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Center conditions for a cubic system with two homogeneous invariant straight lines and exponential factors

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Abstract. In this paper for a cubic differential system with a singular point $O(0,0)$ of a center or a focus type, having two homogeneous invariant straight lines and exponential factors, we determine conditions under which the singular point is a center.
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Condiţii de existenţă a centrului pentru un sistem cubic cu două drepte invariante omogene şi factori exponenţiali

Rezumat. În această lucrare pentru un sistem diferenţial cubic cu punctul singular $O(0,0)$ de tip centru sau focar, care are două drepte invariante omogene şi factori exponenţiali, sunt determinate condiţiile încât punctul singular să fie centru.
Cuvinte-cheie: sistem diferenţial cubic, problema centrului şi focarului, curbă algebrică invariantă, factor exponenţial.

1. Introduction

We consider the cubic differential system of the form

$$\begin{cases} \dot{x} = y + ax^2 + cxy + fy^2 + kx^3 + mx^2y + pxy^2 + ry^3 \equiv P(x,y), \\ \dot{y} = -(x + gx^2 + dxy + by^2 + sx^3 + qx^2y + nxy^2 + ly^3) \equiv Q(x,y), \end{cases} \tag{1}$$

in which $P(x,y)$ and $Q(x,y)$ are real and coprime polynomials in the variables x and y. The origin $O(0,0)$ is a singular point of a center or a focus type for (1). The problem arises of distinguishing between a center and a focus, i.e. of finding the coefficient conditions under which $O(0,0)$ is, for example, a center. These conditions are called *the center conditions* and the problem - *the problem of the center*. When the cubic system (1) contains both quadratic and cubic nonlinearities, the problem of finding a finite number of necessary and sufficient conditions for the center is still open.

It is well known that $O(0,0)$ is a center for system (1) if and only if the Lyapunov quanities $L_1, L_2, \dots, L_k, \dots$ vanish [5], [18].

The problem of finding the center conditions for system (1) has a long history and a variety of methods have been developed. An approach to the problem of the center is to study the local integrability of the system (1) in some neighborhood of the singular point $O(0,0)$. It is known that a singular point $O(0,0)$ is a center for system (1) if and only if it has a holomorphic first integral of the form $F(x,y)=C$ in some neighborhood of $O(0,0)$ [17]. Also, $O(0,0)$ is a center if and only if the system (1) has a holomorphic integrating factor of the form $\mu = 1+\sum \mu_j(x,y)$ in some neighborhood of $O(0,0)$ [1].

The problem of the center was solved for some families of cubic differential systems having invariant algebraic curves (invariant straight lines, invariant conics, invariant cubics) in [5], [7], [9], [10], [12], [13], [16], [19], [20], [21]. Center conditions were determined for some cubic systems having integrating factors in [8], [11], [14], for some reversible cubic systems in [2] and for a few families of the complex cubic system in [15].

In this paper we determine the center conditions for cubic differential system (1) assuming that the system has invariant straight lines and exponential factors. The paper is organized as follows. In Section 2 we present the results concerning the existence of invariant straight lines and exponential factors. In Section 3 we find conditions under which the cubic system has exponential factors. In Section 4 we obtain center conditions for system (1) with two homogeneous invariant straight lines and one exponential factor.

2. Invariant straight lines and exponential factors

We study the problem of the center for cubic differential system (1) assuming that the system has invariant algebraic curves and exponential factors.

Definition 2.1. An algebraic curve $\Phi(x,y)=0$ in $\mathbb{C}^2$ with $\Phi \in \mathbb{C}[x,y]$ is said to be an *invariant algebraic curve* of system (1) if

$$\frac{\partial \Phi}{\partial x}P(x,y)+\frac{\partial \Phi}{\partial y}Q(x,y)=\Phi(x,y)K(x,y), \tag{2}$$

for some polynomial $K(x,y)\in \mathbb{C}[x,y]$, called the *cofactor* of the invariant algebraic curve $\Phi(x,y)=0$.

By the above definition, a straight line

$$C+Ax+By=0, \quad A,B,C\in\mathbb{C}, \quad (A,B)\neq(0,0), \tag{3}$$

is an invariant straight line for system (1) if and only if there exists a polynomial $K(x,y)$ such that the following identity holds

$$A\cdot P(x,y)+B\cdot Q(x,y)\equiv (C+Ax+By)\cdot K(x,y). \tag{4}$$

If the cubic system (1) has complex invariant straight lines then obviously they occur in complex conjugated pairs [5]

$$C + Ax + By = 0 \quad \text{and} \quad \overline{C} + \overline{A}x + \overline{B}y = 0.$$

According to [6] the cubic system (1) cannot have more than four nonhomogeneous invariant straight lines, i.e. invariant straight lines of the form

$$1 + Ax + By = 0, \quad (A, B) \neq (0, 0). \tag{5}$$

As homogeneous invariant straight lines $Ax + By = 0$, the system (1) can have only the lines $x \mp iy = 0, i^2 = -1$.

