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Dedicated to Professor Alexandru Şubă on the occasion of his 70th birthday

Perturbation of singular integral operators with piecewise continuous coefficients

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Abstract. In the paper it is shown that the property of singular integral operators with piecewise continuous coefficients to be Noetherian is stable with respect to their perturbation with certain non-compact operators. An example is constructed showing that the corner points of the integration contour significantly affect the Noetherian property of singular operators with translations. These results are obtained using the symbol of the singular operators on contours with corner points, symbol, which is also determined.

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Perturbarea operatorilor integrali singulari cu coeficienţi continui pe porţiuni

Rezumat. În lucrare se demonstrează că proprietatea operatorilor integrali singulari cu coeficienţi continui pe porţiuni de a fi noetherieni este stabilă în raport cu perturbarea lor cu anumiţi operatori necompacţi. Este construit un exemplu care demonstrează că punctele unghiulare ale conturului de integrare afectează în mod semnificativ proprietatea noetheriană a operatorilor singulari cu translaţii. Aceste rezultate sunt obţinute cu ajutorul simbolului operatorilor singulari pe contururi cu puncte unghiulare, simbol, care de asemenea este determinat.

Cuvinte cheie: operatori integrali singulari, operatori noetherieni, simbol, contur Lyapunov pe porţiuni.

1. Introduction

In the well-known monographs of N. Mushelyishvili and F. Gahov, operators of the form

$$(A\varphi)(t) = a(t)\varphi(t) + \int_\Gamma \frac{k(t,\tau)\varphi(\tau)}{\tau - t} d\tau, \ (t \in \Gamma), \tag{1}$$

are called *complete* singular integral operators. In relation (1) the functions $a(t)$ and $k(t,\tau)$ are functions that verify Holder's conditions on Γ, respectively on $\Gamma \times \Gamma$, and the integral is considered in the principal value sense. The operator A, defined by equality (1) can be represented as

$$A = aI + bS + T,$$

where $b(t) = \pi i k(t,t)$, S is the singular integral operator on the contour Γ,

$$(S\varphi)(t) = \frac{1}{\pi i}\int_{\Gamma}\frac{\varphi(\tau)}{\tau - t}d\tau,$$

and T is the integral operator with the kernel

$$k(t,\tau) = \frac{k(\tau,t) - k(t,t)}{\tau - t}. \tag{2}$$

When $k(t,\tau)$ satisfies Holder's conditions on $\Gamma\times\Gamma$, the kernel (2) has weak singularities and therefore the operator T is compact on the spaces $L_p(\Gamma,\rho)$, where

$$\rho(t) = \prod_{k=1}^{n}|t - t_k|^{\beta_k}\ (1 < p < +\infty,\ -1 < \beta_k < p - 1,\ t_k \in \Gamma).$$

It hence follows that the operator $A = aI + bS + T$ is normally solvable and Noetherian if and only if this property is possessed by its characteristic part, $A_0 = aI + bS$. Moreover

$$dimkerA = dimkerA_0 \text{ and } dimkerA^* = dimkerA_0^*.$$

Based on this statement Noether's theory for singular integral operators developed into the foundation for characteristic singular integral operators. Remarkable results have been obtained in this direction: Noetherian criteria have been established for these operators with continuous piecewise coefficients, with coefficients, which have almost periodic discontinuities, with coefficients arbitrary (measurable and bounded). However, in various problems of mechanics, physics and other fields, which reduce to singular equations, it is not characteristic operators but complete operators that appear. In this contest it arises the necessity to study complete singular operators with discontinuous functions $a(t)$ și $k(t,\tau)$. The main difficulty in this direction consists in the fact that the operator T with kernel (2) may not be compact and (more importantly) may not represent a permissible perturbation for singular characteristic operators.

