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Dedicated to Professor Alexandru Şubă on the occasion of his 70th birthday

Averaging in multifrequency systems with multi-point conditions and a delay

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Abstract. For multifrequency system of differential equations with a discrete and integral delay we find conditions for the existence and uniqueness of the solution. Linear multipoint conditions are set for the solution. An estimate of the error of the averaging method is obtained, which clearly depends on the small parameter.
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Medierea în sistemele multifrecvenţă cu condiţii multi-punct şi întârziere

Rezumat. Pentru sistemul multifrecvenţă de ecuaţii diferenţiale cu o întârziere integrală şi discretă găsim condiţii pentru existenţa şi unicitatea soluţiei. Condiţiile liniare multipunct sunt stabilite pentru soluţie. Se obţine o estimare a erorii metodei de mediere, care depinde în mod clar de parametrul mic.
Cuvinte-cheie: sistem multifrecvenţă, metoda medierii, rezonanţă, întârziere integrală, argument transformat liniar.

1. Introduction

In many cases, mathematical models of oscillating systems are described with differential equations of the form

$$\frac{da}{d\tau} = X(\tau, a, \varphi), \quad \frac{d\varphi}{d\tau} = \frac{\omega(\tau, a)}{\varepsilon} + Y(\tau, a, \varphi), \tag{1}$$

where $0 \le \varepsilon t = \tau$ – slow time, ε – positive small parameter, $a \in \mathbb{D} \subset \mathbb{R}^n$, $\varphi \in \mathbb{R}^m$. The system (1) is rigid, its research and construction of both analytical and numerical solutions is a complex and not always solvable task. Therefore, to simplify the system (1), the averaging procedure for fast variables $\varphi_1, \dots, \varphi_m$ is used, which greatly simplifies it, reducing it to the form

$$\frac{d\overline{a}}{d\tau} = X_0(\tau, \overline{a}), \quad \frac{d\overline{\varphi}}{d\tau} = \frac{\omega(\tau, \overline{a})}{\varepsilon} + Y_0(\tau, \overline{a}). \tag{2}$$

In the general case, the deviation of solutions $\|a(\tau,\varepsilon)-\overline{a}(\tau)\|$ can become $O(1)$ on a finite segment $[0,L]$ or $\mathbb{R}_+=(0,\infty)$ due to frequency resonance, the condition of which is

$$(k,\omega(\tau,a)) := k_1\omega_1(\tau,a)+\cdots+k_m\omega_m(\tau,a)\simeq 0,\quad k\neq 0. \tag{3}$$

Therefore, in order to justify the averaging method, additional conditions are imposed on the frequency vector $\omega(\tau,a)$ for the system to exit from a small circumference of resonance. The works of V. I. Arnold [1], E. O. Grebenikov [2], A. M. Samoilenko and R. I. Petryshyn [3] and many others are devoted to this issue.

The monograph [3] presents a new method of studying multifrequency systems (1), which is based on estimates of the corresponding oscillatory integrals, which made it possible to justify a wide class of multifrequency systems with initial and boundary conditions.

For adequate modeling of oscillating systems, it is also important to take into account informational, technological and other delays. Multifrequency systems with constant and variable delay were studied in the works [4, 5, 6]. In particular, systems in which the delay is specified with a linearly transformed argument of the form $\lambda\tau$, $\tau>0$, $0<\lambda\leq 1$ in [6, 7, 8]. A new resonance condition was obtained, including for systems with linearly transformed arguments and a frequency vector $\omega(\tau)$ in fast variables $\varphi(\theta_\nu\tau)$ of the form

$$\gamma_k(\tau) := \sum_{\nu=1}^{q}\theta_\nu\big(k_\nu,\omega(\theta_\nu\tau)\big)=0. \tag{4}$$

The works [6, 7, 8, 9] are devoted to the substantiation of the averaging method for such systems with initial multipoint and integral conditions.

This article considers systems with both point and integral delay, which allows taking into account the background history of the process at some interval. Parabolic equations with such a delay were studied in [9] for functional differential equations in the monograph [11] and others.

