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Stability of unperturbed motion governed by the ternary differential system of Lyapunov-Darboux type with nonlinearities of degree four

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Abstract. For the ternary differential system of Lyapunov-Darboux type with nonlinearities of degree four, using the Lie algebra admitted by this system, was obtained the analytic first integral, determined the Lyapunov function and the conditions of stability of the unperturbed motion.
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Stabilitatea mişcării neperturbate guvernate de sistemul diferenţial ternar de tip Lyapunov-Darboux cu nelinearităţi de gradul patru

Rezumat. Pentru sistemul difereţial ternar de tip Lyapunov-Darboux cu nelinearităţi de gradul patru, utilizând algebra Lie admisă de acest sistem, s-a obţinut integrala primă analitică, determinată funcţia Lyapunov şi condiţiile de stabilitate a mişcării neperturbate.
Cuvinte-cheie: sistem diferenţial, stabilitatea mişcării neperturbate, comitant şi invariant centroafin, algebră Lie, integrală primă, funcţie Lyapunov.

1. Notion of comitant and invariant for ternary differential system

We examine the differential system of the unperturbed motion [1, 2] with nonlinearities of degree four $s^3(1,4)$, written in the tensorial form [3, 4]

$$\frac{dx^j}{dt} = a^j_\alpha x^\alpha + a^j_{\alpha\beta\gamma\delta} x^\alpha x^\beta x^\gamma x^\delta \quad (j, \alpha, \beta, \gamma, \delta = 1, 2, 3) \tag{1}$$

where $a^j_{\alpha\beta\gamma\delta}$ is a symmetric tensor in lower indices in which the total convolution is done. The centro-affine group $GL(3,\mathbb{R})$ is given by transformations q:

$$\bar{x}^j = q^j_\alpha x^\alpha \quad (\Delta = det(q^j_\alpha) \neq 0) \quad (j, \alpha = 1, 2, 3). \tag{2}$$

In the theory of invariants [5] the vector $x = (x^1, x^2, x^3)$, which is changed by formulas (2), is usually called *contravariant*. The vector $u = (u_1, u_2, u_3)$, which is changed by formulas $\bar{u}_r = p_r^j u_j \ (r, j = 1, 2, 3)$, where $p_j^r q_s^j = \delta_s^r$ is the Kroniker's symbol, is call *covariant*. Any other vector $y = (y^1, y^2, y^3)$, different from x, which is changed by formulas (2) $\bar{y}^j = q_\alpha^j y^\alpha \ (j, \alpha = 1, 2, 3)$, is call *cogradient* with the vector x. The coefficients of the system (1) and the coordinates of the vectors x, u, y take values from the field of real numbers $\mathbb{R}$.

Observe that the transformation (2) preserves the form of the system (1)

$$\frac{d\bar{x}^j}{dt} = \bar{a}_\alpha^j \bar{x}^\alpha + \bar{a}_{\alpha\beta\gamma\delta}^j \bar{x}^\alpha \bar{x}^\beta \bar{x}^\gamma \bar{x}^\delta \quad (j, \alpha, \beta, \gamma, \delta = 1, 2, 3), \tag{3}$$

where the coordinates of the vector $\bar{x} = (\bar{x}^1, \bar{x}^2, \bar{x}^3)$ are determined by the relations (2). The coefficients $\bar{a}_\alpha^j$ şi $\bar{a}_{\alpha\beta\gamma\delta}^j$ from the right–hand sides of (3) are some linear functions in the coefficients of system (1) and rational in the parameters q_α^j of transformations (2).

We will denote the set of coefficients (1) by a and of the system (3) by $\bar{a}$.

Definition 1.1. According to [3, 4, 5], we say that the polynomial $\varkappa(x, y, u, a)$ of the coefficients of system (1) and of the coordinates of vectors x, y and u is called center-affine *mixt comitant* of the system (1) with respect to $GL(3, \mathbb{R})$ group, if the following identity holds

$$\varkappa(\bar{x}, \bar{y}, \bar{u}, \bar{a}) = \Delta^{-g} \varkappa(x, y, u, a) \tag{4}$$

for all q from $GL(3, \mathbb{R})$ and for every coordinates of vectors x, y and u, as well as all the coefficients of the system (1).

Size g is an integer number called *the weight of comitant*.

If the mixt comitant $\varkappa$ does not depend of the coordinates of the vector u, then according to [3, 4, 5], we call it simply *comitant*; but if $\varkappa$ does not depend of the coordinates of the vectors x and y, we call it *contravariant* according to [5]. If $\varkappa$ does not depend of the coordinates of the vectors x, y and u, then we will call it *invariant* of the system (1) with respect to $GL(3, \mathbb{R})$ group.

