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Problems of the theory of invariants and Lie algebras applied in the qualitative theory of differential systems

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Abstract. In this work there were formulated 18 problems from the theory of invariant processes, Lie algebras, commutative graded algebras, generating functions and Hilbert series, orbit theory and Lyapunov stability theory that are important to be solved. There was substantiated the necessity of using the solutions of these problems in the qualitative theory of differential systems.

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Probleme din teoria invarianților şi algebrelor Lie pentru aplicaţii în teoria calitativă a sistemelor diferenţiale

Rezumat. În lucrare au fost formulate 18 probleme importante din teoria proceselor invariante, algebrelor Lie, algebrelor graduate comutative, funcţiilor generatoare şi seriilor Hilbert, teoria orbitelor şi teoria stabilităţii după Lyapunov ce se cer rezolvate. A fost argumentată necesitatea utilizării soluţiilor acestor probleme în teoria caliativă a sistemelor diferenţiale.

Cuvinte-cheie: sistem diferenţial, comitanţi şi invarianţi, algebre Lie şi algebre graduate comutative, funcţii generatoare şi serii Hilbert, teoria orbitelor, stabilitatea mişcării neperturbate.

1. Introduction

Since 1963 in the school of differential equations from Chisinau, Republic of Moldova, under the leadership of the academician C. Sibirsky (1928-1990), there has been founded a new research direction, which later it was formed as "The Method of Algebraic Invariants in the Theory of Differential Equations". This direction was based on the results of the monographs [1]-[4] after which there were published a lot of works of the academician C. Sibirsky and his disciples. Among them we can mention the works of N. Vulpe, M. Popa, Iu. Calin, V. Baltag as well as their students'.

The main results of these works concerned the construction of polynomial bases of invariants and comitants of some classical linear groups (the centro-affine $GL(2,\mathbb{R})$, the rotation $SO(2,\mathbb{R})$ and the orthogonal group $O(2,\mathbb{R})$) with the help of which there were determined some qualitative properties of autonomous polynomial differential systems as well as the geometric behavior of their solutions.

The mentioned direction of investigation has been recognized and has aroused the interest of specialists from Canada, USA, Brazil, Spain, Slovenia, Belarus, France, Algeria and some scientific centers of other countries. This is confirmed by the monograph [5] recently published on this topic by a group of authors from the scientific centers of three countries (Spain, Canada, Moldova).

„*The Method of Algebraic Invariants in the Theory of Differential Equations*" is developed even today quite effectively in the research realized in the Republic of Moldova. The above mentioned direction made it possible since the 90-s of the last century in the course of the next decades to appear and develop together with this direction, researches in the fields of invariant processes, Lie algebras and commutative graded algebras, generating functions and Hilbert series, the theory of orbits, stability of unperturbed motion after Lyapunov, governed by autonomous polynomial differential systems. This direction was confirmed under the name „Differential Equations and Algebras".

The essential results of these researches together with „*The Method of Algebraic Invariants in the Theory of Differential Equations*" were brought in the monographs [6]-[13]. M. Popa, P. Macari, A. Braicov, S. Port, E. Bâcova, E. Staruş (Naidenova), N. Gerştega, O. Cerba (Diaconescu) , V. Orlov, V. Pricop, N. Neagu, V. Repeşco, D. Cozma, contributed to the mentioned research and the ones that followed.

Let us examine the system of autonomous polynomial differential equations (PDS) of the first order in the general form, which contains the maximum possible number of non-zero coefficients

$$\frac{dx^j}{dt} = \sum_{m_i \in \Gamma} P^j_{m_i}(x) \ (j = 1, 2, \dots, n; i = 1, 2, \dots, l), \tag{1}$$

where $\Gamma = \{m_1, m_2, \dots, m_l\}$ is a finite set of non-negative integers and $x = (x^1, x^2, \dots, x^n)$ is the vector of phase variables with n coordinates. We denote by N the maximum number of non-zero coefficients of system (1) and by m_i the degree of homogeneity of the polynomial $P^j_{m_i}(x)$ of system (1) with respect to the coordinates of the phase vector x. Such systems, we will denote by $s^n(\Gamma)$. In the case, when $n = 2$, we will write them simply $s(\Gamma)$. The coefficients and phase variables of PDS (1) take values from the field of real numbers $\mathbb{R}$.

2. The problem of the minimal polynomial basis of centro-affine comitants and invariants

1. Find the minimal polynomial basis of centro-affine comitants and invariants of systems $s(1,2,3)$ and $s(0,1,2,3)$ by tensor method.

