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Extension of linear operators with applications

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Abstract. The article presents a method for solving characteristic singular integral equations perturbed with compact operators. The method consists in reducing the solution of these equations to the solution of the systems of singular (unperturbed) equations, which are solved by factoring the coefficients of the obtained systems. The method presented concerns some results of Gohberg and Krupnik and can be used in solving other classes of functional equations with composite operators that fit into the scheme described by Theorem 1.1.

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Extensii ale operatorilor liniari cu aplicaţii

Rezumat. În lucrare este prezentată o metodă de rezolvare a unor ecuaţii integrale singulare caracteristice perturbate cu operatori compacţi. Metoda constă în reducerea soluţionării acestor ecuaţii la soluţionarea unor sisteme de ecuaţii singulare (neperturbate), care se rezolvă prin factorizarea coeficienţilor sistemelor obţinute. Metoda prezentată are tangenţă cu unele rezultate ale lui Gohberg şi Krupnik şi ar putea fi folosită la rezolvarea altor clase de ecuaţii funcţionale cu operatori compuşi, care se încadrează în schema descrisă de Teorema 1.1.

Cuvinte-cheie: ecuaţii integrale singulare, operator compact, factorizare.

Introduction

In the monographs of Muskhelishvili [1] and Gakhov [2] and in other works it is indicated that the solution of singular integral equations can be found in rare cases. Even in these cases finding the exact solution requires complicated calculations of singular integrals accompanied with combersome theoretical and computational difficulties. The content of this article, as well as the studies of other authors [3], [4], [5], [6], [7], [8], [9] once again confirms the statement of academicians Muskhelishvili and Gakhov.

In this paper we study the problem of solving singular integral equations containing compact terms

$$A\varphi \equiv a(t)\varphi(t) + \frac{b(t)}{\pi i}\int_\Gamma \frac{\varphi(\tau)}{\tau - t}d\tau + \int_\Gamma k(t,\tau)\varphi(\tau) = f(t), \tag{1}$$

where function $k(t,\tau)$ is continuous or with weak singularities. To each operator A, defined by the left hand side of (1), according to the rule described in Theorem 1.1, we associate a matrix operator $\widetilde{A}$

$$\widetilde{A}\psi = C(t)\psi(t) + \frac{D(t)}{\pi i}\int_{\Gamma}\frac{\psi(\tau)}{\tau - t}d\tau, \tag{2}$$

which has the property that both operators A and $\widetilde{A}$ are or are not invertible in the respective spaces. Thus, the solution of the considered equation is reduced to a similar problem for a system of equations, which turns out to be a system of ”ordinary” singular integral equations, without compact terms. The obtained system of singular integral equations is solved by the method of factoring the coefficients, a method developed in the monograph [3] etc. An explicit expression of the solution of the considered equation is obtained through the solution of the system of equations. The method presented in this paper is based on the results of the works of Gohberg and Krupnik [10], and can be used for solving other classes of functional equations with composite operators that fit into the scheme described by Theorem 1.1.

To invert operators of the form (2), where $C(t)$ and $D(t)$ are matrices of continuous functions satisfying the conditions $\det(C(t) \mp D(t)) \neq 0$, it is necessary (see [3]) to factorize the matrix

$$G(t) = (C(t) - D(t))^{-1}(C(t) + D(t)).$$

This means that the matrix $G(t)$ must be represented in the form

$$G(t) = G_{-}(t)\cdot diag\left(t^{k_1}, t^{k_2}, \ldots, t^{k_n}\right)\cdot G_{+}(t),$$

where $G_{+}(z)$ $(G_{-}(z))$ are matrices of functions with analytic elements in the domains $F_{+} = \{z|\ |z| < 1\}$ $(F_{-} = \{z|\ |z| > 1\})$, and $k_1, k_2, \ldots, k_n$ are integers called *partial indices* of the operator $\widetilde{A}$. Depending on the numbers $k_1, k_2, \ldots, k_n$, the operator $\widetilde{A}$ can be invertible, left invertible or right invertible. In particular, if all numbers $k_1, k_2, \ldots, k_n$ are positive, then the operator $\widetilde{A}$ is left invertible, if all are negative, then $\widetilde{A}$ is right invertible, and finally, if all numbers are equal to zero, then $\widetilde{A}$ is invertible. We will apply these results to the inversion of the operator $\widetilde{A}$.