For simplicity, in some cases, we will omit the words "center-affine" or "with respect to $GL(3, \mathbb{R})$ group" for comitants (invariants).

From [5] it is known that the alternation operation, in the case of ternary tensors, is performed by means of the unit trivector ε^{pqr} $(\varepsilon_{\alpha\beta\gamma})$ with coordinates $\varepsilon^{123} = -\varepsilon^{132} = \varepsilon^{312} = -\varepsilon^{321} = \varepsilon^{231} = -\varepsilon^{213} = 1$ $(\varepsilon_{123} = -\varepsilon_{132} = \varepsilon_{312} = -\varepsilon_{321} = \varepsilon_{231} = -\varepsilon_{213} = 1)$ and $\varepsilon^{pqr} = 0$ $(\varepsilon_{\alpha\beta\gamma} = 0)$ $(p, q, r = 1, 2, 3)$ $((\alpha, \beta, \gamma = 1, 2, 3))$ in the other cases.

From [3, 4, 5] results the following assertion

Theorem 1.1. *The expressions obtained by the product of the coefficients of the tensors* a_α^j *and* $a_{\alpha\beta\gamma\delta}^j$, *of system* (1), *as well as the coordinates* x^i, y^j, u_r *of the vectors* x, y, u, *using* *the alternation operation followed by the total convolution, form the basis of the comitants (mixed), contravariants and invariants of the system* (1) *with respect to* $GL(3,\mathbb{R})$ *group.*

Using Theorem 1.1 it is easy to see that the expressions

$$\varkappa_1 = x^\alpha u_\alpha, \quad \varkappa_2 = a^\alpha_\beta x^\beta u_\alpha, \quad \varkappa_3 = a^\alpha_\gamma a^\beta_\alpha x^\gamma u_\beta \tag{5}$$

form *the mixed comitants*, and

$$\delta_1 = a^\alpha_\gamma a^\beta_p a^\gamma_q u_\alpha u_\beta u_r \varepsilon^{pqr} \tag{6}$$

is a *contravariant* of the system (1) with respect to $GL(3,\mathbb{R})$ group.

Likewise the expressions

$$\begin{aligned} \sigma_1 &= a^\alpha_\mu a^\beta_\delta a^\gamma_\alpha x^\delta x^\mu x^\nu \varepsilon_{\beta\gamma\nu}, \\ \eta_1 &= a^\alpha_{\beta\gamma\delta\mu} x^\beta x^\gamma x^\delta x^\mu x^\nu y^\theta \varepsilon_{\alpha\nu\theta}, \end{aligned} \tag{7}$$

are *comitants* of the system (1) with respect to $GL(3,\mathbb{R})$ group.

Some of *the invariants* of the system (1), with respect to $GL(3,\mathbb{R})$ group, are the expressions

$$\theta_1 = a^\alpha_\alpha, \qquad \theta_2 = a^\alpha_\beta a^\beta_\alpha, \qquad \theta_3 = a^\alpha_\gamma a^\beta_\alpha a^\gamma_\beta. \tag{8}$$

We will mention that the expressions $\varkappa_i \ (i = 1,2,3), \delta_1, \sigma_1$ and $\theta_i \ (i = 1,2,3)$ are known from [6, 7].

If we examine the differential system of the first approximation [1, 2] for system (1), written in the expanded form

$$\frac{dx^1}{dt} = a^1_1x^1 + a^1_2x^2 + a^1_3x^3, \quad \frac{dx^2}{dt} = a^2_1x^1 + a^2_2x^2 + a^2_3x^3, \quad \frac{dx^3}{dt} = a^3_1x^1 + a^3_2x^2 + a^3_3x^3, \tag{9}$$

then it can be easily verified the following assertion

Lemma 1.1. *The expression* $\sigma_1 = 0$ *forms a* $GL(3,\mathbb{R})$ *particular invariant integral for system* (9)*.*

The proof follows directly from the equality

$$(a^1_1x^1 + a^1_2x^2 + a^1_3x^3)\frac{\partial\sigma_1}{\partial x^1} + (a^2_1x^1 + a^2_2x^2 + a^2_3x^3)\frac{\partial\sigma_1}{\partial x^2} + (a^3_1x^1 + a^3_2x^2 + a^3_3x^3)\frac{\partial\sigma_1}{\partial x^3} = \theta_1\sigma_1.$$

