Acta et Commentationes, Exact and Natural Sciences ISSN: 2537-6284
Volume 2(14), 2022, Pages 81–87 E-ISSN: 2587-3644
CZU 512.57 DOI: 10.36120/2587-3644.v14i2.81-87

The left product, the right product and the theories of relative torsion

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Abstract. It is demonstrated that any theory of relative torsion is defined by the left and the right products.

2010 Mathematics Subject Classification: 46M15, 18B30.

Keywords: reflexive and coreflective subcategories, the right and left product of two subcategories, the relative torsions theories.

Produsul de stânga, produsul de dreapta şi teorii de torsiune relative

Rezumat. Se demonstrează că orice teorie de torsiuni relative este descrisă de produsele de stânga şi de dreapta.

Cuvinte-cheie: subcategorii reflexive şi coreflexive, produsul de dreapta şi de stânga a două subcategorii, teorii de torsiune relative.

1. Introduction

The paper is a continuation (with notations and terminology) of the article [6] (see also [4]). Note by $C_2\mathcal{V}$ the category of topological vector locally convex Hausdorff spaces (see [9]), where you can also find all the notions referred totopologies. We will use the following notation.

Factorization structures (see [4]):

$\mathbb{B}$ the class of factorization structures;

$(\mathcal{E}pi, \mathcal{M}_f)$ - (the class of epimorphisms, the class of kernels) = (the class of morphisms with dense image, the class of topological inclusions with closed images);

$(\mathcal{E}_u, \mathcal{M}_p)$=(the class of universal epimorphisms, the class of exact monomorphisms)=(the class of surjective morphisms, the class of topological inclusions);

$(\mathcal{E}_p, \mathcal{M}_u)$=(the class of exact epimorphisms, the class of universal monomorphisms);

$(\mathcal{E}_f, \mathcal{M}ono)$=(the class of cokernels, the class of monomorphisms)=(the class of factorial morphisms, the class of injective morphisms);

The properties of factorization structures $(\mathcal{E}_f, \mathcal{M}ono)$ and $(\mathcal{E}pi, \mathcal{M}_f)$ characterize the category $C_2\mathcal{V}$ as a semiabelian category. The factorization structures $(\mathcal{E}_u, \mathcal{M}_p)$

and $(\mathcal{E}_p, \mathcal{M}_u)$ play an important role in the study of the reflective and coreflective subcategories. We need some notions and results from [3], [4] and [6].

We use the following notations for some subcategories of the category $C_2\mathcal{V}$.

$\mathbb{R}$ - the class of non-zero reflective subcategories;

$\mathbb{K}$ - the class of nonzero coreflective subcategories;

Π - the subcategory of complete spaces with a weak topology and with respective functor $\pi : C_2\mathcal{V} \to \Pi$;

$\mathcal{S}$ - the subcategory of spaces endowed with a weak topology, $s : C_2\mathcal{V} \to \mathcal{S}$;

Γ_0 - the subcategory of complete spaces, $g_o : C_2\mathcal{V} \to \Gamma_o$;

Σ - the coreflective subcategory of spaces with the strongest locally convex topology, $\sigma : C_2\mathcal{V} \to \Sigma$;

$\widetilde{\mathcal{M}}$ - the subcategory of spaces endowed with the Mackey topology, $m : C_2\mathcal{V} \to \widetilde{\mathcal{M}}$.

Let $\mathcal{A}$ and $\mathcal{B}$ be two classes of morphisms of the category $C_2\mathcal{V}$. We will use the notations:

1. $\mathcal{A} \circ \mathcal{B} = \{a \cdot b |\, a \in \mathcal{A}, b \in \mathcal{B}$ and there is the composition $a \cdot b\}$.

