if it satisfies the law $xy \cdot zt = tz \cdot yx$ for all $x, y, z, t \in G$.

If a medial guasigroup G contains an element e such that $e \cdot x = x (x \cdot e = x)$ for all x in G, then e is called a *left (right) identity* element of G and G is called a *left (right) medial loop.*

A groupoid G is called a *groupoid Abel-Grassmann* or *AG-groupoid* if it satisfies the left invertive law $(a \cdot b) \cdot c = (c \cdot b) \cdot a$ for all $a, b, c \in G$.

A groupoid G is called *GA-groupoid* if it satisfies the law $xy \cdot z = z \cdot yx$ for all $x, y, z, t \in G$.

A groupoid G is called *AD-groupoid* if it satisfies the law $x \cdot yz = z \cdot yx$ for all $x, y, z, t \in G$.

A groupoid G is called *MC-groupoid* if it satisfies the law $xy \cdot z = y \cdot zx$ for all $x, y, z, t \in G$.

A quasigroup G is called *Ward quasigroup* if there is an element e such that $x \cdot x = e$ for all $x \in Q$, satisfying the identity $(x \cdot y) \cdot z = x \cdot (z \cdot (e \cdot y))$ for all $x, y, z \in G$. Or a quasigroup G is called *Ward quasigroup* if it satisfies the law $(x \cdot z) \cdot (y \cdot z) = x \cdot y$ for all $x, y, z, t \in G$.

Let $\mathbb{N} = \{1, 2, ...\}$ and $\mathbb{Z} = \{..., -3, -2, -1, 0, 1, 2, 3, ...\}$. We shall use the notations and terminology from [1, 2, 3, 4, 5].

3. Concept of the (n, m)-identities

In this section, using the concept of the (n, m)-identities, we prove that if $(G, \cdot)$ is a locally compact paramedial quasigroup, then there exists a unique invariant Haar measure on G.

We recall some important definitions and notations.

Consider a groupoid $(G, +)$. For every two elements a, b from $(G, +)$ we denote:

$$1(a, b, +) = (a, b, +)1 = a + b, \text{ and } n(a, b, +) = a + (n-1)(a, b, +),$$
$$(a, b, +)n = (a, b, +)(n-1) + b$$

for all $n \geq 2$.

If a binary operation $(+)$ is given on a set G, then we shall use the symbols $n(a, b)$ and $(a, b)n$ instead of $n(a, b, +)$ and $(a, b, +)n$.

Definition 3.1. Let $(G, +)$ be a groupoid and let $n, m \geq 1$. The element e of the groupoid $(G, +)$ is called:

- an (n, m)-zero of G if $e + e = e$ and $n(e, x) = (x, e)m = x$ for every $x \in G$;
- an $(n, \propto)$-zero if $e + e = e$ and $n(e, x) = x$ for every $x \in G$;
- an $(\propto, m)$-zero if $e + e = e$ and $(x, e)m = x$ for every $x \in G$.

Clearly, if $e \in G$ is both an $(n, \propto)$-zero and an $(\propto, m)$-zero, then it is also an (n, m)-zero. If $(G, \cdot)$ is a multiplicative groupoid, then the element e is called an (n, m)-identity.

Example 3.1. Let $(G, \cdot)$ be a paramedial groupoid, $e \in G$ and $ex = x$ for every $x \in G$. Then $(G, \cdot)$ is a paramedial groupoid with $(1, 2)$-identity e in G. Indeed, if $x \in G$, then $xe \cdot e = xe \cdot ee = ee \cdot ex = e \cdot x = x$.

Definition 3.2. Let $(G, +)$ be a topological groupoid. A groupoid $(G, \cdot)$ is called a homogeneous isotope of the topological groupoid $(G, +)$ if there exist two topological automorphisms $\varphi, \psi : (G, +) \to (G, +)$ such that $x \cdot y = \varphi(x) + \psi(y)$ for all $x, y \in G$.

For every mapping $f : X \to X$ we denote $f^1(x) = f(x)$ and $f^{n+1}(x) = f(f^n(x))$ for any $n \geq 1$.

Definition 3.3. Let $n, m \leq \infty$. A groupoid $(G, \cdot)$ is called an (n, m)-homogeneous isotope of a topological groupoid $(G, +)$ if there exist two topological automorphisms $\varphi, \psi : (G, +) \to (G, +)$ such that:

1. $x \cdot y = \varphi(x) + \psi(y)$ for all $x, y \in G$;
2. $\varphi\varphi = \psi\psi$;
3. If $n < \infty$, then $\varphi^n(x) = x$ for all $x \in G$;
4. If $m < \infty$, then $\psi^m(x) = x$ for all $x \in G$.