We will illustrate this with an example. Let Γ_0 be the unit circle, $\chi(t)$ be the characteristic function of the semicircle $\Gamma_0^+ = \{t \in \Gamma_0, Imt \geq 0\}$, $k(\tau,t) = \chi(t) - \chi(\tau)$, $\lambda \in C$ and

$$(A\varphi)(t) = \lambda\varphi(t) + \int_{\Gamma}\frac{k(t,\tau)}{\tau - t}\varphi(\tau)d\tau.$$

In this example $k(t,t) = 0$, therefore the characteristic part of the operator A is the scalar operator $(A_0\varphi)(t) = \lambda\varphi(t)$. In this example the operator can be represented as

$$A = \lambda I + \chi S - S\chi I, \tag{3}$$

and it follows that it is contained in the algebra $\sum_p(\Gamma_0)$ generated by singular operators with piecewise continuous coefficients. It is known that this algebra is a symbol algebra. The symbol of the operators is determined from the equalities [5], [9].

$$a(t,\xi) = \begin{Vmatrix} a(t+0)f(\xi) + (1-f(\xi))a(t) & h(\xi)(a(t+0)-a(t)) \\ h(\xi)(a(t+0)-a(t)) & (1-f(\xi))a(t+0) + a(t)f(\xi) \end{Vmatrix}, \tag{4}$$

where

$$f(\xi) = \left[1 - exp\left(-2\pi\left(\xi + \frac{i}{p}\right)\right)\right]^{-1} (-\infty \le \xi \le +\infty),$$

$$S(t,\xi) = \begin{Vmatrix} 1 & 0 \\ 0 & -1 \end{Vmatrix}. \tag{5}$$

In particular, in the case of the operator $A = \lambda I + \chi S - S\chi I$ and $p = 2$ we have:

$$detA(t,\xi) = \lambda^2 \; for \; t \neq \pm 1$$

and

$$detA(t,\xi) = \lambda^2 + 4\frac{e^{\xi}}{1+e^{\xi}} \; (-\infty \le \xi \le +\infty), \text{ for } t = \pm 1.$$

The operator A is Noetherian in the space $L_2(\Gamma_0)$ if and only if $\lambda^2 + 4e^{\xi}/(1+e^{\xi}) \neq 0$ for any ξ, $-\infty \le \xi \le +\infty$. This is equivalent for $\lambda \neq \mu i$, where $\mu \in [-1, 1]$.

Hence, for $\lambda = \mu i$, where $\mu \in [-1, 1] \setminus \{0\}$, the operator A is not Noetherian and its characteristic part $A_0 = \lambda I$ is Noetherian. It follows that the operator $T = A - A_0$ is not a permissible perturbation to the characteristic part of operator A. It also follows that the operator $T = \chi S - S\chi I$ is not compact.

From this reasoning and the examined example, the following problem comes apparent. What (at least necessary) conditions should we impose on the operator kernel T,

$$k(t,\tau) = \frac{k(\tau,t) - k(t,t)}{\tau - t},$$

so that this operator does not influence the Noetherian conditions of the operator $A = aI + bS$, i.e. the operators $A_0 = aI+bS$ and $A = aI+bS+T$ are or are not Noetherian conditions and $IndA_0 = IndA$. If the function $k(t,\tau)$ is continuous or has weak singularities on the integration contour then the operator T is compact and it satisfies the conditions enumerated above. In this paper a class of operators T (non-compact) is described which also possess this property. In the construction of this class of operators an important role will be played by the symbol defined on algebra $\sum_p(\Gamma_0)$.

2. Perturbation of singular operators with operators from the set M

We denote by **M** the set of all operators in algebra $\sum_p(\Gamma)$ with the following properties. If $H \in \mathbf{M}$, then its symbol $H(t,\xi)$ has the form

$$H(t,\xi) = \left\| \begin{array}{cc} 0 & m(t,\xi) \\ n(t,\xi) & 0 \end{array} \right\|,$$

where $m(t,\xi) \cdot n(t,\xi) \equiv 0$ and

$$m(t,\xi) = \frac{(\psi(t,\xi) - 1)h(\xi)}{f(\xi) + (1 - f(\xi))\psi(t,\xi)}$$

and the real function $\psi(t,\xi)$ satisfies the following conditions. If $\psi(t_0,\xi) \neq 1 \ \ (t_0 \in \Gamma)$, then the continuous function $\psi(t_0,\xi)$ is decreasing and $\psi(t_0,\xi) \to +\infty$ for $\xi \to -\infty$ and $\psi(t_0,\xi) \to 0$ for $\xi \to +\infty$. We mention that the set **M** includes all compact operators acting on the space $L_p(\Gamma)$. It will be shown below that some non-compact operators of the algebra $\sum_p(\Gamma)$ also belong to the set **M**. Thus, it will be shown that the conditions for singular integral operators to be Noetherian are stable with respect to their perturbations by non-compact operators. This will follow from the next theorem.