Comments to Problem 1: The number of elements in this basis was considered in [15], where the types and the number of comitants (1170) and invariants (652) of system $s(1,2,3)$ were brought. The expressions for comitants and invariants were constructed by the classical method of transvectants [4], taken from [16], without expressions of mentioned comitants and invariants being published anywhere. A part of expressions of comitants and invariants referred in [15] were constructed earlier in tensorial form by other authors and were brought in the works [1]-[4], [16]-[18]. If we know the tensor expressions from the basis of comitants and invariants of system $s(1,2,3)$, then using the method described in [19], it is easy to build this basis for system $s(0,1,2,3)$.

The necessity to know the elements of the basis of centro-affine comitants and invariants of systems $s(1,2,3)$ and $s(0,1,2,3)$ results from the importance of investigating these systems both theoretically and practically in various scientific centers of the world. The apparatus of the theory of invariants allows us to obtain answer to some problems from the qualitative theory of PDS, which cannot be obtained by other known methods.

2. Find the minimal polynomial bases of centro-affine comitants and invariants of other systems $s(\Gamma)$.

Comments to Problem 2: Until now, the minimal polynomial bases of centro-affine comitants and invariants for the systems $s(0)$, $s(1)$, $s(2)$, $s(3)$, $s(0,1)$, $s(0,2)$, $s(0,3)$, $s(1,2)$, $s(1,3)$, $s(0,1,2)$, $s(0,1,3)$ are known from the papers [1]-[4], [17], [19], [20]. Using the elements of these bases, there were obtained complete, important and surprising results for the mentioned systems.

Remark 2.1. According to [6], [7], [11], [13] the minimal polynomial basis of centro-affine comitants and invariants for PDS forms finitely determined commutative graded algebras of these elements in relation to the unimodular group $SL(2,\mathbb{R})$, which in [11], [13] are called the Sibirsky graded algebras or simply Sibirsky algebras.

3. Construction problems of Hilbert series for Sibirsky graded algebras of unimodular comitants and invariants of systems $s(\Gamma)$

3. Determine a more effective method for solving Cayley's functional equation [6], [7], [11], [13] *for the generalized and ordinary Hilbert series of Sibirsky graded algebras* [6], [7], [11], [13] *of unimodular comitants and invariants of the systems $s(\Gamma)$ from* (1).

4. Determine the maximal degree of the generators of Sibirsky graded algebras of comitants and invariants for any differential system $s(\Gamma)$ *from* (1) *with respect to the coefficients and phase variables of this system, or to indicate a fairly reasonable upper bound for this degree of generators for any system* $s(\Gamma)$ *from* (1).

5. Determine the formula for the number of generators or indicate a fairly reasonable upper bound of this number for Sibirsky algebras of comitants and invariants for all differential systems $s(\Gamma)$ *from* (1).

6. Determine if the Krull dimension [6], [7], [11], [13] *of Sibirsky algebras of comitants* (*invariants*) *of the system* $s(\Gamma)$ *from* (1) *denoted by* $N-1$ $(N-3)$ *can be the upper bound of maximal number of algebraic limit cycles of this system.*

7. *Prove that the projection of Sibirschi algebra of comitants* (*invariants*) *of system* $s(\Gamma)$ *with* $1 \in \Gamma\backslash\{0\}$ *form a graded algebras on the invariant variety of system* $s(\Gamma)$ *from* (1)*, when the matrix elements of the linear part on the main diagonal are equal to zero and the elements on the secondary diagonal are equal to* 1*, and they are with opposite signs. What is the number of algebraically independent elements of this set ?*

Comments to Problems 3-6: In the papers [6], [7] it is shown the tight connection in the construction of centro-affine comitants and invariants of systems $s(\Gamma)$ from (1) with the study of generating functions of generalized and ordinary Hilbert series of Sibirsky graded algebras of unimodular comitants and invariants of the mentioned systems. Here an important role is due to the solution of Cayley's functional equation, for which in the papers [6], [7] it is used the generalized method of J. Silvester. But this method is connected with cumbersome computations and application of supercomputers, which for $s(\Gamma)$ systems (1) with Γ more complicated cannot be realized.

8. Prove the formula

$$H(SI_{1,2k+1,b} = H(SI_1, b)H(S_{2k+1}, u, z)|_{u^2=b,z=b},$$

for system $s(1, 2k+1)$ $(k \geq 1)$*. This formula has been applied to* $s(1,3)$ *and* $s(1,5)$ *systems, for which the generalized Hilbert series are known from* [11]*,* [13].