2. Formulation of the problem

We investigate a system of differential equations with variable delay of the form

$$\frac{da(\tau)}{d\tau}=X\Big(\tau,a(\tau),a_\lambda(\tau),\int\limits_{\Delta\tau}^{\tau}g(s)a(s)ds,\varphi_\Theta(\tau)\Big), \tag{5}$$

$$\frac{d\varphi(\tau)}{d\tau}=\frac{\omega(\tau)}{\varepsilon}+\varepsilon^\beta Y\Big(\tau,a(\tau),a_\lambda(\tau),\int\limits_{\Delta\tau}^{\tau}g(s)a(s)ds,\varphi_\Theta(\tau)\Big), \tag{6}$$

where $\tau \in [0, L]$, $\varepsilon \in (0, \varepsilon_0]$, $a \in \mathbb{D} \subset \mathbb{R}^n$, $\varphi \in \mathbb{R}^m$; $0 < \lambda < 1$, $a_\lambda(\tau) = a(\lambda\tau)$; $\varphi_\Theta = \big(\varphi(\theta_1\tau), \ldots, \varphi(\theta_q\tau)\big)$, $0 < \theta_1 < \cdots < \theta_q \leq 1$, $0 < \Delta < 1$, $\beta > 0$. Vector-functions X and Y are 2π-periodic by components of variables φ_{θ_ν}, $\nu = \overline{1, q}$.

For the solution of the system (5), (6), multipoint conditions are set

$$\sum_{\nu=1}^{r} A_\nu(\varepsilon) a|_{\tau=\tau_\nu} = d_1, \tag{7}$$

$$\sum_{\nu=1}^{r} B_\nu(\varepsilon) \varphi|_{\tau=\tau_\nu} = d_2, \tag{8}$$

where $0 \leq \tau_1 < \tau_2 < \cdots < \tau_r \leq L$, $A_\nu(\varepsilon)$ and $B_\nu(\varepsilon)$ are given matrices of order n and m, respectively, defined at $\varepsilon \in [0, \varepsilon_0]$ and vectors $d_1 \in \mathbb{R}^n$, $d_2 \in \mathbb{R}^m$.

The corresponding system (5), (6) averaged over fast variables on the mq-cube of periods takes the form

$$\frac{d\overline{a}(\tau)}{d\tau} = X_0\Big(\tau, \overline{a}(\tau), \overline{a}_\lambda(\tau), \int_{\Delta\tau}^{\tau} g(s)\overline{a}(s)ds\Big), \tag{9}$$

$$\frac{d\overline{\varphi}(\tau)}{d\tau} = \frac{\omega(\tau)}{\varepsilon} + \varepsilon^\beta Y_0\Big(\tau, \overline{a}(\tau), \overline{a}_\lambda(\tau), \int_{\Delta\tau}^{\tau} g(s)\overline{a}(s)ds\Big) \tag{10}$$

with multipoint conditions

$$\sum_{\nu=1}^{r} A_\nu(\varepsilon) \overline{a}|_{\tau=\tau_\nu} = d_1, \tag{11}$$

$$\sum_{\nu=1}^{r} B_\nu(\varepsilon) \overline{\varphi}|_{\tau=\tau_\nu} = d_2, \tag{12}$$

Now the problem (9), (11) can be solved separately and we can find the solution $\overline{a} = \overline{a}(\tau; \overline{y}, \varepsilon)$, $\overline{a}(0; \overline{y}, \varepsilon) = \overline{y}$. Solving the multipoint problem (11) is reduced to integration if the initial value for the solution component is known $\overline{\varphi} = \overline{\varphi}(\tau; \overline{y}, \overline{\psi}, \varepsilon)$, $\overline{\varphi}(0; \overline{y}, \overline{\psi}, \varepsilon) = \overline{\psi}$.

Suppose that the condition is satisfied:

Condition A. There is a unique solution of the averaged problem (9)–(12), whose component is $\overline{a}(\tau; \overline{y}, \varepsilon)$, $\overline{y} \in \mathbb{D}_1 \subset \mathbb{D}$, at $\tau \in [0, L]$ and $\varepsilon \in (0, \varepsilon_0]$ lies in the area $\mathbb{D}$ with some ρ-circumference.

In the work, sufficient conditions are established, under which there is a unique differentiable solution of the problem (5)–(8). The method of averaging is stipulated and the estimate of the deviation error of the solutions is constructed, which clearly depends on the small parameter ε and has the form

$$u(\tau;\varepsilon) := \|a(\tau;\overline{y}+\mu,\overline{\psi}+\xi,\varepsilon)-\overline{a}(\tau;\overline{y},\varepsilon)\|+ \\ +\|\varphi(\tau;\overline{y}+\mu,\overline{\psi}+\xi,\varepsilon)-\overline{\varphi}(\tau;\overline{y},\overline{\psi},\varepsilon)\| \le c_1\varepsilon^{\alpha}. \tag{13}$$

Here $\alpha = 1/(mq)$, $c_1 > 0$ and does not depend on ε, $a(0;\overline{y}+\mu,\overline{\psi}+\xi,\varepsilon) = \overline{y}+\mu(\varepsilon)$, $\varphi(0;\overline{y}+\mu,\overline{\psi}+\xi,\varepsilon) = \overline{\psi}+\xi(\varepsilon)$.