Definition 3.4. A groupoid $(G, \cdot)$ is called an isotope of a topological groupoid $(G, +)$ if there exist two homeomorphisms $\varphi, \psi : (G, +) \to (G, +)$ such that $x \cdot y = \varphi(x) + \psi(y)$ for all $x, y \in G$.

Under the conditions of Definition 3.4 we shall say that the isotope $(G, \cdot)$ is generated by the homeomorphisms φ, ψ of the topological groupoids $(G, +)$ and write $(G, \cdot) = g(G, +, \varphi, \psi)$.

Example 3.2. Denote by $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z} = \{0, 1, ..., p-1\}$ the cyclic group of order p. Let $(G, +) = (\mathbb{Z}_7, +)$, $\varphi(x) = x$, $\psi(x) = 6x$ and $x \cdot y = x + 6y$. Then $(G, \cdot) = g(G, +, \varphi, \psi)$ is a medial, paramedial, bicommutative and AG-quasigroup and the zero of $(G, +)$ is a $(2, 1)$-identity in $(G, \cdot)$.

Example 3.3. Denote by $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z} = \{0, 1, ..., p-1\}$ the cyclic group of order p. Let $(G, +) = (\mathbb{Z}_{13}, +)$, $\varphi(x) = x$, $\psi(x) = 12x$ and $x \cdot y = x + 12y$. Then $(G, \cdot) = g(G, +, \varphi, \psi)$ is a medial, paramedial, bicommutative and AD-quasigroup and the zero of $(G, +)$ is a $(2, 1)$-identity in $(G, \cdot)$.

Example 3.4. Denote by $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z} = \{0, 1, ..., p-1\}$ the cyclic group of order p. Let $(G, +) = (\mathbb{Z}_{13}, +)$, $\varphi(x) = 12x$, $\psi(x) = x$ and $x \cdot y = 12x + y$. Then $(G, \cdot) = g(G, +, \varphi, \psi)$ is a medial, paramedial, bicommutative and AG-quasigroup and the zero of $(G, +)$ is a $(1, 2)$-identity in $(G, \cdot)$.

Example 3.5. Let $(G, +) = (\mathbb{Z}_{11}, +)$, $\varphi(x) = 2x$, $\psi(x) = 9x$ and $x \cdot y = 2x + 9y$. Then $(G, \cdot) = g(G, +, \varphi, \psi)$ is a medial, paramedial and bicommutative quasigroup and the zero of $(G, +)$ is a $(5, 10)$-identity in $(G, \cdot)$.

Some algebraic and topological properties of groupoids and quasigroups with multiple identities have been studied in [6, 7, 8, 9].

Below we will prove some algebraic properties for CM-groupoids and CM-quasigroups.

Theorem 3.1. *If $(G, \cdot)$ is a multiplicative groupoid, $e \in G$ and the following conditions hold:*

1. $ex = x$ for every $x \in G$,

2. $x^2 = x \cdot x = e$ for every $x \in G$,

3. $xy \cdot z = y \cdot zx$ for all $x, y, z \in G$,

4. for every $a, b \in G$ there exists a unique point $y \in G$ such that $ay = b$, then e is an $(1, 2)$-identity in G and the following algebraic properties are fulfilled:

(i) $ab \cdot e = ba$ for every $a, b \in G$,

(ii) $c \cdot e = ac \cdot a$ for every $a, c \in G$,

(iii) $ab \cdot c = ac \cdot b$ for every $a, b, c \in G$,

(iv) $a \cdot ab = b$ for every $a, b \in G$.

Proof. Fix $x \in G$. Pick $y \in G$ such that $xe \cdot y = x$. By conditions 2 of Theorem 3.1 we have

$$(xe \cdot y) \cdot x = x \cdot x = e. \tag{1}$$

By condition 3 of Theorem 3.1 we get

$$(xe \cdot y) \cdot x = y \cdot (x \cdot xe). \tag{2}$$

From (1) and (2) we obtain

$$y \cdot (x \cdot xe) = e. \tag{3}$$

It is clear that

$$(x \cdot xe) \cdot (x \cdot xe) = e. \tag{4}$$

Thus, from (3) and (4) and condition 3 of Theorem 3.1 we find that

$$y = x \cdot xe = ex \cdot x = xx = e.$$

Hence $y = e$. Therefore $xe \cdot y = xe \cdot e = x$ and e is an $(1, 2)-$ identity in G.