Theorem 2.1. *Let $H \in \mathbf{M}$. The operator*

$$A = aP + Q + H \quad \left(P = \frac{1}{2}(I + S),\ Q = \frac{1}{2}(I - S),\ a \in PC(\Gamma)\right)$$

is Noetherian on the space $L_p(\Gamma)$, if and only if this property is held by the operator $A_0 = aP + Q$.

Proof. The symbol $A(t,\xi)$ of the operator $A = aP + Q + H$ has the form

$$A(t,\xi) = \left\| \begin{array}{cc} a(t+0)f(\xi) + (1 - f(\xi))a(t) & m(t,\xi) \\ h(\xi)(a(t+0) - a(t)) + n(t,\xi) & 1 \end{array} \right\|$$

and

$$detA(t,\xi) = \frac{a(t+0)f(\xi) + (1 - f(\xi))a(t)\psi(t,\xi)}{f(\xi) + (1 - f(\xi))\psi(t,\xi)}. \tag{6}$$

Let the operator A be Noetherian, then $a(t \pm 0) \neq 0$ and

$$a(t+0)f(\xi) + (1 - f(\xi))a(t)\psi(t,\xi) \neq 0 \tag{7}$$

for all $t \in \Gamma$ and $\xi \in \bar{R}$. We admit that the operator $A_0 = aP + Q$ is not Noetherian, then the determinant of its symbol cancels at a point (t_0, ξ_0):

$$a(t_0 + 0)f(\xi_0) + (1 - f(\xi_0))a(t_0) = 0, \tag{8}$$

where $t_0 \in \Gamma$ and $\xi_0 \in \bar{R}$. Therefore $\psi(t_0,\xi) \neq 1$ and ratio $\frac{a(t_0)}{a(t_0+0)}$ can be written as

$$\frac{a(t_0)}{a(t_0+0)} = \exp\Big(2\pi(\xi_0 + i/p)\Big).$$

We will show that in this case $detA(t_0,\xi)$ vanishes at a point $\xi_1 \in R$. Indeed, from the relation (6) and condition (8) we get

$$detA(t_0,\xi) = \frac{(f(\xi)-1)a(t_0)}{f(\xi)+(1-f(\xi))\psi(t_0,\xi)}[e^{2\pi(\xi-\xi_0)} - \psi(t_0,\xi)].$$

From the properties of the function $\psi(t_0,\xi)$ follows that the equation $e^{2\pi(\xi-\xi_0)} - \psi(t_0,\xi)$ possesses a solution $\xi = \xi_1$. Thus, $detA(t_0,\xi_1) = 0$, which is in contradiction with conditions (7). Necessity is proved. Let A not be Noetherian, then or $a(t+0)\cdot a(t) = 0$ at a point $t_0 \in \Gamma$ or that

$$a(t+0)f(\xi) + (1-f(\xi))a(t)\psi(t,\xi) = 0$$

at the point (t_0,ξ_0) $(t_0 \in \Gamma,\ \xi_0 \in \bar{R})$. In the first case the operator A_0 also is not Noetherian, and in the second case we obtain that $\frac{a(t_0)}{a(t_0+0)} = 2\pi/p$. Then $a(t_0+0)f(\xi) + (1-f(\xi))a(t_0) = 0$ at the point $\xi_1 \in R$. Therefore, in this case also the operator A_0 is not Noetherian. □

Corollary 2.1. *The property of singular integral operators with piecewise continuous coefficients to be Noetherian is stable under the perturbation of these operators with the operators of the set* **M**.

3. Example of a non-compact operator from the set M

Let us consider operator H, defined by the equality

$$H = P - \sum_{j=1}^{m}\sum_{k=1}^{n}(t-t_j)^{\alpha_k}P(t-t_j)^{-\alpha_k}I, \qquad (9)$$

where $t_1, t_2, ..., t_m$, are arbitrary, distinct points on Γ and $\alpha_k = \frac{k-1}{n} + \frac{1-n}{np}$.

