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On determination of some exact solutions of the stationary Navier-Stokes equations

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Abstract. In this paper, various exact solutions of the stationary Navier-Stokes equations, which describe the planar flow of an incompressible liquid (or gas), are determined, i.e., solutions containing the components of the velocity of flow - the functions u, v and the created pressure P. We mention that in the paper a series of exact solutions is obtained, in which the viscosity coefficient λ participates explicitly.

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Keywords: stationary two-dimensional Navier-Stokes equations, exact solutions, method of separation of variables, viscosity, pressure, velocity of plane flow of a liquid or gas.

Determinarea unor soluții exacte ale ecuaţiilor staţionare Navier-Stokes

Rezumat. În această lucrare se determină diverse soluții exacte ale ecuațiilor staționare Navier-Stokes, care descriu curgerea plană a unui lichid (sau gaz) incompresibil, și anume soluții ce conțin componentele vitezei fluxului de curgere - funcțiile u, v şi presiunea creată P. Menționăm, că în lucrare sunt obținute un șir de soluții exacte, în care participă în mod explicit coeficientul vâscozității λ.

Cuvinte-cheie: ecuaţii staţionare bidimensionale Navier-Stokes, soluţii exacte, metoda separării variabilelor, vâscozitate, presiune, viteza fluxului de curgere plană a unui lichid sau gaz.

1. Introduction

In the present paper, the Navier-Stokes equations are studied in the two-dimensional case. In this case the Navier-Stokes equations represent a system, containing three differential equations with partial derivatives with three unknown functions. Until today, the examined problem has not been definitively solved even in the case of stationary equations, that is, equations that describe the processes of the planar flow of a liquid or gas that do not vary in time. The complexity of the problem lies in the fact that the first two equations in the system are non-linear. It has been developed a method that would allow us to determine all the solutions of this system. Determining the solutions of the system of Navier-Stokes equations is an important mathematical problem and has various applications in fluid and gas mechanics.

In this paper it is examined the following system of partial differential equations:

$$\begin{cases} \frac{P_x}{\mu} + uu_x + vu_y = \lambda(u_{xx} + u_{yy}) + F_x, \\ \frac{P_y}{\mu} + uv_x + vv_y = \lambda(v_{xx} + v_{yy}) + F_y, \\ u_x + v_y = 0, \end{cases} \tag{1}$$

where $x, y \in R; P = P(x, y); F = F(x; y); u = u(x, y), v = v(x, y); u_x = \frac{\partial u}{\partial x}$.

The system (1) describes the stationary processes of planar flow of an incompressible liquid or gas. Regarding the derivation of the equations of system (1) and the meaning of the physical processes described by this system, consult the works [1], [2], [3].

The unknowns of the system (1) are the following three functions: P, which represents the created pressure; u and v, which represent the components of the flow velocity of a liquid or gas. The given exterior force is F and has a potential nature, that is, its components are equal to the partial derivatives of this force - F_x and F_y. The constants $\lambda > 0$ and $\mu > 0$ are the parameters determined by the viscosity and density of the examined liquid or gas. We mention here, that the viscosity parameter has the form $\lambda = C_0/R_e$, where R_e is the Reinolds number and C_0 is a constant.

Some exact solutions of the system (1) are obtained in the papers [4] - [7]. In [8] a series of solutions of the examined system are indicated only for the components of the flow velocity, without determining the pressure.

2. Equations for determining the velocity and pressure components. Solutions that do not depend on the viscosity parameter

The system (1) is equivalent to the following system:

$$\begin{cases} \frac{P_x}{\mu} - F_x + uu_x + vv_x = \lambda \Delta u - v\left(u_y - v_x\right), \\ \frac{P_y}{\mu} - F_y + uu_y + vv_y = \lambda \Delta v + u\left(u_y - v_x\right), \\ u_x + v_y = 0, \end{cases} \tag{2}$$

where $\Delta u = u_{xx} + u_{yy}$, $\Delta v = v_{xx} + v_{yy}$. Denote

$$G = \frac{1}{\mu} P - F + 0,5\left(u^2 + v^2\right). \tag{3}$$

Then from (2) it follows that

$$\begin{cases} G_x = \lambda \Delta u - v\left(u_y - v_x\right), \\ G_y = \lambda \Delta v + u\left(u_y - v_x\right). \end{cases} \tag{4}$$

