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On regular operators on Banach Lattices

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Abstract. Let E and F be Banach lattices and X and Y be Banach spaces. A linear operator $T : E \to F$ is called regular if it is the difference of two positive operators. $L_r(E, F)$ denotes the vector space of all regular operators from E into F. A continuous linear operator $T : E \to X$ is called M-weakly compact operator if for every disjoint bounded sequence (x_n) in E, we have $lim_{n\to\infty}\|Tx_n\| = 0$. $W^r_M(E, F)$ denotes the regular M-weakly compact operators from E into F. This paper is devoted to the study of regular operators and M-weakly compact operators on Banach lattices. We show that F has a b-property if and only if $L_r(E, F)$ has b-property. Also, $W^r_M(E, F)$ is a KB-space if and only if F is a KB-space.

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Operatori regulari pe latice Banach

Rezumat. Fie E şi F latice Banach, iar X şi Y spaţii Banach. Operatorul linear $T : E \to F$ se numeşte regular dacă reprezintă diferenţa a doi operatori pozitivi. $L_r(E, F)$ este spaţiul vectorial al operatorilor regulari din E în F. Operatorul linear şi continuu $T : E \to X$ se numeşte operator M-slab compact dacă pentru orice şiruri mărginite şi disjuncte (x_n) din E, urmează că $lim_{n\to\infty}\|Tx_n\| = 0$. $W^r_M(E, F)$ reprezintă operatorii regulari M-slab compacţi din E în F. Lucrarea este dedicată studiului operatorilor regulari şi operatorilor M-slab compacţi pe latice Banach. Se demonstrează că F posedă b-proprietate dacă şi numai dacă $L_r(E, F)$ are b-proprietate. La fel, $W^r_M(E, F)$ este KB-spaţiu dacă şi numai dacă F este KB-spaţiu.

Cuvinte-cheie: latice Banach, operatori regulari, operatori M-slab compacţi, norma continue de ordine.

1. Introduction

Let X be a Banach space and E be a Banach lattice. E^+ denotes the positive cone of E. That is, $E^+ = \{x : 0 \leq x\}$. We denote $E^\sim$ by the set of all order bounded linear functionals on E, and $E^{\sim\sim}$ by the set of all second order dual of E, [7]. By X' we denote the set of all continuous linear functionals on X. Since E is a Banach lattice, order dual and continuous dual coincide [1, 5].

A set $[x, y] = \{z \in E : x \leq z \leq y\}$ in a Banach lattice E is called an order interval. Let A be a subset of E. The set A is called order bounded if $A \subseteq [x, y]$ for some $x, y \in E$. A is called a b-order bounded if A is an order bounded in the second order dual E'' of E.

A Banach lattice is said to have b-property if every b-order bounded set is an order bounded set in E [2, 3]. Order dual of a Banach lattice has b-property. The space $C(K)$ of all continuous real valued functions defined on a compact Hausdorff space K has b-property.

A Banach lattice E is called a KB-space if every positive increasing norm bounded sequence in E converges. A Banach lattice E is a KB space if and only if it has an order continuous norm and with property (b) [2]. Reflexive Banach lattice, AL spaces are examples of KB spaces. There are a lot of KB spaces in Banach lattices[1, 5].

A Banach lattice E is said to have an order continuous norm if $x_n \downarrow 0$ in E implies $\| x_n \| \to 0$ as $n \to \infty$. For example, Banach space c_0 of all sequences converging to zero has an order continuous norm. Let E be a Banach lattice. E' is a KB space if and only if E has an order continuous norm.

A Banach lattice E is called Dedekind complete if every non empty subset of E, which is bounded from above, has a supremum. Alternatively, every non empty subset of E, which is bounded from below, has an infimum.

2. KB space of M-weakly compact operators

Definition 2.1. [1, 5] Let X be a Banach space and E be a Banach lattice. A continuous linear operator $T : X \to E$ is called L-weakly compact if $T(ball(X))$ is an L-weakly compact set. A subset A of E is called L-weakly compact if $\| x_n \| \to 0$ as $n \to \infty$ for every disjoint sequence (x_n) in the solid hull of A, where $ball(X)$ denotes the clesed unit ball of X.

By $W_L(E, F)$ we denote the set of all L-weakly compact operators.

Definition 2.2. ([1, 5]) A continuous linear operator $T : E \to X$ is called M-weakly compact if $lim_{n\to\infty} \| Tx_n \| = 0$ for every disjoint sequence (x_n) in the closed unit ball of E.

