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Phase portraits of some polynomial differential systems with maximal multiplicity of the line at the infinity

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Abstract. The present study delves into the investigation of phase portraits of polynomial differential systems, which are systems of differential equations of the form $\frac{dx}{dt} = P(x,y)$, $\frac{dy}{dt} = Q(x,y)$, where x and y are the dependent variables and t is the independent variable. The functions $P(x,y)$ and $Q(x,y)$ are polynomials in x and y. The main objective of this research is to obtain the phase portraits of polynomial differential systems of degree $n \in \{3,4,5\}$ and having an invariant straight line at the infinity of maximal multiplicity.

2010 Mathematics Subject Classification: 34C05.
Keywords: phase portrait, singular point, Poincaré transformation.

Tablouri fazice ale unor sisteme diferențiale polinomiale cu dreapta de la infinit de multiplicitate maximală

Rezumat. Prezentul studiu se aprofundează în investigarea portretelor de fază ale sistemelor diferențiale polinomiale, care sunt sisteme de ecuații diferențiale de forma $\frac{dx}{dt} = P(x,y)$, $\frac{dy}{dt} = Q(x,y)$, unde x și y sunt variabilele dependente și t este variabila independentă. Funcțiile $P(x,y)$ și $Q(x,y)$ au polinoame în x și y. Obiectivul principal al acestei cercetări este obținerea portretelor de fază ale sistemelor diferențiale polinomiale de grad $n \in \{3,4,5\}$ și având o dreaptă invariantă la infinit de multiplicitate maximală.

Cuvinte-cheie: portret fazic, punct singular, transformarea Poincaré.

1. Introduction

Phase portraits are graphical representations of the behaviour of a system of differential equations over time and they can be used to visualize the long-term behaviour of a system. Overall, the phase portrait of a polynomial differential system with maximal multiplicity of the line at infinity can be quite complex and may exhibit a variety of different behaviours.

The study of invariant algebraic curves plays a crucial role in the qualitative analysis of dynamical systems. The problem of determining the maximum number of invariant straight lines present in a polynomial differential system is explored in [1]. Additionally, the utilization of invariant straight lines in the calculation of Darboux first integrals is a significant area of study, as outlined in [2], where it is demonstrated that a Darboux first integral can be calculated for a polynomial differential system if it possesses a sufficient number of invariant straight lines, taking into account their multiplicities.

In this article, we will focus on phase portraits of polynomial differential systems, which are systems of differential equations of the form:

$$\frac{dx}{dt} = P(x, y), \ \frac{dy}{dt} = Q(x, y),$$

where x and y are the dependent variables and t is the independent variable. The functions $P(x, y)$ and $Q(x, y)$ are polynomials in x and y. We will obtain the phase portraits of polynomial differential systems of degree $n \in \{3, 4, 5\}$ and having the invariant straight line at the infinity of the maximal multiplicity.

2. Cubic polynomial differential systems

According to [3], the maximal multiplicity of the line at infinity for the cubic systems is equal to seven and the systems can be brought to the following two forms:

$$\begin{cases} \dot{x} = 1, \\ \dot{y} = x^3 + ax, \end{cases} \qquad a \in \mathbb{R}; \tag{1}$$

and

$$\begin{cases} \dot{x} = -x, \\ \dot{y} = x^3 + 2y. \end{cases} \tag{2}$$

The system (1) does not possess any singular points within the finite region of the phase plane. However, at infinity, there exist $I_{1,2}^{\infty}(0, \pm 1, 0)$, which are multiple singular points. In order to analyze the behavior of trajectories in proximity to these points, we shall employ the first Poincaré transformation. Prior to this, we shall utilize the transformation $x \to y$, $y \to x$ to relocate these points to opposite sides of the Ox axis:

$$\begin{cases} \dot{x} = y^3 + ay, \\ \dot{y} = 1, \end{cases} \qquad a \in \mathbb{R}. \tag{3}$$

By effecting the transformation $x \to 1/y$, $y \to x/y$, the system represented by equation (3) is transformed into the following form:

$$\begin{cases} \dot{x} = -x^4 + ax^2y^2 + y^3, \\ \dot{y} = -xy(x^2 + ay^2). \end{cases} \tag{4}$$

This system possesses a single singular point $I_1(0, 0)$ which corresponds to the singular points $I_{1,2}^{\infty}(\pm 1, 0, 0)$ of the original system represented by equation (3).