Since $G_{xy} = G_{yx}$, we differentiate the first equation from (4) with respect to y and the second one with respect to x. Then we equalize the right hand sides of the obtained equations. As a result, we obtain the following equation for determining the functions u and v:

$$\lambda\Delta(u_y - v_x) - u(u_y - v_x)_x - v(u_y - v_x)_y = 0. \tag{5}$$

In addition, u and v must also check the last equation in the system (2):

$$u_x + v_y = 0. \tag{6}$$

Thus, firstly the components of the flow velocity from the system formed by equations (5) and (6) are determined, then the function G from the system (4), and finally the pressure P from the equation (3).

The following Theorem generates a series of solutions of the system (1).

Theorem 2.1. *Let D be a connected domain in the coordinate plane OXY, and let u, v and P be functions that in this domain admit continuous partial derivatives up to and including the second order. If $f(z)$ is a function of complex variable $z = x + iy$, differentiable at any interior point $(x; y)$ of the domain D, then system (1) admits solutions of the following form in this domain:*

$$u = Imf, v = Ref; P = \left[F - 0,5\left(u^2 + v^2\right) + C\right]\mu, \tag{7}$$

where C is an arbitrary constant.

Proof. May it be $u = Imf, v = Ref, f = v\,(x; y) + iu\,(x; y)$, where $f(z)$ is a function of complex variable $z = x + iy$, differentiable at any interior point $(x; y)$ of the domain D. Then from Cauchy – Riemann conditions [9] we obtain:

$$\begin{cases} v_x = u_y, \\ v_y = -u_x, \end{cases} \Leftrightarrow \begin{cases} u_y - v_x = 0, \\ u_x + v_y = 0. \end{cases} \tag{8}$$

The second equation in (8) coincides with (6), and from the first it follows that these functions verify the equation (5). It remains to determine the pressure P. Since the functions u and v admit continuous derivatives up to the second order, inclusively in D, we have that

$$u_{xy} = u_{yx}, v_{xy} = v_{yx}.$$

Differentiating the first equation from (8) with respect to y and the second one with respect to x, and adding the results, we will obtain that $\Delta u = u_{xx} + u_{yy} = 0$. Then, differentiating the second equation from (8) with respect to y and the first one with respect to x, and subtracting the results, we obtain that $\Delta v = 0$.

Then from (4) we obtain that $\begin{cases} G_x = 0, \\ G_y = 0, \end{cases} \Rightarrow G(x; y) = C - const.$

We substitute this result in (3) and express the pressure P. Theorem 2.1 is proved.

Below we will give one example of determining the solutions of the system (1) according to Theorem 2.1. If $f(z) = (z - z_0)^{-1}, z_0 = x_0 + iy_0,$ then

$$\begin{cases} u = \frac{C_0(y_0-y)}{(x-x_0)^2+(y-y_0)^2}; v = \frac{C_0(x-x_0)}{(x-x_0)^2+(y-y_0)^2}, \\ P = \left[F - \frac{0{,}5C_0{}^2}{(x-x_0)^2+(y-y_0)^2} + C\right]\mu, \end{cases} \quad D = OXY\setminus\{M(x_0; y_0)\}. \tag{9}$$

In the solutions (9), C and C_0 are arbitrary constants.

Next, to determine the solutions of the system (1), we will apply the method of separation of variables.

3. Method of Separation of Variables

We look for the velocity components in the following form:

$$u = g(x) f_1(y); v = f(y)g_1(x), \tag{10}$$

where the functions f and g are differentiable up to the fourth order while the functions f_1 and g_1 up to and including the third order.

From the equation (6) we deduce that

$$g'(x) f_1(y) + g_1(x) f'(y) = 0 \Rightarrow \frac{g'}{g_1} = \frac{-f'}{f_1} = \frac{1}{C} \Rightarrow g_1 = Cg', f_1 = -Cf'.$$

From here we obtain, that:

$$u = -Cg(x) f'(y); v = Cf(y) g'(x); u_y - v_x = -C(gf'' + fg''). \tag{11}$$

In the equations (11) C is an arbitrary non-zero constant.

