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Dedicated to the memory of Academician Mitrofan M. Cioban (1942-2021)

Multiple partial integrals of polynomial Hamiltonian systems

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Abstract. We consider an autonomous real polynomial Hamiltonian ordinary differential system. Sufficient conditions for the construction of additional first integrals on polynomial partial integrals and multiple polynomial partial integrals are obtained. Classes of autonomous polynomial Hamiltonian ordinary differential systems with first integrals which analytically expressed by multiple polynomial partial integrals are identified. Also we present examples that illustrate the theoretical results.

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Integrale particulare multiple ale sistemelor Hamiltoniene polinomiale

Rezumat. Se consideră sistemele autonome, reale, polinomiale și Hamiltoniane de ecuaţii diferenţiale ordinare. Sunt obţinute unele condiţii suficiente de construire a integralelor prime pe baza integralelor particulare polinomiale simple şi multiple şi sunt identificate sistemele diferenţiale ce au astfel de integrale prime. Rezultatele teoretice sunt ilustrate prin exemple.

Cuvinte cheie: sistem Hamiltonian, integrabilitate Darboux, integrală particulară, multiplicitate.

1. Introduction

Consider a canonical Hamiltonian ordinary differential system

$$\frac{dq_i}{dt} = \partial_{p_i} H(q,p), \qquad \frac{dp_i}{dt} = -\,\partial_{q_i} H(q,p), \quad i = 1,\dots,n, \tag{1}$$

with n degrees of freedom, where $q = (q_1,\dots,q_n) \in \mathbb{R}^n$ and $p = (p_1,\dots,p_n) \in \mathbb{R}^n$ are the generalized coordinates and momenta, respectively, the independent variable $t \in \mathbb{R}$, and the Hamiltonian function $H\colon \mathbb{R}^{2n} \to \mathbb{R}$ is a polynomial of degree $h \geqslant 2$.

The fundamental problem for differential systems is to solve whether a given differential system is integrable. Integrability in this context usually means the existence of as

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many functionally independent first integrals as the dimension of the phase space. Many different methods have been used for studying the existence of first integrals for differential systems. Some of these methods based on (see, for example, the monographs [1, 2, 3, 4, 5, 6, 7]): Noether symmetries, the Darboux theory of integrability, the Lie symmetries, the Painlevé analysis, the use of Lax pairs, the direct method, the linear compatibility analysis method, the Carleman embedding procedure, etc.

In this paper, using the Darboux theory of integrability, we study the existence of additional first integrals of the Hamiltonian system (1) in the space $\mathbb{R}^{2n}$. Notice that according to our knowledge, A.J. Maciejewski and M. Przybylska were the first researchers who applied [8] the Darboux theory of integrability to the study of integrability of polynomial Hamiltonian systems. More precisely, they considered natural polynomial Hamiltonian systems, i.e., the differential systems (1) with the polynomial Hamiltonian

$$H(q,p) = \frac{1}{2}\sum_{i=1}^{n}\mu_i p_i^2 + V(q), \quad \mu_i \in \mathbb{C}, \quad V \in \mathbb{C}[q], \ \deg \mathrm{V} > 2.$$

To avoid ambiguity, we stipulate the following notation and definitions.

The *Poisson bracket* of functions $u, v \in C^1(G),\ G \subset \mathbb{R}^{2n}$, is the function

$$\big[u(q,p), v(q,p)\big] = \sum_{i=1}^{n}\big(\partial_{q_i}u(q,p)\,\partial_{p_i}v(q,p) - \partial_{p_i}u(q,p)\,\partial_{q_i}v(q,p)\big) \text{ for all } (q,p)\in G.$$

The following rules hold for any functions $u, v, w, w_1, \ldots, w_s \in C^1(G)$ and $\alpha, \beta \in \mathbb{R}$:

1) anticommutativity $[u,v] = -\,[v,u]$;

2) bilinearity $[u, \alpha v + \beta w] = \alpha[u,v] + \beta[u,w]$ and $[\alpha u + \beta v, w] = \alpha[u,w] + \beta[v,w]$;

