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Schröder T-quasigroups of generalized associativity

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Abstract. We prolong research of Schröder quasigroups and Schröder T-quasigroups [14].

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Schröder T-cvasigrupuri de asociativitate generalizată

Rezumat. Se extinde cercetarea quasigrupurilor de tip Schröder şi a T-quasigrupurilor de tip Schröder [14].

Cuvinte-cheie: quasigrup, buclă, grupoid, quasigrupuri de tip Schröder, identitatea Schröder.

1. Introduction

Necessary definitions can be found in [1, 3, 2, 7, 10, 15].

Definition 1.1. Binary groupoid $(Q, \circ)$ is called a left quasigroup if for any ordered pair $(a, b) \in Q^2$ there exist the unique solution $x \in Q$ to the equation $a \circ x = b$ [1].

Definition 1.2. Binary groupoid $(Q, \circ)$ is called a right quasigroup if for any ordered pair $(a, b) \in Q^2$ there exist the unique solution $y \in Q$ to the equation $y \circ a = b$ [1].

Definition 1.3. A quasigroup $(Q, \cdot)$ with an element $1 \in Q$, such that $1 \cdot x = x \cdot 1 = x$ for all $x \in Q$, is called a *loop*.

Definition 1.4. Binary groupoid $(Q, \cdot)$ is called medial if this groupoid satisfies the following medial identity:

$$xy \cdot uv = xu \cdot yv \tag{1}$$

for all $x, y, u, v \in Q$ [1].

We recall

Definition 1.5. Quasigroup $(Q, \cdot)$ is a T-quasigroup if and only if there exists an abelian group $(Q, +)$, its automorphisms φ and ψ, and a fixed element $a \in Q$ such that $x \cdot y = \varphi x + \psi y + a$ for all $x, y \in Q$ [8].

A T-quasigroup with the additional condition $\varphi\psi = \psi\varphi$ is medial.

Definition 1.6. Garrett Birkhoff [2] has defined an equational quasigroup as an algebra with three binary operations $(Q, \cdot, /, \backslash)$ that satisfies the following six identities:

$$x \cdot (x \backslash y) = y, \tag{2}$$

$$(y/x) \cdot x = y, \tag{3}$$

$$x \backslash (x \cdot y) = y, \tag{4}$$

$$(y \cdot x)/x = y, \tag{5}$$

$$x/(y \backslash x) = y, \tag{6}$$

$$(x/y) \backslash x = y. \tag{7}$$

Ernst Schröder (a German mathematician mainly known for his work on algebraic logic) introduced and studied the following identity of generalized associativity [13]:

$$(y \cdot z) \backslash x = z(x \cdot y). \tag{8}$$

In the quasigroup case the identity (8) is equivalent to the following identity [11]:

$$(y \cdot z) \cdot (z \cdot (x \cdot y)) = x \tag{9}$$

If in the idempotent quasigroup $(Q; \cdot)$, the identity (9), we put $x = y$, then we obtain the following standard Schröder's identity:

$$(x \cdot y) \cdot (y \cdot x) = x. \tag{10}$$

Definition 1.7. Any quasigroup with the identity (10) is called a Schröder quasigroup.

So we have different objects that have name Schröder. Namely,
(i) the following identity of generalized associativity on groupoids [13]:
$(y \cdot z) \backslash x = z(x \cdot y)$ (8);
(ii) the Schröder's identity of generalized associativity in quasigroups (9);
(iii) the Schröder's identity (Schröder's 2-nd identity [12]) $(x \cdot y) \cdot (y \cdot x) = x$ (10);
(iv) identity

$$(x \cdot y) \cdot (y \cdot x) = y \tag{11}$$

is named by Albert Sade [12] as Stein's 3-rd identity.

Many information about these identities is given in articles [4, 5]. We tried do not repeat information from these articles here.

Each of these identities deserves a separate study in the class of groupoids, left (right) quasigroups; in the classes of quasigroups and of T-quasigroups.

1.1. Schröder's identity of generalized associativity in quasigroups

It is convenient to call this identity the Schröder's identity of generalized associativity.

Often various variants of associative identity, which are true in a quasigroup, guarantee that this quasigroup is a loop.

It is not so in the case with the identity. We give an example of quasigroup which is not a loop with the identity (9) [11]. See also [15]. A quasigroup from this example does not have left and right identity element.

Quasigroups with Schröder's identity of generalized associativity are not necessary idempotent and associative. See the following example [11].

$\cdot$	0	1	2	3	4	5	6	7
0	1	4	7	0	6	5	2	3
1	5	2	3	6	0	1	4	7
2	0	7	4	1	5	6	3	2
3	6	3	2	5	1	0	7	4
4	4	1	0	7	3	2	5	6
5	3	6	5	2	4	7	0	1
6	7	0	1	4	2	3	6	5
7	2	5	6	3	7	4	1	0

The left cancellation (left division) groupoid with the identity (9) and with the identity $(x/x = y/y)$ (in a quasigroup this identity guarantees existence of the left identity element) is a commutative group of exponent two [11].