Lemma 1.2. *Let* $\delta_1 \equiv 0$ *in* (6)*. Then we obtain the following relations between the coefficients of the system* (9)*:*

$$a)\ a^2_1 = a^3_1 = 0; \quad a^3_2 \neq 0; \quad a^1_3 = \frac{a^1_2(a^3_3 - a^1_1)}{a^3_2}; \quad a^2_3 = \frac{(a^1_1 - a^2_2)(a^1_1 - a^3_3)}{a^3_2};$$

$$b)\ a_1^3 = a_2^3 = 0;\quad a_1^2 \neq 0;\quad a_2^1 = \frac{(a_1^1 - a_3^3)(a_2^2 - a_3^3)}{a_1^2};\quad a_3^1 = \frac{a_3^2(a_1^1 - a_3^3)}{a_1^2};$$

$$c)\ a_1^2 = a_1^3 = a_2^3 = 0;\quad a_2^1 \neq 0;\quad a_1^1 = a_3^3;;\quad a_3^2 = \frac{a_3^1(a_2^2 - a_3^3)}{a_2^1}$$

$$d)\ a_1^2 = a_3^2 = a_1^3 = a_2^3 = 0;\quad a_2^1 \neq 0;\quad a_2^2 = a_3^3;$$

$$e)\ a_2^1 = a_1^2 = a_1^3 = a_2^3 = 0;\quad a_3^1 \neq 0;\quad a_1^1 = a_2^2;$$

$$f)\ a_2^1 = a_1^2 = a_3^2 = a_1^3 = a_2^3 = 0;\quad a_3^1 \neq 0;\quad a_2^2 = a_3^3;$$

$$g)\ a_2^1 = a_3^1 = a_1^2 = a_1^3 = a_2^3 = 0;\quad a_3^2 \neq 0;\quad a_1^1 = a_2^2;$$

$$h)\ a_2^1 = a_3^1 = a_1^2 = a_1^3 = a_2^3 = 0;\quad a_3^2 \neq 0;\quad a_1^1 = a_3^3;$$

$$i)\ a_2^1 = a_3^1 = a_1^2 = a_3^2 = a_1^3 = a_2^3 = 0;\quad a_1^1 = a_2^2;$$

$$j)\ a_2^1 = a_3^1 = a_1^2 = a_3^2 = a_1^3 = a_2^3 = 0;\quad a_1^1 = a_3^3;$$

$$k)\ a_2^1 = a_3^1 = a_1^2 = a_3^2 = a_1^3 = a_2^3 = 0;\quad a_2^2 = a_3^3. \tag{10}$$

Proof. From $\delta_1 \equiv 0$ we get the equalities:

$$(a_2^1)^2 a_3^2 - a_2^1 a_3^1 a_2^2 + a_2^1 a_3^1 a_3^3 - (a_3^1)^2 a_2^3 = 0,$$

$$-2a_1^1 a_2^1 a_3^2 + a_1^1 a_3^1 a_2^2 - a_1^1 a_3^1 a_3^3 + a_2^1 a_3^1 a_1^2 + a_2^1 a_2^2 a_3^2 + a_2^1 a_3^2 a_3^3 + (a_3^1)^2 a_1^3 - a_3^1 (a_2^2)^2 +$$
$$+ a_3^1 a_2^2 a_3^3 - 2a_3^1 a_3^2 a_2^3 = 0,$$

$$(a_1^1)^2 a_3^2 - a_1^1 a_3^1 a_1^2 - a_1^1 a_2^2 a_3^2 - a_1^1 a_3^2 a_3^3 - a_2^1 a_1^2 a_3^2 + 2a_3^1 a_1^2 a_2^2 - a_3^1 a_1^2 a_3^3 + 2a_3^1 a_3^2 a_1^3 +$$
$$+ a_2^2 a_3^2 a_3^3 - (a_3^2)^2 a_2^3 = 0,$$

$$a_1^1 a_2^1 a_2^2 - a_1^1 a_2^1 a_3^3 + 2a_1^1 a_3^1 a_2^3 - (a_2^1)^2 a_1^2 - a_2^1 a_3^1 a_1^3 - a_2^1 a_2^2 a_3^3 + 2a_2^1 a_3^2 a_2^3 + a_2^1 (a_3^3)^2 -$$
$$- a_3^1 a_2^2 a_2^3 - a_3^1 a_2^3 a_3^3 = 0,$$

$$-(a_1^1)^2 a_2^2 + (a_1^1)^2 a_3^3 + a_1^1 a_2^1 a_1^2 - a_1^1 a_3^1 a_1^3 + a_1^1 (a_2^2)^2 - a_1^1 (a_3^3)^2 - a_2^1 a_1^2 a_2^2 - 3a_2^1 a_3^2 a_1^3 + 3a_3^1 a_1^2 a_2^2 +$$
$$+ a_3^1 a_1^3 a_3^3 - (a_2^2)^2 a_3^3 + a_2^2 a_3^2 a_2^3 + a_2^2 (a_3^3)^2 - a_3^2 a_2^3 a_3^3 = 0,$$