In this section we prove that the operator H belongs to the set **M** and is not completely continous on the space $L_p(\Gamma)$.

Lemma 3.1. *Operator*

$$H_{j,k} = (t-t_j)^{\alpha_k}P(t-t_j)^{-\alpha_k}I$$

is bounded on the space $L_p(\Gamma)$.

Proof. First we are going to show that the operator $H_{j,k}$ is bounded on the space $L_p(\Gamma)$ if and only if the operator P is bounded on the space L_p with the weight $\rho(t) = |t-t_j|^{\alpha_k p}$. Since $-1 < \alpha_k p < p-1$, then, from Theorem of B. Khvedelidze [8], the operator P is bounded on the space $L_p(\Gamma, |t-t_j|^{\alpha_k p})$. □

Theorem 3.1. *The operator*

$$H_{j,k} = (t - t_j)^{\alpha_k} P (t - t_j)^{-\alpha_k} I$$

belongs to algebra $\sum_p(\Gamma)$ *and its symbol has the form*

$$H(t,\xi) = \begin{cases} \left\| \begin{matrix} 1 & 0 \\ 0 & 0 \end{matrix} \right\|, & \text{for } t \neq t_j, \\ \left\| \begin{matrix} 1 & -h(\xi)\frac{1-\exp(2\pi i \alpha_k)}{(1-f(\xi))(1-\exp(2\pi i\alpha_k))} \\ 0 & 0 \end{matrix} \right\|, & \text{for } t = t_j. \end{cases} \tag{10}$$

Proof. Let $\varphi_{i,k}(z)$ be a fixed branch of the function $z^{-\alpha_k}$, defined on the complex plane with the cut connecting the point 0 with ∞ and intersecting the contour Γ at a single point t_j. The function $\varphi_{j,k}(t)$ is continuous at every point $\Gamma \setminus \{t_j\}$ and

$$\frac{\varphi_{i,k}(t_j)}{\varphi_{i,k}(t_j+0)} = e^{-2\pi i \alpha_k}.$$

Since $1/p - 1 < \alpha_k < 1/p$, the function $\varphi_{j,k}(t)$ admits a factorization on the space $L_p(\Gamma)$ in the form

$$\varphi_{j,k}(t) = (t - t_j)^{-\alpha_k} \left(\frac{t - t_j}{t} \right)^{\alpha_k}.$$

Let us consider the operator $B_{j,k} = \varphi_{j,k}(t) P + Q$. This operator $B_{j,k}$ is invertible in space $L_p(\Gamma)$ and its inverse is defined by the equality

$$B_{j,k}^{-1} = (t - t_j)^{\alpha_k} P \left(\frac{t - t_j}{t} \right)^{-\alpha_k} I + \left(\frac{t - t_j}{t} \right)^{\alpha_k} Q \left(\frac{t - t_j}{t} \right)^{-\alpha_k} I.$$

It follows from this that

$$PB_{j,k}^{-1} = (t - t_j)^{\alpha_k} P \left(\frac{t - t_j}{t} \right)^{-\alpha_k} I.$$

Therefore,

$$(t - t_j)^{\alpha_k} P \left(\frac{t - t_j}{t} \right)^{-\alpha_k} I = PB_{j,k}^{-1} t^{-\alpha_k} I = P(P + t^{\alpha_k} Q)^{-1}. \tag{11}$$

Because the operator $P(P+t^{\alpha_k}Q)^{-1}$ belongs to algebra $\sum_p(\Gamma)$, then from (11) it results that the operator $H_{j,k} = (t-t_j)^{\alpha_k} P (t-t_j)^{-\alpha_k} I$ also belongs to algebra $\sum_p(\Gamma)$. By direct calculations, taking into account the equality (11), we make sure that the operator symbol $H_{j,k}$ coincides with the right-hand side of the equality (10). □

Corollary 3.1. *The operator H, defined by equality (9), belongs to algebra $\sum_p(\Gamma)$ and its symbol has the form*

$$H_{j,k}(t,\xi)=\begin{cases}\left\|\begin{matrix}0 & 0\\ 0 & 0\end{matrix}\right\|, & for\ t\neq t_j,\\ \left\|\begin{matrix}1 & h(\xi)\frac{1-\exp(2\pi(n-1)\xi)}{(1-f(\xi))(1-\exp(2\pi(n\xi+i/p)))}\\ 0 & 0\end{matrix}\right\|, & for\ t=t_j.\end{cases} \tag{12}$$

From the equality (12) it results.