We will prove algebraic properties (i) - (iv).

According to condition 3 of Theorem 3.1 we obtain $ab \cdot c = b \cdot ca$. Let $c = e$. Hence, we have property (i), $ab \cdot e = b \cdot ea = ba$.

By conditions 2 and 3 we get property (ii), $c \cdot e = c \cdot (aa) = ac \cdot a$.

We will show property $ab \cdot c = ac \cdot b$. Indeed, by conditions 1 and 3 of Theorem 3.1 and property (i) we obtain property (iii)

$$ab \cdot c = b \cdot ca = e(b \cdot ca) = (ca \cdot e) \cdot b = ac \cdot b. \tag{5}$$

We will prove property (iv). Indeed, by properties (i) and (iii) we obtain

$$a \cdot ab = a \cdot (ba \cdot e) = a \cdot (be \cdot a) = (aa) \cdot (be) = e \cdot be = (ee) \cdot b = b. \quad (6)$$

The proof is complete. □

Theorem 3.2. *If $(G, \cdot)$ is a CM-multiplicative groupoid, $e \in G$ and the following conditions hold:*

1. $ex = x$ for every $x \in G$,

2. $x^2 = x \cdot x = e$ for every $x \in G$,

3. if $xa = ya$, then $x = y$ for all $x, y, a \in G$,

then $(G, \cdot)$ is a CM-quasigroup with an $(1, 2)$-identity e.

Proof. By Theorem 3.1 e is an $(1, 2)$-identity.
Consider the equation $ax = b$. Then $ax \cdot a = b \cdot a$ or $x \cdot aa = ba$. Thus $x \cdot e = ba$ and $xe \cdot e = ba \cdot e$. Therefore $x = ba \cdot e = ab$ is a solution of the equation.

Since $ax = b$ we can verify that $a \cdot ab = b$. By property 4 from Theorem 3.1 we get $a \cdot ab = a \cdot (ba \cdot e) = b$. In this case the element $x = ab$ is a unique solution of the equation $ax = b$.

Now we consider the equation $ya = b$. Then $ya \cdot e = b \cdot e$ or by property 3 from Theorem 3.1 we have $ay = be$. Thus, using the solution of the equation $ax = b$ in the last identity, we get $y = a \cdot be$. Hence, $y = a \cdot be$ is a solution of the equation $ya = b$ and we can verify that

$$(a \cdot be) \cdot a = (be) \cdot (aa) = (be) \cdot e = b.$$

In this case the element $y = a \cdot be$ is a unique solution of the equation $ya = b$. Thus, $(G, \cdot)$ is a CM-quasigroup with an $(1, 2)$-identity e. The proof is now complete. □

Corollary 3.1. *If $(G, \cdot)$ is a CM-quasigroup with an $(1, 2)$-identity e and $x^2 = e$, then solutions of the equations $ax = b$ and $ya = b$ are, respectively, $x = ab$ and $y = a \cdot be$ for all $x, y, a, b \in G$.*

4. On a method of construction medial topological left loops

In Section 4 we prove a new method of constructing non-associative medial topological quasigroups with left identity.

Theorem 4.1. *Let $(G, +, \tau)$ be a commutative topological group. For (x_1, y_1) and (x_2, y_2) in $G \times G$ define*

$$(x_1, y_1) \circ (x_2, y_2) = (x_1 + y_1 + x_2 + y_1, y_1 + y_2)$$

Then $(G \times G, \circ, \tau_G)$, relative to the product topology τ_G, is a medial, non-paramedial and non-associative topological quasigroup with left identity. Moreover, if (G, τ) is $T_i - space$, then $(G \times G, \tau_G)$ is $T_i - space$ too, where $i = 1, 2, 3, 3.5$.