1. Extension of linear operators

Let V be some Banach algebra of linear bounded operators acting in a Banach space B, and $V^{(m)}$ be a Banach algebra of elements of the form $\left\|A_{jk}\right\|_{j,k=1}^{m}$, where $A_{jk} \in V$. If $B^{(m)}$ is a Banach space of vectors $X = \lfloor x_1, \ldots, x_m \rfloor$ with elements $x_j \in B$ and with the norm $\|X\| = \max_k \|x_k\|$, then $V^{(m)}$ is a Banach algebra of linear bounded operators in the space $B^{(m)}$. Denote by I and I_m the unit operators acting in the spaces V and, respectively, $V^{(m)}$. Suppose also that $I \in V$ and $I_m \in V^{(m)}$. Assume that

$$A = \sum_{j=1}^{r} A_{j1} A_{j2} \cdots A_{js}, \tag{3}$$

where $A_{jk} \in V$. The operator $\widetilde{A} \in V^{(m)}$ is called a *linear extension o*f the operator A (of order m) if:

1) the elements of the matrix $\widetilde{A}$ are linear combinations of the elements A_{jk} and the unit operator;

2) there exist invertible operators X and Z from the algebra $V^{(m)}$ such that

$$\widetilde{A} = Y \cdot \begin{pmatrix} I_{m-1} & 0 \\ 0 & A \end{pmatrix} \cdot Z. \tag{4}$$

It is easy to see that the operator $A = \sum_{j=1}^{r} A_{j1} A_{j2} \cdots A_{js}$ and its linear extension $\widetilde{A}$ (if it exists) are Noetherian (or are not Noetherian) simultaneously in the spaces B and $B^{(m)}$, respectively, and

$$dimker A = dimker \widetilde{A} \quad \text{and} \quad dimcoker A = dimcoker \widetilde{A}.$$

The following Theorem holds

Theorem 1.1. *Each element A from the algebra V of the form $A = \sum_{j=1}^{r} A_{j1} A_{j2} \cdots A_{js}$ $(A_{jk} \in V)$ admits the linear expansion (of order $m \le r(s+1)+1$).*

Proof. Let us compose the following matrix of order $r(s+1)$

$$M = \begin{pmatrix} I_r & B_1 & 0 & \cdots & 0 \\ 0 & I_r & B_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & B_s \\ 0 & 0 & 0 & \cdots & I_r \end{pmatrix},$$

where

$$B_k = \begin{pmatrix} A_{1k} & 0 & \cdots & 0 \\ 0 & A_{2k} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & A_{kk} \end{pmatrix}.$$

Denote by F a column of the length $r(s+1)$, whose top rs elements are equal to zero and the bottom r elements are equal to the identity operator. Let also $G = \| \underbrace{I, \ldots, I}_{r}, \underbrace{0, \cdots, 0}_{rs} \|$.

It is easy to verify the validity of the expansion

$$\begin{pmatrix} M & F \\ G & 0 \end{pmatrix} = \begin{pmatrix} I_{m-1} & 0 \\ H & I \end{pmatrix} \cdot \begin{pmatrix} I_{m-1} & 0 \\ 0 & A \end{pmatrix} \cdot \begin{pmatrix} I_{m-1} & F \\ 0 & I \end{pmatrix}, \tag{5}$$

with $m = r\,(s+1)+1$, $H = \|M_0, M_1, \dots, M_s\|$, where $M_0 = \|\underbrace{I, \dots, I}_{r}\|$ and

$$M_k = \|A_{11}A_{12}\dots A_{1j}, A_{21}A_{22}\dots A_{2j}, \dots, A_{k1}A_{k2}\dots A_{rk}\| \quad (k = 1, 2, \dots, s).$$

Note that the operators

$$Y = \begin{pmatrix} I_{m-1} & 0 \\ H & I \end{pmatrix}, \quad Z = \begin{pmatrix} I_{m-1} & F \\ 0 & I \end{pmatrix}$$

are invertible in the space $B^{(m)}$ and their inverse operators are of the form

$$Y^{-1} = \begin{pmatrix} I_{m-1} & 0 \\ -H & I \end{pmatrix}, \quad Z^{-1} = \begin{pmatrix} I_{m-1} & -F \\ 0 & I \end{pmatrix},$$

respectively. Therefore, the operator

$$\widetilde{A} = \begin{pmatrix} M & F \\ G & 0 \end{pmatrix}$$

is a linear extension of the operator A. Theorem 1.1 is proved. □

Note that the extreme factors on the right hand side of equality (5) are triangular matrices with unity on the main diagonal, therefore, they are invertible. This implies that the operator A is normally solvable. It is Noetherian or invertible if and only if the operator $\widetilde{A}$ is of such type.

