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Centers of cubic differential systems with the line at infinity of maximal multiplicity

Alexandru Şubă

Abstract. We classify all cubic differential systems with a center-focus critical point and the line at infinity of maximal multiplicity. It is proved that the critical point is of the center type if and only if the divergence of the vector field associated to differential system vanishes.

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Keywords: cubic differential system, multiple invariant line, center problem.

Centre în sistemele diferenţiale cubice ce au linia de la infinit de multiplicitate maximală

Rezumat. Sunt clasificate sistemele diferenţiale cubice ce au puncte critice de tip centru-focar şi infinitul e de multiplicitate maximală. Se arată că în punctul critic avem centru, dacă şi numai dacă divergenţa câmpului vectorial asociat sistemului diferenţial se anulează.

Cuvinte-cheie: sistem diferenţial cubic, linii invariante multiple, problema centrului.

1. Introduction

Consider the real cubic system of differential equations

$$\begin{cases} \dot{x} = y + ax^2 + cxy + fy^2 + kx^3 + mx^2y + pxy^2 + ry^3 \equiv p(x,y), \\ \dot{y} = -(x + gx^2 + dxy + by^2 + sx^3 + qx^2y + nxy^2 + ly^3) \equiv q(x,y), \\ \gcd(p,q) = 1, sx^4 + (k+q)x^3y + (m+n)x^2y^2 + (l+p)xy^3 + ry^4 \not\equiv 0. \end{cases} \tag{1}$$

The critical point $(0,0)$ of the system (1) is either a focus or a center. The problem of distinguishing between a center and a focus is called *the center problem*. It is well known that $(0,0)$ is a center if and only if the Lyapunov quantities $L_1, L_2, ..., L_j, ...$ vanish (see, for example, [2], [6], [7]). Also, the critical point $(0,0)$ is a center if the system (1) has an analytic in $(0,0)$ first integral $F(x,y)$.

The homogeneous system associated to the system (1) has the form

$$\begin{cases} \dot{x} = yZ^2 + (ax^2 + cxy + fy^2)Z + kx^3 + mx^2y + pxy^2 + ry^3 \equiv P(x,y,Z), \\ \dot{y} = -(xZ^2 + (gx^2 + dxy + by^2)Z + sx^3 + qx^2y + nxy^2 + ly^3) \equiv Q(x,y,Z). \end{cases}$$

Denote $\mathbb{X} = p(x,y)\frac{\partial}{\partial x} + q(x,y)\frac{\partial}{\partial y}$, $\mathbb{X}_\infty = P(x,y,Z)\frac{\partial}{\partial x} + Q(x,y,Z)\frac{\partial}{\partial y}$ and $E_\infty = P \cdot \mathbb{X}_\infty(Q) - Q \cdot \mathbb{X}_\infty(P)$. The polynomial E_∞ has the form $E_\infty = C_2(x,y) + C_3(x,y)Z + C_4(x,y)Z^2 + C_5(x,y)Z^3 + C_6(x,y)Z^4 + C_7(x,Y)Z^5 + C_8(x,y)Z^6$, where $C_j(x,y)$, $j = 2, ..., 8$, are polynomial in x and y. For example,

$$C_j(x,y) = D_j(a,b,c,d,f,g,k,l,m,n,p,q,r,s,x,y) \\ +D_j(b,a,d,c,g,f,l,k,n,m,q,p,s,r,y,x),\ j = 5,6, \tag{2}$$

where

$$D_5(a,b,c,d,f,g,k,l,m,n,p,q,r,s,x,y) = \\ (adg - cg^2 + 3ak + dk - 2gm + aq - 2cs - 2gs)x^5 \\ +(a^2d + ad^2 + 2abg - acg - cdg - 2fg^2 + 2bk + 2ck - gk \\ +2am - dm \quad + 2an - 4gp - cq - gq - 4as - ds - 4fs)x^4y \\ +(2a^2b + 3abd + acd - c^2g - 2afg - 3dfg + dk + fk + 3al \\ +cm - 2gm + ap - 3dp - 2aq - 3fq - 6gr - 5cs)x^3y^2,$$

$$D_6(a,b,c,d,f,g,k,l,m,n,p,q,r,s,x,y) = \\ (a^2 + ad - 2cg - g^2 - m - 2s)x^4 \\ +(2ab + ac - cd - 2ag - dg - 4fg + k - 2p - q)x^3y - 3(cg + r)x^2y^2$$