3) Leibniz's rule (product identity) $[u, vw] = w\,[u,v] + v\,[u,w]$;

4) the Poisson bracket of composite function

$$\big[u(q,p), v(w_1(q,p),\ldots,w_s(q,p))\big] =$$
$$= \sum_{k=1}^{s}\partial_{w_k}v(w_1,\ldots,w_s)_{\big|w=w(q,p)}\ \big[u(q,p), w_k(q,p)\big] \ \text{ for all } (q,p)\in G.$$

The functions u and v are *in involution* on the domain G if the Poisson bracket

$$\big[u(q,p), v(q,p)\big] = 0 \ \text{ for all } (q,p)\in G.$$

A function $F \in C^1(G)$ is called a *first integral on the domain* G of the Hamiltonian system (1) if the functions F and H are in involution, i.e.,

$$\big[F(q,p), H(q,p)\big] = 0 \quad \text{for all } (q,p)\in G \subset \mathbb{R}^{2n}. \tag{2}$$

The Hamiltonian system (1) is *completely integrable* (in the Liouville sense) if it has n functionally independent first integrals which are in involution. Notice that the Hamiltonian H is a first integral of the Hamiltonian differential system (1).

A set of functionally independent on a domain $G \subset \mathbb{R}^{2n}$ first integrals $F_l \in C^1(G)$, $l = 1, \dots, k$, of the Hamiltonian system (1) is called a *basis of first integrals* (or *integral basis*) on the domain G of system (1) if any first integral $F \in C^1(G)$ of system (1) can be represented on the domain G in the form

$$F(q,p) = \Phi\big(F_1(q,p), \dots, F_k(q,p)\big) \ \text{ for all } (q,p) \in G,$$

where Φ is some continuously differentiable function. The number k is said to be the *dimension* of basis of first integrals on the domain G for the Hamiltonian system (1).

The autonomous Hamiltonian system (1) on a domain without equilibrium points has an integral basis (autonomous) of dimension $2n - 1$ [9, pp. 167 – 169].

2. Darboux method of integrability

The Darboux theory of integrability (or the theory of partial integrals) was established by the French mathematician *Jean-Gaston Darboux* [10] in 1878, which provided a link between the existence of first integrals and invariant algebraic curves (or partial integrals, or Darboux polynomials) for polynomial autonomous differential systems. For the polynomial differential systems, the Darboux theory of integrability is one of the best theories for studying the existence of first integrals (see, for example, some recent works in this field [11, 12, 1, 13, 14, 15, 7, 16, 17, 18, 19] and the references therein).

Recall some facts from the Darboux theory of integrability, described in [5, 1, 16], and applied it to polynomial Hamiltonian differential systems.

Definition 2.1. A real polynomial

$$w\colon (q,p) \to w(q,p) \quad \text{for all } (q,p) \in \mathbb{R}^{2n} \tag{3}$$

is a *partial integral* of the polynomial Hamiltonian system (1) if the Poisson bracket

$$\big[w(q,p), H(q,p)\big] = w(q,p)\, M(q,p) \quad \text{for all } (q,p) \in \mathbb{R}^{2n},$$

where the polynomial M such that $\deg M \leqslant h - 2$. Moreover, the polynomial M is called the *cofactor* of the partial integral (3) of the Hamiltonian system (1).

Suppose (3) be a partial integral of the Hamiltonian system (1). Then the algebraic hypersurface $\{(q,p)\colon w(q,p) = 0\}$ is invariant by the flow of the Hamiltonian system (1) and if the cofactor M of the partial integral (3) is zero, then it is a polynomial first integral. We say that (3) is a proper partial integral if its cofactor M is not zero.

The classical Darboux method of construction first integrals for the polynomial Hamiltonian differential system (1) can be formulated as the following statement.