9. Find the generalized Hilbert series of Sibirsky algebra $S_{1,2,3}$ *of comitants for system* $s(1,2,3)$ *from* (1).

Comments to Problem 9: We mention that using the residue method in [11], [13] the ordinary Hilbert series of Sibirsky algebras of comitants $S_{1,2,3}$ and invariants $SI_{1,2,3}$ for system $s(1,2,3)$ were constructed, as well as for other systems $s(\Gamma)$ from (1). But for the construction of generalized Hilbert series, this method couldn't be used.

10. Suppose that the Hilbert series $H(S_m, u, z_1)$ and $H(S_n, u, z_2)$ $(m \neq n)$ are known. Can the Hilbert series $H(S_{m,n}, u, z_1, z_2)$ be constructed using these series without solving Cayley's equation

$$\varphi_\Gamma(u) - u^{-2}\varphi_\Gamma(u^{-1}) = \varphi_\Gamma^{(0)}(u)$$

known from [6], [7], [11], [13] *?*

Comments to Problem 10: The problem was formulated in [6], [7], but until now no positive or negative answer has been given to this problem.

4. Problems of construction the Hammond's functions for syzygies (definition relations), related to generators of Sibirsky algebras

11. a) Determine the Hammond series of differential systems for the generators of Sibirsky algebras of invariants $SI_{1,2,3}$ and comitants $S_{1,2,3}$.

b) Determine the type of syzygies. Carry out their construction and show their irreducibility.

Comments to Problem 11: It is known from [11], [12], [13] that any finitely determined algebra A can be written as follows

$$A =< a_1, a_2, \ldots, a_m | f_1 = 0, f_2 = 0, \ldots, f_n = 0 > (m, n < \infty), \tag{2}$$

where m is the number of generators, and n is the number of definition relations (syzygies). These numbers are related by the formula

$$n = m - \varrho(A). \tag{3}$$

where $\varrho(A)$ is the Krull dimension of algebra A.

Using the formula of Hammond series from [6], [7] and the generators from [17], [20], Problem 11 can be solved.

5. Problems of constructing the generating functions for centro-affine comitants and invariants of the systems $s^n(\Gamma)$ $(n \geq 3)$

12. Determine the generating functions of comitants and invariants for systems $s^3(1)$ and $s^3(0,1)$. Generalize this result for any system $s^3(\Gamma)$.

Comments to Problem 12: In papers [21]-[23] some centro-affine comitants and invariants necessary for the research carried out within the systems $s^n(\Gamma)$ were constructed for various values of $n \geq 3$. However, the generating functions, which determine the dimensions of linear spaces of these invariant polynomials according to their type, are not known. These dimensions play an important role in the construction of the polynomial bases of comitants and invariants for the mentioned systems, which are very important in the qualitative investigation of these systems. In the two-dimensional case, these functions were constructed in [6], [7], [11], [13]. The idea of constructing the generating functions for this case could also help us for systems $s^n(\Gamma)$ with $n \geq 3$. Here, the work of the famous German mathematician Paul Gordan of the 19th-20th centuries, who constructed generating functions for the comitants and invariants of ternary forms, could be useful. This work could serve as a point of inspiration for the formulated problem.

6. Lie algebras admitted by PDS, which govern the comitants and invariants of PDS

13. Determine the Lie algebra of operators with or without representations admitted by the Lorenz generalized system:

$$\dot{x} = gx + hy + kz + ayz,$$
$$\dot{y} = px + qy + rz + bxz,$$
$$\dot{z} = sx + my + nz + cxy.$$

Investigate the integrability of this system and the behavior of its solutions.

Does Problem 13 *remain open, when the system can be generalized by adding or excluding some terms from the right-hand sides of the system, provided that the classical Lorenz system indicated below can be obtained ?*

Comments to Problem 13: If we denote $g = -\sigma$, $h = \sigma$, $a = 0$, $p = r$, $q = -1$, $b = -1$, $s = m = 0$, $n = -\beta$, $c = 1$, then we obtain the classical form of Lorenz's system [24]. The determining equations of Lie algebras and the formula of the Lie integrating factor can be found in papers [21], [22].