3. Auxiliary Statements

Lemma 3.1. *Let the matrix* $B(\varepsilon) := \sum\limits_{\nu=1}^{r} B_\nu(\varepsilon)$ *be nondegenerate for* $\varepsilon \in [0,\varepsilon_0]$*. Then there is a unique solution to the problem* (10)*,* (12)*.*

Proof. From the equation (10) we have

$$\varphi(\tau_\nu;\overline{y},0,\varepsilon) = \int_0^{\tau_\nu} \Big(\frac{\omega(s)}{\varepsilon} + Y_0(s,\overline{a},\overline{a}_\lambda,\overline{v}_\Delta)\Big)ds,$$

where

$$\overline{v}_\Delta(\tau,\varepsilon) = \int_{\Delta\tau}^{\tau} g(s)\overline{a}(s;\overline{y},\varepsilon)ds.$$

It follows from the condition (12) that

$$B(\varepsilon)\overline{\psi} = d_2 - \sum_{\nu=1}^{r}\varphi(\tau_\nu;\overline{y},0,\varepsilon),$$

wherefrom we find the initial value of $\overline{\psi}(\overline{y},\varepsilon)$. The solution to the problem (10), (12) takes the form

$$\varphi(\tau;\overline{y},\overline{\psi},\varepsilon) = \overline{\psi}(\overline{y},\varepsilon) + \varphi(\tau;\overline{y},0,\varepsilon).$$

□

Lemma 3.2. *Let*

1) number $d \ge 0$, $\lambda,\Delta \in (0,1)$*;*

2) f_1, f_2 *and g – continuous functions on [0,L] with value in* $\mathbb{R}_+ = [0,\infty)$*;*

$$0 \le u(\tau) \le d + \int_0^{\tau} f_1(s)u(s)ds + \int_0^{\lambda\tau} f_2(s)u(s)ds + \int_0^{\tau}\Big(\int_{\Delta s}^{s} g(z)u(z)dz\Big)ds. \tag{14}$$

Then

$$u(\tau) \le d\cdot exp\Big(\int_0^{\tau}\big(f_1(s)+\lambda f_2(s)\big)ds + \int_0^{\tau}\big(\int_{\Delta s}^{s} g(z)dz\big)ds\Big), \quad 0 \le \tau \le L. \tag{15}$$

Proof. We denote by $w(\tau)$ the right-hand side of the inequality (14). Then $w(0) = d$, $u(\tau) \le w(\tau)$ and $w^{'}(\tau) \ge 0$ for $\tau \in [0, L]$.

Then we have

$$v^{'}(\tau) = f_1(\tau)u(\tau) + \lambda f_2(\lambda\tau)u(\lambda\tau) + \int\limits_{\lambda\tau}^{\tau} g(s)u(s)ds \le$$

$$\le f_1(\tau)v(\tau) + \lambda f_2(\lambda\tau)v(\lambda\tau) + \int\limits_{\lambda\tau}^{\tau} g(s)v(s)ds \le$$

$$\le \Big(f_1(\tau) + \lambda f_2(\lambda\tau) + \int\limits_{\lambda\tau}^{\tau} g(s)ds\Big)v(\tau).$$

After integrating the inequality, we obtain the solution (15) of the integral inequality (14) □

The article [10] substantiates the averaging method for a system of equations of a more general form than (5), (6) with initial conditions. The following condition is the condition for exiting the (4) resonance small circumference.

Condition B. Let $\omega \in \mathbb{C}^{mq}[0, L]$ and be constructed according to the mq system of functions $\big\{\omega(\theta_1\tau), \ldots, \omega(\theta_q\tau)\big\}$ Wronskian

$$W(\varphi_\Theta) \ne 0, \quad \tau \in [0, L].$$

Theorem 3.1. *Suppose that:*

1) *vector function $F(\tau, a, a_\lambda, w_\Delta, \varphi_\Theta) := \big(X, Y\big)$ is twice continuously differentiable over all arguments in the area $G = G_1 \times \mathbb{R}^m q$, $G_1 = [0, L] \times \mathbb{D} \times \mathbb{D} \times \mathbb{D}_\nu$, 2π-periodic in the components of the vectors φ_ν, $\nu = \overline{1, q}$ and bounded together with the derivatives by the constant σ_1;*
2) *conditions A and B are satisfied;*
3) *for the Fourier coefficients F_k in the area G_1 the evaluation is performed:*