$$-(a_1^1)^2 a_2^3 + a_1^1 a_2^1 a_1^3 + a_1^1 a_2^2 a_2^3 + a_1^1 a_2^3 a_3^3 - 2a_2^1 a_1^2 a_2^3 + a_2^1 a_2^2 a_1^3 - 2a_2^1 a_1^3 a_3^3 + a_3^1 a_1^3 a_2^3 -$$
$$- a_2^2 a_2^3 a_3^3 + a_3^2 (a_2^3)^2 = 0,$$

$$a_1^1 a_1^2 a_3^2 - a_3^1 (a_1^2)^2 - a_1^2 a_3^2 a_3^3 + (a_3^2)^2 a_1^3 = 0,$$

$$-a_1^1 a_1^2 a_2^2 + a_1^1 a_1^2 a_3^3 + a_1^1 a_3^2 a_1^3 + a_2^1 (a_1^2)^2 - 2a_3^1 a_1^2 a_1^3 + a_1^2 a_2^2 a_3^3 + a_1^2 a_3^2 a_2^3 - a_1^2 (a_3^3)^2 -$$
$$-2a_2^2 a_3^2 a_1^3 + a_3^2 a_1^3 a_3^3 = 0,$$

$$-a_1^1 a_1^2 a_2^3 - a_1^1 a_2^2 a_1^3 + a_1^1 a_1^3 a_3^3 + 2a_2^1 a_1^2 a_1^3 - a_3^1 (a_1^3)^2 - a_1^2 a_2^2 a_2^3 + 2a_1^2 a_2^3 a_3^3 + (a_2^2)^2 a_1^3 -$$
$$- a_2^2 a_1^3 a_3^3 - a_3^2 a_1^3 a_2^3 = 0,$$

$$-a_1^1 a_1^3 a_2^3 + a_2^1 (a_1^3)^2 - a_1^2 (a_2^3)^2 + a_2^2 a_1^3 a_2^3 = 0. \tag{11}$$

Without loss of generality we can assume that

$$a_1^3 = 0, \tag{12}$$

because, otherwise, we can obtain this equality by transformation

$$\bar{x}^1 = x^2, \quad \bar{x}^2 = x^1 + \frac{a_2^3}{a_1^3}x^2, \quad \bar{x}^3 = x^3. \tag{13}$$

Substituting $a_1^3 = 0$ in (11), from the last equality, we get $a_1^2 a_2^3 = 0$. This implies the following cases: 1) $a_1^3 = a_1^2 = 0,\ a_2^3 \neq 0$; 2) $a_1^3 = a_2^3 = 0,\ a_1^2 \neq 0$; 3) $a_1^3 = a_1^2 = a_2^3 = 0$. Calculating the other coefficients by means of the equalities (11) from 1), we obtain the case a) from (10). From 3) we get the cases $c) - k)$, from (10).

Lemma 1.2 is proved. □

Lemma 1.3. *Assume that* $\sigma_1 \equiv 0$ *in* (7)*. Then we get the relation* (10).

The proof of Lemma 1.3 is analogous to the proof of Lemma 1.2.

Using Lemmas 1.2 and 1.3, it is obtained

Theorem 1.2.

$$\sigma_1(x) \equiv 0 \Leftrightarrow \delta_1(u) \equiv 0 \tag{14}$$

and conversely

$$\sigma_1(x) \neq 0 \Leftrightarrow \delta_1(u) \neq 0. \tag{15}$$

2. Notions of stability of unperturbed motion and the Lyapunov function

Let the differential system of the perturbed motion [2] be given in the form (1). Then, according to [2], the zero values of the variables x^j $(j = 1, 2, 3)$ correspond to the unperturbed motion.

Definition of stability by Lyapunov [2]**.** Let for any small number ε, there exists a positive number δ such that for any perturbation $x^j(t_0)$ is satisfied the condition

$$\sum_{j=1}^{3}(x^j(t_0))^2 \leq \delta, \tag{16}$$

and for any $t \geq t_0$ is satisfied the condition

$$\sum_{j=1}^{3}(x^j(t))^2 < \varepsilon.$$

Then the unperturbed motion $x^j = 0$ $(j = 1, 2, 3)$ is called *stable*, otherwise *unstable*.

If the unperturbed motion is stable and the value δ can be found however small such that for any perturbed motions satisfying (16) the condition

$$\lim_{t\to+\infty}\sum_{j=1}^{3}(x^j(t))^2 = 0$$

is valid, then the unperturbed motion is called *asymptotically stable*.