2. The class $\mathcal{A}$ is called $\mathcal{B}$-hereditary if from the fact that $f \cdot g \in \mathcal{A}$ and $f \in \mathcal{B}$, it follows that $g \in \mathcal{A}$. The class $\mathcal{E}pi$ is $\mathcal{M}_u$-hereditary ([4], Lemma 2.6);

2*. The class $\mathcal{A}$ is called $\mathcal{B}$-cohereditary if from the fact that $f \cdot g \in \mathcal{A}$ and $g \in \mathcal{B}$, it follows that $f \in \mathcal{A}$.

If $\mathcal{R} \in \mathbb{R}$, then $(\mathcal{P}''(\mathcal{R}), \mathcal{I}''(\mathcal{R})) = ((\varepsilon\mathcal{R}) \circ \mathcal{E}_p, (\varepsilon\mathcal{R})^{\perp} \cap \mathcal{M}_u)$.

If $\mathcal{K} \in \mathbb{K}$, then $(\mathcal{P}'(\mathcal{K}), \mathcal{I}'(\mathcal{K})) = ((\mu\mathcal{K})^{\top} \cap \mathcal{E}_u, \mathcal{M}_p \circ (\mu\mathcal{K}))$ (see [5]).

We will show the application of left and right products to the description of relative torsion theories.

2. The right and left product of two subcategories

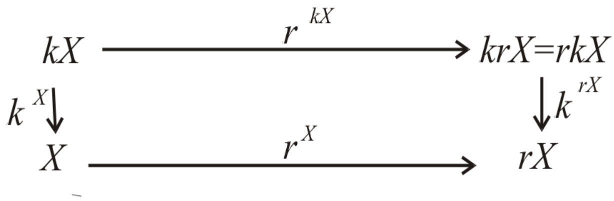
Definition 2.1 ([1])**.** Let $\mathcal{K}$ be a coreflective subcategory, and $\mathcal{R}$ a reflective subcategory of category C. The pair $(\mathcal{K}, \mathcal{R})$ is called relative torsion theory (TTR), that is, relative to the subcategory $\mathcal{K} \cap \mathcal{R}$, if the functors $k : C \to \mathcal{K}$ and $r : C \to \mathcal{R}$ verify the following two relations:

1. The functors k and r commute: $k \cdot r = r \cdot k$;

2. For any object X of category C the square

$$r^X \cdot k^X = k^{rX} \cdot r^{kX} \tag{1}$$

is puschout and pullback.



Remark 2.1. In abelian categories a theory of torsion $(\mathcal{T},\mathcal{F})$ is a TTR relative to intersections $\mathcal{T} \cap \mathcal{F} = 0$ [2].

Theorem 2.1. *([3], Theorem 2.1). Let $\mathcal{K}$ be a non-zero coreflective subcategories, and $\mathcal{R}$ be a non-zero reflective subcategories of category $C_2\mathcal{V}$ and $\mathcal{R} \in \mathbb{R}(\mathcal{M}_p)$. The pair $(\mathcal{K},\mathcal{R})$ forms a TTR if and only if the coreflector functor $k : C_2\mathcal{V} \longrightarrow \mathcal{K}$ and reflector $r : C_2\mathcal{V} \longrightarrow \mathcal{R}$ commute: $k \cdot r = r \cdot k$.*

In the work [3] this theorem is without proof, therefore, for completeness, the proof will be included here.

Proof. Let the respective functors commute: $k \cdot r = r \cdot k$ and we will prove that for any object X of the category $C_2\mathcal{V}$ the square

$$r^X \cdot k^X = k^{rX} \cdot r^{kX} \tag{2}$$

is puschout and pullback. Indeed, either

$$u^X \cdot k^X = v^X \cdot r^{kX} \tag{3}$$

the puschout built on the morphisms k^X and r^{kX}. Then

$$r^X = t^X \cdot u^X, \tag{4}$$

$$k^{rX} = t^X \cdot v^X \tag{5}$$

for a morphism t^X. Since r^{kX} is an epi, according to construction, we deduce that u^X is also an epi. Moreover, $r^X \in \mathcal{M}_p$, $u^X \in \mathcal{E}pi$ and the class $\mathcal{M}_p$ is $\mathcal{E}pi$-cohereditary. So from equality (5) it turns out that $t^X \in \mathcal{M}_p$. Also $k^{rX} \in \mathcal{E}_u$. Thus from equality (5) we deduce as $t^X \in \mathcal{E}_u$. Finally $v^X \in \mathcal{E}_u \cap \mathcal{M}_p = \mathcal{I}so$.