$$\sum_{k\ne 0}\Big(\sup_{G_1}\|F_k\| + \frac{1}{\|k\|_\Theta}\big(\sup_{G_1}\Big\|\frac{\partial F_k}{\partial \tau}\Big\| + \sup_{G_1}\Big\|\frac{\partial F_k}{\partial a}\Big\| + \sup_{G_1}\Big\|\frac{\partial F_k}{\partial a_\lambda}\Big\|$$

$$+(1-\Delta_\nu)\sup_{G_1}\Big\|\frac{\partial F_k}{\partial v_\lambda}\frac{\partial v_\lambda}{\partial \tau}\Big\|\big)\Big) \le \sigma_2$$

where $\|k\|_\Theta = \sum\limits_{\nu=1}^{q}\theta_\nu\|k_\nu\|$.

Then for sufficiently small $\varepsilon_1 \in (0, \varepsilon_0]$ *there exists a unique solution* $\big(a(\tau; \overline{y}, \overline{\psi}, \varepsilon),$ $\varphi(\tau; \overline{y}, \overline{\psi}, \varepsilon)\big)$ *with initial conditions* $(\overline{y}, \overline{\psi})$ *and the evaluation is performed*

$$u(\tau; \varepsilon) := \|a - \overline{a}\| + \|v - \overline{v}\| \leq c_2 \varepsilon^{\alpha}, \tag{16}$$

for all $(\tau, \varepsilon) \in [0, L] \times (0, \varepsilon_1]$, $\alpha = (mq)^{-1}$, $c_2 > 0$ *and does not depend on* ε.

Remark 3.1. *If the vector functions* X *and* Y *are continuously differentiable* mq *once over the variable* τ *and* $mq + 1$ *the other time over the other variables, then condition 3) of Theorem 3.1 is satisfied and the estimate of the form* (16) *is correct for the derivatives of the deviation of the solutions for the initial variables* y *and* ψ *with the constant* c_2.

4. Justification of the Averaging Method

Let

$$A(\varepsilon) = \sum_{\nu=1}^{r} A_{\nu}(\varepsilon) \frac{\partial \overline{a}(\tau_{\nu}; \overline{y}, \varepsilon)}{\partial y}.$$

Theorem 4.1. *Suppose that:*

1) *condition 1) of Theorem 3.1 and conditions A and B are satisfied;*
2) *matrices* $A_{\nu}(\varepsilon)$, $B_{\nu}(\varepsilon)$, $\nu = \overline{1, r}$ *are continuous at* $\varepsilon \in (0, \varepsilon_0]$, $A(\varepsilon)$, $B(\varepsilon)$ *are non-degenerate and* $\|A^{-1}(\varepsilon)\| \leq \sigma_2$, $\|B^{-1}(\varepsilon)\| \leq \sigma_3$;
3) $g \in \mathbb{C}[0, L]$.

Then there exists such $\varepsilon^* \in (0, \varepsilon_0]$ *that for each* $\varepsilon \in [0, \varepsilon^*]$ *there is a unique solution to the problem* (5)–(8) *in the class* $\mathbb{C}^1[0, L]$ *and for all* $(\tau, \varepsilon) \in [0, L] \times (0, \varepsilon^*]$ *evaluation is performed* (13).

Besides

$$\|\mu\| \leq c_3 \varepsilon^{\alpha}, \quad \|\xi\| \leq c_4 \varepsilon^{\alpha}, \quad \alpha = 1/(mq). \tag{17}$$

Proof. Let

$$2c_1 \varepsilon^{\alpha} \leq \rho, \quad \|\mu\| \leq c_3 \varepsilon^{\alpha} \leq \rho/2, \tag{18}$$

where the constant $c_3 > 0$ and will be determined further. Then based on the estimate (16) for all $\psi \in \mathbb{R}^m$, $(\tau, \varepsilon) \in (0, \varepsilon_2]$, $\varepsilon_2 = min\Big(\varepsilon_1, \big(\frac{\rho}{2c_2}\big)^{mp}, \big(\frac{\rho}{2c_3}\big)^{mp}\Big)$

$$\|a(\tau; \overline{y} + \mu, \psi) - \overline{a}(\tau; \overline{y} + \mu, \varepsilon)\| \leq c_2 \varepsilon^{\alpha}. \tag{19}$$