The singular point I_1 is a multiple one, thus we shall utilize the blow-up method. This results in the following differential system:

$$\begin{cases} \dot{x} = -\frac{1}{4}x\sin y\,(2(a+1)x + 2x(a-1)\cos(2y) - 3\cos y - \cos(3y)), \\ \dot{y} = \cos^4 y. \end{cases} \tag{5}$$

The system at hand possesses the singular points $S_1(0, \frac{\pi}{2})$ and $S_2(0, \frac{3\pi}{2})$, both of which are multiple. Thus, we shall utilize the transformation $x \to x$, $y \to y - \pi/2$ to relocate the point S_1 to the origin. Then by expanding the right-hand sides in a Taylor series about $y = 0$, and retaining only a subset of the first monomials, we obtain the following system:

$$\begin{cases} \dot{x} = -xy^3 + x^2\left(-ay^2 + \frac{3y^2}{2} - 1\right), \\ \dot{y} = y^4 - \frac{2y^6}{3}. \end{cases} \tag{6}$$

Using the blow-up method we get that the singular point S_1 decompose in 4 singular points $N_1(0,0)$, $N_2(0, \frac{\pi}{2})$, $N_3(0,\pi)$ and $N_4(0, \frac{3\pi}{2})$ of the following system:

$$\begin{cases} \dot{x} = -\frac{1}{6}x\left(6ax^2\sin^2 y\cos^3 y + 4x^4\sin^7 y - 6x^2\sin^5 y + 6x^2\sin^3 y\cos^2 y - \right. \\ \quad \left. -9x^2\sin^2 y\cos^3 y + 6\cos^3 y\right), \\ \dot{y} = \frac{1}{6}\sin y\cos y\left(6ax^2\sin^2 y\cos y - 4x^4\sin^5 y + 12x^2\sin^3 y - \right. \\ \quad \left. -9x^2\sin^2 y\cos y + 6\cos y\right). \end{cases} \tag{7}$$

The singular points N_1 and N_3 are of saddle type, while the singular points N_2 and N_4 are multiple. By utilizing the transformation $x \to x$, $y \to y - \pi/2$, we reposition the point N_2 to the origin of coordinates. Subsequently, by expanding the right-hand sides in a Taylor series in the vicinity of $y = 0$, and discarding all terms of higher order, we arrive at the system:

$$\begin{cases} \dot{x} = xy^3 + x^3\left(1 - \frac{7y^2}{2}\right), \\ \dot{y} = \frac{1}{6}y\left(6ax^2y - 9x^2y + 4x^4 - 12x^2 + 6y\right). \end{cases} \tag{8}$$

By utilizing the blow-up method, we obtain the following system:

$$\begin{cases} \dot{x} = -\frac{1}{24}x\left(-6ax^2\sin y\sin^2(2y) + 3x^2\sin y\sin^2(2y) + 68x^3\sin^2 y\cos^4 y + \right. \\ \quad \left. +12x\sin^2(2y) - 24x\cos^4 y - 24\sin^3 y\right), \\ \dot{y} = \frac{1}{6}\sin y\cos y\left(6ax^2\sin y\cos^2 y - 6x^2\sin^3 y + 4x^3\cos^4 y + 21x^3\sin^2 y\cdot \right. \\ \quad \left. \cdot\cos^2 y - 9x^2\sin y\cos^2 y - 18x\cos^2 y + 6\sin y\right). \end{cases} \tag{9}$$

By solving the equation $Q(0, y) = 0$, it follows that the point N_2 decomposes into the following four points $P_1(0,0)$, $P_2(0, \frac{\pi}{2})$, $P_3(0,\pi)$, $P_4(0, \frac{3\pi}{2})$, wherein P_1 and P_3 are multiple, while P_2 and P_4 are saddles.

Once more, we shall expand the right-hand sides in a Taylor series in the vicinity of the point $y = 0$, and subsequently apply the blow-up method. This results in the following system:

$$\begin{cases} \dot{x} = -\frac{1}{6}x\left(-6ax^2\sin^3 y\cos^2 y + 20x^2\sin^2 y\cos^3 y + 3x^2\sin^3 y\cos^2 y - \right. \\ \quad \left. -6\sin^3 y - 6\cos^3 y + 18\sin^2 y\cos y\right) \\ \dot{y} = \frac{1}{6}\sin y\cos y\left(6ax^2\sin y\cos^2 y - 6x^2\sin^3 y + 4x^2\cos^3 y - 9x^2\sin y\cdot \right. \\ \quad \left. \cdot\cos^2 y + 24x^2\sin^2 y\cos y + 6\sin y - 24\cos y\right), \end{cases} \tag{10}$$

which possesses 6 singular points. The coordinates and types of these singular points are listed in Table 1.