Theorem 2.1 (**The Darboux theorem**). *Suppose the Hamiltonian system* (1) *has partial integrals*

$$w_l\colon (q,p) \to w_l(q,p) \quad \text{for all } (q,p) \in \mathbb{R}^{2n}, \quad l = 1,\dots,s, \tag{4}$$

such that the following identities hold

$$\big[w_l(q,p), H(q,p)\big] = w_l(q,p)\, M_l(q,p) \quad \textit{for all } (q,p) \in \mathbb{R}^{2n},\ \ l = 1,\dots,s. \tag{5}$$

Then the scalar function

$$F\colon (q,p) \to \prod_{l=1}^{s} w_l^{\gamma_l}(q,p) \quad \textit{for all } (q,p) \in G \subset \mathbb{R}^{2n}, \qquad \gamma_l \in \mathbb{R},\ \sum_{l=1}^{s} |\gamma_l| \neq 0,$$

is a first integral on the domain G of the Hamiltonian system (1) *if and only if the cofactors* M_l *of the partial integrals* w_l *such that* $\sum\limits_{l=1}^{s} \gamma_l M_l(q,p) = 0$ *for all* $(q,p) \in \mathbb{R}^{2n}$.

For instance, the polynomial Hamiltonian system given by

$$H(q,p) = \frac{1}{2}\sum_{i=1}^{n} \mu_i p_i^2 - f(p)\sum_{i=1}^{n} \alpha_i p_i q_i \quad \text{for all } (q,p) \in \mathbb{R}^{2n},\ \ \mu_i, \alpha_i \in \mathbb{R}, \tag{6}$$

where $f(p)$ is an arbitrary polynomial. Using the partial integrals $w_l\colon (q,p) \to p_l$ for all $(q,p) \in \mathbb{R}^{2n}$ with cofactors $M_l\colon (q,p) \to \alpha_l f(p)$ for all $(q,p) \in \mathbb{R}^{2n}$, $l = 1,\dots,n$, we can build the first integrals of the Hamiltonian differential system (6)

$$F_{1\xi}\colon (q,p) \to p_1^{\gamma_{1\xi}} p_\xi^{\gamma_\xi} \quad \text{for all } (q,p) \in \mathbb{R}^n \times G_\xi,\ \ G_\xi \subset \mathbb{R}^n, \quad \xi = 2,\dots,n,$$

where the real numbers $\gamma_{1\xi}$ and γ_ξ are solutions to the linear homogeneous equations $\alpha_1\gamma_{1\xi} + \alpha_\xi\gamma_\xi = 0$ under the conditions $|\gamma_{1\xi}| + |\gamma_\xi| \neq 0$, $\xi = 2,\dots,n$.

The functionally independent first integrals H and $F_{1\xi}$, $\xi = 2,\dots,n$, are in involution. Therefore, the polynomial Hamiltonian system (6) is completely integrable (in the Liouville sense) [2, p. 83] on a domain $\mathbb{R}^n \times G$, $G \subset G_\xi$, $\xi = 2,\dots,n$.

Definition 2.2. A partial integral w with cofactor M of the Hamiltonian system (1) is said to be *multiple* with multiplicity $\varkappa = 1 + \sum\limits_{\xi=1}^{\varepsilon} r_\xi$ if there exist natural numbers f_ξ and polynomials $Q_{f_\xi g_\xi}\colon \mathbb{R}^{2n} \to \mathbb{R}$, $g_\xi = 1,\dots,r_\xi$, $\xi = 1,\dots,\varepsilon$, such that on the domain

$G \subset \{(q,p) \colon w(q,p) \neq 0\}$ the identities hold

$$\Bigl[\frac{Q_{f_\xi g_\xi}(q,p)}{w^{f_\xi}(q,p)}, H(q,p)\Bigr] = R_{f_\xi g_\xi}(q,p), \ \ g_\xi = 1,\dots,r_\xi, \ \xi = 1,\dots,\varepsilon,$$

where the polynomials $R_{f_\xi g_\xi} \colon \mathbb{R}^{2n} \to \mathbb{R}$ have degrees at most $h-2$.

Note that a similar point of view on multiplicity of partial integral for polynomial differential systems was presented by J. Llibre and X. Zhang in [13].