From equation (9) we have

$$v(\tau, \mu, \varepsilon) := \|\overline{a}(\tau; \overline{y} + \mu, \varepsilon) - \overline{a}(\tau; \overline{y}, \varepsilon)\| \leq$$

$$\leq \|\mu\| + \sigma_1 \int_0^{\tau} v(s, \mu, \varepsilon) ds + \sigma_1 \int_0^{\tau} v(\lambda s, \mu, \varepsilon) ds + \sigma_1 \int_0^{\tau} \int_{\Delta s}^{s} |g(z)| v(z, \mu, \varepsilon) dz ds.$$

Applying the estimate (15) gives

$$v(\tau,\mu,\varepsilon) \le \|\mu\| \exp\Big(2\sigma_1 + \int\limits_0^{\tau} \int\limits_{\Delta s}^{s} |g(z)| dz ds\Big)\tau.$$

So for $\tau \in [0, L]$

$$v(\tau,\mu,\varepsilon) \le c_5 \varepsilon^{\alpha},$$

where $c_5 = c_3 \exp\Big(2\sigma_1 + \int\limits_0^{L} \int\limits_{\Delta s}^{s} |g(z)| dz ds\Big)L.$

The solution $a(\tau; \overline{y} + \mu, \psi, \varepsilon)$ under the conditions (18) lies in the ρ circumference of the solution $\overline{a}(\tau; \overline{y}, \varepsilon)$ and the evaluation is performed

$$\overline{w}(\tau;\mu,\psi,\varepsilon) := \|a(\tau;\overline{y}+\mu,\psi,\varepsilon) - \overline{a}(\tau;\overline{y},\varepsilon)\| \le$$
$$\le \|a(\tau;y+\mu,\psi,\varepsilon) - \overline{a}(\tau;\overline{y}+\mu,\varepsilon)\| + v(\tau,\mu,\varepsilon) \le c_6\varepsilon^{\alpha},$$

where $c_6 = c_2 + c_5$.

We will show that there is μ that satisfies the condition (18) such that the solution a of the equation (5) satisfies the condition (7).

From the conditions (7) and (11) we have

$$\sum_{\nu=1}^{r} A_\nu(\varepsilon)\Big(\overline{a}(\tau_\nu;\overline{y}+\mu,\varepsilon) - \overline{a}(\tau_\nu;\overline{y},\varepsilon)\Big) =$$
$$-\sum_{\nu=1}^{r} A_\nu(\varepsilon)\Big(\big(a(\tau_\nu;\overline{y}+\mu,\overline{\psi}+\xi,\varepsilon) - \overline{a}(\tau_\nu;\overline{y}+\mu,\varepsilon)\big) + R_{1,\nu}(\mu,\varepsilon)\Big), \tag{20}$$

where

$$R_{1,\nu}(\mu,\varepsilon) = \overline{a}(\tau_\nu;\overline{y}+\mu,\varepsilon) - \overline{a}(\tau_\nu;\overline{y},\varepsilon) - \frac{\partial \overline{a}(\tau_\nu;y,\varepsilon)}{\partial \overline{y}}\mu.$$

From (19) we obtain the equation for μ:

$$\mu = \Phi_1(\mu,\xi,\varepsilon), \tag{21}$$

where $\Phi_1(\mu,\xi,\varepsilon) =$

$$= -A^{-1}(\varepsilon)\Big(\sum_{\nu=1}^{r}\Big(A_\nu(\varepsilon)\big(a(\tau_\nu;\overline{y}+\mu,\overline{\psi}+\xi,\varepsilon) - \overline{a}(\tau_\nu;\overline{y}+\mu,\varepsilon)\big) + R_{1,\nu}(\mu,\varepsilon)\Big)\Big)$$

It follows from the differentiability of the solution $\overline{a}(\tau;\mu,\varepsilon)$ over the variable $\overline{y}$ that

$$\|R_{1,\nu}(\mu,\varepsilon)\| \le c_{7,\nu}\|\mu\|^2, \quad \Big\|\frac{\partial R_{1,\nu}}{\partial \mu}\Big\| \le c_{8,\nu}\|\mu\|. \tag{22}$$

Considering the estimates (16) and (19), when $(\tau,\varepsilon) \in [0,L] \times (0,\varepsilon_0]$ we obtain

$$\big\|\Phi_1(\mu,\xi,\varepsilon)\big\| \le \sigma_2 \sum_{\nu=1}^{r} \max_{[0,\varepsilon_0]} \big\|A_\nu(\varepsilon)\big\|\Big(c_2\varepsilon^{\alpha} + c_{7,\nu}\|\mu\|^2\Big) = c_9\varepsilon^{\alpha} + c_{10}\|\mu\|^2.$$

Let in condition (18) be

$$c_3 = 2c_9, \quad c_3^2 c_{10}\varepsilon_3^\alpha \le c_9.$$

Then

$$\left\|\Phi_1(\mu,\xi,\varepsilon)\right\| \le 2c_9\varepsilon^\alpha$$

for $\mu \le 2c_9\varepsilon^\alpha$, $\xi \in \mathbb{R}^m$ and $\varepsilon \in (0,\varepsilon_3]$.