This is how we proved that the square (2) is puschout. The class $\mathcal{A}$ is called $\mathcal{B}$-hereditary if from the fact that $f \cdot g \in \mathcal{A}$ and $f \in \mathcal{B}$, it follows that $g \in \mathcal{A}$. Let's prove that it is also pullback. Let

$$r^X \cdot l^X = k^{rX} \cdot m^X \tag{6}$$

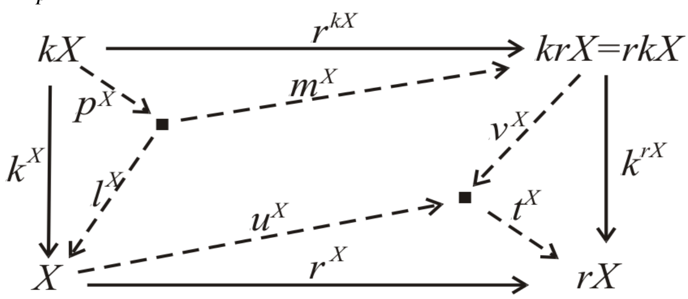
the pullback built on morphisms r^X and k^{rX}. Then

$$k^X = l^X \cdot p^X, \tag{7}$$

$$r^{kX} = m^X \cdot p^X \tag{8}$$

for un morphism p^X. Since $k^{rX} \in \mathcal{E}_u \cap \mathcal{M}ono$, it turns out that $l^X \in \mathcal{E}_u \cap \mathcal{M}ono$.

Thus in equality (7) the morphisms k^X and l^X belong to the class $\mathcal{E}_u \cap \mathcal{M}ono$. So also p^X belongs to this class. From equality (8) it follows that $p^X \in \mathcal{M}_p$, because $r^{kX} \in \mathcal{M}_p$. So $p^X \in \mathcal{E}_u \cap \mathcal{M}_p = \mathcal{I}so$.



Remark 2.2. Regarding examples of TTR (see [1-3]).

Since $\mathcal{K}$-coreplica for any object of the category $C_2\mathcal{V}$ is a bijective application, we get:

Lemma 2.1. *Let $\mathcal{R} \in \mathbb{R}^s(\mu\mathcal{K})$. Then for any object (E, u) of it and any locally convex topology v with the property $u \leq v \leq k(u)$, where $(E, k(u))$ is $\mathcal{K}$-core replica of the object (E, u), the object (E, v) also belongs to the subcategory $\mathcal{R}$.*

Lemma 2.2. *For the subcategories $\mathcal{K}$ and $\mathcal{R}$ of the category $C_2\mathcal{V}$ the following statements are equivalent:*

1. $\mathcal{K} *_s \mathcal{R} = \mathcal{K}$.

2. $\mathcal{K} \in \mathbb{K}_f(\varepsilon\mathcal{R})$.

If the subcategory $\mathcal{K}$ contains the subcategory $\widetilde{\mathcal{M}}$ of spaces with Mackey topology, then the previous conditions are equivalent to the condition:

3. The subcategory $\mathcal{K}$ is $\mathcal{I}''(\mathcal{R})$-coreflective.

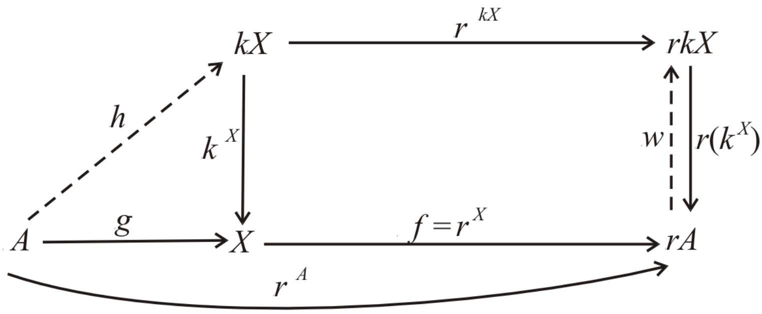
Proof. $1 \Rightarrow 2$. Let $A \in |\mathcal{K}|$ and

$$r^A = f \cdot g \tag{9}$$

be a decomposition of the morphism r^A with g as an epi. We will prove that $X \in |\mathcal{K}|$. Since g is an epi, we deduce that f is the $\mathcal{R}$-replica of object X. Let k^X be the $\mathcal{K}$-coreplica of the object X. We have $A \in |\mathcal{K}|$, so

$$g = k^X \cdot h \tag{10}$$

for a morphism h. We examine the left product diagram for the object X.



Thus we have equality

$$r^X \cdot k^X = r(k^X) \cdot r^{kX} \tag{11}$$

For the morphism $r^{kX} \cdot h$ there is a morphism w as follows

$$w \cdot r^A = r^{kX} \cdot h \tag{12}$$

We have

$$r(k^X) \cdot w \cdot r^A = (din(14)) = r(k^X) \cdot r^{kX} \cdot h = (din(13)) = r^X \cdot k^X \cdot h =$$

$$= (din(12)) = r^X \cdot g = r^A$$

i.e.

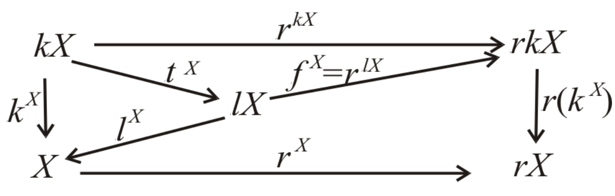
$$r(k^X) \cdot w \cdot r^A = r^A \tag{13}$$

Since r^A is an epi, it follows that

$$r(k^X) \cdot w = 1 \tag{14}$$

According to the first hypothesis, the square (10) is pullback, and the morphism $r(k^X)$ is a retraction, it turns out that k^X is the same. But k^X is also a mono. Thus we proved that $X \in |\mathcal{K}|$.

$2 \Rightarrow 1$. Let X be an arbitrary object of the category $C_2\mathcal{V}$. We construct the left product diagram for it.



We examine the equality

$$r^{kX} = f^X \cdot t^X. \tag{15}$$

Because the class $\mathcal{E}pi$ is $\mathcal{M}_u$-hereditary ([4], Lemma 2.6), the morphism t^X is an epi. Thus according to hypothesis (2) lX is an object of the subcategory $\mathcal{K}$. Therefore, t^X is an iso, and $\mathcal{K} *_s \mathcal{R} = \mathcal{K}$.

3 $\Longleftrightarrow$ 1. For an arbitrary object of the category $C_2\mathcal{V}$ we examine the commutative square:

$$\begin{array}{ccc} kX & \xrightarrow{\quad r^{kX} \quad} & rkX \\ {\scriptstyle k^X}\downarrow & & \downarrow{\scriptstyle r(k^X)} \\ X & \xrightarrow{\quad r^X \quad} & rX \end{array}$$

$$r^X \cdot k^X = r(k^X) \cdot r^{kX} \tag{16}$$

Since $\widetilde{\mathcal{M}} \subset \mathcal{K}$, it follows that $\mathcal{K}$ is a $\mathcal{M}_u$-coreflective subcategory. Thus $k^X \in \mathcal{M}_u$. According to Theorem 2.12 [4] the square (16) is pullback if and only if $k^X \in \mathcal{I}''(\mathcal{R})$.

We formulate the dual statement.