Lemma 2.1. *The cubic system* (1) *has the invariant straight lines* $x \mp iy = 0$ *if and only if the following set of conditions holds*

$$d = f - a, \quad c = g - b, \quad k - l = p - q, \quad r + s = m + n. \tag{6}$$

Proof. By Definition 2.1, the straight lines $x \mp iy = 0$ are invariant for (1) if and only if

$$P(x, y) \mp iQ(x, y) \equiv (x \mp iy)(c_{00} + c_{10}x + c_{01}y + c_{20}x^2 + c_{11}xy + c_{02}y^2). \tag{7}$$

Identifying the coefficients of the monomials $x^j y^h$ in (7), we find that

$$\begin{aligned} &c_{00} = \pm i,\ c_{10} = a \pm ig,\ c_{02} = p - k - q \pm i(m + n - s),\\ &c_{01} = c - g \pm i(a + d),\ c_{20} = k \pm is,\ c_{11} = m - s \pm i(k + q) \end{aligned}$$

and

$$f - a - d \pm i(b + c - g) = 0, \quad r + s - m - n \pm i(l - k + p - q) = 0.$$

The last identities yield the set of conditions (6). □

The cofactors of the invariant straight lines $x \mp iy = 0$ are

$$\begin{aligned} K_1(x, y) = {} & i + (a + i(b + c))x + (-b + i(a + d))y + (k + is)x^2 + \\ & + (m - s + i(k + q))xy + (p - k - q + i(m + n - s))y^2, \\ K_2(x, y) = {} & \overline{K_1(x, y)}. \end{aligned} \tag{8}$$

Denote $k = u + l,\ p = u + q,\ s = v - r,\ n = v - m$, where u, v are some real parameters. Assume that the conditions (6) are fulfilled, then system (1) can be written as follows

$$\begin{aligned} \dot{x} = {} & y + ax^2 + (g - b)xy + fy^2 + (u + l)x^3 + mx^2y + \\ & + (u + q)xy^2 + ry^3 \equiv P(x, y), \\ \dot{y} = {} & -(x + gx^2 + (f - a)xy + by^2 + (v - r)x^3 + qx^2y + \\ & + (v - m)xy^2 + ly^3) \equiv Q(x, y). \end{aligned} \tag{9}$$

The problem of the center was solved for system (9) with: one invariant straight line $1 + Ax + By = 0$ in [20], two invariant straight lines of the form (5) in [5], one invariant conic $a_{20}x^2 + a_{11}xy + a_{02}y^2 + a_{10}x + a_{01}y + 1 = 0$ in [7]. By using the method of Darboux integrability and rational reversibility, the center conditions were found for (9) in [8].

In this Section, we investigate the problem of the existence of exponential factors for cubic differential system (9).

Definition 2.2. Let $h,\ g \in \mathbb{C}[x, y]$ be relatively prime in the ring $\mathbb{C}[x, y]$. The function $\Phi = \exp(g/h)$ is called an *exponential factor* of a system (1) if for some polynomial $K \in \mathbb{C}[x, y]$ of degree at most two it satisfies the equation

$$\frac{\partial \Phi}{\partial x} P(x, y) + \frac{\partial \Phi}{\partial y} Q(x, y) \equiv \Phi K(x, y). \tag{10}$$

As before, we say that K is the cofactor of the exponential factor $\exp(g/h)$.

This means that if we have a cubic differential system (1) with an exponential factor of the form $\exp(g/h)$, then there is a 1-parameter perturbation of system (1), given by a small ε, with two invariant algebraic curves, namely $h = 0$ and $h + \varepsilon g = 0$. Hence, when $\varepsilon = 0$, these two curves coalesce giving the exponential factor $\exp(g/h)$ for the system with $\varepsilon = 0$ (the invariant algebraic curve $h = 0$ has geometric multiplicity larger than one), as well as the invariant algebraic curve $h = 0$ which does not disappear [3].

Since the exponential factor cannot vanish, it does not define invariant curves of the cubic system (1). The next theorem, proved in [3], gives the relationship between the notion of invariant algebraic curve and exponential factor.

Theorem 2.1. *If* $\exp(g/h)$ *is an exponential factor with cofactor* K *for a cubic system* (1) *and if* h *is not a constant, then* $h = 0$ *is an invariant algebraic curve with cofactor* K_h*, and* g *satisfies the equation* $\mathcal{X}(g) = gK_h + hK$.

Eventually $\Phi = \exp(g)$ can be an exponential factor coming from the multiplicity of the infinite invariant straight line.

3. Cubic differential systems with exponential factors

In this Section, we consider the cubic differential system (9) with two homogeneous invariant straight lines $x \mp iy = 0$. We determine the conditions under which the system (9) has exponential factors of the form

$$\Phi = \exp\left(g(x, y)\right), \quad \Phi = \exp\left(\frac{g(x, y)}{x^2 + y^2}\right), \tag{11}$$

where $g(x, y)$ is a real polynomial with degree$(g) \leq 2$.