So, $\Phi_1 : S_1 \to S_1$, $S_1 = \{\mu : \|\mu\| \le c_3\varepsilon^\alpha\}$.

Then we have

$$\frac{\partial \Phi_1}{\partial \mu} = -A^{-1}(\varepsilon)\sum_{\nu=1}^{r} A_\nu(\varepsilon)\frac{\partial}{\partial \mu}\big(a(\tau;\overline{y}+\mu,\overline{\psi}+\xi,\varepsilon) - \overline{a}(\tau;\overline{y}+\mu,\varepsilon)\big) -$$
$$-A^{-1}(\varepsilon)\sum_{\nu=1}^{r} A_\nu(\varepsilon)\frac{\partial R_{1,\nu}(\mu,\varepsilon)}{\partial \mu}.$$

From Theorem 3.1, condition 2) of Theorem 4.1 and estimates (22), we obtain

$$\left\|\frac{\partial \Phi_1}{\partial \mu}\right\| \le \sigma_2 c_2 \sum_{\nu=1}^{r}\max_{[0,\varepsilon_0]}\left\|A_\nu(\varepsilon)\right\|\varepsilon^\alpha + \sigma_2 c_3 \sum_{\nu=1}^{r}\max_{[0,\varepsilon_0]}\left\|A_\nu(\varepsilon)\right\| = c_{11}\varepsilon^\alpha < \frac{1}{4}, \tag{23}$$

if $\varepsilon \le \varepsilon_4 = (4c_{11})^{mq}$.

Similarly, we have

$$\left\|\frac{\partial \Phi_1}{\partial \psi}\right\| = \left\|A^{-1}(\varepsilon)\sum_{\nu=1}^{r} A_\nu(\varepsilon)\frac{\partial}{\partial \psi}\big(a(\tau;\overline{y}+\mu,\overline{\psi}+\xi,\varepsilon) - \overline{a}(\tau;\overline{y}+\mu,\varepsilon)\big)\right\| \le$$
$$\le \sigma_2 c_2 \sum_{\nu=1}^{r}\max_{[0,\varepsilon_0]}\left\|A_\nu(\varepsilon)\right\|\varepsilon^\alpha = c_{12}\varepsilon^\alpha < \frac{1}{4},$$

if $\varepsilon \le \varepsilon_5 = (4c_{12})^{mq}$.

Now from the conditions (8) and (12) we find

$$\xi = \Phi_2(\mu,\xi,\varepsilon),$$

where

$$\Phi_2(\mu,\xi,\varepsilon) = -B^{-1}(\varepsilon)\sum_{\nu=1}^{r} B_\nu(\varepsilon)\Big(\big(\varphi(\tau_\nu;\overline{y}+\mu,\overline{\psi}+\xi,\varepsilon) - \overline{\varphi}(\tau_\nu;\overline{y}+\mu,\varepsilon)\big) +$$
$$+\big(\overline{\varphi}(\tau_\nu;\overline{y}+\mu,\varepsilon) - \overline{\varphi}(\tau_\nu;\overline{y},\varepsilon)\big)\Big).$$

Based on the estimates (15) and (19) and condition 2) of Theorem 4.1, we obtain

$$\left\|\Phi_2(\mu,\xi,\varepsilon)\right\| \le \Big(\sigma_3 c_2 \sum_{\nu=1}^{r} \max_{[0,\varepsilon_0]} \left\|B_\nu(\varepsilon)\right\| + \sigma_3 c_5 \sum_{\nu=1}^{r} \max_{[0,\varepsilon_0]} \left\|B_\nu(\varepsilon)\right\|\Big)\varepsilon^\alpha \le$$

$$\le \sigma_3(c_2+c_5)\Big(\sum_{\nu=1}^{r} \max_{[0,\varepsilon_0]} \left\|B_\nu(\varepsilon)\right\|\Big)\varepsilon^\alpha = c_{13}\varepsilon^\alpha.$$