By $W_M(E, F)$ we denote the set of all M-weakly compact operators from a Banach lattice E into a Banach lattice F. If F is a Dedekind complete Banach lattice, then it is a Banach lattice. We denote the linear span of the positive operators in $W_M(E, F)$ by $W_M^r(E, F)$. If F is a Dedekind complete Banach lattice, then $W_M^r(E, F)$ is a Dedekind complete Banach lattice under the regular norm.

Adjoint of an M-weakly compact operator is an L-weakly compact and adjoint of an L-weakly compact operator is an M-weakly compact. Every L-weakly compact and M-weakly compact operators are weakly compact.

Definition 2.3. A Banach lattice E is called an AL space if the norm

$$\| x + y \| = \| x \| + \| y \|$$

holds for every $x, y \in E$.

A linear operator T from a E Banach lattice to a Banach lattice F is called regular if it is the difference of two positive operators from E into F. By $L_r(E, F)$ we denote the vector space of all regular operators from E into F, [6]. Every positive linear operator from a Banach lattice E into a Banach lattice F is continuous. By $L(E, F)$ we denote the vector space of all linear continuous operators from E into F.

A linear operator T from a Banach lattice E into a Banach lattice F is called order bounded if it sends an order bounded set in E to an order bounded set in F. $L_b(E, F)$ denotes the vector space of all order bounded linear operators from E into F. The following inclusions hold: $L_r(E, F) \subseteq L_b(E, F) \subseteq L(E, F)$.

Let $T \in L_r(E, F)$. The regular operator norm of T is given by

$$\|T\| = inf\{\|S\| : |T| \leq S \quad \text{for} \quad 0 \leq S \in L_r(E, F)\}.$$

Theorem 2.1. *(*[2, 3]*) Let E, F be Banach lattices with F Dedekind complete. Then, $L_r(E, F)$ has b-property if and only if F has b-property.*

Proof. Take a sequence (x_n) in F with the property $x_n \uparrow y$ in F''. Let us choose $0 \neq f \in E'$. We define the map $\psi : F \to L_r(E, F)$, $\psi(y) = f \otimes y$, which is given by $(f \otimes y)(x) = f(x)y$ for all $x \in E$. $\psi(x_n)$ is b-order bounded in $L_r(E, F)$. So, there is a $T \in L_r(E, F)$ such that $0 \leq f \otimes x_n \leq T$. There is an $x \in E$ such that $0 \leq x_n \leq T(x)$ in F. It means F has b-property.

Suppose that F has b-property. Assume that (T_n) in $L_r(E, F)$ such that $0 \leq T_n \uparrow T$ in $L_r(E, F)''$. Let us choose $0 \neq x \in E^+$. We define a map $\varphi : F' \to L_r(E, F)'$ which is defined by $\varphi(f)T = f(Tx)$ for $T \in L_r(E, F)$.This mapping is one-one and positive. From here, for every $x \in E^+$, we have that $T_n(x)$ is b-order bounded in F. That is, $T_n(x)$ is order bounded in F. By Kantorovich lemma, we extend the mapping defined by $T(x) = sup\{T_n(x) : n = 1, 2, 3, ...\}$. Therefore, $L_r(E, F)$ has b-property.

Let E be a Banach lattice. By E^a, we denote the maximal ideal space on which the norm is order continuous. Equivalently,

$$E^a = \{x : any \quad monotone \quad sequence \quad (x_n) \quad in \quad [0, | x |] \quad converges\}.$$

Theorem 2.2. *([4]) Let E, F be Banach lattices with $(E')^a \neq \{0\}$. Then, $W_M^r(E, F)$ has order continuous norm if and only if F has an order continuous norm.*

Let X and Y be Banach spaces and $T : X \to Y$ be a continuous linear operator.The adjoint operator T' of T is defined from Y' into X' by $T'(f)(x) = f(Tx)$ for every $f \in Y'$ and for every $x \in X$.

Theorem 2.3. *Let E, F be Banach lattices. Then, $W_M^r(E, F)$ has b-property if and only if F has b-property.*

Proof. Proof is similar to the proof of Theorem 2.4. So, it is omitted.

Theorem 2.4. *Let E, F be Banach lattices. Then, $W_M^r(E, F)$ is a KB space if and only if F is a KB space.*

Proof. It is proved this result by using the fact that a Banach lattice is a KB space if and only if it has an order continuous norm and it has b-property [2].

$W_L^r(E, F)$ denotes the vector space of all regular L-weakly compact operators. It is a Banach lattice.

Theorem 2.5. *Let E, F be Banach lattices and $(E')^a \neq \{0\}$ and $F^a \neq \{0\}$. Then, the following claims are equivalent:*

(i) $W_L^r(E, F)$ has order continuous norm.

(ii) E' has order continuous norm.

(iii) $W_M^r(F', E')$ is a KB space.

Proof. Proof is done by using [4].

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