Lemma 3.1. *The cubic differential system* (9) *has an exponential factor of the form* $\Phi = \exp(a_{10}x + a_{01}y)$ *if and only if one the following two sets of conditions holds:*

(i_1) $m = r,\ q = l,\ l = (ra_{10})/a_{01},\ v = (ra_{01}^2 + ua_{01}a_{10} + ra_{10}^2)/a_{01}^2;$
(i_2) $m = r = 0,\ q = l,\ u = -l.$

Proof. By Definition 2.2, the function $\Phi = \exp(a_{10}x + a_{01}y)$ is an exponential factor for system (9) if there exists numbers $c_{10}, c_{01}, c_{20}, c_{11}, c_{02}$ such that

$$a_{10}P(x, y) + a_{01}Q(x, y) = c_{10}x + c_{01}y + c_{20}x^2 + c_{11}xy + c_{02}y^2. \tag{12}$$

Substituting $P(x, y), Q(x, y)$ in (12) and identifying the coefficients of the monomials $x^i y^j$, $i + j = 1, 2, 3$, we find that $c_{01} = a_{10},\ c_{10} = -a_{01},\ c_{20} = aa_{10} - ga_{01},\ c_{11} = (a - f)a_{01} + (g - b)a_{10},\ c_{02} = fa_{10} - ba_{01}$ and $a_{10},\ a_{01}$ satisfy the system of equations:

$$\begin{aligned} U_{30} &\equiv (l + u)a_{10} + (r - v)a_{01} = 0,\\ U_{21} &\equiv ma_{10} - qa_{01} = 0,\\ U_{12} &\equiv (q + u)a_{10} + (m - v)a_{01} = 0,\\ U_{03} &\equiv ra_{10} - la_{01} = 0. \end{aligned} \tag{13}$$

Assume that $a_{01} \neq 0$. Then the equations of (13) yield

$$m = r,\ q = l,\ l = (ra_{10})/a_{01},\ v = (ra_{01}^2 + ua_{01}a_{10} + ra_{10}^2)/a_{01}^2.$$

We obtain the set of conditions (i_1) of Lemma 3.1. The system (9) has the exponential factor $\Phi = \exp(a_{10}x + a_{01}y)$ with cofactor $K(x, y) = (aa_{10} - ga_{01})x^2 + (aa_{01} - fa_{01} - ba_{10} + ga_{10})xy + (fa_{10} - ba_{01})y^2 - a_{01}x + a_{10}y$.

Assume that $a_{01} = 0$, then $a_{10} \neq 0$. In this case the equations of (13) imply $m = r = 0,\ q = l,\ u = -l$. We obtain the set of conditions (i_2) of Lemma 3.1. The system (9) has the exponential factor $\Phi = \exp(x)$ with cofactor $K(x, y) = ax^2 - bxy + gxy + fy^2 + y$. □

Lemma 3.2. *The cubic system* (9) *has an exponential factor of the form*

$$\Phi = \exp(a_{20}x^2 + a_{11}xy + a_{02}y^2 + a_{10}x + a_{01}y + a_{00}) \tag{14}$$

if and only if one the following three sets of conditions holds:

(i_1) $c = d = 0, f = a, g = b, k = p = q = l, r = m, s = n;$
(i_2) $c = d = 0,\ f = a,\ g = b,\ k = p,\ p = l + u,\ r = m,\ m = (al)/b,\ s = n,\ n = ((l + u)b)/a, q = l;$
(i_3) $c = g - b, d = f - a, k = l = p = q = 0, m = n = s = r.$

Proof. By Definition 2.2, the function (14) is an exponential factor for cubic system (9) if there exist numbers $c_{20}, c_{11}, c_{02}, c_{10}, c_{01}$ such that

$$\begin{aligned}&(2a_{20}x + a_{11}y + a_{10})P(x, y) + (2a_{02}y + a_{11}x + a_{01})Q(x, y) =\\ &= c_{20}x^2 + c_{11}xy + c_{02}y^2 + c_{10}x + c_{01}y.\end{aligned} \tag{15}$$

Substituting $P(x, y)$ and $Q(x, y)$ in (15) and identifying the coefficients of the monomials $x^i y^j$, we reduce this identity to a system of fourteen equations

$$\{U_{ij} = 0, \ \ i + j = 1, 2, 3, 4\} \tag{16}$$

for the unknowns a_{ij}, c_{ij} and the coefficient of system (9).

When $i + j = 1, 2$, from the equations of (16) we get

$$c_{01} = a_{10}, c_{10} = -a_{01}, c_{02} = a_{11} - ba_{01} + fa_{10},$$
$$c_{20} = aa_{10} - ga_{01} - a_{11}, c_{11} = (a - f)a_{01} + 2a_{20} - 2a_{02} + (g - b)a_{10}.$$

1. Assume that $a_{11} \neq 0$. In this case the equations $U_{40} = 0, U_{22} = 0$ and $U_{04} = 0$ of (16) yield $v = (ra_{11} + 2la_{20} + 2ua_{20})/a_{11}, r = (2la_{02})/a_{11}, m = ((l + q)a_{02} + (l - q)a_{20})/a_{11}$. The resultant of the polynomials U_{31} and U_{13} with respect to q is $Res(U_{31}, U_{13}, q) = 2uf_1 f_2$, where $f_1 = 4a_{02}a_{20} - a_{11}^2$, $f_2 = (a_{02} - a_{20})^2 + a_{11}^2 \neq 0$.