Proof. **1.** We will prove that $(G\times G, \circ)$ is a quasigroup. To this end, we will show that the equations $y \circ a = b$ and $x \circ a = b$ have unique solutions in $(G \times G, \circ)$. Let $y = (y_1, y_2)$, $x = (x_1, x_2)$, $a = (a_1, a_2)$ and $b = (b_1, b_2)$. Since $y \circ a = b$ we have

$$(y_1, y_2) \circ (a_1, a_1) = (b_1, b_2). \tag{7}$$

According to the conditions of the Theorem

$$(y_1, y_2) \circ (a_1, a_2) = (y_1 + y_2 + a_1 + y_2, a_2 + y_2). \tag{8}$$

From (7) and (8) we get

$$y_1 + y_2 + a_1 + y_2 = b_1 \tag{9}$$

and

$$a_2 + y_2 = b_2. \tag{10}$$

From (10) and (9) we obtain

$$y_2 = b_2 - a_2. \tag{11}$$

and

$$y_1 = b_1 - a_1 - 2(b_2 - a_2). \tag{12}$$

Hence, $y_1 = b_1 - a_1 - 2(b_2 - a_2)$ and $y_2 = b_2 - a_2$ are solutions of the equation $y \circ a = b$. It is easy to show that any other solutions of that equation coincide with y_1 and y_2.

In this case

$$l((a_1, a_2), (b_1, b_2)) = (b_1 - a_1 - 2(b_2 - a_2), b_2 - a_2)$$

and $l((a_1, a_2), (b_1, b_2)) \circ (a_1, a_2) = (b_1, b_2)$.

Similarly it is shown that the equation $a \circ x = b$ or

$$(a_1, a_2) \circ (x_1, x_2) = (b_1, b_2) \tag{13}$$

has a unique solutions $x_1 = b_1 - a_1 - 2a_2$ and $x_2 = b_2 - a_2$. It is clear that

$$r((a_1, a_2), (b_1, b_2)) = (b_1 - a_1 - 2a_2, b_2 - a_2)$$

and $(a_1, a_2) \circ r((a_1, a_2), (b_1, b_2)) = (b_1, b_2)$.

Any other solutions of the equation $a \circ x = b$ coincides with x_1 and x_2. Thus $(G \times G, \circ)$ is a quasigroup.

2. We will prove that associativity

$$((x_1, y_1) \circ (x_2, y_2)) \circ (x_3, y_3) = (x_1, y_1) \circ ((x_2, y_2) \circ (x_3, y_3)) \tag{14}$$

does not hold in $(G \times G, \circ)$.

Indeed, for the first side of law (14) we obtain

$$((x_1, y_1) \circ (x_2, y_2)) \circ (x_3, y_3) = (x_1 + y_1 + x_2 + y_1, y_1 + y_2) \circ (x_3, y_3) =$$

$$= (x_1 + y_1 + x_2 + y_1 + y_1 + y_2 + x_3 + y_1 + y_2, y_1 + y_2 + y_3) =$$

$$= (x_1 + 4y_1 + x_2 + 2y_2 + x_3, y_3 + y_1 + y_2). \tag{15}$$

Similarly, for the second side of law (14) we have

$$(x_1, y_1) \circ ((x_2, y_2) \circ (x_3, y_3)) = (x_1, y_1) \circ (x_2 + y_2 + x_3 + y_2, y_3 + y_2) =$$

$$= (x_1 + y_1 + x_2 + y_2 + x_3 + y_2 + y_1, y_2 + y_3 + y_1 =$$

$$= (x_1 + x_2 + x_3 + 2y_1 + 2y_2, y_2 + y_3 + y_1). \tag{16}$$

From (15) and (16) it is clear that associativity does not hold in $(G \times G, \circ)$.

3. We will show that $(G \times G, \circ)$ is medial quasigroup that is, the property $xy \cdot zt = = xz \cdot yt$ holds.

Let $x = (x_1, y_1)$, $y = (x_2, y_2)$, $z = (x_3, y_3)$, $t = (x_4, y_4)$, then

$$((x_1, y_1) \circ (x_2, y_2)) \circ ((x_3, y_3) \circ (x_4, y_4)) = ((x_1, y_1) \circ (x_3, y_3)) \circ ((x_2, y_2) \circ (x_4, y_7)). \tag{17}$$

According to the Theorem for the first side of law (17) we have

$$((x_1, y_1) \circ (x_2, y_2)) \circ ((x_3, y_3) \circ (x_4, y_4)) =$$

$$= ((x_1 + y_1 + x_2 + y_1, y_2 + y_1)) \circ ((x_3 + y_3 + x_4 + y_3, y_4 + y_3)) =$$

$$= (x_1 + x_2 + 2y_1 + y_1 + y_2 + x_3 + 2y_3 + x_4 + y_1 + y_2, y_1 + y_2 + y_3 + y_4) =$$

$$= (x_1 + x_2 + 4y_1 + 2y_2 + x_3 + 2y_3 + x_4, y_1 + y_2 + y_3 + y_4). \tag{18}$$