Corollary 1.1. *The operator A is invertible in the space B if and only if the operator*

$$\widetilde{A} = \begin{pmatrix} M & F \\ G & 0 \end{pmatrix}$$

is invertible in the space $B^{n(N+1)+1}$.

Corollary 1.2. *Let A_0, C_k, $D_k \in L(B)$ $(k = 1, 2, \dots, n)$ and $\widetilde{A}$ be an operator defined by the equality*

$$\widetilde{A} = \begin{pmatrix} I & 0 & \dots & 0 & D_1 \\ 0 & I & \dots & 0 & D_2 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & I & D_n \\ C_1 & C_2 & \dots & C_n & A_0 \end{pmatrix}. \tag{6}$$

In this case, the following statements are true:

$$\widetilde{A} \in GL\left(B^{n+1}\right) \Leftrightarrow A = A_0 - \sum_{k=1}^{n} C_k D_k \in GL(B).$$

Indeed, we note that

$$A_0 - \sum_{k=1}^{n} B_k C_k = det\widehat{A}$$

and the validity of the following equality is directly verified

$$\begin{pmatrix} I & 0 & \dots & 0 & D_1 \\ 0 & I & \dots & 0 & D_2 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & I & D_n \\ C_1 & C_2 & \dots & C_n & A_0 \end{pmatrix} = \begin{pmatrix} I & 0 & \dots & 0 & 0 \\ 0 & I & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & I & 0 \\ C_1 & C_2 & \dots & C_n & I \end{pmatrix} \times$$

$$\times \begin{pmatrix} I & 0 & \dots & 0 & D_1 \\ 0 & I & \dots & 0 & D_2 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & I & D_n \\ 0 & 0 & \dots & 0 & A_0 - \Delta \end{pmatrix} \times \begin{pmatrix} I & 0 & \dots & 0 & D_1 \\ 0 & I & \dots & 0 & D_2 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & I & D_n \\ 0 & 0 & \dots & 0 & I \end{pmatrix}, \qquad (7)$$

where $\Delta = \sum_{k=1}^{n} C_k D_k$. Since the left and right factors of equality (7) are invertible operators in the space B^{n+1}, then $\widetilde{A} \in GL\left(B^{n+1}\right) \Leftrightarrow A_0 - \sum_{k=1}^{n} C_k D_k$.

Corollary 1.3. *If the vector $\varphi = (\varphi_1, \dots, \varphi_{n+1}) \in B^{n+1}$ is a solution of the equation $\widetilde{A}\varphi = \widetilde{\psi}$ with the right hand side $\widetilde{\psi} = (0, 0, \dots, \psi)$, then the equality $A\varphi_{n+1} = \psi$ holds. That is, the coordinate standing on $n+1$ place of the solution of the equation $\widetilde{A}\varphi = \widetilde{\psi}$ with the right hand side $\widetilde{\psi} = (0, 0, \dots, \psi)$ is the solution of the equation $Af = \psi$. Solutions of this type exhaust all solutions of the equation $Af = \psi$.*

Indeed, from equality (6) it follows that the equation $\widetilde{A}\varphi = \widetilde{\psi}$ is equivalent to the system of equations:

$$\begin{cases} \varphi_1 + D_1\varphi_{n+1} = 0, \\ \varphi_2 + D_2\varphi_{n+1} = 0, \\ \quad \dots \\ \varphi_n + D_n\varphi_{n+1} = 0, \\ A\varphi_{n+1} = \psi. \end{cases}$$

This implies the assertion of Corollary 1.3.

Remark 1.1. It is clear that Theorem 1.1 and Corollary 1.3 can be effectively applied only in the cases when the solvability criteria for the operators of the form (4) are known. This is done in the cases when the operator $\widetilde{A}$ is a singular integral operator.

2. Application to the solution of singular equations

We apply Theorem 1.1 and Corollary 1.3 to solve singular integral equations perturbed by compact operators. Such equations are also called *complete singular integral equations* (see [3]). We noted above that singular integral equations are solved in rather rare cases. This problem becomes more complicated (see [1]) in the case of systems of singular equations being related to the problem of the factorization of functional matrices and the solution of the corresponding Riemann problem.Taking into account these difficulties, we will study equations that can be reduced to systems of equations whose coefficients can be effectively factorized.