Let $u = 0$. Then $U_{31} = 0$ and $U_{13} = 0$ imply $q = l$. From the equations $U_{30} = 0, U_{03} = 0$ of (16) we express f, g and calculate the resultant of the polynomials U_{21} and U_{12} with respect to b. We obtain that $Res(U_{21}, U_{12}, b) = -4a_{11}g_1g_2g_3$, where $g_1 = aa_{11} - la_{01}$, $g_2 = 4a_{02}a_{20} - a_{11}^2$, $g_3 = (a_{02} - a_{20})^2 + a_{11}^2 \neq 0$.

If $g_1 = 0$, then $a_{01} = (aa_{11})/l$ and $a_{10} = (ba_{11})/l$. In this case we obtain the set of conditions (i_1) of Lemma 3.2. The exponential factor is

$$\Phi = \exp(2bx + 2ay + 2lxy + nx^2 + my^2)$$

having the cofactor $K(x, y) = 2(-ax + by - lx^2 + ly^2 - mxy + nxy)$.

If $g_1 \neq 0$ and $g_2 = 0$, then $a_{20} = a_{11}^2/(4a_{02})$, $a_{10} = [(la_{01} - aa_{11} + 2ba_{02})a_{11}]/(2la_{02})$. This case is contained in Lemma 3.2, (i_1) $(n = l^2/m)$.

Assume that $u \neq 0$ and let $f_1 = 0$. Then $U_{31} = 0$ and $U_{13} = 0$ yield $q = l$. From the equations $U_{30} = 0, U_{03} = 0$ we express a, f and calculate the resultant of the polynomials U_{21} and U_{12} with respect to b. We obtain that $Res(U_{21}, U_{12}, b) = a_{11}uh_1h_2$, where $h_1 = a_{01}a_{11} - 2a_{02}a_{10}$, $h_2 = 4a_{02}^2 + a_{11}^2 \neq 0$, $a_{11}u \neq 0$.

Let $h_1 = 0$. Then $a_{10} = (a_{01}a_{11})/(2a_{02})$ and $g = b$. In this case we get the set of conditions (i_2) of Lemma 3.2. The exponential factor is

$$\Phi = \exp((bx + ay)(2ba_{01} + ba_{11}x + aa_{11}y)/(2ab))$$

having the cofactor $K(x, y) = (by - ax)(ba_{01} + ba_{11}x + aa_{11}y)/(ab)$.

2. Assume that $a_{11} = 0$ and let $a_{02} = 0$. This case is contained in Lemma 3.2, (i_1) $(l = n = 0)$.

3. Assume that $a_{11} = 0$ and let $a_{02} \neq 0$. In this case $l = 0$ and $U_{40} \equiv ua_{20} = 0$.

If $u = 0$, then $a_{20} = a_{02}$, $v = m + r$. The equations $U_{ij} = 0, i + j = 3$ of (16) yield $m = r$, $q = 0$, $a_{01} = (2aa_{02})/r$, $a_{10} = (2ba_{02})/r$. We obtain the set of conditions (i_3) of Lemma 3.2. The exponential factor is

$$\Phi = \exp(2bx + 2ay + rx^2 + ry^2)$$

with cofactor $K(x, y) = 2(ax - by)(ay + bx - fy - gx - 1)$.

If $a_{20} = 0$ and $u \neq 0$, then $q = 0, m = v = r$. The equations $U_{ij} = 0, i + j = 3$ of (16) yield $a_{10} = 0$, $b = g = 0$, $f = a$. In this case $P(x, y) = -x$ is not a cubic polynomial. □

Lemma 3.3. *The cubic system* (9) *has an exponential factor of the form*

$$\Phi = \exp((a_{10}x + a_{01}y + a_{00})/(x^2 + y^2)) \tag{17}$$

if and only if the following set of conditions holds

$$l = b(a+f), m = 2a^2+2af-2b^2-2bg+r, q = -3ab-2ag-bf, u = ag-bf,$$
$$v = m + r - a^2 - af + b^2 + bg.$$

Proof. By Definition 2.2, the function (17) is an exponential factor for cubic system (9) if there exist numbers c_{10}, c_{01} such that

$$\begin{aligned}&(-a_{10}x^2 + a_{10}y^2 - 2a_{01}xy - 2a_{00}x)P(x, y)+\\ &+(a_{01}x^2 - a_{01}y^2 - 2a_{10}xy - 2a_{00}y)Q(x, y) = (x^2 + y^2)^2(c_{10}x + c_{01}y).\end{aligned} \tag{18}$$

Substituting $P(x, y), Q(x, y)$ in (18) and identifying the coefficients of the monomials $x^i y^j$, we reduce this identity to a system of nine equations

$$\{U_{ij} = 0, \quad i + j = 1,\ 2,\ 3\} \tag{19}$$

for the unknowns $a_{10}, a_{01}, a_{00}, c_{10}, c_{01}$ and the coefficient of system (9).