Table 1. Blow up for point P_1

S.P.	$O_1(0,0)$	$O_2(0,\operatorname{arctg}4)$	$O_3(0,\frac{\pi}{2})$	$O_4(0,\pi)$	$O_5(0,\pi+\operatorname{arctg}4)$	$O_6(0,\frac{3\pi}{2}$
$\lambda_{1,2}$	$-4;1$	$\frac{1}{\sqrt{17}};\frac{4}{\sqrt{17}}$	± 1	$-1;4$	$-\frac{4}{\sqrt{17}};-\frac{1}{\sqrt{17}}$	± 1
Type	S	N^u	S	S	N^s	S

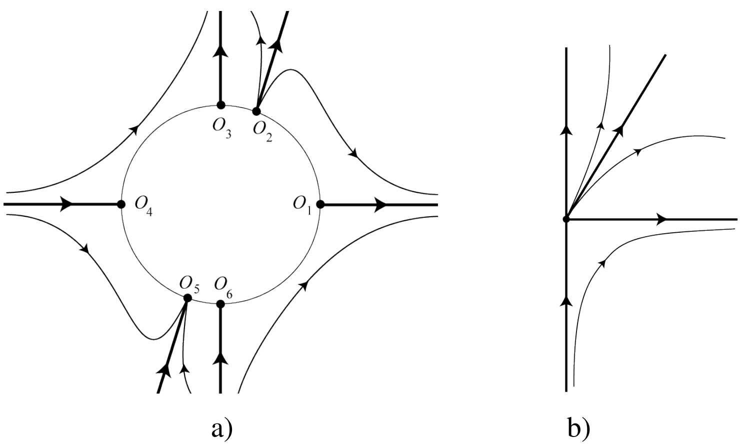


Figure 1. Blow-up for the point P_1 $(0,0)$.

By constructing all these points on a circle, and plotting their behaviour in proximity to them (Figure 1 a)), followed by compressing the circle into a single point, we can obtain the behaviour of the trajectories in the vicinity of the singular point P_1 of the system represented by equation (9). To determine the type of the singular point N_2 of the system represented by equation (7), we shall only utilize the portion of the phase plane corresponding to $x > 0$ (as depicted in Figure 1 b)).

By utilizing P_1, we can construct the phase portrait for the singular point N_2 (Figure 2 a), b)). The phase portrait of the singular point $N_4(0, \frac{3\pi}{2})$ is also depicted in Figure 2

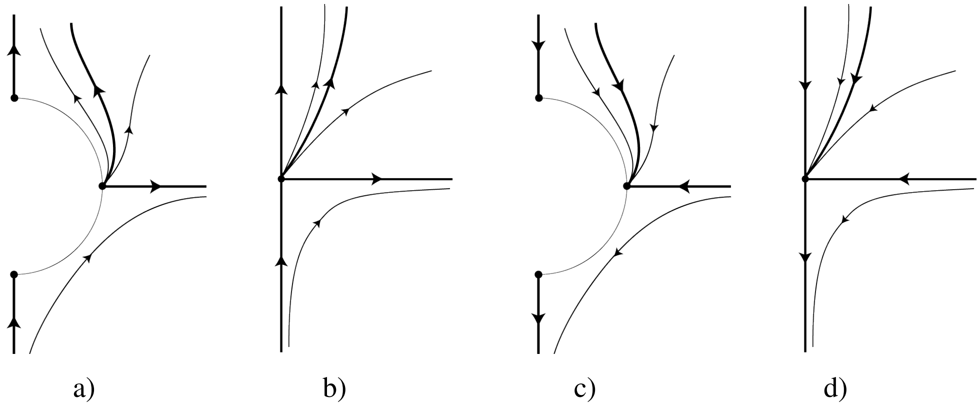


Figure 2. Blow-up for the points $N_2\left(0, \frac{\pi}{2}\right)$ and $N_4(0, \frac{3\pi}{2})$.

c) d), which is obtained in an analogous manner, with the only difference being that the direction of the trajectories is inverted.

We can now construct the trajectories for the singular point S_1, which was decomposed into the points P_1, P_2, P_3 and P_4 (see Figure 3 a)). Since we only require the half-plane $x > 0$, we obtain Figure 3 b).