Corollary 3.2. *The operator H is not compact. Moreover*

$$\psi(t,\xi)=\begin{cases}1, & for\ t\neq t_j,\\ \exp(-2\pi(n-1)\xi), & for\ t=t_j.\end{cases}$$

From this and from Theorem 2.1 it results.

Theorem 3.2. *The operator*

$$A=(a+1)P+Q-\sum_{j=1}^{m}\sum_{k=1}^{n}(t-t_j)^{\alpha_k}P(t-t_j)^{-\alpha_k}I$$

is Noetherian on the space $L_p(\Gamma)$ if and only if the operator $A_0=aP+Q$ has the same property.

4. The symbol of singular operators on contours with angular points

Let the contour Γ consist of two semi-axes starting at the point $z=0$. We denote by α $(0<\alpha\leq\pi)$ the angle formed by these half lines. We will assume that one of these half lines corresponds to the half axis $R^+=[0,+\infty)$ and that the contour Γ is oriented in a such way that the orientation on $\Gamma\cap R^+$ coincide with the orientation on R^+.

Let $B=L_p(\Gamma,|t|^\beta)$ $(-1<\beta<p-1)$. We denote by $\lambda_0(\Gamma)$ the set of constant piecewise functions that receives on Γ two values: a value on R^+ and another value on $\Gamma\setminus R^+$. If $h\in\lambda_0$, then we denote

$$h(t)=\begin{cases}h_1, & \text{for } t\in R^+,\\ h_2, & \text{for } t\in\Gamma\setminus R^+,\end{cases}\quad h_j\in\mathbb{C}.$$

Thus, $h(0)=h_2,\ h(0+0)=h_1,\ h(\infty-0)=h_1,\ h(\infty+0)=h_2$.

The contour Γ will be considered compactified with a point at infinity, whose neighborhoods are complements of the neighborhoods of $z_0=0$. Evidently, the contour Γ is homeomorphic with an bounded contour $\tilde{\Gamma}$, having two angular points.

We denote by K_α the Banach algebra generated by the singular integration operator S_Γ and all multiplication operators to the functions $h \in \lambda_0(\Gamma)$. By K^+ we denote the subalgebra of algebra $L(L_p(R^+, |t|^\beta))$ generated by singular integral operators $aI + bS$ $(S = S_{R^+})$ with constant coefficients on R^+. As K^+ is commutative, then it possesses [5] a sufficient system of multiplicative functionals. The operator ν is linear and bounded

$$(\nu\varphi)(x) = (\varphi(x), \varphi(e^{i\alpha}x)) \ (x \in R^+),$$

acting from the space $L_p(\Gamma, |t|^\beta)$ on the space $L_p^2(R^+, t^\beta)$.

Let $\varphi \in L_p(\Gamma, |t|^\beta)$ and consider the equation

$$A\varphi = a\varphi + bS_\Gamma\varphi = \psi,$$

$$a(t) = \begin{cases} a_1, \text{ for } t \in R^+, \\ a_2, \text{ for } t \in \Gamma \setminus R^+, \end{cases} \quad b(t) = \begin{cases} b_1, \text{ for } t \in R^+, \\ b_2, \text{ for } t \in \Gamma \setminus R^+, \end{cases} \quad a_j, \ b_j \in \mathbb{C}.$$