Similarly, for the other side of law (17) we get

$$((x_1, y_1) \circ (x_3, y_3)) \circ ((x_2, y_2) \circ (x_4, y_4)) =$$

$$= (x_1 + y_1 + x_3 + y_1, y_1 + y_3) \circ (x_2 + y_2 + x_4 + y_2, y_2 + y_4) =$$

$$= (x_1 + 2y_1 + x_3 + y_1 + y_3 + x_2 + 2y_2 + x_4 + y_1 + y_3, y_1 + y_3 + y_2 + y_4) =$$

$$= (x_1 + 4y_1 + x_3 + 2y_3 + x_2 + 2y_2 + x_4, y_1 + y_3 + y_2 + y_4). \tag{19}$$

From (18) and (19) we obtain that both sides are equal and $(G \times G, \circ)$ is a medial quasigroup. Similarly, it is shown that paramediality does not hold in $(G \times G, \circ)$.

4. We will show that there is a left identity $ex = x$.

Let $x = (x_1, x_2)$ and $e = (e, e)$, then $(e, e) \circ (x_1, x_2) = (x_1, x_2)$.

Indeed, according to the Theorem

$$(e, e) \circ (x_1, x_2) = (e + e + x_1 + e, e + x_2) =$$

$$= (e + x_1 + e, e + x_2) = (e + x_1, e + x_2) = (x_1, x_2). \tag{20}$$

Hence, (e, e) is a left identity. Similarly, it is shown that the right identity does not exist.

Therefore, the quasigroup $(G \times G, \circ)$ is medial with left identity. Multiplication $(\circ)$ and divisions $l(a, b)$ and $r(a, b)$ are jointly continuous relative to the product topology. Consequently, $(G \times G, \circ, \tau_G)$ is a topological medial quasigroup with left identity.

If (G, τ) is T_i-space, then according to Theorem 2.3.11 in [4], a product of T_i-spaces is a T_i-spaces, where $i = 1, 2, 3, 3.5$. The proof is complete. □

Theorem 4.2. *Let $(G, +, \tau)$ be a commutative topological group. For (x_1, y_1) and (x_2, y_2) in $G \times G$ define*

$$(x_1, y_1) \circ (x_2, y_2) = (x_1 - y_1 + x_2 - y_2, y_1 + y_2).$$

Then $(G \times G, \circ, \tau_G)$, relative to the product topology τ_G, is a non-associative, medial, paramedial, bicommutative and GA-topological quasigroup. Moreover, if (G, τ) is $T_i -$ space, then $(G \times G, \tau_G)$ is $T_i - space$ too, where $i = 1, 2, 3, 3.5$.

Proof. The proof is analogous to that of Theorem 4.1. □

Example 4.1. Let $G = \{0, 1, 2\}$. We define the binary operation " + ".

(+)	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

Then $(G, +)$ is a commutative group. Define a binary operation $(\circ)$ on the set $G \times G$ by $(x_1, y_1) \circ (x_2, y_2) = (x_1 + y_1 + x_2 + y_1, y_1 + y_2)$ for all $x_1, y_1, x_2, y_2 \in G \times G$. If we label the elements as follows $(0, 0) \leftrightarrow 0$, $(0, 1) \leftrightarrow 1$, $(0, 2) \leftrightarrow 2$, $(1, 0) \leftrightarrow 3$, $(1, 1) \leftrightarrow 4$, $(1, 2) \leftrightarrow 5$, $(2, 0) \leftrightarrow 6$, $(2, 1) \leftrightarrow 7$, $(2, 2) \leftrightarrow 8$, then we obtain:

(+)	0	1	2	3	4	5	6	7	8
0	0	1	2	3	4	5	6	7	8
1	7	8	6	1	2	0	4	5	3
2	5	3	8	8	6	7	2	0	1
3	3	4	5	6	7	8	0	1	2
4	1	2	0	4	5	3	7	8	6
5	8	6	7	2	0	1	5	3	4
6	6	7	8	0	1	2	3	4	5
7	4	5	3	7	8	6	1	2	0
8	2	0	1	5	3	4	8	6	7

Then $(G \times G, \circ)$ is a medial quasigroup with left identity.