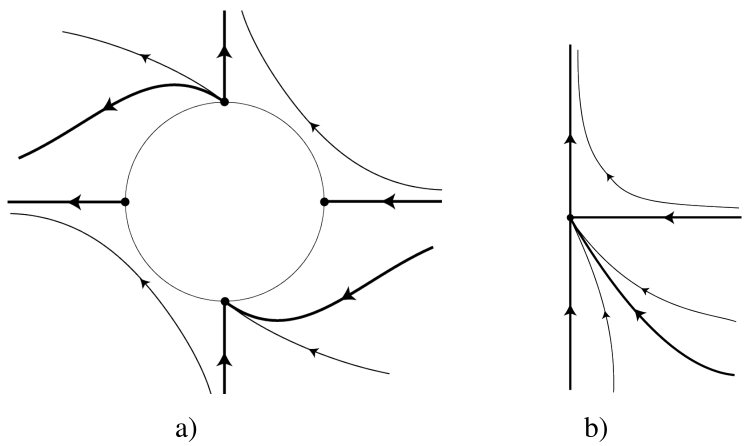
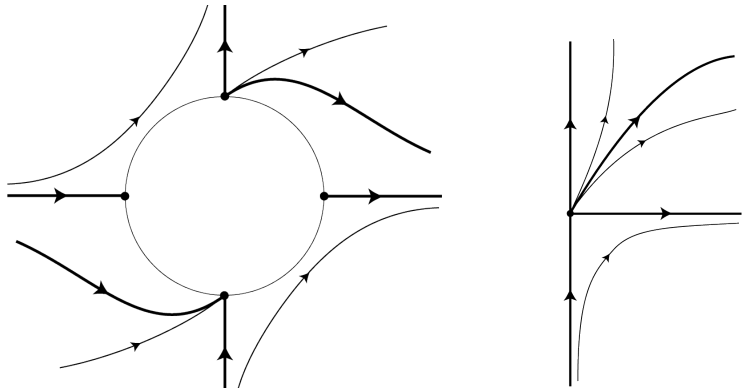


Figure 3. Blow-up for the point $S_1\left(0, \frac{\pi}{2}\right)$.

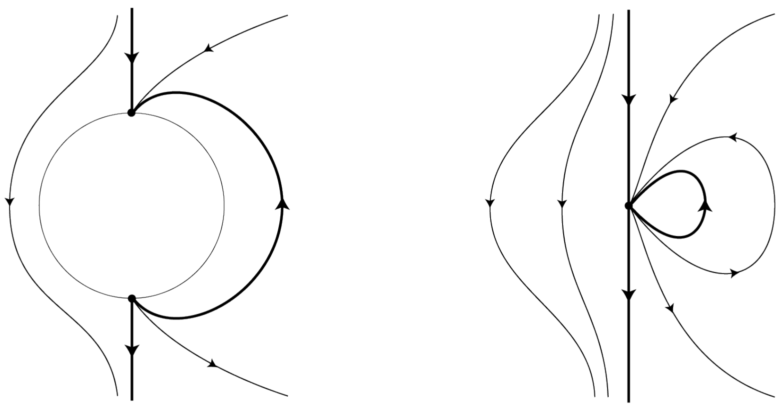
By executing the same procedure (applying the blow-up method three times) for the point $S_2(0, \frac{3\pi}{2})$, we obtain the representation illustrated in Figure 4.

Ultimately, by utilizing S_1 and S_2, we can construct the phase portrait for the point $I_1(0, 0)$ of the system represented by equation (4) (see Figure 5).



a) b)

Figure 4. Blow-up for the point $S_2\left(0, \frac{3\pi}{2}\right)$.



a) b)

Figure 5. Blow-up for the point $I_1\ (0,0)$.

Having now determined the behaviour of trajectories in the proximity of $I^{\infty}_{1,2}$, we can construct the phase portrait for the polynomial differential system represented by equation (3). By applying the transformation $x \to y$, $y \to x$, we can obtain the phase portrait for the system represented by equation (1) (see Figure 6 a)). By following a similar procedure, we can construct the phase portrait for the system represented by equation (2) (see Figure 6 b)).