We say that the line at infinity $Z = 0$ has *multiplicity* ν if $C_2(x,y) \equiv 0, ..., C_\nu(x,y) \equiv 0$, $C_{\nu+1}(x,y) \not\equiv 0$, i.e. $\nu - 1$ is the greatest positive integer such that $Z^{\nu-1}$ divides E_∞. In particular, $Z = 0$ has multiplicity five if the following identity and non-identity in Z:

$$C_2(x,y) + C_3(x,y)Z + C_4(x,y)Z^2 + C_5(x,y)Z^3 \equiv 0,\ C_6(x,y) \not\equiv 0, \tag{3}$$

holds, i.e. $C_2(x,y) \equiv 0$, $C_3(x,y) \equiv 0$, $C_4(x,y) \equiv 0$, $C_5(x,y) \equiv 0$ and $C_6(x,y) \not\equiv 0$. If $C_2(x,y) \not\equiv 0$, then we say that $Z = 0$ has the multiplicity one. Denote by $m(Z_\infty)$ the multiplicity of the line at infinity $Z = 0$.

The algebraic line $f(x,y) = 0$ is called *invariant* for (1) if there exists a polynomial $K_f \in \mathbb{C}[x,y]$ such that the identity $\mathbb{X}(f) \equiv f \cdot K_f(x,y)$ holds. In particular, *a straight line* $\mathcal{L} \equiv \alpha x + \beta y + \gamma = 0$, $\alpha, \beta, \gamma \in \mathbb{C}$ is called *invariant* for the system (1) if there exists a polynomial $K_\mathcal{L} \in \mathbb{C}[x,y]$ such that the identity $\alpha p(x,y) + \beta q(x,y) \equiv (\alpha x + \beta y + \gamma)K_\mathcal{L}(x,y)$, $(x,y) \in \mathbb{R}^2$, i.e. $\mathbb{X}(\mathcal{L}) \equiv \mathcal{L}(x,y)K_\mathcal{L}(x,y)$, $(x,y) \in \mathbb{R}^2$, holds. Some notions on multiplicity (algebraic, integrable, infinitesimal, geometric) of an invariant algebraic line and its equivalence for polynomial differential systems are given in [1].

The cubic differential systems with multiple invariant straight lines (including the line at infinity) were studied in [11], [14], and the center problem for (1) with invariant straight lines was considered in [2], [3], [4], [5], [8], [10], [12], [13], [15].

2. Cubic systems (1) with the line at infinity of maximal multiplicity

Let $\mathcal{X} = (x, y)$, $\mathcal{A}_2 = (a, b, c, d, f, g)$, $\mathcal{A}_3 = (k, l, m, n, p, q, r, s)$, $\mathcal{U} = (u, v)$, $\mathcal{B}_2 = (A, B, C, D, F, G)$, $\mathcal{B}_3 = (K, L, M, N, P, Q, R, S)$ and $\mathcal{X} = 2^{-1}M_1\mathcal{U}$,

$$\mathcal{A}_2 = 2^{-3}\mathcal{M}_2\mathcal{B}_2,\ \mathcal{A}_3 = 2^{-4}\mathcal{M}_3\mathcal{B}_3, \tag{4}$$

where

$$\mathcal{M}_1 = \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}, \quad \mathcal{M}_2 = \begin{pmatrix} -i & i & -i & i & -i & i \\ -1 & -1 & 1 & 1 & -1 & -1 \\ -2 & -2 & 0 & 0 & 2 & 2 \\ -2i & 2i & 0 & 0 & 2i & -2i \\ i & -i & -i & i & i & -i \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix},$$

$$\mathcal{M}_3 = \begin{pmatrix} -i & i & -i & i & -i & i & -i & i \\ i & -i & -i & i & i & -i & -i & i \\ -3 & -3 & -1 & -1 & 1 & 1 & 3 & 3 \\ -3 & -3 & 1 & 1 & 1 & 1 & -3 & -3 \\ 3i & -3i & -i & i & -i & i & 3i & -3i \\ -3i & 3i & -i & i & i & -i & 3i & -3i \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix},$$

$\det\mathcal{M}_1 = -2i$, $\det\mathcal{M}_2 = -2^9 i$, $\det\mathcal{M}_3 = 2^{16}$, $i^2 = -1$.