5. Some examples of quasigroups with multiple identities

Example 5.1. Denote by $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z} = \{0, 1, ..., p-1\}$ the cyclic group of order p. Let $(G, +) = (\mathbb{Z}_{11}, +)$, $\varphi(x) = 5x$, $\psi(x) = 6x$ and $x \cdot y = 5x + 6y$. Then $(G, \cdot) = g(G, +, \varphi, \psi)$ is a medial, paramedial, bicommutative and the zero of $(G, +)$ is a $(10, 5)$-identity in $(G, \cdot)$.

Example 5.2. Denote by $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z} = \{0, 1, ..., p-1\}$ the cyclic group of order p. Let $(G, +) = (\mathbb{Z}_{11}, +)$, $\varphi(x) = 6x$, $\psi(x) = 5x$ and $x \cdot y = 6x + 5y$. Then $(G, \cdot) = g(G, +, \varphi, \psi)$ is a medial, paramedial, bicommutative and the zero of $(G, +)$ is a $(5, 10)$-identity in $(G, \cdot)$.

Example 5.3. Denote by $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z} = \{0, 1, ..., p-1\}$ the cyclic group of order p. Let $(G, +) = (\mathbb{Z}_{11}, +)$, $\varphi(x) = 10x$, $\psi(x) = x$ and $x \cdot y = 10x + y$. Then $(G, \cdot) = g(G, +, \varphi, \psi)$ is a medial, paramedial, bicommutative and AG-quasigroup and the zero of $(G, +)$ is an $(1, 2)$-identity in $(G, \cdot)$.

Example 5.4. Let $(G, +) = (\mathbb{Z}_{11}, +)$, $\varphi(x) = 9x$, $\psi(x) = 2x$ and $x \cdot y = 9x + 2y$. Then $(G, \cdot) = g(G, +, \varphi, \psi)$ is a medial, paramedial and bicommutative quasigroup and the zero of $(G, +)$ is a $(10, 5)$-identity in $(G, \cdot)$.

Example 5.5. Let $(G, +) = (\mathbb{Z}_{11}, +)$, $\varphi(x) = x$, $\psi(x) = 10x$ and $x \cdot y = x + 10y$. The next Cayley table describes the structure of a finite quasigroup $(G, \cdot) = g(G, +, \varphi, \psi)$:

$(\cdot)$	0	1	2	3	4	5	6	7	8	9	10
0	0	10	9	8	7	6	5	4	3	2	1
1	1	0	10	9	8	7	6	5	4	3	2
2	2	1	0	10	9	8	7	6	5	4	3
3	3	2	1	0	10	9	8	7	6	5	4
4	4	3	2	1	0	10	9	8	7	6	5
5	5	4	3	2	1	0	10	9	8	7	6
6	6	5	4	3	2	1	0	10	9	8	7
7	7	6	5	4	3	2	1	0	10	9	8
8	8	7	6	5	4	3	2	1	0	10	9
9	9	8	7	6	5	4	3	2	1	0	10
10	10	9	8	7	6	5	4	3	2	1	0

Then $(G, \cdot) = g(G, +, \varphi, \psi)$ is a medial, paramedial, bicommutative, *AD* and Ward quasigroup and the zero of $(G, +)$ is a $(2, 1)$-identity in $(G, \cdot)$.

Example 5.6. Let $(\mathbb{R}, +)$ be the topological Abelian group of real numbers.

1. If $\varphi(x) = x$, $\psi(x) = 7x$ and $x \cdot y = x + 7y$, then $(\mathbb{R}, \cdot) = g(\mathbb{R}, +, \varphi, \psi)$ is a commutative locally compact medial quasigroup. By Theorem 7 from [1] there exists a left (but no right) invariant Haar measure on $(\mathbb{R}, \cdot)$.

2. If $\varphi(x) = 5x$, $\psi(x) = 5x$ and $x \cdot y = 5x + 5y$, then $(\mathbb{R}, \cdot) = g(\mathbb{R}, +, \varphi, \psi)$ is a commutative locally compact paramedial quasigroup and on $(\mathbb{R}, \cdot)$. As above, by Theorem 5.1 from [8], there does not exist any left or right invariant Haar measure.

Example 5.7. Consider the commutative group $(G, +) = (\mathbb{Z}, +)$, $\varphi(x) = x$, $\psi(x) = -x + 1$ and $x \cdot y = x - y + 1$. Then $(G, \cdot) = g(G, +, \varphi, \psi, 0, 1)$ is a medial and paramedial quasigroup and $(G, \cdot)$ does not contain (n, m)-identities. There exists an invariant Haar measure on $(G, \cdot)$.

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