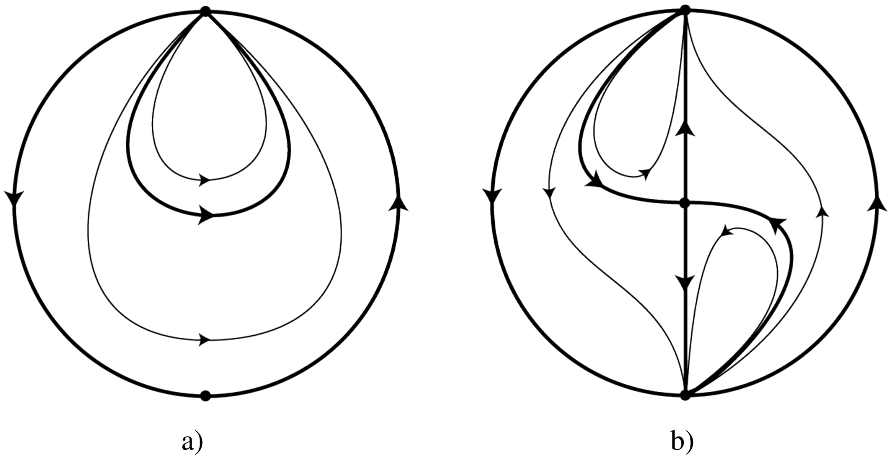


Figure 6. The phase portraits for the cubic differential systems (1) and (2).

3. Quartic polynomial differential systems

According to [4], a quartic polynomial differential system with maximal multiplicity can be brought into the following form:

$$\begin{cases} \dot{x} = -3x + ay^4, \\ \dot{y} = y, \quad a > 0. \end{cases} \tag{11}$$

This system has an invariant line at infinity with multiplicity equal to 10. By referring to [5], we can construct its phase portrait. However, we first need to relocate the singular points at infinity to be situated at the ends of the Oy axis by applying the transformation $x \to y$, $y \to x$ (see Figure 7).

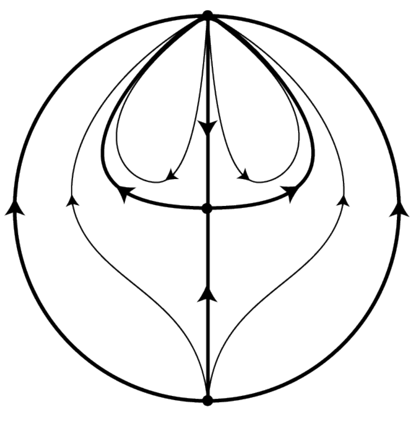


Figure 7. The phase portrait for the quartic differential system (11).

4. Quintic polynomial differential systems

As stated in [6], a quintic polynomial differential system with the line at infinity of maximal multiplicity can be transformed into the following form:

$$\begin{cases} \dot{x} = x, \quad a \neq 0, \\ \dot{y} = -4y + ax^5. \end{cases} \tag{12}$$

The transformations $x \to x$, $y \to -y$, $a \to -a$ do not alter the form of the system, in order to maintain its generality, the condition $a > 0$ is imposed. Furthermore, it is apparent that the transformations $x \to -x$, $y \to -y$ does not affect the form of the system, thus the trajectories of the system are symmetric with respect to the origin of coordinates.

By applying the transformation $x \to y$, $y \to x$, the system (12) can be transformed into the following system:

$$\begin{cases} \dot{x} = -4x + ay^5, \\ \dot{y} = y, \quad a \neq 0. \end{cases} \tag{13}$$

The system (13) can be transformed into the following form by applying the Poincaré transformation $x \to \frac{1}{x}$, $y \to \frac{y}{x}$:

$$\begin{cases} \dot{x} = x(4x^4 - ay^5), \\ \dot{y} = y(5x^4 - ay^5) \end{cases} \tag{14}$$

By utilizing a blow-up transformation on the system (13), we are able to perform a detailed analysis of the phase space behaviour in the vicinity of the multiple singular point at the origin for the system (14), which corresponds to the points $I_{1,2}^{\infty}(\pm 1, 0, 0)$ of the system (13).

$$\begin{cases} \dot{x} = x\left(ax\sin^7 y + ax\sin^5 y\cos^2 y - 4\cos^6 y - 5\sin^2 y\cos^4 y\right), \\ \dot{y} = \sin y\cos^5 y \end{cases} \tag{15}$$

Specifically, by solving the equation $Q(0, y) = 0$, we can identify the coordinates and topological classification of the resulting singular points M_1, M_2, M_3, and M_4. Of these, M_1 and M_3 are classified as nodal singularities, with M_1 being an unstable node and M_3 being a stable node. On the other hand, M_2 and M_4 are multiple singular points.