This equation can be written as a system of two equations: in one equation $t \in \Gamma \setminus R^+$ and in the second equation $t \in R^+$. We get,

$$\begin{cases} a(t)\varphi(t) + \dfrac{b(t)}{\pi i}\displaystyle\int_{R^+} \dfrac{\varphi(\tau)}{\tau - t}d\tau + \dfrac{b(t)}{\pi i}\displaystyle\int_{\Gamma\setminus R^+} \dfrac{\varphi(\tau)}{\tau - t}d\tau = \psi(t), \ t \in R^+, \\ a(t)\varphi(t) + \dfrac{b(t)}{\pi i}\displaystyle\int_{R^+} \dfrac{\varphi(\tau)}{\tau - t}d\tau + \dfrac{b(t)}{\pi i}\displaystyle\int_{\Gamma\setminus R^+} \dfrac{\varphi(\tau)}{\tau - t}d\tau = \psi(t), \ t \in \Gamma \setminus R^+. \end{cases}$$

In the integral $\int_{\Gamma\setminus R^+} \frac{\varphi(\tau)}{\tau - t}d\tau$ we change the variable $\tau \longrightarrow e^{i\alpha}\tau$ and in the second equation of the obtained system, we change t by $e^{i\alpha}t$. We obtain:

$$\begin{cases} a_1\varphi_1(t) + \dfrac{b_1}{\pi i}\displaystyle\int_{R^+} \dfrac{\varphi_1(\tau)}{\tau - t}d\tau - \dfrac{b_1}{\pi i}\displaystyle\int_{R^+} \dfrac{\varphi_2(\tau)}{\tau - e^{-i\alpha}t}d\tau = \psi_1(t), \ t \in R^+, \\ a_2\varphi_2(t) + \dfrac{b_2}{\pi i}\displaystyle\int_{R^+} \dfrac{\varphi_2(\tau)}{\tau - e^{i\alpha}t}d\tau - \dfrac{b_2}{\pi i}\displaystyle\int_{R^+} \dfrac{\varphi_2(\tau)}{\tau - t}d\tau = \psi_2(t), \ t \in \Gamma \setminus R^+, \end{cases}$$

in which the notations can be used: $f_1(t) = f(t), \ f_2(t) = f(e^{i\alpha}t) \ (t \in R^+)$.

Thus, the operator $\nu A\nu^{-1}$ has the form

$$\nu A \nu^{-1} = \begin{Vmatrix} a_1I + b_1S & -b_1M \\ -b_2N & a_2I + b_2S \end{Vmatrix}$$

where

$$(S\varphi)(t) = \frac{1}{\pi i}\int_{R^+} \frac{\varphi(\tau)}{\tau - t}d\tau, \ (M\varphi)(t) = \frac{1}{\pi i}\int_{R^+} \frac{\varphi(\tau)}{\tau - e^{-i\alpha}t}d\tau,$$

$$(N\varphi)(t) = \frac{1}{\pi i}\int_{R^+} \frac{\varphi(\tau)}{\tau - e^{i\alpha}t}d\tau, \ t \in \Gamma.$$

From the results of the works [6], [7], [10], it appears that operators M and N belong to the algebra K^+ generated by the operator S $(= S_{R^+})$ and the multiplication operators on constant functions. Therefore, $\nu K_\alpha \nu^{-1} \subset (K^+)^{2\times 2}$.

Let $\{\gamma_M\}$ be the system of homeomorphisms determining the symbol on the algebra K^+. For any operator $A \in K_\alpha$ we put

$$\tilde{\gamma}_M(A) = \|\gamma_M(A_{jk})\|_{j,k=1}^2, \text{ where } |A_{jk}\|_{j,k=1}^2 = \nu A \nu^{-1}.$$

Theorem 4.1. *The operator $A \in K$ is Noetherian on the space $L_p(\Gamma, |t|^\beta)$ if and only if*

$$det\tilde{\gamma}_M(A) \neq 0.$$

Proof. The factor algebra $\hat{K}^+$ is commutative with respect to all compact operators in $L(L_p(R^+, t^\beta))$. Therefore, the elements of the matrix operator $||A_{jk}||_{j,k=1}^2 = \nu A \nu^{-1}$ commute with the exactness of a compact. Then according to [1] the operator $||A_{jk}||_{j,k=1}^2$ is Noetherian in $L_p(R^+, t^\beta)$ if and only if the operator $\Delta = det||A_{jk}||$ is Noetherian in $L_p(R^+, t^\beta)$. But the operator $det||A_{jk}||$ is Noetherina if and only if $\gamma_M(det||(A_{jk})||) \neq 0$. As $\gamma_M(det||(A_{jk})||) = det||\gamma_M(A_{jk})||$, it follows that A is Noetherian, if and only if $det\tilde{\gamma}_M(A) \neq 0$. □