Before we pass to solving the proposed equations, we pay our attention to an unexpected result that is obtained by means of Theorem 1.1. It is known that the theory of singular integral equations

$$a(t)\,\varphi(t) + \frac{b(t)}{\pi i}\int_{\Gamma}\frac{\varphi(\tau)}{\tau - t}d\tau = f(t) \tag{8}$$

is well developed and presented in monographs [1]-[3] and others. Usually, the contour of integration Γ is assumed to be of Lyapunov type, and in the case of a contour with angular points, certain difficulties appear. Let Γ be a contour having an angular point of size $\frac{\pi}{2}$ and consider the equation (8). After certain integral (equivalent) transformations, the operator

$$V\varphi \equiv a(t)\,\varphi(t) + \frac{b(t)}{\pi i}\int_{\Gamma}\frac{\varphi(\tau)}{\tau - t}d\tau,$$

determined by the left hand side of equation (8), turns into the operator

$$W\psi = \tilde{a}(t)\,\psi(t) + \frac{\tilde{b}(t)}{\pi i}\int_{\widetilde{\Gamma}}\frac{\psi(\tau)}{\tau - t}d\tau + \frac{\tilde{b}(t)}{2\pi i}\left[\int_{\widetilde{\Gamma}}\left(\frac{\sqrt{t+1}}{(\tau - t)\sqrt{\tau + 1}} - \frac{1}{\tau - t}\right)\psi(\tau)\,d\tau\right],$$

where $\tilde{a}(t) = a\sqrt{t+1}$, $\tilde{b}(t) = b\sqrt{t+1}$ and $\widetilde{\Gamma}$ is already a Lyapunov contour! The operator W satisfies the conditions of Theorem 1.1 and Corollary 1.3 by means of which (we do not dwell on the details) we have that the operator V is Noetherian if and only if the following operator is Noetherian

$$W_0\psi = \tilde{a}(t)\,\psi(t) + \frac{\tilde{b}(t)}{\pi i}\int_{\widetilde{\Gamma}}\frac{\psi(\tau)}{\tau - t}d\tau.$$

Thus, the study of a singular operator, in the case of the contour with an angular point, is reduced to the study of a similar operator on a Lyapunov type contour. From these results it follows that the Noetherian conditions of the operator W_0 do not change being perturbed by the operators of the form

$$H\psi = \frac{\tilde{b}(t)}{2\pi i}\left[\int_{\widetilde{\Gamma}}\left(\frac{\sqrt{t+1}}{(\tau-t)\sqrt{\tau+1}} - \frac{1}{\tau-t}\right)\right]\psi(\tau)\,d\tau,$$

which is not compact!

Let $\Gamma = \{t \in \mathbb{C} : \ |t| = 1\}$. In space $B = L_p(\Gamma)$ $(p > 1)$, we consider the equation

$$\frac{1}{\pi i}\int_{\Gamma}\frac{\tau^3 - t^3}{(\tau-t)^2}\varphi(\tau)d\tau = \psi(t). \tag{9}$$

The left hand side of equation (9) corresponds to the operator, which can be written in the following form

$$(A\varphi)(t) = \frac{3t^2}{\pi i}\int_{\Gamma}\frac{\varphi(\tau)}{\tau-t}d\tau + (T\varphi)(t),$$

where

$$(T\varphi)(t) = \frac{1}{\pi i}\int_{\Gamma}(\tau+2t)\varphi(\tau)d\tau.$$

The operator T, being an integral operator with a continuous kernel, is compact in $L_p(\Gamma)$. In the case of studying the Noetherian properties and the index of the operator A, the operator T can be neglected, i.e. the operator T does not affect the Noetherian properties of the operator A. However, this does not happen if the operator A is inverted or in the case of solving the equation $A\varphi = \psi$.

Let

$$(S\varphi)(t) = \frac{1}{\pi i}\int_{\Gamma}(\tau-t)^{-1}\varphi(\tau)d\tau, \quad (B\varphi)(t) = t\varphi(t), \tag{10}$$

then the operator A can be written as follows

$$A = SB^2 + BSB + B^2S,$$

and the corresponding operator $\widetilde{A}$, defined by equality (6), of the operator A has the form

$$\widetilde{A} = \begin{pmatrix} I & 0 & B^2 \\ 0 & I & B \\ -S & -BS & B^2S \end{pmatrix}.$$

By virtue of Corollary 1.3, any solution of equation (9) can be obtained as the last coordinate φ_3 of the solution of equation $\widetilde{A}\widetilde{\varphi} = \widetilde{\psi}$ $(\widetilde{\varphi} = (\varphi_1, \ \varphi_2, \varphi_3), \ \widetilde{\psi} = (0, 0, \psi))$.