We translate the singular point M_2 to the origin of coordinates, then we expand the right-hand sides in a Taylor series in the neighbourhood of $y = 0$, and retain only the terms of low degree, we obtain the system:

$$\begin{cases} \dot{x} = \frac{5}{2}ax^2y^2 - ax^2 + xy^4\left(5 - \frac{65ax}{24}\right), \\ \dot{y} = \frac{4y^7}{3} - y^5, \end{cases} \tag{16}$$

Subsequently, by utilizing the blow-up method, we derive the following system:

$$\begin{cases} \dot{x} = \frac{1}{24}x\left(-65ax^4\sin^4 y\cos^3 y + 60ax^2\sin^2 y\cos^3 y - 24a\cos^3 y + \right. \\ \quad \left. +32x^5\sin^8 y - 24x^3\sin^6 y + 120x^3\sin^4 y\cos^2 y\right), \\ \dot{y} = \frac{1}{24}\sin y\cos y\left(65ax^4\sin^4 y\cos y - 60ax^2\sin^2 y\cos y + \right. \\ \quad \left. + 24a\cos y + 32x^5\sin^6 y - 144x^3\sin^4 y\right), \end{cases} \tag{17}$$

We resolve the equation $Q(0, y) = 0$, resulting in the identification of the singular points $N_1(0, 0)$, $N_2(0, \frac{\pi}{2})$, $N_3(0, \pi)$ and $N_4(0, \frac{3\pi}{2})$. The points N_1 and N_3 are classified as saddle singularities, while N_2 and N_4 are classified as compound singularities.

Subsequently, we effect a translation of N_2 to the origin of coordinates, followed by an expansion in a Taylor series in the vicinity of $y = 0$, and a blow-up transformation. This results in the decomposition of the singularity into four distinct points: $R_1(0, 0)$, $R_2(0, \frac{\pi}{2})$, $R_3(0, \pi)$ and $R_4(0, \frac{3\pi}{2})$. The points R_2 and R_4 are classified as saddle singularities, while the points R_1 and R_3 are compound singularities.

Employing the blow-up technique on R_1, results in the decomposition of this point into four distinct singularities: $S_1(0, 0)$, $S_2(0, \frac{\pi}{2})$, $S_3(0, \pi)$ and $S_4(0, \frac{3\pi}{2})$. The points S_2 and S_4 are classified as hyperbolic saddle singularities, while the singularities S_1 and S_3 are classified as non-hyperbolic multiple singularities.

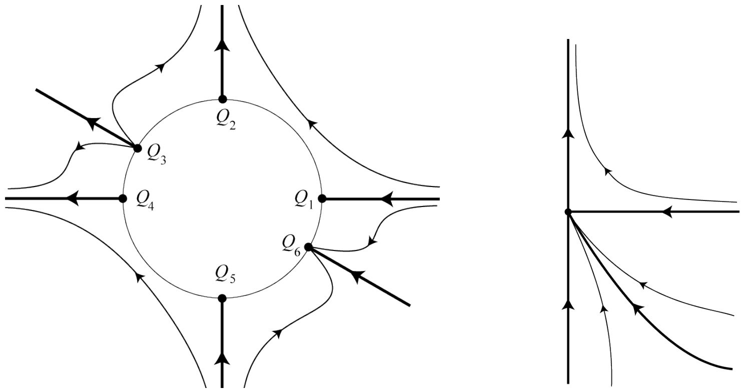
Finally, by utilizing the blow-up method on S_1, we obtain a further decomposition of this point into six distinct singularities: $Q_1(0, 0)$, $Q_2(0, \frac{\pi}{2})$, $Q_3(0, \pi - arctg\frac{9}{a})$, $Q_4(0, \pi)$, $Q_5(0, \frac{3\pi}{2})$ and $Q_6(0, arctg\frac{9}{a})$, their eigenvalues and types are tabulated in Table 2.