Conclusion 4.1. *Theorem 4.1 allows us to define the symbol on algebra K. Namely, the matrix $\tilde{\gamma}_M(A)$ is called the symbol of the operators $A \in K$. Then Theorem 4.1 can be formulated as follows.*

Theorem 4.2. *The operator $A \in K$ is Noetherian on the space $L_p(\Gamma, |t|^\beta)$ if and only if the determinant of its symbol is non-zero.*

Taking into account the results of the work [7], the symbol of the operators $H = hI$ $(h \in \lambda_0(\Gamma))$ and S_Γ has the form:

$$\tilde{\gamma}_M = \left\| \begin{matrix} h_1 & 0 \\ 0 & h_2 \end{matrix} \right\|, \ \tilde{\gamma}_M(S_\Gamma) = \left\| \begin{matrix} z & (z-1)^{1-\frac{\alpha}{2\pi}}(z+1)^{\frac{\alpha}{2\pi}} \\ (z-1)^{\frac{\alpha}{2\pi}}(z+1)^{1-\frac{\alpha}{2\pi}} & -z \end{matrix} \right\|.$$

The symbol of the operator S_Γ can be written in a more convenient form. For this we put

$$z = \frac{e^{2\pi(\xi+i\gamma)}+1}{e^{2\pi(\xi+i\gamma)}-1} = cth(\pi(\xi+i\gamma)) \ \left(-\infty \leq \xi \leq +\infty, \ \gamma = \frac{1+\beta}{p} \right).$$

Then

$$(z-1)^{1-\frac{\alpha}{2\pi}}(z+1)^{\frac{\alpha}{2\pi}} = 2\frac{e^{\alpha(\xi+i\gamma)}}{e^{\alpha(\xi+i\gamma)}-1} = \frac{e^{(\alpha-\pi)(\xi+i\gamma)}}{sh\pi(\xi+i\gamma)},$$
$$(z-1)^{\frac{\alpha}{2\pi}}(z+1)^{1-\frac{\alpha}{2\pi}} = 2\frac{e^{(2\pi-\alpha)(\xi+i\gamma)}}{e^{\alpha(\xi+i\gamma)}-1} = \frac{e^{(\pi-\alpha)(\xi+i\gamma)}}{sh\pi(\xi+i\gamma)}.$$

Therefore the symbol of the operator can be written in the form

$$\tilde{\gamma}_M(S_\Gamma) = \left\| \begin{array}{cc} \mathrm{cth}(\pi(\xi + i\gamma)) & \frac{e^{(\alpha-\pi)(\xi+i\gamma)}}{\mathrm{sh}\,\pi(\xi+i\gamma)} \\ \frac{e^{(\alpha-\pi)(\xi+i\gamma)}}{\mathrm{sh}\,\pi(\xi+i\gamma)} & -\,\mathrm{cth}\,\pi(\xi + i\gamma) \end{array} \right\|.$$

Remark 4.1. *Let $\alpha = \pi$, i.e. the contour Γ satisfies the Lyapunov conditions at the point $z_0 = 0$. Then the symbol of the operator $H = hI$ does not change and the symbol of the operator S_Γ has the form* [1], [11]

$$\tilde{\gamma}_M(S_\Gamma) = \left\| \begin{array}{cc} z & -\sqrt{z^2-1} \\ \sqrt{z^2-1} & -z \end{array} \right\| = \left\| \begin{array}{cc} \mathrm{cth}\,\pi(\xi + i\gamma) & -(\mathrm{sh}\,\pi(\xi + i\gamma))^{-1} \\ (\mathrm{sh}\,\pi(\xi + i\gamma))^{-1} & -\,\mathrm{cth}\,\pi(\xi + i\gamma) \end{array} \right\|.$$

Now we can define the symbol of singular integral operators with coefficients from $CP(\Gamma)$ in the case of a piecewise Lyapunov contour.