We remark that, in general, the elements $\mathcal{U}$, $\mathcal{B}_2$, $\mathcal{B}_3$ are complex and $v = \overline{u}$,

$$B = \overline{A},\ D = \overline{C},\ G = \overline{F},\ L = \overline{K},\ N = \overline{M},\ Q = \overline{P},\ S = \overline{R}. \tag{5}$$

In $u, v, A, B, ..., R, S$ the identity (3), up to a non zero factor, looks as

$$M_2(u, v) + M_3(u, v)Z + M_4(u, v)Z^2 + M_5(u, v)Z^3 \equiv 0,$$

where

$$M_j(u, v) = 2^{j-12}(N_j(u, v) + \overline{N_j(u, v)}),\ j = 2, 3, 4, 5,$$
$$N_5(u, v) = u^3((A^2D - ACG - 2CK + 4GK - 2AM - 2AS)u^2$$
$$+(2A^2B + ACD - C^2G + ADG - 2AFG - CG^2 + 10DK - 4FK$$
$$-4CM - 2GM - 4AP + 4AQ - 8CS - 10GS)uv + (3ABC + AD^2$$
$$+2ABG - CDG - 3CFG - 2FG^2 + 16BK + 4DM - 6FM$$
$$+10AN - 6CP - 8GP - 2CQ - 8GQ - 6AR - 8DS - 14FS)v^2),$$

and for $C_6(x,y)$ we have $M_6(u,v) = 2^{-5}(N_6(u,v) + \overline{N_6(u,v)})$,

$$N_6(u,v) = u^2((AC-2K)u^2 + (C^2 - 3AD + 2AF + 3CG + 2G^2 + 2M + 10S)uv \\ +3(CF - AB + FG + 2P)v^2).$$

Solving the series of identities

$$\{M_2(u,v) \equiv 0,\ M_3(u,v) \equiv 0,\ M_4(u,v) \equiv 0\}$$

the following Theorem is obtained in [9]:

Theorem 2.1. *The line at infinity has for cubic system* (1) *the multiplicity:*

- at least two ($m(Z_\infty) \geq 2: \quad M_2(u,v) \equiv 0$) *if and only if the coefficients of* (1) *verify one of the following three sets of conditions:*

2.1) $K = L = R = S = 0,\ P = \alpha M, Q = N/\alpha,\ MN \neq 0,\ \alpha \in \mathbb{C},\ \alpha\overline{\alpha} = 1;$

2.2) $M = N = P = Q = 0,\ R = \beta K,\ S = L/\beta,\ KL \neq 0, \beta \in \mathbb{C},\ \beta\overline{\beta} = 1;$

2.3) $P = \gamma N,\ Q = M/\gamma,\ R = \gamma L,\ S = K/\gamma,\ KLMN \neq 0,\ \gamma \in \mathbb{C},\ \gamma\overline{\gamma} = 1;$

-at least three ($m(Z_\infty) \geq 3: \quad \{M_2(u,v) \equiv 0, M_3(u,v) \equiv 0\}$) *iff*

3.1) $K = L = R = S = 0, F = B/\alpha, G = \alpha A, N = \alpha^2 M,$
$P = \alpha M, Q = \alpha M, M \neq 0, \alpha\overline{\alpha} = 1;$

3.2) $K = L = R = S = 0,\ D = CN/(\alpha M),\ F = \alpha BM/N,\ G = AN/(\alpha M),$
$P = \alpha M, Q = N/\alpha, M(N - \alpha^2 M) \neq 0, \alpha\overline{\alpha} = 1;$

3.3) $M = N = P = Q = 0,\ C = \beta DK/L,\ F = \beta BK/L,$
$G = AL/(\beta K),\ R = \beta K,\ S = L/\beta,\ \beta\overline{\beta} = 1;$

3.4) $M = N = P = Q = 0,\ F = D + (\beta^2 BK^2 - CL^2)/(\beta KL),$
$G = C + (AL^2 - \beta^2 DK^2)/(\beta KL),\ R = \beta K,\ S = L/\beta,$
$L^3 - \beta^4 K^3 = 0,\ \beta\overline{\beta} = 1;$