We will examine the system (9). The characteristic equation of this system is

$$\varrho^3 + L_1\varrho^2 + L_2\varrho + L_3 = 0, \tag{17}$$

where the coefficients of this equation are expressed by center-affine invariants (8), and have the form

$$L_1 = -\theta_1,\quad L_2 = \frac{1}{2}(\theta_2 - \theta_1^2),\quad L_3 = -\frac{1}{6}(\theta_1^3 - 3\theta_1\theta_2 + 2\theta_3). \tag{18}$$

Using the Lyapunov's theorems on stability of unperturbed and perturbed motion in the first approximation [2], and the Hurwitz's theorem [2], we obtain the following theorems:

Theorem 2.1. *Assume that the center-affine invariants* (18) *of the system* (1) *satisfy the inequalities*

$$L_1 > 0,\quad L_2 > 0,\quad L_3 > 0,\quad L_1L_2 - L_3 > 0,$$

then the unperturbed motion $x^1 = x^2 = x^3 = 0$ *of the system* (1) *is asymptotically stable.*

Theorem 2.2. *If at least one of the center-affine invariant expressions* (18) *of system* (1) *has the sign less than zero, then the unperturbed motion* $x^1 = x^2 = x^3 = 0$*, of the system* (1)*, is unstable.*

Following [2], we consider the real function $V(x) = V(x^1, x^2, x^3)$, which is defined in the domain

$$\sum_{j=1}^{3}(x^j)^2 \le \mu, \tag{19}$$

where μ is a positive numerical constant.

In this domain, the function $V(x)$ is unique and continuous and is vanishing for $x^1 = x^2 = x^3 = 0$, i.e.

$$V(0) = 0. \tag{20}$$

If in the domain (19) this function takes values of the same sign, then it is called of *constant sign* (respectively *positive* or *negative*). If the function of constant sign vanishes only when x^1, x^2, x^3 are zero, then V is called of *determined sign*. The introduction of such functions V, in the research of the stability of motion, are called *Lyapunov functions*.

Later on, we will use the following Lyapunov Theorem:

Theorem 2.3. [1, 2] *Let for equations of the perturbed motion can be found a function* $V(x) = V(x^1, x^2, x^3)$ *of the determined sign such that its derivative* $\dot{V}$*, by virtue of the system (45) from* [1]*, with* $s = 1$*, would be of constant sign, opposite to the sign of the function V or identically zero. Then the unperturbed motion is stable.*

3. Invariant conditions for obtaining the Lyapunov form of differential system (1)

Lemma 3.1. *Suppose that* $\sigma_1 \not\equiv 0$ *in* (7)*. Then system* (1)*, by means of a centro-affine transformation, can be brought to the form*

$$\begin{aligned} \frac{dx^1}{dt} &= x^2 + a^1_{\alpha\beta\gamma\delta}x^\alpha x^\beta x^\gamma x^\delta, \\ \frac{dx^2}{dt} &= x^3 + a^2_{\alpha\beta\gamma\delta}x^\alpha x^\beta x^\gamma x^\delta, \\ \frac{dx^3}{dt} &= -L_3x^1 - L_2x^2 - L_1x^3 + a^3_{\alpha\beta\gamma\delta}x^\alpha x^\beta x^\gamma x^\delta, \end{aligned} \tag{21}$$

where L_i $(i = 1, 2, 3)$ *are from* (18)*.*

Proof. Consider the substitution

$$\bar{x}^1 = \varkappa_1, \quad \bar{x}^2 = \varkappa_2, \quad \bar{x}^3 = \varkappa_3, \tag{22}$$

where $\varkappa_i$ $(i = 1, 2, 3)$ are given in (5). From (22), by means of expressions $\varkappa_i$, it is obtained (in $\varkappa_2$ the index α is renotated by α_1)

$$\Delta \equiv det(\varkappa_1, \varkappa_2, \varkappa_3) = \begin{vmatrix} u_1 & u_2 & u_3 \\ a^{\alpha_1}_1 u_{\alpha_1} & a^{\alpha_1}_2 u_{\alpha_1} & a^{\alpha_1}_3 u_{\alpha_1} \\ a^\alpha_1 a^\beta_\alpha u_\beta & a^\alpha_2 a^\beta_\alpha u_\beta & a^\alpha_3 a^\beta_\alpha u_\beta \end{vmatrix} = \delta_1, \tag{23}$$