The operator $\widetilde{A}$ represents the characteristic singular operator with matrix coefficients:

$$\widetilde{A} = \begin{pmatrix} 1 & 0 & t^2 \\ 0 & 1 & t \\ -1 & -t & t^2 \end{pmatrix} P + \begin{pmatrix} 1 & 0 & t^2 \\ 0 & 1 & t \\ 1 & t & -t^2 \end{pmatrix} Q =$$

$$= \begin{pmatrix} 1 & 0 & t^2 \\ 0 & 1 & t \\ 1 & t & -t^2 \end{pmatrix} \left[\frac{1}{3} \begin{pmatrix} 1 & -2t & 2t^2 \\ -2t^{-1} & 1 & 2t \\ 2t^{-2} & 2t^{-1} & 1 \end{pmatrix} P + Q \right],$$

where $P = \frac{1}{2} diag(I+S)$ and $Q = \frac{1}{2} diag\,(I-S)$.

The matrix that is the coefficient of the operator P can be factorized:

$$\frac{1}{3} \begin{pmatrix} 1 & -2t & 2t^2 \\ -2t^{-1} & 1 & 2t \\ 2t^{-2} & 2t^{-1} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -2t^{-1} & 1 & 0 \\ 2t^{-2} & 2t^{-1} & 1 \end{pmatrix} \begin{pmatrix} 1/3 & -2t/3 & 2t^2/3 \\ 0 & -1 & 2t \\ 0 & 0 & 3 \end{pmatrix} = c_- \cdot c_+.$$

Since the partial indices under this factorization are equal to zero, the operator $\widetilde{A}$ is invertible in B^3 [1] and its inverse operator is defined by the following equality:

$$\widetilde{A}^{-1} = \left[\begin{pmatrix} 1/3 & -2t/3 & 2t^{-2}/3 \\ 0 & -1 & 2t \\ 0 & 0 & 3 \end{pmatrix}^{-1} P + \begin{pmatrix} 1 & 0 & 0 \\ -2t^{-1} & 1 & 0 \\ 2t^{-2} & -2t^{-1} & 1 \end{pmatrix} Q \right] \times$$

$$\times \begin{pmatrix} 1 & 0 & 0 \\ -2t^{-1} & 1 & 0 \\ 2t^{-2} & -2t^{-1} & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 & t^2 \\ 0 & 1 & t \\ 1 & t & t^2 \end{pmatrix} =$$

$$= \left[\begin{pmatrix} 3 & -2t & 2t^2/3 \\ 0 & -1 & 2t/3 \\ 0 & 0 & 1/3 \end{pmatrix} P + \begin{pmatrix} 1 & 0 & 0 \\ -2t^{-1} & 1 & 0 \\ 2t^{-2} & -2t^{-1} & 1 \end{pmatrix} Q \right] \cdot \begin{pmatrix} 2/3 & -t/3 & 1/3 \\ t^{-1} & 0 & t^{-1} \\ t^{-2} & t^{-1} & t^{-2} \end{pmatrix}.$$

According to the scheme of inversion of the singular operator A, given in Corollary 1.3, we find

$$A^{-1} \begin{pmatrix} 0 \\ 0 \\ \psi \end{pmatrix} = \begin{pmatrix} 3P+Q & -2tP & 2t^2/3P \\ -2/tQ & -P+Q & 2t/3P \\ 2/t^2Q & -2/tQ & 1/3P+Q \end{pmatrix} \begin{pmatrix} 1/3\psi \\ 1/t\psi \\ 1/t^2\psi \end{pmatrix}.$$

Hence

$$A^{-1}\psi = \begin{pmatrix} \frac{2}{t^2}Q & \frac{-2}{t}Q & \frac{1}{3}P+Q \end{pmatrix} \begin{pmatrix} 1/3\psi \\ 1/t\psi \\ 1/t^2\psi \end{pmatrix} =$$

$$= \left(-\frac{1}{3} SB^{-2} + B^{-1} SB^{-1} - \frac{2}{3} B^{-2} S \right) \psi .$$

Thus, equation (9) is uniquely solvable and its solution is found by the formula

$$\varphi(t) = \frac{1}{3\pi i} \int_\Gamma \frac{3\tau t - 2\tau^2 - t^2}{\tau^2 t^2 (\tau - t)} \psi(\tau)\, d\tau. \tag{11}$$