Lemma 2.3. *For the subcategories $\mathcal{K}$ and $\mathcal{R}$ of the category $C_2\mathcal{V}$ the following statements are equivalent:*

1. $\mathcal{K} *_d \mathcal{R} = \mathcal{R}$.

2. $\mathcal{R} \in \mathbb{R}^s(\mu\mathcal{K})$.

If the subcategory $\mathcal{R}$ contains the subcategory $\mathcal{S}$ of spaces with weak topology, then the previous conditions are equivalent to the condition:

3. The subcategory $\mathcal{R}$ is $\mathcal{E}'(\mathcal{K})$-reflective.

The proven Lemmas allow us to formulate the following result. From Theorem 2.1 and Lemmas 2.2, 2.3 we obtain:

Theorem 2.2. *Let $\mathcal{K} \in \mathbb{K}(\mathcal{M}_u)$, i.e. $\mathcal{K}$ is a $\mathcal{M}_u$- coreflective subcategory of the category $C_2\mathcal{V}$ ($\widetilde{\mathcal{M}} \subset \mathcal{K}$), and $\mathcal{R} \in \mathbb{R}(\mathcal{M}_p)$, i.e. $\mathcal{R}$ is a $\mathcal{M}_p$-reflective subcategory ($\Gamma_0 \subset \mathcal{R}$). Then:*

1. The subcategory $\mathcal{K}$ is closed in relation to $(\mathcal{E}pi \cap \mathcal{M}_p)$-factorobjects. In other words, the subcategory $\mathcal{K}$ is closed in relation to the extensions.

2. $\mathcal{R} \in \mathbb{R}^s(\mu\mathcal{K})$.

Remark 2.3. For some subcategories $\mathcal{K}$ of the class $\mathbb{K}(\mathcal{M}_u)$, in particular, for the subcategory $\widetilde{\mathcal{M}}$, it is well known that they are closed in relation to extensions ([9], Assertion IV.3.5.)

2. Any fully convex local space (E, t) remains complete in any topology u finer than t and compatible with the same duality:

$$t \leq u \leq m(t),$$

910], VI, Proposition 5). This result was generalized for any $\mathcal{M}_p$-reflective subcategory by D. Botnaru and O. Cerbu [6], Theorem 1.12.

3. The theories of relative torsion

Theorem 3.1. *Let $\mathcal{K}$ be a coreflective subcategory, and $\mathcal{R}$ - a nonzero reflective subcategory of the category $C_2\mathcal{V}$. The following statements are equivalent:*

1. The pair $(\mathcal{K}, \mathcal{R})$ forms a TTR.

2. a) The coreflector function $k : C_2\mathcal{V} \longrightarrow \mathcal{K}$ and reflector $r : C_2\mathcal{V} \longrightarrow \mathcal{R}$ commute $k \cdot r = r \cdot k$; b) $\mathcal{K} *_s \mathcal{R} = \mathcal{K}$; c) $\mathcal{K} *_d \mathcal{R} = \mathcal{R}$.

3. a) The functors k and r commute $k \cdot r = r \cdot k$; b) $\mathcal{K} \in \mathbb{K}_f(\varepsilon\mathcal{R})$; c) $\mathcal{R} \in \mathbb{R}^s(\mu\mathcal{K})$.

If $\widetilde{\mathcal{M}} \subset \mathcal{K}$ and $\mathcal{S} \subset \mathcal{R}$ then the preceding conditions are equivalent to the following:

4. a) The functors k and r commute $k \cdot r = r \cdot k$; b) The subcategory $\mathcal{K}$ is $\mathcal{I}''(\mathcal{R})$-coreflective; c) The subcategory $\mathcal{R}$ is $\mathcal{E}'(\mathcal{K})$-reflective.

Remark 3.1. In the previous Theorem p.2 and p.3 condition a) is not a consequence of conditions b) and c).

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Received: November 01, 2022 *Accepted: December 16, 2022*

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