3.5) $D = C/\gamma, F = B\gamma, G = A/\gamma, P = \gamma N, R = \gamma L,$
$Q = M/\gamma, S = K/\gamma, KM \neq 0, \gamma\overline{\gamma} = 1;$

3.6) $D = (CL\gamma^3 + (F - B\gamma)(K - M\gamma))/(L\gamma^4),\ Q = M/\gamma, S = K/\gamma,$
$G = (K(B\gamma - F) + AL\gamma^2)/(L\gamma^3),\ N = (-K + M\gamma + L\gamma^4)/\gamma^3,$
$P = (-K + M\gamma + L\gamma^4)/\gamma^2, R = L\gamma, M(F - B\gamma) \neq 0, \gamma\overline{\gamma} = 1.$

(In the cases 3.1) − 3.4) the multiplicity is exactly three);

- *at least four* ($m(Z_\infty) \geq 4$: $\{M_2(u,v) \equiv 0, M_3(u,v) \equiv 0, M_4(u,v) \equiv 0\}$) *iff*

4.1) $D = CS/K, F = BK/S, G = AS/K, L = -S^4/K^3,$
$M = S, N = R = -S^3/K^2, Q = -P = S^2/K;$

4.2) $A = 2(K^3L + S^4)/(S^2(BK - FS)) - S(BK - 2FS)/(KL),$
$R = KL/S,\ C = 2(K^3L + S^4)/(KS(BK - FS))$
$-(BK^4L - 2FK^3LS - FS^5)/(K^2LS^2),$
$D = (FK^2L + BS^3)/(K^2L) + 2(K^3L + S^4)/(K^2(BK - FS)),$
$G = FS^3/(K^2L) + 2(K^3L + S^4)/(KS(BK - FS)),$
$M = (K^3L + 2S^4)/S^3, N = (2K^3L + S^4)/(K^2S),$
$P = (2K^3L + S^4)/(KS^2), Q = (K^3L + 2S^4)/(KS^2).$

Solving in each of conditions 4.1) and 4.2) the identity $M_5(u,v) \equiv 0$, we obtain

Theorem 2.2. *The system* (1) *has the line at infinity of multiplicity five if and only if its coefficients verify one of the following three sets of conditions:*

$$B = -AS^3/K^3, C = D = 0, F = -AS^2/K^2, G = AS/K, L = -S^4/K^3,\\ M = S, N = -S^3/K^2, P = -S^2/K, R = -S^3/K^2, Q = S^2/K; \tag{6}$$

$$A = 5F^3/B^2, C = -6F^2/B,\ D = 2F,\ G = -3F^2/B,\ K = F^5/B^3,\\ L = BF,\ M = -3F^4/B^2,\ N = -3F^2,\ P = 3F^3/B,\ Q = 3F^3/B,\\ R = -F^2,\ S = -F^4/B^2,\ F \neq 0; \tag{7}$$

$$A = (F^3K^2 + 8BKS^2 + 4FS^3)/(B^2K^2),\ D = 2F,\ L = S^4/K^3,\\ C = 2(F^2K^2 + 4S^3)/(BK^2),\ G = (F^2K^2 + 4S^3)/(BK^2),\ M = 3S,\\ N = 3S^3/K^2,\ P = 3S^2/K,\ Q = 3S^2/K,\ R = S^3/K^2,\\ K^2(BK - FS)^2 + 4S^5 = 0. \tag{8}$$

Theorem 2.3. *In the class of cubic differential systems of the form* (1) *the maximal multiplicity of the line at infinity is five.*

Indeed, under the conditions (6), (7) and (8) the polynomial $M_6(u,v)$ becomes, respectively:

$$(Ku - Sv)(Ku + Sv)(K^2u^2 - 6KSuv + S^2v^2)/K^3 \not\equiv 0;$$
$$F^2u(Bv - 4Fu)(Bv - Fu)^2/B^3 \not\equiv 0;$$
$$(B^3K^5 - F^5K^4 - 8BF^2K^3S^2 - 8F^3K^2S^3 - 32BKS^5 - 16FS^6)u^4\\ -4B(F^4K^4 + 2B^2K^4S - 4BFK^3S^2 + 10F^2K^2S^3 + 24S^6)u^3v\\ +6B^2K^2(BKS^2 - F^3K^2 - 4FS^3)u^2v^2 - 4B^3F^2K^4uv^3\\ -B^3K(BFK^3 - S^4)v^4 \not\equiv 0.$$

In the expressions of $M_6(u,v)$ we have neglected non-zero numerical factors.