14. a) Determine the form of polynomial systems of type (1) $(j = 2, 3, 4, 5)$*, which admit Lie operators with degree coordinates* ≥ 2 *besides the partial derivatives of the phase variables. Determine the comitants of these systems with respect to the Lie algebra admitted by examined system. Carry out the qualitative investigation of these systems.*

b) Determine the form of polynomial systems of type (1) $(j = 2, 3, 4, 5)$*, which admit Lie operators with rational function coordinates besides the partial derivatives of the phase variables. Determine the invariants and comitants of this Lie algebra and carry out the qualitative investigation of these systems.*

Comments to Problem 14: The author did not know any examples that would give a positive or negative answer to this problem.

15. a) Study the proprieties of the factorized systems $s(1,2)/GL(2,\mathbb{R})$ *and* $s(0,1,2,3)/GL(2,\mathbb{R})$ *from* [20].

b) Complete the classification of the dimension of the orbits for system $s(0,1,2,3)$ determining more successfully the invariant elements, which form the Krull dimension of Sibirsky algebra of comitants for this system, which can separate all non-singular invariant variety [6]-[8], [10].

Comments to Problem 15: The factorized systems [20] belong to non-singular invariant varieties that contain $GL(2,\mathbb{R})$ - orbits of the maximal dimension [6]-[8], [10]. These varieties are closed sets and the differential systems, which are on these orbits, are the richest in qualitative properties.

16. a) Obtain the classification of $GL(3,\mathbb{R})$ – orbits for system (1) *of the Darboux type $s^3(1,2)$ şi $s^3(1,3)$ and determine their factored systems.*

b) Carry out investigation on the stability of the Lyapunov unperturbed motion governed by systems 16 a).

Comments to Problem 16: The factorized systems defined in [20] for the two-dimensional case can also be extended to the ternary case. But here it is necessary to build the algebraic base of $GL(3,\mathbb{R})$ - comitants and invariants for these systems.

17. *Let the ternary linear system be given*

$$\frac{dx^j}{dt} = a^j_\alpha x^\alpha (j,\alpha = 1,2,3). \tag{4}$$

a) *Prove that the centro-affine invariants*

$$\theta_1 = a^\alpha_\alpha,\ \theta_2 = a^\alpha_\beta a^\beta_\alpha,\ \theta_3 = a^\alpha_\gamma a^\beta_\alpha a^\gamma_\beta \tag{5}$$

form the polynomial base of system (4).

b) *Prove that the centro-affine comitants*

$$\sigma_1 = a^\alpha_\mu a^\delta_\beta a^\gamma_\alpha x^\delta x^\mu x^\nu \varepsilon_{\beta\gamma\nu},\ \chi_1 = x^\alpha u_\alpha,\ \chi_2 = a^\alpha_\beta x^\beta u_\alpha,$$
$$\chi_3 = a^\alpha_\gamma a^\beta_\alpha x^\gamma u_\beta,\ \delta_4 = a^\alpha_\gamma a^\beta_p a^\gamma_q u_\alpha u\beta u_r \varepsilon_{pqr} \tag{6}$$

together with the invariants (5) *forms the polynomial base of comitants and invariants for system* (4).

Comments to Problem 17: The centro-affine invariants (5) and comitants (6) of system (4) were studied in [21]. Here is brought the syzygy

$$\chi_1(l\chi_1 + m\chi_2)^2 + [n(l\chi_1 + m\chi_2) - m\chi_3](\chi_2^2 + \chi_1\chi_3) + l\chi_2(\chi_2^2 -$$
$$-3\chi_1\chi_3) + \chi_3(n\chi_2 - \chi_3)^2 + \delta_4\sigma_1 = 0,$$

where

$$l = \frac{1}{6}(\theta_1^3 - 3\theta_1\theta_2 + 2\theta_3),\ m = \frac{1}{2}(\theta_2 - \theta_1^2),\ n = \theta_1.$$

But, until now, the answer to Problem 17 is not known.

7. The Center and Focus Problem

18. *Prove that the number of essential Poincaré-Lyapunov quantities* [11], [13] *for system* (1) *with* $\Gamma = \{1, m_1, m_2, \ldots, m_l\}$ *is equal to the Krull dimension of Sibirsky algebra of invariants of system* (1) *with* $\Gamma = \{m_1, m_2, \ldots, m_l\}$.

Comments to Problem 18: This statement is true for systems $s(1,2)$ and $s(1,3)$. We suppose that the proof of this hypothesis came from the study of many invariants and comitants of systems (1) with $\Gamma = \{1, m_1, m_2, \ldots, m_l\}$ and $\Gamma = \{m_1, m_2, \ldots, m_l\}$ with respect to the groups $GL(2,\mathbb{R}) \supset SL(2,\mathbb{R}) \supset SO(2,\mathbb{R})$.

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