where δ_1 is from (6) and

$$\begin{aligned} x^1 &= \frac{1}{\delta_1}[(a^{\alpha_1}_2 a^\alpha_3 a^\beta_\alpha u_{\alpha_1} u_\beta - a^{\alpha_1}_3 a^\alpha_2 a^\beta_\alpha u_{\alpha_1})\bar{x}^1 + (a^\alpha_2 a^\beta_\alpha u_\alpha u_\beta - a^\alpha_3 a^\beta_\alpha u_\beta u_2)\bar{x}^2 + \\ &\quad +(a^{\alpha_1}_3 u_{\alpha_1} u_2 - a^{\alpha_1}_2 u_{\alpha_1} u_3)\bar{x}^3], \\ x^2 &= \frac{1}{\delta_1}[(a^{\alpha_1}_3 a^\alpha_1 a^\beta_\alpha u_{\alpha_1} u_\beta - a^{\alpha_1}_1 a^\alpha_3 a^\beta_\alpha u_{\alpha_1} u_\beta)\bar{x}^1 + (a^\alpha_3 a^\beta_\alpha u_\beta u_1 - a^\alpha_1 a^\beta_\alpha u_\beta u_3)\bar{x}^2 + \\ &\quad +(a^{\alpha_1}_1 u_{\alpha_1} u_3 - a^{\alpha_1}_3 u_{\alpha_1} u_1)\bar{x}^3], \\ x^3 &= \frac{1}{\delta_1}[(a^{\alpha_1}_1 a^\alpha_2 a^\beta_\alpha u_{\alpha_1} u_\beta - a^{\alpha_1}_2 a^\alpha_1 a^\beta_\alpha u_{\alpha_1} u_\beta)\bar{x}^1 + (a^\alpha_1 a^\beta_\alpha u_\beta u_2 - a^\alpha_2 a^\beta_\alpha u_\beta u_1)\bar{x}^2 + \\ &\quad +(a^{\alpha_1}_2 u_{\alpha_1} u_1 - a^{\alpha_1}_1 u_{\alpha_1} u_2)\bar{x}^3]. \end{aligned} \tag{24}$$

Considering (5) and substitutions (22)-(24), then from the system (1) we obtain the system (21) with $\delta_1 \not\equiv 0$, which according to Theorem 1.2 is equivalent to $\sigma_1 \not\equiv 0$. Lemma 3.1 is proved. □

Using Lemma 1.3, it can easily be verified that the following assertion is proved:

Remark 3.1. If for ystem (9) of the first approximation, the condition $\sigma_1 \equiv 0$ holds from (7), then the characteristic equation (17) has only real roots.

Taking into consideration Remark 3.1, it can easily be verified that the following assertion is proved:

Lemma 3.2. *The characteristic equation of system* (21)*, with* $\sigma_1 \not\equiv 0$*, has purely imaginary eigenvalues if and only if the system has the form*

$$\begin{aligned}\frac{dx^1}{dt} &= x^2 + a^1_{\alpha\beta\gamma\delta}x^\alpha x^\beta x^\gamma x^\delta,\\ \frac{dx^2}{dt} &= x^3 + a^2_{\alpha\beta\gamma\delta}x^\alpha x^\beta x^\gamma x^\delta,\\ \frac{dx^3}{dt} &= -L_1L_2x^1 - L_2x^2 - L_1x^3 + a^3_{\alpha\beta\gamma\delta}x^\alpha x^\beta x^\gamma x^\delta, \quad (L_1, L_2 > 0),\end{aligned} \tag{25}$$

where L_i $(i = 1, 2)$ *are of the form* (18).

Theorem 3.1. *Let* $\sigma_1 \not\equiv 0$ *in* (7)*. Then, by a centro-affine transformation, the system* (21) *can be brought to the form*

$$\begin{aligned}\frac{dx^1}{dt} &= -\lambda x^2 + a^1_{\alpha\beta\gamma\delta}x^\alpha x^\beta x^\gamma x^\delta,\\ \frac{dx^2}{dt} &= \lambda x^1 + a^2_{\alpha\beta\gamma\delta}x^\alpha x^\beta x^\gamma x^\delta,\\ \frac{dx^3}{dt} &= x^2 - L_1x^3 + a^3_{\alpha\beta\gamma\delta}x^\alpha x^\beta x^\gamma x^\delta,\end{aligned} \tag{26}$$

with the linear parts of the first two equations in the Lyapunov form, where L_1, L_3 *are from* (18) *and* $\lambda^2 = L_3$ $(L_1, L_3 > 0)$.