For example, the polynomial Hamiltonian differential system given by

$$H(q,p) = -q_1^2 + 6q_1q_2 + (2p_1 + p_2)q_1 + 2q_2p_2 + 3p_2^2 \ \text{ for all } (q,p) \in \mathbb{R}^4 \tag{7}$$

has the multiple (multiplicity not less than two) partial integral

$$w \colon (q,p) \to 3q_1 + 2p_2 \quad \text{for all } (q,p) \in \mathbb{R}^4$$

with the cofactor $M(q,p) = -2$ and the polynomials

$$Q_{11}(q,p) = \frac{1}{32}(17q_1 + 12q_2 + 8p_1), \quad R_{11}(q,p) = 1,$$

such that $\deg M = \deg R_{11} = 0 \leqslant h - 2 = 0$. Indeed, the Poisson brackets

$$\bigl[w(q,p), H(q,p)\bigr] = \bigl[3q_1 + 2p_2, H(q,p)\bigr] = 3\bigl[q_1, H(q,p)\bigr] + 2\bigl[p_2, H(q,p)\bigr] =$$

$$= 3\partial_{p_1}H(q,p) - 2\partial_{q_2}H(q,p) = 6q_1 - 2(6q_1 + 2p_2) = -2w(q,p) \ \text{ for all } (q,p) \in \mathbb{R}^4,$$

$$\Bigl[\frac{Q_{11}(q,p)}{w(q,p)}, H(q,p)\Bigr] = \frac{1}{w(q,p)}\Bigl(\bigl[Q_{11}(q,p), H(q,p)\bigr] + 2Q_{11}(q,p)\Bigr) =$$

$$= \frac{1}{32\,w(q,p)}\Bigl((62q_1 - 24q_2 - 16p_1 + 64p_2) + 2(17q_1 + 12q_2 + 8p_1)\Bigr) = 1$$

$$\text{for all } (q,p) \in G \subset \{(q,p) \in \mathbb{R}^4 \colon 3q_1 + 2p_2 \neq 0\}.$$

Property 2.1. *Suppose the real polynomial partial integral* (3) *of the Hamiltonian differential system* (1) *such that the following identity holds*

$$\bigl[w(q,p), H(q,p)\bigr] = w^{m+1}(q,p)\,P(q,p) \quad \textit{for all } \ (q,p) \in \mathbb{R}^{2n}, \tag{8}$$

where m is some natural number and $P \colon \mathbb{R}^{2n} \to \mathbb{R}$ is some polynomial. Then the partial integral (3) *of the Hamiltonian system* (1) *is multiple* (*multiplicity at least two*).

Proof. Let the function $K_{m1}(q,p) = \dfrac{\lambda}{w^m(q,p)}$ for all $(q,p) \in G$, $\lambda \in \mathbb{R} \setminus \{0\}$, $m \in \mathbb{N}$. Then on a domain $G \subset \{(q,p) \colon w(q,p) \neq 0\}$ the Poisson bracket

$$\big[K_{m1}(q,p), H(q,p)\big] = -m\lambda w^{-m-1}(q,p)\,\big[w(q,p), H(q,p)\big] = -m\lambda P(q,p).$$

From the identity (8), we have $\deg P < h-2$. By Definition 2.2, the real polynomial partial integral (3) of the Hamiltonian system (1) is multiple. □

3. Main Results

By the Definitions 2.1 and 2.2, the partial integrals (4) of the Hamiltonian system (1) are multiple with multiplicities $\varkappa_l = 1 + \sum\limits_{\xi_l=1}^{\varepsilon_l} r_{\xi_l}$, $l = 1,\dots,s$, if and only if there exist polynomials $Q_{lf_{\xi_l}g_{\xi_l}}\colon \mathbb{R}^{2n} \to \mathbb{R}$ and $R_{lf_{\xi_l}g_{\xi_l}}\colon \mathbb{R}^{2n} \to \mathbb{R}$ such that the following identities on domains $G_{0l} \subset \{(q,p)\colon w_l(q,p) \neq 0\}$ hold

$$\Big[K_{lf_{\xi_l}g_{\xi_l}}(q,p), H(q,p)\Big] = R_{lf_{\xi_l}g_{\xi_l}}(q,p) \quad \text{for all } (q,p) \in G_{0l},$$