We will consider the case when the functions $g(x)$ and $f(y)$ are not constant because, if one of these functions is constant, then from (10) it follows that one of the functions u or v is equal to zero. In this case, the well-known Poiseuille or Couette ([2], [10]) type flows are obtained.

By replacing (10) into equation (5), we get:

$$\lambda\left(g^{(4)}f + 2g''f'' + gf^{(4)}\right) + Cgf'\left(g'f'' + g^{(3)}f\right) - Cg'f(gf^{(3)} + g''f') = 0 \Rightarrow$$

$$\lambda\left[\frac{g^{(4)}}{g} + \frac{2g''f''}{gf} + \frac{f^{(4)}}{f}\right] + C\left[g'\left(\frac{f'f''}{f} - f^{(3)}\right) + f'\left(g^{(3)} - \frac{g'g''}{g}\right)\right] = 0. \tag{12}$$

We will study firstly the case when the expressions from the square brackets in the equation (12) are equal to zero. After we equalize to zero the expression next to C and applying the method of separation of variables, we get:

$$g'\left(\frac{f'f''}{f}-f^{(3)}\right)+f'\left(g^{(3)}-\frac{g'g''}{g}\right)=0 \qquad \Rightarrow$$

$$\frac{f}{f'}\left(\frac{f'f''-ff^{(3)}}{f^2}\right)+\frac{g}{g'}\left(\frac{gg^{(3)}-g'g''}{g^2}\right)=0 \qquad \Rightarrow$$

$$\frac{g}{g'}\left(\frac{g''}{g}\right)'=\frac{f}{f'}\left(\frac{f''}{f}\right)'=k-\ \text{const.} \tag{13}$$

From (13) it follows that the functions $g(x)$ and $f(y)$ are determined in the same way. Let us first examine the case $k=0$. From (13) we deduce, that

$$\begin{cases} g''=ag, \\ f''=bf, \end{cases} \quad \Rightarrow \quad \begin{cases} g^{(4)}=ag'', \\ f^{(4)}=bf'', \end{cases} \tag{14}$$

where a and b are arbitrary constants.

We replace (14) in the equality (12) while the expression in the second square bracket cancels, and in the first square bracket in (12) we obtain $a^2+2ab+b^2$. Thus, the equation (12) takes the following form: $\lambda(a^2+2ab+b^2)=0$. Because $\lambda>0$, it follows that $b=-a$. Therefore, the following three situations are possible:

(1) $a=c^2, b=-c^2, c\neq 0.$ Then $\begin{cases} g''=c^2g, \\ f''=-c^2f, \end{cases} \Rightarrow \begin{cases} g=a_1e^{cx}+a_2e^{-cx}, \\ f=b_1\cos(cy)+b_2\sin(cy). \end{cases}$

(2) $a=-c^2, b=c^2, c\neq 0.$ Then $\begin{cases} g''=-c^2g, \\ f''=c^2f, \end{cases} \Rightarrow \begin{cases} g=a_1\cos(cx)+a_2\sin(cx), \\ f=b_1e^{cy}+b_2e^{-cy}. \end{cases}$

(3) $a=0, b=0.$ Then $\begin{cases} g''=0, \\ f''=0, \end{cases} \Rightarrow \begin{cases} g=a_1x+a_2, \\ f=b_1y+b_2. \end{cases}$

In the all three cases c, a_1, a_2, b_1, b_2 are arbitrary constants.

According to the equalities (10), we have that

$$\begin{cases} u=-Cg(x)f'(y), \\ v=Cg'(x)f(y), \end{cases} \tag{15}$$

with $g(x)$ and $f(y)$ determined in cases (1), (2) and (3).

From (15) and (14) we obtain in these cases, that $u_y-v_x=0, \Delta u=0, \Delta v=0$. Then, from (4) we easily determine that $G=C_0$ – constant and then, from relation (3) we find that the pressure is

$$P=\left(F-0,5C^2\left[(gf')^2+(g'f)\,2\right]+C_0\right)\mu. \tag{16}$$

Thus, in the case $k = 0$ the solutions of the system (1) are given in the formulas (15) and (16).