Proof. We will examine the system (21). According to Lyapunov system (45) from [1], the linear part of this system must have the form

$$\frac{dX^1}{dt} = -\lambda X^2 + \dots, \quad \frac{dX^2}{dt} = \lambda X^1 + \dots, \quad \frac{dX^3}{dt} = aX^1 + bX^2 + cX^3 + \dots, \tag{27}$$

where by dots we mean the homogeneities of the fourth order with respect to X^1, X^2, X^3. The coefficients λ, a, b, c are expressions in L_i $(i = 1, 2, 3)$ and the new variables

X^1, X^2, X^3 have the form

$$X^1 = \alpha_1 x^1 + \alpha_2 x^2 + \alpha_3 x^3,\ X^2 = \beta_1 x^1 + \beta_2 x^2 + \beta_3 x^3,\ X^3 = \gamma_1 x^1 + \gamma_2 x^2 + \gamma_3 x^3, \quad (28)$$

where

$$\Delta = \begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{vmatrix} \neq 0. \quad (29)$$

Unde the conditions (29), we observe that the substitution (28) forms a centro-affine transformation. Substituting (28) in the linear part of the system (27) and comparing with the system (25), we obtain a system of nine algebraic equation in 12 unknowns $a, b, c, \alpha_i, \beta_j, \gamma_k\ \ (i, j, k = 1, 2, 3)$. Solving this system, we have

$$X^1 = -L_1^2\lambda x^1 + \lambda x^3,\ X^2 = L_1 L_2 x^1 + (L_1^2 + L_2)x^2 + L_1 x^3,\ X^3 = 2L_2 x^1 + L_1 x^2 + x^3,$$

where $\lambda^2 = L_2$, and the determinant of this transformation is

$$\Delta = -2L_2\lambda(L_1^2 + L_2) \neq 0 \quad (L_2 > 0).$$

This transformation brings the system (25) to a system with the linear part in the Lyapunov form (26) for which the initial notations of the phase variables are preserved. The form of the fourth-degree homogeneity does not change, apart from the coefficients and the phase variables. Theorem 3.1 is proved. □

4. Lyapunov-Darboux form of system (1) and stability conditions of unperturbed motion

Remark 4.1. For $\eta_1 \equiv 0$, from (7), the system (1) will get the following Darboux form

$$\frac{dx^j}{dt} = a^j_\alpha x^\alpha + 4x^j R(x) \quad (j = 1, 2, 3), \quad (30)$$

where $R(x)$ is a homogeneous polynomial of the third degree with respect to the vector coordinates $x = (x^1, x^2, x^3)$.

Remark 4.2. The system (30) has as $GL(3, \mathbb{R})$-invariant integral the expression $\sigma_1 \not\equiv 0$.

This affirmation results from the identity

$$[a^1_\alpha x^\alpha + 4x^1 R(x)]\frac{\partial \sigma_1}{\partial x^1} + [a^2_\alpha x^\alpha + 4x^2 R(x)]\frac{\partial \sigma_1}{\partial x^2} + [a^3_\alpha x^\alpha + 4x^3 R(x)]\frac{\partial \sigma_1}{\partial x^3} = [\theta_1 + 12R(x)]\sigma_1,$$

where θ_1 is from (8).

Taking into consideration that the system (26) was obtained using the invariant condition $\sigma_1 \not\equiv 0$ by means of the centro-affine transformations (22), and the Darboux system (30) is governed by the invariant condition $\eta_1 \equiv 0$, we obtain

Lemma 4.1. *Let* $\sigma_1 \not\equiv 0,\ \eta_1 \equiv 0$ *in* (7) *and* $L_1, L_2 > 0$ *in* (18). *Then system* (1), *by the centro-affine transformations, can be brought to the following Lyapunov-Darboux form*

$$\frac{dx}{dt} = -\lambda y + 4xR(x,y,z),$$
$$\frac{dy}{dt} = \lambda x + 4yR(x,y,z), \tag{31}$$
$$\frac{dz}{dt} = y - L_1 z + 4zR(x,y,z),$$

where $x = x^1, y = x^2, z = x^3$, *and*

$$R(x,y,z) = a_1x^3 + a_2y^3 + a_3z^3 + 3a_4x^2y + 3a_5x^2z + 3a_6xy^2 + 3a_7xz^2 + \\ +3a_8xyz + 3a_9y^2z + 3a_{10}yz^2. \tag{32}$$