From the equations $U_{30} = 0$, $U_{03} = 0$ of (19), we find that

$$c_{01} = la_{01} + ra_{10}, c_{10} = (r - v)a_{01} - (l + u)a_{10}.$$

When $i + j = 1$, we obtain that $a_{10} = -2ba_{00}$ and $a_{01} = -2aa_{00}$. Then the equations $U_{20} = 0, U_{11} = 0, U_{02} = 0$ of (19) yield

$$l = b(a + f), u = ag - bf, v = m + r - a^2 - af + b^2 + bg.$$

The equations $U_{21} = 0$, $U_{12} = 0$ of (19) imply

$$m = 2a^2 + 2af - 2b^2 - 2bg + r, q = -3ab - 2ag - bf.$$

In this case we determine the exponential factor

$$\Phi = \exp((1 - 2bx - 2ay)/(x^2 + y^2))$$

with cofactor $K_3(x, y) = 2(a^2 + af + r)(ax - by)$. □

Lemma 3.4. *The cubic system* (9) *has an exponential factor of the form*

$$\Phi = \exp((a_{20}x^2 + a_{11}xy + a_{02}y^2 + a_{10}x + a_{01}y + a_{00})/(x^2 + y^2)) \tag{20}$$

if and only if one of the following two sets of conditions holds:

(i_1) $l = bf,\ m = r + af - bg,\ q = -ag,\ v = 2r,\ u = ag - bf$;
(i_2) $f = -a,\ g = -b,\ q = -l,\ v = 2r,\ u = 0$.

Proof. By Definition 2.2, the function (20) is an exponential factor for cubic differential system (9) if there exist numbers $c_{20}, c_{11}, c_{02}, c_{10}, c_{01}$ such that

$$\begin{aligned}&(2a_{20}xy^2 - 2a_{02}xy^2 - a_{11}x^2y + a_{11}y^3 - a_{10}x^2 - 2a_{01}xy +\\ &+ a_{10}y^2 - 2a_{00}x)P(x,y) + (a_{11}x^3 + 2a_{02}x^2y - 2a_{20}x^2y - a_{11}xy^2 +\\ &+ a_{01}x^2 - a_{01}y^2 - 2a_{10}xy - 2a_{00}y)Q(x,y) =\\ &= (x^2+y^2)(c_{20}x^2 + c_{11}xy + c_{02}y^2 + c_{10}x + c_{01}y).\end{aligned} \tag{21}$$

Substituting the polynomials $P(x,y)$, $Q(x,y)$ from (9) in (21) and identifying the coefficients of the monomials x^iy^j, we reduce (21) to a system of fourteen equations

$$\{U_{ij} = 0,\ \ i+j = 1,\ 2,\ 3,\ 4\} \tag{22}$$

for the unknowns $a_{20}, a_{11}, a_{02}, a_{10}, a_{01}, a_{00}, c_{20}, c_{11}, c_{02}, c_{10}, c_{01}$ and the coefficient of system (9). From the equations $U_{04} = 0, U_{13} = 0, U_{40} = 0$ of (22), we obtain that

$$c_{02} = ra_{11},\ c_{11} = 2r(a_{20} - a_{02}) + (l + q + u)a_{11},\ c_{20} = (r - v)a_{11}$$

and from the equations $U_{03} = 0, U_{30} = 0$, we get that

$$c_{01} = la_{01} + ra_{10} + fa_{11},\ c_{10} = (r - v)a_{01} - (l + u)a_{10} - ga_{11}.$$

The equations $U_{10} = 0, U_{01} = 0$ and $U_{20} = 0$ yield

$$a_{01} = -2aa_{00},\ a_{10} = -2ba_{00},\ a_{11} = 2a_{00}(ab + ag - l - u)$$

and the equations $U_{02} = 0, U_{11} = 0$ imply

$$u = ag - bf,\ a_{02} = (a^2 + af - b^2 - bg - m - r + v)a_{00} + a_{20}.$$

Then the system of equations (22) becomes

$$U_{21} = 0, U_{12} = 0, U_{31} = 0, U_{22} = 0. \tag{23}$$

The resultant of the polynomials U_{22}, U_{31} with respect to m is $Res(U_{22}, U_{31}, m) = f_1f_2$, where $f_1 = ab + bf - l,\ f_2 = (ag - bf + l + q)^2 + (2r - v)^2$.

1. Assume that $f_1 = 0$, then $l = b(a + f)$ and $U_{31} \equiv g_1g_2 = 0$, where

$$g_1 = a^2 + af - b^2 - bg - m - r + v,\ g_2 = 2r - v.$$

When $g_1 = 0$ we obtain that $U_{31} \equiv 0$, $U_{22} \equiv 0$. The resultant of the polynomials U_{21}, U_{12} with respect to q is $Res(U_{21}, U_{12}, q) = h_1h_2$, where

$$h_1 = 2a^2 + 2af - 2b^2 - 2bg - m + r,\ h_2 = a^2 + b^2.$$

If $h_1 = 0$, then we have the exponential factor $\Phi = \exp((1 - 2bx - 2ay)/(x^2 + y^2))$ obtained in Lemma 3.3. If $h_1 \neq 0$ and $a = b = 0$, then the right hand sides of (9) have a common factor $rx^2 + ry^2 + qxy + gx + fy + 1$.