$$f_{\xi_l} \in \mathbb{N},\ \ g_{\xi_l} = 1,\dots,r_{\xi_l},\ \ \xi_l = 1,\dots,\varepsilon_l,\ \ l = 1,\dots,s, \tag{9}$$

where the scalar functions

$$K_{lf_{\xi_l}g_{\xi_l}}(q,p) = \frac{Q_{lf_{\xi_l}g_{\xi_l}}(q,p)}{w_l^{f_{\xi_l}}(q,p)} \quad \text{for all } (q,p) \in G_{0l}$$

and $\deg R_{lf_{\xi_l}g_{\xi_l}} \leqslant h-2,\ f_{\xi_l} \in \mathbb{N},\ g_{\xi_l} = 1,\dots,r_{\xi_l},\ \xi_l = 1,\dots,\varepsilon_l,\ l = 1,\dots,s.$

Theorem 3.1. *Suppose the polynomial Hamiltonian system* (1) *has the partial integrals* (4) *with multiplicities* $\varkappa_l = 1 + \sum\limits_{\xi_l=1}^{\varepsilon_l} r_{\xi_l}$, $l = 1,\dots,s$, *such that the identities* (5) *and* (9) *hold. Then the function*

$$F\colon (q,p) \to \prod_{l=1}^{s} w_l^{\gamma_l}(q,p)\,\exp\sum_{l=1}^{s}\sum_{\xi_l=1}^{\widetilde{\varepsilon}_l}\sum_{g_{\xi_l}=1}^{\widetilde{r}_{\xi_l}} \alpha_{lf_{\xi_l}g_{\xi_l}} K_{lf_{\xi_l}g_{\xi_l}}(q,p) \tag{10}$$

is an additional first integral of the Hamiltonian system (1) *if and only if*

$$\sum_{l=1}^{s} \gamma_l M_l(q,p) + \sum_{l=1}^{s}\sum_{\xi_l=1}^{\widetilde{\varepsilon}_l}\sum_{g_{\xi_l}=1}^{\widetilde{r}_{\xi_l}} \alpha_{lf_{\xi_l}g_{\xi_l}} R_{lf_{\xi_l}g_{\xi_l}}(q,p) = 0, \tag{11}$$

where numbers γ_l *and* $\alpha_{lf_{\xi_l}g_{\xi_l}}$ *such that* $\sum\limits_{l=1}^{s}|\gamma_l| + \sum\limits_{l=1}^{s}\sum\limits_{\xi_l=1}^{\widetilde{\varepsilon}_l}\sum\limits_{g_{\xi_l}=1}^{\widetilde{r}_{\xi_l}} \Big|\alpha_{lf_{\xi_l}g_{\xi_l}}\Big| \neq 0.$

Proof. Taking into account the identities (5) and (9), we calculate the Poisson bracket of the function (10) and the Hamiltonian H:

$$\big[F(q,p),H(q,p)\big]=\Bigg[\prod_{l=1}^{s} w_l^{\gamma_l}(q,p)\exp\Bigg(\sum_{l=1}^{s}\sum_{\xi_l=1}^{\widetilde{\varepsilon}_l}\sum_{g_{\xi_l}=1}^{\widetilde{r}_{\xi_l}}\alpha_{lf_{\xi_l}g_{\xi_l}}K_{lf_{\xi_l}g_{\xi_l}}(q,p)\Bigg),H(q,p)\Bigg]=$$

$$=\exp\Bigg(\sum_{l=1}^{s}\sum_{\xi_l=1}^{\widetilde{\varepsilon}_l}\sum_{g_{\xi_l}=1}^{\widetilde{r}_{\xi_l}}\alpha_{lf_{\xi_l}g_{\xi_l}}K_{lf_{\xi_l}g_{\xi_l}}(q,p)\Bigg)\cdot\Bigg[\prod_{l=1}^{s} w_l^{\gamma_l}(q,p),H(q,p)\Bigg]+$$

$$+F(q,p)\cdot\Bigg[\sum_{l=1}^{s}\sum_{\xi_l=1}^{\widetilde{\varepsilon}_l}\sum_{g_{\xi_l}=1}^{\widetilde{r}_{\xi_l}}\alpha_{lf_{\xi_l}g_{\xi_l}}K_{lf_{\xi_l}g_{\xi_l}}(q,p),H(q,p)\Bigg]=$$