By means of the determining equations, from [7], we construct the Lie algebra admitted by the system (31)-(32). Using Lie algebra for the mentioned system, we obtain the analytic first integral of the form

$$F(x,y,z) \equiv \frac{h_1^3}{(J+h_2)^2} = 0 \tag{33}$$

governed by the condition

$$J(J+h_2) \neq 0, \tag{34}$$

where

$$h_1 = x^2 + y^2,\quad J = -L_1\lambda^2(4L_1^2+\lambda^2)(L_1^2+4\lambda^2),$$

$$h_2 = \lambda[4(8a_3L_1^2 + 24a_{10}L_1^3 + 12a_5L_1^4 + 24a_9L_1^4 + 8a_2L_1^5 + 12a_4L_1^5 - 24a_7L_1^2\lambda + 22a_3\lambda^2 - \\ -12a_8L_1^3\lambda + 66a_{10}L_1\lambda^2 + 75a_5L_1^2\lambda^2 + 78a_9L_1^2\lambda^2 + 34a_2L_1^3\lambda^2 + 51a_4L_1^3\lambda^2 - 36a_7\lambda^3 - \\ -3a_8L_1\lambda^3 + 18a_5\lambda^4 + 18a_9\lambda^4 + 8a_2L_1\lambda^4 + 12a_4L_1\lambda^4)x^3 - 4L_1(12a_7L_1^2 + 12a_8L_1^3 + \\ +8a_1L_1^4 + 12a_6L_1^4 + 10a_3\lambda + 30a_{10}L_1\lambda - 24a_5L_1^2\lambda + 24a_9L_1^2\lambda - 12a_7\lambda^2 + \\ +3a_8L_1\lambda^2 + 34a_1L_1^2\lambda^2 + 51a_6L_1^2\lambda^2 - 6a_5\lambda^3 + 6a_9\lambda^3 + 8a_1\lambda^4 + 12a_6\lambda^4)y^3 + \\ +4a_3\lambda(4L_1^2+\lambda^2)(L_1^2+4\lambda^2)z^3 - 12a_1L_1(4L_1^2+\lambda^2)(L_1^2+4\lambda^2)x^2y + \\ +12\lambda(12a_5L_1^4 + 12a_7L_1^2\lambda + 12a_8L_1^3\lambda + 10a_3\lambda^2 + 30a_{10}L_1\lambda^2 + 27a_5L_1^2\lambda^2 + \\ +24a_9L_1^2\lambda^2 - 12a_7\lambda^3 + 3a_8L_1\lambda^3 + 6a_5\lambda^4 + 6a_9\lambda^4)x^2z + 12(4a_3L_1^2 + 12a_{10}L_1^3 + \\ +12a_9L_1^4 + 4a_2L_1^5 - 18a_7L_1^2\lambda - 12a_8L_1^3\lambda + 6a_3\lambda^2 + 18a_{10}L_1\lambda^2 + 24a_5L_1^2\lambda^2 + \\ +27a_9L_1^2\lambda^2 + 17a_2L_1^3\lambda^2 - 12a_7\lambda^3 - 3a_8L_1\lambda^3 + 6a_5\lambda^4 + 6a_9\lambda^4 + 4a_2L_1\lambda^4)xy^2 + \\ +12\lambda(6a_7L_1^2 + a_3\lambda + 3a_{10}L_1\lambda)(L_1^2+4\lambda^2)xz^2 + 12L_1\lambda(12a_7L_1^2 + 12a_8L_1^3 + \\ +10a_3\lambda + 30a_{10}L_1\lambda - 24a_5L_1^2\lambda + 24a_9L_1^2\lambda - 12a_7\lambda^2 + 3a_8L_1\lambda^2 - 6a_5\lambda^3 + 6a_9\lambda^3)xyz +$$

$$+12\lambda(4a_3L_1^2 + 12a_{10}L_1^3 + 12a_9L_1^4 - 18a_7L_1^2\lambda - 12a_8L_1^3\lambda + 6a_3\lambda^2 + 18a_{10}L_1\lambda^2+$$
$$+24a_5L_1^2\lambda^2 + 27a_9L_1^2\lambda^2 - 12a_7\lambda^3 - 3a_8L_1\lambda^3 + 6a_5\lambda^4 + 6a_9\lambda^4)y^2z+$$
$$+12L_1\lambda(2a_3 + 6a_{10}L_1 - 3a_7\lambda)(L_1^2 + 4\lambda^2)yz^2].$$

Analyzing the first integral (33), we notice that if the inequality (34) holds, then the function $F(x, y, z)$ forms *the Lyapunov function*. According to Theorem 2.3, we have

Theorem 4.1. *Let for system of the Lyapunov-Darboux type* (31)-(32) *the inequality* (34) *holds. Then the unperturbed motion $x = y = z = 0$, governed by this system, is stable.*

Remark 4.3. For the first time, a problem analogous to that examined in this paper was investigated for ternary system with quadratic nonlinearities in [8]. Here, the invariant centro-affine conditions of stability or instability of unperturbed motion were obtained.

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