Assume that $g_1 \neq 0$ and let $g_2 = 0$. Then $v = 2r$, $U_{31} \equiv 0$ and the equation $U_{22} = 0$ yields $q = -a(b + g)$. The resultant of the polynomials U_{21}, U_{12} with respect to m is

$$Res(U_{21}, U_{12}, m) = -2ab((a + f)^2 + (b + g)^2).$$

If $f = -a$ and $g = -b$, then we obtain the set of conditions (i_2) $(l = 0)$, Lemma 3.4.

Suppose that $(a+f)^2+(b+g)^2 \neq 0$. If $a = 0$, then $U_{21} = 0, U_{12} = 0$ imply $m = r - bg$ and we get set of conditions (i_1) $(a = 0)$. If $a \neq 0$ and $b = 0$, then we have the set of conditions (i_1) $(b = f = 0)$.

2. Assume that $f_1 \neq 0$ and let $f_2 = 0$. Then $q = bf - l - ag$ and $v = 2r$. In this case $U_{22} \equiv 0$, $U_{31} \equiv 0$ and the resultant of the polynomials U_{21}, U_{12} with respect to m is $Res(U_{21}, U_{12}, m) = e_1 e_2$, where $e_1 = l - bf$, $e_2 = (a + f)^2 + (b + g)^2$.

If $e_1 = 0$, then $m = r + af - bg$. We get the condition (i_1). The exponential factor is

$$\Phi = \exp((b^2x^2 - a^2x^2 + 2abxy - 2bx - 2ay + 1)/(x^2 + y^2))$$

and have the cofactor $K_3(x, y) = 2r(by - ax)(ay + bx - 1)$.

If $e_1 \neq 0$ and $e_2 = 0$, then $f = -a$, $g = -b$. We obtain the set of conditions (i_2), Lemma 3.4. The exponential factor is

$$\Phi = \exp((mx^2 - rx^2 - 2lxy - 2bx - 2ay + 1)/(x^2 + y^2))$$

and have the cofactor $K_3(x, y) = 2r(ax - by + lx^2 - ly^2 + mxy - rxy)$. □

4. The problem of the center

We are interested in the algebraic integrability of a cubic differential system (1) with two homogeneous invariant straight lines $x \mp iy = 0$ and an exponential factor of the form (11), called *the Darboux integrability* [4], [22]. It consists in constructing of a first integral or an integrating factor of the Darboux form

$$f_1^{\alpha_1} f_2^{\alpha_2} \Phi^{\alpha_3}, \tag{24}$$

where $\alpha_j \in \mathbb{C}$, $f_1 = x - iy$, $f_2 = x + iy$ and Φ is of the form (11).

By [18, pag. 141], if for the cubic system (1) we can construct an integrating factor (a first integral) of the form (24), then $O(0, 0)$ is a center.

Definition 4.1. An integrating factor for system (1) on some open set U of $\mathbb{R}^2$ is a C^1 function μ defined on U, not identically zero on U such that

$$P(x, y)\frac{\partial \mu}{\partial x} + Q(x, y)\frac{\partial \mu}{\partial y} + \mu \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) \equiv 0. \tag{25}$$

Lemma 4.1. *The following three sets of conditions are sufficient conditions for the origin to be a center for system* (1)*:*

(i) $c = g - b,\ d = l = m = q = r = 0,\ f = a,\ k = p = a(g - b),\ s = n;$
(ii) $c = g - b,\ d = f - a,\ g = (abf - bf^2 + fl - al + br)/(a^2 - af + r),\ s = n,$ $n = (pl)/r,\ k = p = [r(ab - bf + l)]/(a^2 - af + r),\ q = l,\ m = r;$
(iii) $c = g - b,\ d = f - a,\ k = m = p = r = 0,\ q = l,\ l = bf - ag,\ s = n,$ $(ag - bf)(b - g) + (a - f)n = 0.$

Proof. Let the conditions (i_1) and (i_2) of Lemma 3.1 be fulfilled. By Definition 4.1, the cubic system (1) has an integrating factor of the form (24) if and only if the identity (25) holds. Identifying the coefficients of the monomials $x^i y^j$ in (25), we obtain that $\alpha_1 = \alpha_2$ and α_2, α_3 are the solutions of the system

$$\{F_{ij} = 0,\quad i + j = 1, 2\}. \tag{26}$$

In Case (i_1), from the equations $F_{10} = 0$ and $F_{02} = 0$ of (26) we express α_3 and u. Then we reduce the equations $F_{20} = 0, F_{11} = 0$ by g from $F_{01} = 0$. If $r = a(f - a)$, then $f = a$ and we get the condition (i) of Lemma 4.1. We obtain the exponential factor $\Phi = \exp\left((aby - agy - nx)/(a(b - g))\right)$ and the system (1) has the integrating factor

$$\mu = (x^2 + y^2)^{(4bg - 3b^2 - g^2 - 2n)/(2(b^2 - bg + n))} \exp\left(\frac{(b - g)(nx - aby + agy)}{b^2 - bg + n}\right).$$