$$=\exp\Bigg(\sum_{l=1}^{s}\sum_{\xi_l=1}^{\widetilde{\varepsilon}_l}\sum_{g_{\xi_l}=1}^{\widetilde{r}_{\xi_l}}\alpha_{lf_{\xi_l}g_{\xi_l}}K_{lf_{\xi_l}g_{\xi_l}}(q,p)\Bigg)\sum_{l=1}^{s}\gamma_l w_l^{\gamma_l-1}(q,p)\prod_{\substack{k=1,\\ k\neq l}}^{s} w_k^{\gamma_k}(q,p)\cdot$$

$$\cdot\Big[w_l(q,p),H(q,p)\Big]\;+\;F(q,p)\cdot\Bigg(\sum_{l=1}^{s}\sum_{\xi_l=1}^{\widetilde{\varepsilon}_l}\sum_{g_{\xi_l}=1}^{\widetilde{r}_{\xi_l}}\alpha_{lf_{\xi_l}g_{\xi_l}}\cdot\Big[K_{lf_{\xi_l}g_{\xi_l}}(q,p),H(q,p)\Big]\Bigg)=$$

$$=F(q,p)\Bigg(\sum_{l=1}^{s}\gamma_l M_l(q,p)+\sum_{l=1}^{s}\sum_{\xi_l=1}^{\widetilde{\varepsilon}_l}\sum_{g_{\xi_l}=1}^{\widetilde{r}_{\xi_l}}\alpha_{lf_{\xi_l}g_{\xi_l}}R_{lf_{\xi_l}g_{\xi_l}}(q,p)\Bigg).$$

By the definition of first integral for the polynomial Hamiltonian system (1), the identity (2) is true if and only if the identity (11) is true. □

Theorem 3.2. *Suppose the polynomial Hamiltonian system* (1) *has the partial integrals* (4) *with multiplicities* $\varkappa_l = 1+\sum\limits_{\xi_l=1}^{\varepsilon_l} r_{\xi_l}$, $l=1,\dots,s$, *such that the identities* (5) *and* (9) *are true with the polynomials*

$$M_l(q,p)=\lambda_l M(q,p)\quad \text{for all } (q,p)\in\mathbb{R}^{2n},\ \ \lambda_l\in\mathbb{R},\ l=1,\dots,s, \tag{12}$$

$$\begin{gathered} R_{lf_{\xi_l}g_{\xi_l}}(q,p)=\lambda_{lf_{\xi_l}g_{\xi_l}}M(q,p),\quad \lambda_{lf_{\xi_l}g_{\xi_l}}\in\mathbb{R},\ f_{\xi_l}\in\mathbb{N},\ g_{\xi_l}=1,\dots,\widetilde{r}_{\xi_l},\\ \widetilde{r}_{\xi_l}\leqslant r_{\xi_l},\ \ \xi_l=1,\dots,\widetilde{\varepsilon}_l,\ \ \widetilde{\varepsilon}_l\leqslant\varepsilon_l,\ \ l=1,\dots,s, \end{gathered} \tag{13}$$

where $M\colon \mathbb{R}^{2n}\to\mathbb{R}$ *is some polynomial. Then the function* (10) *is an additional first integral of the Hamiltonian differential system* (1) *if real numbers* γ_l *and* $\alpha_{lf_{\xi_l}g_{\xi_l}}$ *are an* *nontrivial solution to the linear homogeneous equation*

$$\sum_{l=1}^{s} \lambda_l \gamma_l + \sum_{l=1}^{s} \sum_{\xi_l=1}^{\widetilde{\varepsilon}_l} \sum_{g_{\xi_l}=1}^{\widetilde{r}_{\xi_l}} \lambda_{l f_{\xi_l} g_{\xi_l}} \alpha_{l f_{\xi_l} g_{\xi_l}} = 0. \tag{14}$$