Therefore, the following theorem is proved:

Theorem 3.1. *If the functions f and g are differentiable up to the fourth order on any interval of the real axis, then system (1) admits solutions of form (15), (16) where the functions $g(x)$ and $f(y)$ are determined according to cases (1), (2) or (3) above.*

We will investigate the case $k \neq 0$. Let us find $g(x)$. Regarding the solution of different ordinary differential equations, consult [11]. From (13) we deduce that

$$\left(\frac{g''}{g}\right)' = k\frac{g'}{g} \quad \Rightarrow \quad \frac{g''}{g} = k\ln g + c.$$

Denote $g' = p(g)$, then $g'' = p'p$. As a result we obtain a first order differential equation:

$$p'p = k{\cdot}g\ln g + cg \Rightarrow 0,5p^2 = 0,5k\left(g^2\ln g - \frac{g^2}{2}\right) + 0,5cg^2 + c_1.$$

If $c_1 = 0$, then $g' = \pm g\sqrt{k(\ln g - 0,5) + c} \Rightarrow g = e^{[\frac{k}{4}(c_2 \pm x)^2 + c_3]}$.

Analogously, we obtain that $f = e^{[\frac{k}{4}(c_4 \pm y)^2 + c_5]}$.

But in this case the equality (12) in not fulfilled because in the equality (12) the expression in the second square bracket cancels and the expression in the parenthesis next to λ is different of zero. Thus, in the case $k \neq 0$ the solutions of the equation (12) cannot be determined.

4. The case when one of the functions $g(x)$ or $f(y)$ is linear. Solutions in which the viscosity parameter participates explicitly

We return to the equality (12) and now we are studying the situation when the expressions in the square brackets are not equal to zero. We examine the case when the derivative of the function f or the derivative of the function g is constant. That is, we will examine the case when one of the functions $g(x)$ or $f(y)$ is linear.

Let $f(y) = by + m$, with $b \neq 0$ and m arbitrary constants, then $f'(y) = b$. From the equality (12) we obtain a fourth-order non-linear differential equation, containing only the function $g(x)$:

$$\lambda g^{(4)} + Cb\left(gg^{(3)} - g'g''\right) = 0. \qquad (17)$$

We integrate the equation (17), taking into account that $(gg'')' = g'g'' + gg^{(3)}$, and we obtain the following third-order differential equation:

$$\lambda g^{(3)} + Cb\left(gg'' - 2\int g'g''dx\right) = C_1 \Rightarrow$$

$$\lambda g^{(3)} + Cb\left(gg'' - (g')^2\right) = C_1 \tag{18}$$

where C_1 is an arbitrary constant.

We notice that in the case $C_1 = -Cba^2$ this equation admits solutions of the form $g = ax + d$ for any reals constants a and b. Then, according to the formulas (15), the solutions of system (1) will be:

$$u = -Cb\,(ax + d)\,; \quad v = Ca\,(by + m)\,, \tag{19}$$

with a, b, C, m, d – arbitrary constants. The pressure P is determined from the relation (16):

$$P = \left(F - 0,5C^2\left[b^2(ax + d)^2 + a^2(by + m)^2\right] + C_0\right)\mu. \tag{20}$$

In the formula (20) and in all formulas that follow, C_0 is an arbitrary constant. Thus, we obtain solutions of the system (1) given by (19) and (20).

Next, we will look for solutions of the equation (18) of the form $g = a(x + d)^n$ with the constants a, d, n, $n \neq 0$, $n \neq 1$. Substituting in (18), we find that

$$n = -1, C_1 = 0, a = \frac{6\lambda}{Cb} \Rightarrow g\,(x) = \frac{6\lambda}{Cb\,(x + d)}.$$

Substituting the obtained function $g(x)$ and $f(y) = by + m$ into (15), we get the following solutions of the system (1):

$$\begin{cases} u = -\frac{6\lambda}{x+d}, \\ v = -\frac{6\lambda(by+m)}{b(x+d)^2}, \end{cases} \tag{21}$$

with the arbitrary constants b, d, m. We determine the function G from the system (4):

$$G = \frac{18\lambda^2(by + m)^2}{b^2(x + d)^4} + \frac{6\lambda^2}{(x + d)^2} + C_0.$$

Then, we substitute the determined function G in (3) and find the pressure in the case of the solutions (21):

$$P = \left(F - \frac{12\lambda^2}{(x + d)^2} + C_0\right)\mu \tag{22}$$

As a result, we obtain solutions of the system (1) in the form of the formulas (21), (22). Next, we will look for solutions of the equation (14) of the form $g = a + ne^{kx}$ with the constants a, n, k. Substituting into (18), we obtain

$$C_1 = 0, a = -\frac{k\lambda}{Cb} \Rightarrow g\,(x) = -\frac{k\lambda}{Cb} + ne^{kx},$$

with k, b, C, n – arbitrary real constants.