If $r \neq a(f - a)$, then $\alpha_2 = (4af - 3a^2 - f^2 - 2r)/[(a^2 - af + r)]$. In this case we determine the condition (ii) of Lemma 4.1. We have the exponential factor $\Phi = \exp\left((lx + ry)/r\right)$ and the system (1) has the integrating factor

$$\mu = (x^2 + y^2)^{(4af - 3a^2 - f^2 - 2r)/(2(a^2 - af + r))} \exp\left(\frac{(a - f)(lx + ry)}{a^2 - af + r}\right).$$

In Case (i_2), the equations $F_{20} = 0,\ F_{02} = 0,\ F_{01} = 0$ of (26) yield $\alpha_3 = l/a$, $\alpha_2 = (f - 3a)/(2a)$, $l = f(b - a)$. In this case we get the condition (iii) of Lemma 4.1. The system (1) has the exponential factor $\Phi = \exp(x)$ and the functon

$$\mu = (x^2 + y^2)^{(f - 3a)/(2a)} \exp\left(\frac{(bf - ag)x}{a}\right).$$

is an integrating factor for system (1). □

Lemma 4.2. *The following four sets of conditions are sufficient condition for the origin to be a center for system* (1).

(i) $c = d = 0,\ f = a,\ g = b,\ k = p = q = l,\ r = m,\ s = n;$

(ii) $c = g - b,\ d = f - a,\ k = l,\ q = p,\ n = 2r - s,\ m = 2s - r,\ p = -3l,\ l = b(a + f),$ $s = a^2 + af - b^2 - bg + r,\ ag - bf = 0;$

(iii) $c = g - b,\ d = f - a,\ k = ag,\ l = bf,\ m = af - bg + r,\ n = -m,\ p = -bf,$ $q = -ag,\ s = r;$

(iv) $c = -2b,\ d = -2a,\ f = -a,\ g = -b,\ k = l,\ n = 2r - m,\ p = -l,\ q = -l,\ s = r.$

Proof. In each of the cases (i) – (iv) the cubic differential system (1) has two homogeneous invariant straight lines $x \mp iy = 0$ and an exponential factor Φ.

In Case (i), we find the exponential factor $\Phi = \exp(2bx + 2ay + 2lxy + nx^2 + my^2)$ and the system (1) has the first integral

$$(x^2 + y^2)\exp(2bx + 2ay + 2lxy + nx^2 + my^2) = C.$$

In Case (ii), we have $\Phi = \exp((1 - 2bx - 2ay)/(x^2 + y^2))$. We can construct an integrating factor of the form

$$\mu = (x^2 + y^2)^{-3}\exp\left(\frac{(3b + g)(1 - 2bx - 2ay)}{2b(a^2 + af + r)(x^2 + y^2)}\right).$$

In Case (iii), we determine $\Phi = \exp((b^2x^2 - a^2x^2 + 2abxy - 2bx - 2ay + 1)/(x^2 + y^2))$. We can construct an integrating factor of the form

$$\mu = (x^2 + y^2)^{-3/2}\exp\left(\frac{g(b^2x^2 - a^2x^2 + 2abxy - 2bx - 2ay + 1)}{2br(x^2 + y^2)}\right).$$

In Case (iv), we have $\Phi = \exp((mx^2 - rx^2 - 2lxy - 2bx - 2ay + 1)/(x^2 + y^2))$. We can construct the first integral

$$(x^2 + y^2)^{-r}\exp\left(\frac{mx^2 - rx^2 - 2lxy - 2bx - 2ay + 1}{x^2 + y^2}\right) = C.$$

□

Theorem 4.1. *The cubic system* (1) *with two invariant straight lines* $x \mp iy = 0$ *and an exponential factor of the form* (14) *has a center at the origin* $O(0, 0)$ *if and only if the first Lyapunov quantity vanishes.*

Proof. We compute the first Lyapunov quantities L_1 for cubic system (9) assuming that the conditions of Lemma 3.2 hold.

In Case (i_1) the first Lyapunov quantity vanishes. We have Lemma 4.2, (i).

In Case (i_2) we find that $L_1 = u \neq 0$. Therefore, the origin is a focus.

In Case (i_3) the first Lyapunov quantity is $L_1 = ag - bf$. If $L_1 = 0$, then we have Lemma 2.2.2, (iv) $(l = 0, n = r)$ from [5] and the origin is a center. □

Theorem 4.2. *The cubic system* (1) *with two invariant straight lines* $x \mp iy = 0$ *and an exponential factor* $\Phi = \exp((1 - 2bx - 2ay)/(x^2 + y^2))$ *has a center at the origin* $O(0,0)$ *if and only if the first two Lyapunov quantities vanish.*

Proof. We compute the first two Lyapunov quantities L_1, L_2 for cubic system (9) assuming that the set of conditions of Lemma 3.3 is fulfilled. The vanishing of the first Lyapunov quantity gives $u = ag - bf$. The second Lyapunov quantity looks

$$L_2 = 48(a^2 + af + r)(ag - bf).$$

Let $r = -a^2 - af$. Then the right hand sides of (1) have a common factor $h(x, y) = ay + bx + fy + gx + 1$. Assume that $r \neq -a^2 - af$ and let $ag - bf = 0$. In this case $L_2 = 0$ and we have Lemma 4.2, (ii). □

Theorem 4.3. *The cubic system* (1) *with two invariant straight lines* $x \mp iy = 0$ *and an exponential factor of the form* (20) *has a center at the origin* $O(0,0)$ *if and only if the first two Lyapunov quantities vanish.*

Proof. We compute the first two Lyapunov quantities L_1, L_2 for cubic system (9) assuming that the conditions of Lemma 3.4 hold.