Table 2. Blow up for point S_1

S.P.	Q_1	Q_2	Q_3	Q_4	Q_5	Q_6
$\lambda_{1,2}$	$-1, 9$	$\pm a$	$\frac{a}{\sqrt{81+a^2}}, \frac{9a}{\sqrt{81+a^2}}$	$-9, 1$	$-a, a$	$-\frac{a}{\sqrt{81+a^2}}, -\frac{9a}{\sqrt{81+a^2}},$
Type	S	S	N^u	S	S	N^s

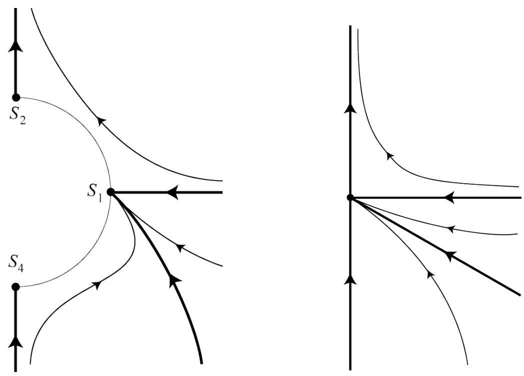
By leveraging the data presented in Table 2, we are able to construct the phase portrait in the vicinity of the singular point S_1 (Figure 8). Utilizing this information, we can then graphically depict the qualitative behaviour of the trajectories in the immediate vicinity of R_1 (as illustrated in Figure 9).

Utilizing the information obtained from the analysis of the type of singularity at R_1, we are able to construct the local phase portrait for the point N_2 (as depicted in Figure 10 a), b)). By applying a similar procedure, we are able to construct the phase portrait for the point N_4 (depicted in Figure 10 c), d)).



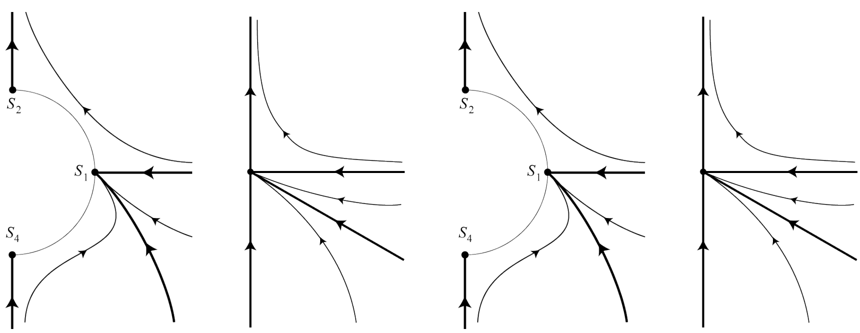
a) b)

Figure 8. Blow-up for the point S_1 $(0, 0)$.



a) b)

Figure 9. Blow-up for the point R_1 $(0, 0)$.



a) b) c) d)

Figure 10. Blow-up for the points N_2 $\left(0, \frac{\pi}{2}\right)$ and N_4 $\left(0, \frac{3\pi}{2}\right)$.

Through the utilization of blow-up techniques, we previously decomposed the singular point $M_2(0, \frac{\pi}{2})$ into the points N_1, N_2, N_3 and N_4. Given that N_1 and N_4 are classified as saddle singularities, we are able to construct the phase portrait for M_2 as depicted in Figure 11.

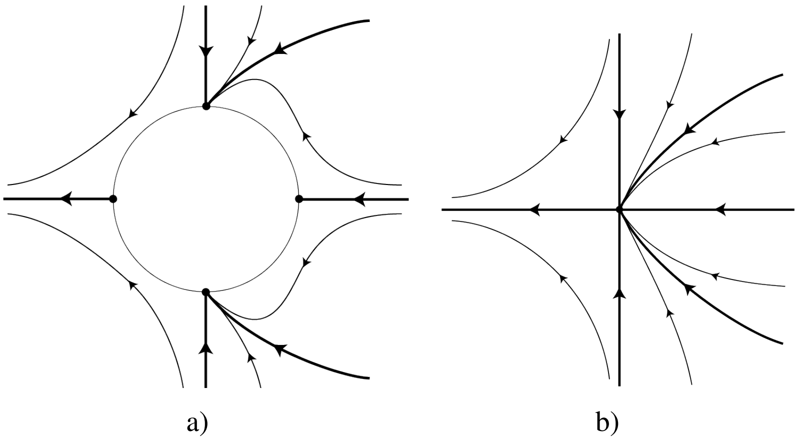


Figure 11. Blow-up for the points $M_2\left(0, \frac{\pi}{2}\right)$.

By applying similar techniques, we can construct the phase portrait for the point M_4 (as illustrated in Figure 12).