Substituting into (15), we obtain the following solutions of the system (1):

$$\begin{cases} u = \lambda k - Cbne^{kx}, \\ v = C(by+m)nke^{kx}, \end{cases} \tag{23}$$

C, b, m, n, k - arbitrary constants. We find the function G from the system (4):

$$G = \frac{(Cnk)^2(by+m)^2 e^{2kx}}{2} - \lambda Cbnke^{kx} + C_0.$$

Then, we substitute the determined function G in (3) and find the pressure corresponding to the case of the solutions (23):

$$P = \left[F - 0,5\left[(Cbn)^2 e^{2kx} + (\lambda k)^2\right] + C_0\right]\mu \tag{24}$$

Thus, we obtain solutions for the system (1) given by (23) and (24).

Now let $g(x) = bx+m$, with $b \neq 0$ and m arbitrary constants; then $g'(x) = b$. From (12) we obtain a forth-order nonlinear differential equation which contains only the function $f(y)$:

$$\lambda f^{(4)} - Cb\left(ff^{(3)} - f'f''\right) = 0. \tag{25}$$

We integrate the equation (25) and we obtain the following equation of order 3:

$$\lambda f^{(3)} - Cb\left(ff'' - (f')^2\right) = C_1, \tag{26}$$

where C_1 is an arbitrary constant.

We notice that in the case $C_1 = Cba^2$ the equation (25) admits solutions of the form $f = ay + d$ for any real constants a and d. According to the formulas (15), the solutions of the system (1) are:

$$u = -Ca\,(bx+m)\,; \quad v = Cb\,(ay+d) \tag{27}$$

with the arbitrary constants a, b, C, m, d. The pressure P in this case is:

$$P = \left(F - 0,5C^2\left[a^2(bx+m)^2 + b^2(ay+d)^2\right] + C_0\right)\mu. \tag{28}$$

Looking for solutions of this equation of the form $f = a(y+d)^n$ with constants a, d and n, we will obtain that

$$n = -1, \quad a = -\frac{6\lambda}{Cb} \quad \Rightarrow \quad f(y) = -\frac{6\lambda}{Cb\,(y+d)}.$$

Substituting the determined function $f(y)$ and $g(x) = bx + m$ into (15), we obtain the following solutions:

$$\begin{cases} u = -\frac{6\lambda(bx+m)}{b(y+d)^2}, \\ v = -\frac{6\lambda}{y+d}, \end{cases} \tag{29}$$

with the arbitrary constants b, d, m. The pressure, corresponding to the solution (26), is

$$P = \left(F - \frac{12\lambda^2}{(y+d)^2} + C_0\right)\mu. \tag{30}$$

Thus, we obtain solutions for the system (1) given by (29) and (30).
Looking further for solutions of the equation (11) of the form $f = a + ne^{ky}$ with the constants a, n, k, we will find out that

$$C_1 = 0, a = \frac{k\lambda}{Cb} \Rightarrow f(y) = \frac{k\lambda}{Cb} + ne^{ky}.$$

Substituting into (15), we obtain the following solutions for the system (1):

$$\begin{cases} u = -C\,(bx+m)\,kne^{ky}, \\ v = \lambda k + Cbne^{ky}, \end{cases} \tag{31}$$

where C, b, m, n, k are arbitrary constants.