So $\Phi_2 : S_2 \to S_2$, $S_2 = \{\varphi : \|\varphi - \overline{\varphi}\| \le c_{13}\varepsilon^\alpha\}$, if

$$\|\xi\| \le c_{13}\varepsilon^\alpha. \tag{24}$$

Then we have

$$\frac{\partial \Phi_2}{\partial \mu} = -B^{-1}(\varepsilon)\sum_{\nu=1}^{r} B_\nu(\varepsilon)\Big(\frac{\partial}{\partial \mu}\big(\varphi(\tau_\nu;\overline{y}+\mu,\overline{\psi}+\xi,\varepsilon) - \overline{\varphi}(\tau_\nu;\overline{y}+\mu,\overline{\psi}+\xi,\varepsilon)\big)+$$

$$+\frac{\partial}{\partial \mu}\overline{\varphi}(\tau_\nu;\overline{y}+\mu,\overline{\psi}+\xi,\varepsilon)\Big),$$

$$\left\|\frac{\partial \Phi_2}{\partial \mu}\right\| \le \sigma_3 \sum_{\nu=1}^{r} \max_{[0,\varepsilon_0]} \left\|B_\nu(\varepsilon)\right\|\left(c_2\varepsilon^\alpha + c_{13}\varepsilon^\beta\right) \le c_{14}\varepsilon^\gamma < \frac{1}{4},$$

if $\varepsilon \le \varepsilon_6 = (4c_{14})^{-\gamma}$, $\gamma = \min(\alpha,\beta)$.

Let $\Phi = col(\Phi_1,\Phi_2)$, $\eta = col(\mu,\psi)$. Then

$$\left\|\frac{\partial \Phi}{\partial \eta}\right\| < 1,$$

from which, according to the fixed point theorem [13], it follows that there is a single fixed point (μ^*,ψ^*) if $\varepsilon < \varepsilon^* = \min_{\nu=1,6}\varepsilon_\nu$. Therefore, there exists a unique solution $\big(a(\tau;\overline{y}+\mu^*,\overline{\psi}+\xi^*,\varepsilon),\varphi(\tau;\overline{y}+\mu^*,\overline{\psi}+\xi^*,\varepsilon)\big)$ of the system (5), (6), which satisfies the conditions (7), (8).

From the equation (9), estimates (16), (24) we obtain

$$\overline{\overline{w}}(\tau;\mu,\xi,\varepsilon) = \|\overline{\varphi}(\tau;\overline{y}+\mu,\overline{\psi}+\xi,\varepsilon) - \overline{\varphi}(\tau;\overline{y},\overline{\psi},\varepsilon)\| \le$$

$$\le \|\xi\| + \sigma_1\int_0^\tau \Big(w(s,\mu,\xi,\varepsilon) + w(\lambda s,\mu,\xi,\varepsilon) + \int_{\Delta s}^{s} |g(z)|w(z,\mu,\xi,\varepsilon)dz\Big)ds.$$

So,

$$\overline{\overline{w}}(\tau;\mu,\xi,\varepsilon) = \Big(c_{12} + \sigma_1 c_6 \int_0^L \big(2 + (1-\Delta)|g(s)|\big)ds\Big)\varepsilon^\alpha = c_{14}\varepsilon^\alpha. \tag{25}$$

Based on evaluations (18) we get

$$u(\tau,\varepsilon) \le \overline{w}(\tau;\mu,\xi,\varepsilon) + \overline{\overline{w}}(\tau;\mu,\xi,\varepsilon) +$$
$$+\|a(\tau;\overline{y}+\mu,\overline{\psi}+\xi,\varepsilon) - a(\tau;\overline{y},\overline{\psi},\varepsilon)\| + \|\varphi(\tau;\overline{y}+\mu,\overline{\psi}+\xi,\varepsilon) - \overline{\varphi}(\tau;\overline{y},\overline{\psi},\varepsilon)\| \le$$
$$\le (c_5+c_{15})\varepsilon^{\alpha} + c_2\varepsilon^{\alpha} = c_1\varepsilon^{\alpha}, \quad (\tau,\varepsilon) \in [0,L]\times(0,\varepsilon^*),$$

where $c_1 = c_2 + c_5 + c_{15}$. □

Remark 4.1. *If* $\beta = 0$ *and no other conditions are imposed on the system* (5), (6) *or the conditions* (7), (8), *then it is possible to prove only the existence of a solution based on the Brouwer's theorem* [13].

5. Model Example

Consider a single-frequency system

$$\frac{da(\tau)}{d\tau} = a(\lambda\tau) + \int_{\lambda\tau}^{\tau} a(s)ds + \cos(k\varphi(\tau) + l\varphi(\theta\tau)),$$
$$\frac{d\varphi(\tau)}{d\tau} = \frac{1+2\tau}{\varepsilon}, \quad 0 \le \tau \le 1,$$

where $0 < \lambda < 1$, $0 < \theta < 1$; $k, l \in \mathbb{Z} \setminus \{0\}$, $k + l\theta = 0$.