Consider two more equations

$$\frac{t^2+1}{t} \varphi(t) + \frac{1}{\pi i} \int_\Gamma \frac{\tau t - 1}{\tau(\tau - t)} \varphi(\tau)\, d\tau = \psi(t) \tag{12}$$

and

$$\frac{t^2+1}{t} f(t) + \frac{1}{\pi i} \int_\Gamma \frac{1 - \tau t}{\tau(\tau - t)} f(\tau)\, d\tau = \psi(t). \tag{13}$$

Let A and C be operators defined by the left hand sides of equalities (12) and (13), respectively. It is directly verified that in this case the operators A and C differ from the characteristic singular operators by compact terms, i.e., equations (12) and (13) are complete singular equations. With the notation (10), the operators A and B can be written in the following form

$$A = B + B^{-1} + BS - SB^{-1}.$$

Since $S^* = S$ and $C^* = C^{-1}$, then $C = A^*$. As operators $\widetilde{A}$ and $\widetilde{C}$, appearing in Corollary 1.2, we can take

$$\widetilde{A} = \begin{pmatrix} I & B^{-1} \\ S & B + B^{-1} + BS \end{pmatrix}, \quad \tilde{C} = \begin{pmatrix} I & B^{-1} \\ -S & B + B^{-1} - BS \end{pmatrix}.$$

The operators $\widetilde{A}$ and $\widetilde{C}$ (as in the previous example) are characteristic singular operators with matrix coefficients:

$$\widetilde{A} = \begin{pmatrix} 1 & t^{-1} \\ -1 & t^{-1} \end{pmatrix} \left[\begin{pmatrix} 0 & -t \\ t & 1+t^2 \end{pmatrix} P + Q \right],$$

$$\widetilde{C} = \begin{pmatrix} 1 & t^{-1} \\ 1 & 2t + t^{-1} \end{pmatrix} \left[\begin{pmatrix} 1+t^{-2} & t^{-1} \\ -t^{-1} & 0 \end{pmatrix} P + Q \right].$$

However, unlike the previous example, the matrices-coefficients of P have non-zero partial indices. In the case of the operator $\widetilde{A}$ this index is equal to 2, and in the case of $\widetilde{C}$ it is equal to -2. This results from the factorization of the coefficients of the operator P :

$$\begin{pmatrix} 0 & -t \\ t & 1+t^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -t^{-1} & 1 \end{pmatrix} \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & t \end{pmatrix},$$

$$\begin{pmatrix} 1+t^{-2} & t^{-1} \\ -t^{-1} & 0 \end{pmatrix} = \begin{pmatrix} t^{-1} & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} t^{-1} & 0 \\ 0 & t^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}.$$

By virtue of well-known results from the theory of singular equations with matrix coefficients, the operator $\widetilde{A}$ is left invertible, while the operator $\widetilde{C}$ is left invertible. This implies the operator A to be left invertible and the operator C to be right invertible. The general solution of the equation $\widetilde{C}\varphi = 0$ is of the form (see [1]):

$$\begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = \begin{pmatrix} \alpha\left(1 - t^{-1} - t^{-2}\right) \\ \beta(1 - t + t^{-1}) \end{pmatrix}$$

and the particular solution of the equation $\widetilde{C}\varphi = \psi$ has de form

$$\begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ (t^2 P + Q)t^{-1}\psi(t) \end{pmatrix}.$$

Thus, equation (13) is solvable for any right hand side and its general solution is of the form

$$f(t) = \beta\left(1 - t + t^{-1}\right) + \frac{t^2+1}{4}\psi(t) + \frac{t^2-1}{4\pi i}\int_\Gamma \frac{\psi(\tau)}{\tau(\tau - t)}d\tau,$$

where $\beta \in \mathbb{C}$. Equation (12) is not solvable for any right hand side. Since the operator A is left invertible, it is normally solvable. For its solvability it is necessary and sufficient that the right hand side of ψ be orthogonal to each solution of the equation $C\varphi = 0$, i.e. to fulfill the condition

$$\int_\Gamma (1 - t + t^{-1})\psi(t)|dt| = 0.$$

If this condition is satisfied, then equation (12) has a unique solution, which can be found by formula

$$\varphi(t) = \frac{t+1}{4t}\psi(t) + \frac{1}{4\pi i}\int_\Gamma \frac{\tau + t + \tau t - \tau^2 t}{\tau^2 t(\tau - t)}\psi(\tau)\,d\tau.$$

This solution is obtained according to the scheme proposed in Corollaries 1.2 and 1.3.

3. Solution of integral equations by the regularization method

Let A be some Noetherian operator. If the regularizing operator M for A is known, then the solution of the equation

$$A\varphi = f \tag{14}$$

can be reduced to solving the equation

$$MA\varphi = Mf, \tag{15}$$

in which the operator $MA - I$ is completely continuous. To equation (15) may be applied many methods developed for inverting operators of the form $I - T$, where T is a completely continuous operator.