Proof. If the polynomial Hamiltonian system (1) has the partial integrals (4) with multiplicities $\varkappa_l = 1 + \sum\limits_{\xi_l=1}^{\varepsilon_l} r_{\xi_l}$, $l = 1, \ldots, s$, such that the identities (5) and (9) under the conditions (12) and (13) are true, then the Poisson bracket of the function (10) and the Hamiltonian H is the function

$$\bigl[F(q,p), H(q,p)\bigr] = \Biggl(\sum_{l=1}^{s} \gamma_l M_l(q,p) + \sum_{l=1}^{s} \sum_{\xi_l=1}^{\widetilde{\varepsilon}_l} \sum_{g_{\xi_l}=1}^{\widetilde{r}_{\xi_l}} \alpha_{l f_{\xi_l} g_{\xi_l}} R_{l f_{\xi_l} g_{\xi_l}}(q,p) \Biggr) F(q,p) =$$

$$= \Biggl(\sum_{l=1}^{s} \lambda_l \gamma_l + \sum_{l=1}^{s} \sum_{\xi_l=1}^{\widetilde{\varepsilon}_l} \sum_{g_{\xi_l}=1}^{\widetilde{r}_{\xi_l}} \lambda_{l f_{\xi_l} g_{\xi_l}} \alpha_{l f_{\xi_l} g_{\xi_l}} \Biggr) M(q,p)\, F(q,p).$$

This yields that if numbers γ_l and $\alpha_{l f_{\xi_l} g_{\xi_l}}$ are a solution to the equation (14), then the scalar function (10) is a first integral of the polynomial Hamiltonian system (1). $\square$

Corollary 3.1. *Under the conditions of Theorem 3.2, we have*

$$F_{\zeta l}\colon (q,p) \to w_\zeta^{\gamma_\zeta}(q,p) \exp\Bigl(\alpha_{l f_{\xi_l} g_{\xi_l}} K_{l f_{\xi_l} g_{\xi_l}}(q,p)\Bigr), \ \ \zeta = 1, \ldots, s, \ l = 1, \ldots, s,$$

are first integrals of the polynomial Hamiltonian system (1), *where fixed numbers* $f_{\xi_l} \in \mathbb{N}$, $g_{\xi_l} \in \bigl\{1, \ldots, \widetilde{r}_{\xi_l}\bigr\}$, $\xi_l \in \bigl\{1, \ldots, \widetilde{\varepsilon}_l\bigr\}$, $l = 1, \ldots, s$, *the numbers* γ_ζ *and* $\alpha_{l f_{\xi_l} g_{\xi_l}}$ *are solutions to the linear homogeneous equations*

$$\lambda_\zeta \gamma_\zeta + \lambda_{l f_{\xi_l} g_{\xi_l}} \alpha_{l f_{\xi_l} g_{\xi_l}} = 0 \ \ \textit{under} \ \ |\gamma_\zeta| + \Bigl|\alpha_{l f_{\xi_l} g_{\xi_l}}\Bigr| \neq 0, \ \zeta, l = 1, \ldots, s.$$

Theorem 3.3. *Suppose the Hamiltonian system* (1) *has the Darboux polynomials* (4) *with multiplicities* $\varkappa_l = 1 + \sum\limits_{\xi_l=1}^{\varepsilon_l} r_{\xi_l}$, $l = 1, \ldots, s$, *such that the identities* (9) *hold and there exist numbers* $\xi_l \in \{1, \ldots, \varepsilon_l\}$, $g_{\xi_l} \in \{1, \ldots, r_{\xi_l}\}$, $l = 1, \ldots, s$, *such that the polynomials*

$$R_{l f_{\xi_l} g_{\xi_l}}(q,p) = \lambda_l M(q,p), \quad \lambda_l \in \mathbb{R}, \ \ l = 1, \ldots, s. \tag{15}$$