Taking into account (4) and (5), the equalities (6) give us the following four series of conditions in the real coefficient of system (1):

$$a=b=c=f=g=k=l=0,\ m=n=p=r=s=0,\ q\neq 0; \tag{9}$$

$$\begin{aligned} &b=-as/k,\ c=a(k^2-s^2)/(ks),\ d=a(k^2-s^2)/k^2,\\ &f=-a, g=as/k,\ l=-k,\ m=(2k^2-s^2)/s,\ n=(k^2-2s^2)/s,\\ &p=k(k^2-2s^2)/s^2,\ q=(2k^2-s^2)/k,\ r=-k^2/s; \end{aligned} \tag{10}$$

$$a=b=d=f=g=k=l=0,\ m=n=q=r=s=0,\ p\neq 0; \tag{11}$$

$$\begin{aligned} &a=-br/l,\ c=b(l^2-r^2)/l^2,\ d=b(l^2-r^2)/(lr),\\ &f=br/l,\ g=-b, k=-l,\ m=(l^2-2r^2)/r,\ n=(2l^2-r^2)/r,\\ &p=(2l^2-r^2)/l,\ q=l(l^2-2r^2)/r^2,\ s=-l^2/r, \end{aligned} \tag{12}$$

and the equalities (8) give us eight real series of conditions:

$$b=c=0,\ d=2a,\ f=k=l=m=n=p=q=r=0,\ s=a^2,\ a\neq 0; \tag{13}$$

$$\begin{aligned} &a=0,\ b=-gk^2/s^2,\ c=-2gk^2/s^2,\ d=0,\ f=-2gk^3/s^3,\ l=k^3/s^2,\\ &m=n=3k^2/s,\ p=3k^3/s^2,\ q=3k,\ r=k^4/s^3,\ g^2k^2-k^2s-s^3=0; \end{aligned} \tag{14}$$

$$\begin{aligned} &c=2b=-2as/k,\ f=-a(k^2+2s^2)/s^2,\ g=(k^2+a^2s)/(ak),\\ &m=n=3k^2/s,\ p=3l=3k^3/s^2,\ q=3k,\ r=k^4/s^3,\ d=2a,\\ &k^4-a^2k^2s-a^2s^3=0; \end{aligned} \tag{15}$$

$$\begin{aligned} &b=k(-agk+k^2+a^2s+s^2)/(s(as-gk)),\ c=2k(2as-gk)/s^2,\\ &d=2a,\ f=k^2(3as-2gk)/s^3,\ l=k^3/s^2,\ m=3k^2/s,\ n=3k^2/s,\\ &p=3k^3/s^2,\ q=3k,\ r=k^4/s^3,\ g^2k^2-2agks-k^2s+a^2s^2-s^3=0. \end{aligned} \tag{16}$$

$$a=d=0,\ c=2b,\ g=k=l=m=n=p=q=s=0,\ r=a^2,\ b\neq 0; \tag{17}$$

$$\begin{aligned} &b=c=0,\ a=-fl^2/r^2,\ d=-2fl^2/r^2,\ g=-2fl^3/r^3,\ k=l^3/r^2,\\ &m=n=3l^2/r,\ p=3l,\ q=3l^3/r^2,\ s=l^4/r^3,\ f^2l^2-l^2r-r^3=0; \end{aligned} \tag{18}$$

$$\begin{aligned} &d=2a=-2br/l,\ f=(l^2+b^2r)/(bl),\ g=-b(l^2+2r^2)/r^2,\\ &m=n=3l^2/r,\ p=3l,\ q=3k=3l^3/r^2,\ s=l^4/r^3,\ c=2b,\\ &l^4-b^2l^2r-b^2r^3=0; \end{aligned} \tag{19}$$

$$\begin{aligned} &a=l(-bfl+l^2+b^2r+r^2)/(r(br-fl)),\ d=2l(2br-fl)/r^2,\\ &c=2b,\ g=l^2(3br-2fl)/r^3,\ k=l^3/r^2,\ m=3l^2/r,\ n=3l^2/r,\\ &p=3l,\ q=3l^3/r^2,\ s=l^4/r^3,\ f^2l^2-2bflr-l^2r+b^2r^2-r^3=0. \end{aligned} \tag{20}$$

Remark 2.1. 1) The set of equalities (7) is not satisfied in the real coefficients of cubic system (1).