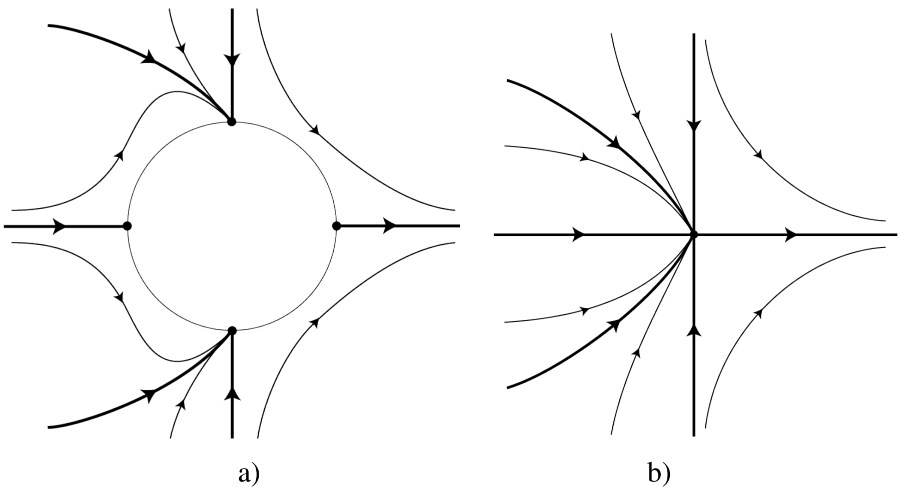
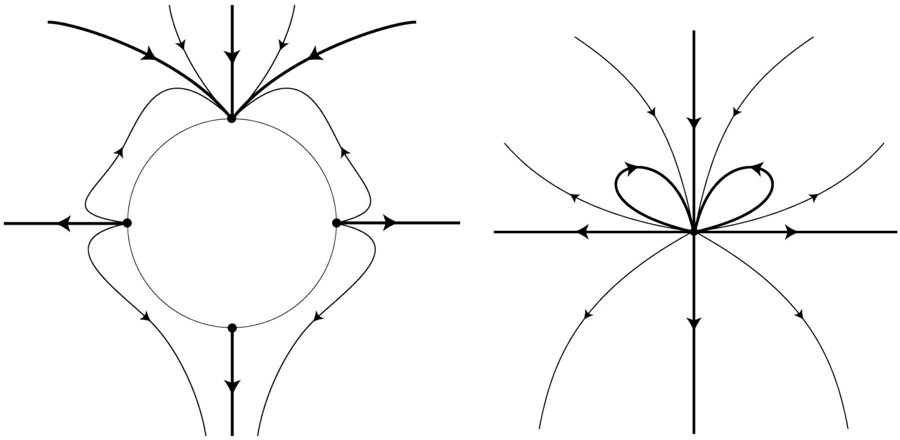


Figure 12. Blow-up for the point $M_4\left(0, \frac{3\pi}{2}\right)$.

By utilizing the information obtained from the phase portraits of M_2 and M_4 in conjunction with the fact that M_1 is an unstable node and M_2 is a stable node, we can construct the local phase portrait (as depicted in Figure 13) in the vicinity of the origin of coordinates for the system (14), which corresponds to the points $I^{\infty}_{1,2}(\pm 1, 0, 0)$ of the system (13).



a) b)

Figure 13. Blow-up for the points $I^{\infty}_{1,2}$ $(\pm 1, 0, 0)$.

Taking into account that the singular point in the finite portion of the phase space is of saddle type, upon application of the transformation $x \to y$, $y \to x$, we obtain the phase portrait (Figure 14) for the quintic differential system (12) which possesses a line of maximal multiplicity at infinity.

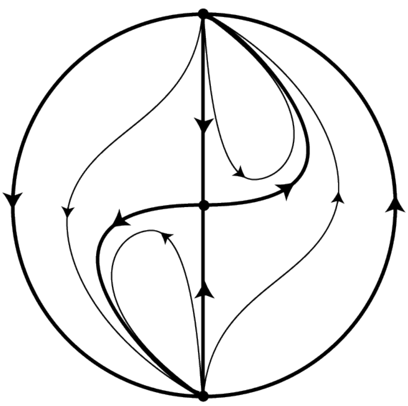


Figure 14. The phase portrait for the quintic differential system (12).

5. Conclusion

This article has provided an in-depth analysis of the phase portraits of polynomial differential systems of degree at most five and having the invariant straight line at the infinity of the maximal multiplicity. By using the blow-up method, we were able to decompose singular points, transform the systems, and obtain the phase portraits for the systems. This understanding of polynomial differential systems is valuable in many fields including physics, engineering, and biology.

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