The pressure corresponding to the solution (30) is

$$P = \left[F - 0,5\left[(Cbn)^2 e^{2ky} + (\lambda k)^2\right] + C_0\right]\mu. \tag{32}$$

As a result, we obtain solutions of the system (1) in the form of the formulas (31), (32).
Based on what has been proved in this section, the following theorem results:

Theorem 4.1. *If the function $f(y)$ is linear, i.e. $f(y) = by + m$, $b \neq 0$, then the function $g(x)$ is the solution of equation (18). In this case, the system (1) admits the exact solutions (19), (20); (21), (22) and (23), (24). If $g(x)$ is linear, i.e. $g(x) = ax + d$, $a \neq 0$, then the function $f(y)$ is the solution of equation (26). In this case, the system (1) admits the exact solutions (27), (28); (29), (30) and (31), (32).*

Remark 4.1. Unlike the solutions obtained in Theorems 2.1 and 3.1, Theorem 4.1 mentions solutions for the system (1) in which the viscosity parameter λ is explicitly indicated.

Remark 4.2. Equations (18) and (26) can be reduced to second-order differential equations.

Let us illustrate what was said, for example, for the equation (18). Then, making the substitution $g' = p(g)$, $g'' = p'p$, we arrive at a second-order nonlinear differential equation for determining the function $p(g)$:

$$\lambda\left(pp'' + (p')^2\right) + Cb\,(gp' - p) = c_1 p^{-1}. \tag{33}$$

However, the problem of determining the solutions of equation (33) is not simpler than the problem of determining the solutions of equation (18). We observe in the case of

$c_1 \neq 0$, that particular solutions of equation (33) are the following constants:

$$p = a, C_1 = -Cba^2; \Rightarrow g(x) = ax + d.$$

In this case, we obtain solutions for the system (1) in the form of (19), (20).

In the case $c_1 = 0$, we will look for particular solutions for the equation (33) of the form $p(g) = ag^2 + mg + k$, where a, m, k are constants and $a \neq 0$ or $m \neq 0$. Substituting into (33), we get:

$$\lambda \left[2a\left(ag^2 + mg + k\right) + (2ag + m)^2\right] + cb\left[g(2ag + m) - ag^2 - mg - k\right] = 0. \quad \Rightarrow$$

$$g^2\left[6\lambda a^2 + Cba\right] + g\left[\lambda am\right] + \left[\lambda\left(2ak + m^2\right) - Cbk\right] = 0 \tag{34}$$

Because the function $g(x)$ is not constant, the equality (34) can be fulfilled only when all the expressions in the square brackets cancel, i.e. the following equalities are true:

$$6\lambda a^2 + Cba = 0 \quad \text{and} \quad \lambda am = 0 \quad \text{and} \quad \lambda\left(2ak + m^2\right) - Cbk = 0. \tag{35}$$

The equalities (35) take place simultaneously in the following two cases:

(1) $a = 0, m \neq 0$. Then the first two equalities are satisfied and from the third one we have that $k = \dfrac{\lambda m^2}{Cb} \Rightarrow p(g) = g' = mg + k$. From here we get

$$g' - mg = k \Rightarrow g(x) = ne^{mx} - \frac{k}{m} \Rightarrow g(x) = ne^{mx} - \frac{\lambda m}{Cb}.$$

In this case we obtain the solutions of the form (23), (24).

(2) $m = 0, a \neq 0$. The second equality in (35) is satisfied. Then from the first and the third equalities we find that $a = -\frac{Cb}{6\lambda}, k = 0 \Rightarrow p(g) = ag^2$. In this case we have that

$$p = g' = ag^2 \Rightarrow g(x) = \frac{6\lambda}{Cb(C_0 + x)}.$$

Thus, we obtain the solutions of the form (21), (22).

Analogously reducing the equation (26) to a second-order differential equation and studying it in the same way as the equation (33), we will obtain the solutions of the system (1) of the form (27), (28); (29), (30) and (31), (32).

5. Conclusions

In the current article are determined a lot of exact solutions for the stationary bidimensional Navier-Stokes equations. We mention, that the solutions obtained in Theorems 2.1 and 3.1 do not explicitly depend on the viscosity. This result occurs because the expression $u_y - v_x$ (the rotor), corresponding to these solutions, is constant and equal to zero.

In the other solutions obtained in Theorem 4.1 it is indicated the explicit dependence on the viscosity coefficient λ. In the case of these solutions, the rotor is not constant. We also mention that several arbitrary constants participate in the expressions containing the found solutions. The values of these constants can be determined based on initial conditions and boundary conditions of the given physical problems.

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