A special interest is represented by the case when equations (14) and (15) are equivalent for any vector f, i.e., equations (14) and (15) are simultaneously solvable or unsolvable, and in the case of solvability, they have the same solutions. This happens to be if and only if $KerM = 0$. Indeed, if $MA\varphi = 0$, then $A\varphi = h$, where $h \in KerM$.

Assume that equations (14) and (15) are equivalent, then either $KerM = \{0\}$, or $dim\, KerM > 0$ and $KerM \bigcap ImA = \{0\}$. The last assertion is impossible, since in this case the equations $A\varphi = f \quad (f \in KerM)$ and $MA\varphi = Mf = 0$ are not equivalent. Conversely, if $KerM = \{0\}$, then it is obvious that equations (14) and (15) are equivalent.

We say that an operator A admits equivalent regularization if it has a regularizing operator M for which equations (14) and (15) are equivalent for any vector f. In this case, the operator M is called an *equivalent regularizing operator* for A.

It follows from the above that an operator M is an equivalent regularizer for A if it is a regularizer for A and also is left invertible.

Theorem 3.1. *(see* [11]*) Operator A admits an equivalent regularization if and only if*

$$IndA \geq 0. \tag{16}$$

Indeed, if M is an equivalent regularizer for A, then it is left invertible and, therefore, $IndM \leq 0$. Since $IndMA = IndM + IndA = 0$, then $IndA \geq 0$. Let condition (16) be satisfied and M_1 be a regularizer for A. Then M_1 is Noetherian and $IndM_1 + IndA = 0$. Hence, $IndM_1 \leq 0$. According to the results of [12], the operator M_1 can be represented as $M_1 = M + T$, where M is left invertible. Obviously, M is an equivalent regularizing operator for A. Theorem 3.1 is proved.

We now consider the case when the Noetherian operator A does not admit equivalent regularization, that is, the following condition holds

$$IndA < 0. \tag{17}$$

Let M_1 be a regularizer for A. Since $IndM_1 > 0$, then according to the results of work [12] the operator M_1 can be represented as $M_1 = M + T$, where M is right invertible. The operator M is also a regularizing operator for A and all solutions of the equation

$$A\varphi = f \quad (f \in ImA)$$

can be obtained by formula $\varphi = M\psi$, where ψ runs through all solutions of the equation

$$AM\psi = f.$$

As an example, to illustrate the stated theory, let us regularize (see [2]) the following singular integral equation

$$A\varphi \equiv \left(t+t^{-1}\right)\varphi(t)+\frac{t-t^{-1}}{\pi i}\int_\Gamma \frac{\varphi(\tau)}{\tau-t}d\tau-\frac{1}{2\pi i}\int_\Gamma\left(t+t^{-1}\right)\left(\tau+\tau^{-1}\right)\varphi(\tau)d\tau=2t^2, \tag{18}$$

in various ways, where Γ is the unit circle.

The regular part of the kernel is degenerate. Therefore, in the same way that is used in solving the Fredholm equations with the degenerate kernel, the equation (17) can be reduced to a combination of the characteristic equation and a linear algebraic equation and, it can be solved in the closed form. Thus, there is no necessity for regularization here, but the equation under consideration is convenient for illustrating general methods on it. Here all the calculations can be carried out to the end.

For further reasoning, we first solve this equation denoting

$$\frac{1}{2\pi i}\int_\Gamma\left(\tau+\tau^{-1}\right)\varphi(\tau)d\tau=C, \tag{19}$$

We write it in the characteristic form:

$$\left(t+t^{-1}\right)\varphi(t)+\frac{t-t^{-1}}{\pi i}\int_\Gamma\frac{\varphi(\tau)}{\tau-t}d\tau=2t^2+C\left(t+t^{-1}\right).$$

For the corresponding Riemann boundary value problem [2]

$$\Phi^+(t)=t^{-2}\Phi^-(t)+t+\frac{C}{2}(1+t^{-2})$$

the index $\kappa=-2$ and the solvability conditions will be satisfied only for $C=0$. In this case

$$\Phi^+(z)=z,\quad \Phi^-(z)=0.$$

From here we obtain the solution of equation (18):

$$\varphi(t)=\Phi^+(t)-\Phi^-(t)=t.$$

Putting the last expression into the equality (19), we make sure that it is satisfied at $C=0$. Therefore, this equation is solvable and has a unique solution $\varphi(t)=t$.