In Case (i_1) we have $L_1 = 0$ and the second Lyapunov quanity is $L_2 = 48r(ag - bf)$.

If $r = 0$, then the right hand sides of (1) have a common factor $h(x, y) = gx + fy + 1$. Assume that $r \neq 0$ and let $ag - bf = 0$. In this case $L_2 = 0$ and we have Lemma 4.2, (iii).

In Case (i_2) we find that $L_1 = L_2 = 0$. Then Lemma 4.2, (iv). □

References

[1] Amel'kin, V.V., Lukashevich, N.A., Sadovskii, A.P. *Non-linear oscillations in the systems of second order*. Belarusian University Press, Belarus, 1982 (in Russian).

[2] Arcet, B., Romanovski, V.G. On Some Reversible Cubic Systems. *Mathematics*, 2021, vol. 9, 1446. https:// doi.org/10.3390/math9121446

[3] Christopher C. Invariant algebraic curves and conditions for a centre. *Proc. Roy. Soc. Edinburgh Sect. A*, 1994, vol. 124, 1209–1229.

[4] Christopher, C., Llibre, J., Pantazi, C., Zhang, X. Darboux integrability and invariant algebraic curves for planar polynomial systems. *J. Phys. A.*, 2002, vol. 35, no. 10, 2457–2476.

[5] Cozma, D. *Integrability of cubic systems with invariant straight lines and invariant conics*. Ştiinţa, Chişinău, 2013.

[6] Cozma, D., Şubă, A. Conditions for the existence of four invariant straight lines in a cubic differential system with a singular point of a center or a ficus type. Bul. Acad. de Şt. a Republicii Moldova. Matematica, 1993, no. 3, 54–62.

[7] Cozma, D. The problem of the centre for cubic differential systems with two homogeneous invariant straight lines and one invariant conic. *Annals of Differential Equations*, 2010, vol. 26, no. 4, 385–399.

[8] Cozma D. Darboux integrability and rational reversibility in cubic systems with two invariant straight lines. *Electronic Journal of Differential Equations*, 2013, vol. 2013, no. 23, 1–19.

[9] Cozma, D., Dascalescu, A. Center conditions for a cubic system with a bundle of two invariant straight lines and one invariant cubic. *ROMAI Journal*, 2017, vol. 13, no. 2, 39–54.

[10] Cozma, D., Dascalescu, A. Integrability conditions for a class of cubic differential systems with a bundle of two invariant straight lines and one invariant cubic. *Buletinul Academiei de Ştiinţe a Republicii Moldova. Matematica*, 2018, vol. 86, no. 1, 120–138.

[11] Cozma, D., Matei, A. Center conditions for a cubic differential system having an integrating factor. *Bukovinian Math. Journal*, 2020, vol. 8, no. 2, 6–13.

[12] Cozma, D., Matei, A. Integrating factors for a cubic differential system with two algebraic solutions. *ROMAI Journal*, 2021, vol. 17, no. 1, 65–86.

[13] Cozma, D. Darboux integrability of a cubic differential system with two parallel invariant straight lines. *Carpathian J. Math.*, 2022, vol. 38, no. 1, 129–137.

[14] Cozma, D. Center conditions for a cubic differential system with an invariant conic. *Bukovinian Mathematical Journal*, 2022, vol. 10, no. 1, 22–32.

[15] Dukarić M. On integrability and cyclicity of cubic systems. *Electr. J. Qual. Theory Differ. Equ.*, 2020, vol. 55, 1–19.

[16] Llibre, J. On the centers of cubic polynomial differential systems with four invariant straight lines. *Topological Methods in Nonlinear Analysis*, 2020, vol. 55, no. 2, 387–402.

[17] Lyapunov, A.M. *The general problem of stability of motion*. Gostekhizdat, Moscow, 1950 (in Russian).

[18] Romanovski, V.G., Shafer, D.S. *The center and cyclicity problems: a computational algebra approach*, Birkhäuser, Boston, Basel, Berlin, 2009.

[19] Sadovskii, A.P., Scheglova, T.V. Solution of the center-focus problem for a nine-parameter cubic system. *Diff. Equations*, 2011, vol. 47, 208–223.

[20] Şubă, A., Cozma, D. Solution of the problem of the center for cubic systems with two homogeneous and one non–homogeneous invariant straight lines. *Bull. Acad. Sci. of Moldova, Mathematics*, 1999, vol. 29, no. 1, 37–44.

[21] Şubă, A. Center problem for cubic differential systems with the line at infinity of multiplicity four. *Carpathian J. Math.*, 2022, vol. 38, no. 1, 217–222.

[22] Xiang Zhang. *Integrability of Dynamical Systems: Algebra and Analysis*, Springer Nature Singapure, Singapure, 2017.

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