If $\varphi(0) = 0$, then $\varphi(\tau) = \tau(1+\tau)/\varepsilon$, $k\varphi(\tau) + l\varphi(\theta\tau) = \kappa\tau^2/\varepsilon$, $\kappa = k + l\theta^2 \ne 0$.

At the point $\tau = 0$, the resonance condition is satisfied, since $\gamma_{kl} = 2\tau\kappa$.

Let us set the boundary condition

$$\alpha_0 a|_{\tau=0} + \alpha_1 a|_{\tau=1} = d, \quad |\alpha_0| + |\alpha_1| \ne 0. \tag{26}$$

The averaged equation for the slow variable

$$\frac{d\overline{a}(\tau)}{d\tau} = \overline{a}(\lambda\tau) + \int_{\lambda\tau}^{\tau} \overline{a}(s)ds \tag{27}$$

with a boundary condition of the form (26) has a solution

$$\overline{a}(\tau;\overline{y}) = \overline{y}e^{\tau}, \quad \overline{y} = d/(\alpha_0 + \alpha_1 e).$$

Let $v(\tau;\mu,\varepsilon) = a(\tau;\overline{y}+\mu,\varepsilon) - \overline{a}(\tau;\overline{y}+\mu)$. Then

$$v(\tau;\mu,\varepsilon) = \int_0^{\tau} v(\lambda\tau;\mu,\varepsilon)ds + \int_0^{\tau}\int_{\lambda s}^{s} v(z;\mu,\varepsilon)dzds + \int_0^{\tau} \cos\frac{\kappa s^2}{\varepsilon}ds.$$

Applying the estimate of the Fresnel integral [12] we obtain

$$\int_0^{\tau} \cos\frac{\kappa s^2}{\varepsilon}ds = \frac{\sqrt{\varepsilon}}{\sqrt{\kappa}}\int_0^{\sqrt{\pi}\tau/\sqrt{\varepsilon}} \cos x^2 dx = \frac{\sqrt{\pi}}{2\sqrt{2\kappa}}\sqrt{\varepsilon} + O(\sqrt[4]{\varepsilon^3}) \le c_{16}\sqrt{\varepsilon},$$

where $c_{16} = \sqrt{\pi}/\sqrt{2\kappa}$, $\varepsilon \le 4\kappa/\pi^2$.

From the estimate for $v(\tau;\mu,\varepsilon)$ and $\tau \in [0,1]$ it follows

$$|v(\tau;\mu,\varepsilon)| \le \sqrt{\varepsilon}c_{16}\exp\big(1 + (1-\lambda)\tau/2\big)\tau \le c_{17}\sqrt{\varepsilon},$$

where $c_{17} = c_{16}\exp(3-\lambda)/2$.

From the boundary conditions for the solutions $a(\tau;\overline{y}+\mu,\varepsilon)$ and $\overline{a}(\tau;\overline{y})$ we find

$$\mu = -\big(\alpha_1/(\alpha_0+\alpha_1 e)\big)\big(a(1;\overline{y}+\mu,\varepsilon) - \overline{a}(1;\overline{y}+\mu,\varepsilon)\big),$$

hence it follows

$$|\mu| \le \big(\alpha_1 c_{17}/(\alpha_0+\alpha_1 e)\big)\sqrt{\varepsilon}.$$

Based on the estimates for $v(\tau;\mu,\varepsilon)$ and μ, we obtain

$$|a(\tau;\overline{y}+\mu,\varepsilon) - \overline{a}(\tau;\overline{y})| \le |v(\tau;\mu,\varepsilon)| + |\overline{a}(\tau;\overline{y}+\mu) - \overline{a}(\tau;\overline{y})| \le c_{18}\sqrt{\varepsilon},$$

where $c_{18} = c_{17}\big(1 + \alpha_1/(\alpha_0+\alpha_1 e)\big)$.

6. Conclusions

In the article the existence and uniqueness of the solution in $\mathbb{C}^1[0,L]$ is proved for the system of equations (5), (6) with linear multipoint conditions (7), (8) and an estimate of the error of the method of averaging and deviation of the initial conditions for slow and fast variables of order ε^{α}, $\alpha = 1/(mq)$ was obtained. The same result can be obtained by more complex technical transformations for an arbitrary finite number of arguments $a_{\lambda_1},\ldots,a_{\lambda_p}$ and $v_{\Delta_1},\ldots,v_{\Delta_r}$ in vector functions X and Y.

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