Regularization on the left. Since the index of the equation is $\kappa=-2<0$, then any of its regularizing operators will have eigenfunctions (at least two). Therefore, regularization on the left leads, generally speaking, to an equation that is not equivalent to the original one (regularization is not equivalent).

Consider first the regularization on the left using the regularizer R:

$$(Rh)(t)\equiv\left(t+t^{-1}\right)h(t)-\frac{t-t^{-1}}{\pi i}\int_\Gamma\frac{h(\tau)}{\tau-t}d\tau. \tag{20}$$

The corresponding Riemann boundary value problem

$$H^{+}(t) = t^{2} H^{-}(t)$$

has now the index $\kappa = 2$. Finding the eigenfunctions of the operator R, we obtain that

$$\lambda_1(t) = 1 - t^{-2}, \quad \lambda_2(t) = t - t^{-1}.$$

Based on the general theory, the regular equation $RA\varphi = Rf$ will be equivalent to the singular equation

$$A\varphi = f + \alpha_1 \lambda_1 + \alpha_2 \lambda_2, \tag{21}$$

where α_1, α_2 are some constants, which can be either arbitrary or defined. Taking into account (19), we write equation (21) in the characteristic form:

$$\left(t + t^{-1}\right) \varphi(t) + \frac{t - t^{-1}}{\pi i} \int_{\Gamma} \frac{\varphi(\tau)}{\tau - t} d\tau = 2t^{2} + C\left(t + t^{-1}\right) + \alpha_1 \left(1 - t^{-2}\right) + \alpha_2 \left(t - t^{-1}\right).$$

The corresponding Riemann boundary value problem for this equation is

$$\Phi^{+}(t) = t^{-2} \Phi^{-}(t) + t + \frac{C}{2}\left(1 + t^{-2}\right) + \frac{\alpha_1}{2}\left(t^{-1} - t^{-3}\right) + \frac{\alpha_2}{2}\left(1 - t^{-2}\right).$$

Its solution can be presented in the form

$$\Phi^{+}(z) = z + \frac{1}{2}C + \frac{1}{2}\alpha_2, \quad \Phi^{-}(t) = \frac{1}{2} z^{2}\left[\alpha_1 z^{-3} + (\alpha_2 - C) z^{-2} - \alpha_1 z^{-1}\right].$$

The solvability condition will give $\alpha_1 = 0,\ \alpha_2 = C$. Then, the solution of equation (21) is determined by formula $\varphi(t) = \Phi^{+}(t) - \Phi^{-}(t) = t + C$.

Substituting the found value of φ into equality (19), we obtain the identity $C = C$. Therefore, the constant $\alpha_2 = C$ remains to be arbitrary and the regularized equation is not equivalent to the original equation, but to the equation

$$A\varphi = f + \alpha_2 \lambda_2,$$

having the solution $\varphi(t) = t + C$, where C is an arbitrary constant. The last solution satisfies the original equation only at $C = 0$.

Regularization on the right. As a regularizer on the right, we take the operator R, defined by equality (20). Assuming that

$$\varphi(t) = (Rh)(t) \equiv \left(t + t^{-1}\right) h(t) - \frac{t - t^{-1}}{\pi i} \int_{\Gamma} \frac{h(\tau)}{\tau - t} d\tau \tag{22}$$

we obtain the Fredholm equation for the function $h(t)$:

$$(ARh)(t) \equiv$$

$$h(t) - \frac{1}{4\pi i} \int_{\Gamma} \left[t\left(\tau^{2} - 1 + \tau^{-2}\right) + 2\tau^{-1} + t^{-1}\left(\tau^{2} + 3 + \tau^{-2}\right) - 2t^{-2}\tau^{-1}\right] h(\tau)\, d\tau = \frac{t^{2}}{2}.$$

Solving the last equation as a degenerate one, we have

$$h(t) = \frac{t^2}{2} + \alpha\left(t - t^{-1}\right) + \beta\left(1 - t^{-2}\right),$$

where $\alpha,\ \beta$ are arbitrary constants.

Thus, the regularized equation has two linearly independent solutions with respect to $h(t)$ while the original equation (18) was solved uniquely. Substituting the found value $h(t)$ in formula (22), we obtain that

$$\varphi(t) = R\left[\frac{t^2}{2} + \alpha\left(t - t^{-1}\right) + \beta\left(1 - t^{-2}\right)\right] = t$$

is the solution of the original singular equation. The result agrees with the general theory, since the regularization on the right is equivalent for a negative index.

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