2) The transformation $x \leftrightarrow y$, $t \to -t$ reduce the system $\{(1), (11)\}$ (respectively, $\{(1), (12)\}$ $\{(1), (17)\}$ $\{(1), (18)\}$ $\{(1), (19)\}$ $\{(1), (20)\}$ to the system $\{(1), (9)\}$ (respectively, $\{(1), (10)$ $\{(1),(13)\}$ $\{(1), (14)\}$ $\{(1), (15)\}$ $\{(1), (16)\}$).

Theorem 2.4. *The real cubic system* (1) *has the line at infinity of multiplicity five if and only if one of the following twelve sets of conditions* (13) − (20) *holds.*

3. Solution of the center problem for cubic systems with the line at infinity of maximal multiplicity.

In each of the series of conditions (6), (7) and (8) we calculate the first Lyapunov quantity L_1. In the cases (7) and (8) this quantity vanishes and the divergence of vector field $\mathbb{X}$ associated to system (1) also vanishes.

In the case (6) we have $L_1 = 2iS^2/K \neq 0$ (see, (5), (6)) and therefore, $(0, 0)$ is a focus.

In this way we prove the statements of the following two theorem.

Theorem 3.1. *The cubic system* (1) *with the line at infinity of maximal multiplicity has a center at the origin* $(0, 0)$ *if and only if the first Lyapunov quantity vanishes* $L_1 = 0$.

Theorem 3.2. *The cubic system* (1) *with the line at infinity of maximal multiplicity has a center at the origin* $(0, 0)$ *if and only if the divergence of the vector field* $\mathbb{X}$ *associated to system* (1) *vanishes, i.e. iff* (1) *has a polynomial first integral.*

In the cases of real conditions (9)-(12) we have, respectively, $L_1 = q \neq 0$, $L_1 = -(k^2 + s^2)^2/(ks^2) \neq 0, L_2 = -p \neq 0$, $L_1 = (l^2 + r^2)^2/(lr^2) \neq 0$. Therefore, in each of the cases (9)-(12) the origin is a focus for (1).

In each of the cases (13)-(20) the first Lyapunov quantity and the divergence vanishes.

The first integrals $\mathcal{F}$ of the systems $\{(1),(13)\} - \{(1),(20)\}$ are, respectively,

$$\mathcal{F} = 6(x^2 + y^2) + 4gx^3 + 12ax^2y + 3a^2x^4;$$

$$\mathcal{F} = 6s^3(x^2 + y^2) + (sx + ky)^2(4gsx - 8gky + 3s^2x^2 + 6ksxy + 3k^2y^2);$$

$$\mathcal{F} = 6aks^3(x^2 + y^2) + 4s^3(k^2 + a^2s)x^3 + 12a^2s^3xy(kx - sy)$$
$$-4a^2ks(k^2 + 2s^2)y^3 + 3ak(sx + ky)^4;$$

$$\mathcal{F} = 6s^3(x^2 + y^2) + (sx + ky)^2(4(gsx - 2gky + 3asy) + 3(sx + ky)^2);$$

$$\mathcal{F} = 6(x^2 + y^2) + 12bxy^2 + 4fy^3 + 3b^2y^4;$$

$$\mathcal{F} = 6r^3(x^2+y^2) - (lx+ry)^2(8flx - 4fry - 3l^2x^2 - 6lrxy - 3r^2y^2);$$

$$\mathcal{F} = 6blr^3(x^2+y^2) - 4b^2lr(l^2+2r^2)x^3 - 12b^2r^3xy(rx-ly) \\ +4r^3(l^2+b^2r)y^3 + 3bl(lx+ry)^4;$$

$$\mathcal{F} = 6r^3(x^2+y^2) + (lx+ry)^2(4(3brx - 2flx + fry) + 3(lx+ry)^2).$$

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(Alexandru Şubă) Institute of Mathematics and Computer Sciences "V. Andrunachievici",
5 Academiei St., Chişinău, Republic of Moldova;
Ion Creangă State Pedagogical University, 1 Ion Creangă St., Chişinău, Republic of Moldova
E-mail address: alexandru.suba@math.md