Then an additional first integral of the Hamiltonian system (1) *is the function*

$$F\colon (q,p) \to \sum_{l=1}^{s} \alpha_l K_{l f_{\xi_l} g_{\xi_l}}(q,p) \quad \textit{for all } (q,p) \in G, \tag{16}$$

where the numbers α_l, $l = 1, \ldots, s$, *are a solution to the linear homogeneous equation* $\sum\limits_{l=1}^{s} \lambda_l \alpha_l = 0$ *under the condition* $\sum\limits_{l=1}^{s} \alpha_l^2 \neq 0$, *and* $M\colon \mathbb{R}^{2n} \to \mathbb{R}$ *is some polynomial.*

Proof. If the identities (9) under (15) hold, then the Poisson bracket

$$\big[F(q,p), H(q,p)\big] = \sum_{l=1}^{s} \alpha_l \Big[K_{l\, f_{\xi_l} g_{\xi_l}}(q,p), H(q,p)\Big] = \sum_{l=1}^{s} \lambda_l \alpha_l M(q,p).$$

If numbers α_l, $l = 1, \ldots, s$, are a solution of the equation $\sum\limits_{l=1}^{s} \lambda_l \alpha_l = 0$ under $\sum\limits_{l=1}^{s} \alpha_l^2 \neq 0$, then the function (16) is a first integral of the Hamiltonian system (1). □

Corollary 3.2. *Under the conditions of Theorem 3.3, we see that*

$$F_{\zeta\varrho}\colon (q,p) \to \ \alpha_\zeta K_{\zeta f_{\xi_\zeta} g_{\xi_\zeta}}(q,p) + \alpha_\varrho K_{\varrho f_{\xi_\varrho} g_{\xi_\varrho}}(q,p), \quad \zeta, \varrho = 1, \ldots, s, \ \ \zeta \neq \varrho,$$

are additional first integrals of the polynomial Hamiltonian system (1)*, where the numbers* α_ζ *and* α_ϱ *are solutions to the linear homogeneous equations*

$$\lambda_\zeta \alpha_\zeta + \lambda_\varrho \alpha_\varrho = 0 \ \ \textit{under} \ \ \alpha_\zeta^2 + \alpha_\varrho^2 \neq 0, \ \zeta = 1, \ldots, s, \ \varrho = 1, \ldots, s, \ \zeta \neq \varrho.$$

As an example, the Hamiltonian differential system (7) has the multiple partial integral (multiplicity of at least two) $w_1\colon (q,p) \to 3q_1 + 2p_2$ for all $(q,p) \in \mathbb{R}^4$ with

$$M_1(q,p) = -2, \quad K_{1,11}(q,p) = \frac{17q_1 + 12q_2 + 8p_1}{32(3q_1 + 2p_2)}, \quad R_{1,11}(q,p) = 1,$$

and the multiple partial integral $w_2\colon (q,p) \to q_1$ for all $(q,p) \in \mathbb{R}^4$ with

$$M_2(q,p) = 2, \quad K_{2,11}(q,p) = \frac{2q_2 + 3p_2}{16q_1}, \quad R_{2,11}(q,p) = -1.$$

Using Theorem 3.2 (or Corollary 3.1) and Theorem 3.3 (or Corollary 3.2), we can construct the additional first integrals of the Hamiltonian differential system (7)

$$F_1\colon (q,p) \to \ (3q_1 + 2p_2)\exp\left(\frac{17q_1 + 12q_2 + 8p_1}{16(3q_1 + 2p_2)}\right) \quad \text{for all } (q,p) \in G_1,$$

$$F_2\colon (q,p) \to \ q_1 \exp\left(\frac{2q_2 + 3p_2}{8q_1}\right) \quad \text{for all } (q,p) \in G_2 \subset \{(q,p)\colon q_1 \neq 0\},$$

$$F_3\colon (q,p) \to \ \frac{17q_1 + 12q_2 + 8p_1}{32(3q_1 + 2p_2)} + \frac{2q_2 + 3p_2}{16q_1} \quad \text{for all } (q,p) \in G \subset G_1 \cap G_2,$$

where a domain $G_1 \subset \{(q,p)\colon 3q_1 + 2p_2 \neq 0\}$. The functionally independent first integrals F_1, F_2 and F_3 of the Hamiltonian differential system (7) are an autonomous integral basis of the Hamiltonian differential system (7) on any domain $G \subset G_1 \cap G_2$.

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