The similar results are true for the right case [11]. In this case we use the identity

$$(x\backslash x = y\backslash y).$$

It is clear that this result is true for any quasigroup with the left or right identity element.

Notice, any 2-group $(G, +)$ (in such group $x + x = 0$ for any $x \in G$) satisfies Schröder's identity of generalized associativity.

2. Schröder's identity of generalized associativity in T-quasigroups

Theorem 2.1. *In T-quasigroup $(Q, \cdot)$ of the form $x \cdot y = \varphi x + \psi y$ Schröder's identity of generalized associativity is true if and only if $\varphi x = \psi^{-2}x$, $\varepsilon = \varphi^7$, $\varepsilon = \psi^{14}$, $\varphi\psi z + \psi\varphi z = 0$.*

Proof. We rewrite identity (9) in the following form:

$$\varphi^2 y + \psi^3 y + \varphi\psi z + \psi\varphi z + \psi^2\varphi x = x. \qquad (12)$$

If we substitute in equality (12) $y = z = 0$, then we have

$$\varphi x = \psi^{-2}x. \tag{13}$$

If we substitute in equality (12) $x = z = 0$, then we have

$$\varphi^2 y + \psi^3 y = 0. \tag{14}$$

Taking into consideration equality (13), we can re-write equality (14) in the form

$$\psi^{-4} y + \psi^3 y = 0, \tag{15}$$

or in the form

$$\psi^3 = I\psi^{-4}, \tag{16}$$

where $Ix = -x$ for all $x \in Q$. Notice, the permutation I is an automorphism of the group $(Q, +)$ here. Therefore, we can rewrite previous equalities in the form

$$\varepsilon = I\psi^{-7}, I = \psi^{-7}, \varepsilon = \psi^{-14}, \varepsilon = \psi^{14}, \varepsilon = \varphi^7. \tag{17}$$

If we substitute in equality (12) $x = y = 0$ then we have

$$\varphi\psi z + \psi\varphi z = 0. \tag{18}$$

Converse. If we substitute in identity (9) the expression $x \cdot y = \varphi x + \psi y$, then we obtain equality (12), which is true taking into consideration the equalities (13), (14), (18). Then we obtain, that identity (9) is true in this case. □

Corollary 2.1. *In medial quasigroup $(Q, \cdot)$ of the form $x \cdot y = \varphi x + \psi y$ Schröder's identity of generalized associativity is true if and only if the group $(Q, +)$ is an abelian 2-group (i.e. $x + x = 0$ for any $x \in Q$), $\varphi x = \psi^{-2}x$, $\varepsilon = \varphi^7$, $\varepsilon = \psi^{14}$.*

Proof. From the identity of mediality it follows that $\varphi\psi z + \psi\varphi z = 2 \cdot \varphi\psi z = 0$ for all $z \in Q$, i.e., the group $(Q, +)$ is an abelian 2-group. □

Example 2.1. We present elements of the group $(Z_2^3, +)$ in the following form: $1 = (000), 2 = (001), 3 = (010), 4 = (011), 5 = (100), 6 = (101), 7 = (110), 8 = (111)$.

+	1	2	3	4	5	6	7	8
1	1	2	3	4	5	6	7	8
2	2	1	4	3	6	5	8	7
3	3	4	1	2	7	8	5	6
4	4	3	1	1	8	7	6	5
5	5	6	7	8	1	2	3	4
6	6	5	8	7	2	1	4	3
7	7	8	5	6	3	4	1	2
8	8	7	6	5	4	3	2	1

We can see on the group $Aut(Z_2^3, +)$ as on the group $GL(3, 2)$. This group is the group of non-degenerate matrices of size 3×3 over the field of order 2 relatively to standard multiplication of matrices [7].

The group $PSL(2, 7)$ is the group of non-degenerate matrices of size 2×2 over the field of order 7. These groups are isomorphic, i.e., $Aut(Z_2^3, +) \cong GL(3, 2) \cong PSL(2, 7)$. Notice $|(GL(3, 2)| = 168 = 3 \times 7 \times 8$ [7].

We have the following automorphisms of the group $Aut(Z_2^3, +) \cong GL(3, 2)$:

$$\varphi = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \ \psi = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

Notice that $\varphi^7 = \psi^7 = \varepsilon, \varphi = \psi^{-2}, \varphi\psi = \psi\varphi$. Therefore, Schröder's medial quasigroup $(Q, \circ)$ of generalized associativity can have the form $x \circ y = \varphi x + \psi y$:

$\circ$	1	2	3	4	5	6	7	8
1	1	4	8	5	3	2	6	7
2	3	2	6	7	1	4	8	5
3	6	7	3	2	8	5	1	4
4	8	5	1	4	6	7	3	2
5	7	6	2	3	5	8	4	1
6	5	8	4	1	7	6	2	3
7	4	1	5	8	2	3	7	6
8	2	3	7	6	4	1	5	8

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