Let Γ be a piecewise closed Lyapunov contour. We denote by $t_1, ..., t_n$ all the corner points with angles $\alpha_k (0 < \alpha < \pi)(k = 1, ..., n)$ and

$$\rho(t) = \prod_{k=1}^{n} |t - t_k|^{\beta_k} \ \ (1 < p < \infty, \ -1 < \beta_k < p - 1).$$

We denote by $\sum(\Gamma, \rho)(\subset L(L_p(\Gamma, \rho))$ the algebra generated by operators $(H\varphi)(t) = h(t)\varphi(t), \ h(t) \in CP(\Gamma)$ and the operator S_Γ. We mention, that the ideal formed by compact operators acting on the space $L_p(\Gamma, \rho)$ is contained in the algebra $\sum(\Gamma, \rho)$. Let us define the symbol of the operator from $\sum(\Gamma, \rho)$. For this it is sufficient to define the symbol of the operator hI $(h \in CP(\Gamma))$ and the operator S_Γ. We will use the local principle of Simonenko [12]. The symbol $H(t, \xi)$ $(t \in \Gamma, \ \xi \in R)$ of the operator hI will be defined as follows:

$$H(t, \xi) = \left\| \begin{array}{cc} h(t+0) & 0 \\ 0 & h(t-0) \end{array} \right\|. \tag{13}$$

We define the symbol $S_\Gamma(t, \xi)$ of the operator S_Γ as follows:

$$S(t, \xi) = \left\| \begin{array}{cc} \mathrm{cth}\,\pi(\xi + i\gamma) & -\frac{\exp((\alpha(t)-\pi)(\xi+i\gamma(t)))}{sh\pi(\xi+i\gamma(t))} \\ \frac{\exp((\pi-\alpha(t))(\xi+i\gamma(t)))}{sh\pi(\xi+i\gamma)} & -\,\mathrm{cth}\,\pi(\xi + i\gamma(t)) \end{array} \right\|, \tag{14}$$

where

$$\alpha(t) = \begin{cases} \alpha_k, \ for \ t = t_k \ (k = 1, 2, ..., n), \\ \pi, \ for \ t \in \Gamma \setminus \{t_1, t_2, ..., t_n\}, \end{cases} \quad \text{and}$$

$$\gamma(t) = \begin{cases} \frac{1+\beta_k}{p}, \ for \ t = t_k \ (k = 1, 2, ..., n), \\ \frac{1}{p}, \ for \ t \in \Gamma \setminus \{t_1, t_2, ..., t_n\}. \end{cases}$$

Theorem 4.3. *Let $A \in \sum(\Gamma, \rho)$ and $A(t, \xi)$ be its symbol. The operator A is Noetherian on the space $L_p(\Gamma, \rho)$ if and only if*

$$detA(t, \xi) \neq 0 \ (t \in \Gamma, \ -\infty \leq \xi \leq +\infty). \tag{15}$$

5. Singular operators with a shift on a piecewise Lyapunov contour

Let Γ be a closed piecewise Lyapunov contour, $\nu : \Gamma \to \Gamma$ and $(V\varphi)(t) = \varphi(\nu(t))$. On the space $L_p(\Gamma)$, we consider a linear singular integral operator with a shift $\nu(t)$ of the form

$$(A\varphi)(t) = a(t)\varphi(t) + \frac{b(t)}{\pi i}\int_\Gamma \frac{\varphi(\tau)}{\tau - t}d\tau + c(t)\varphi(\nu(t)) + \frac{d(t)}{\pi i}\int_\Gamma \frac{\varphi(\nu(t))}{\tau - t}d\tau, \tag{16}$$

where $a(t), \ b(t), \ c(t)$ and $d(t)$ are bounded measurable functions on Γ. Assume that the mapping ν satisfies the following conditions:

(1) Carleman conditions: $\nu(\nu(t)) = t$;

(2) the derivative $\nu'(t)$ has a finite number of discontinuity points of the first kind on Γ , and on the arcs l_k connecting the discontinuity points it satisfies Hölder's condition, $\nu'(t) \in H(l_k)$;

(3) $\nu'(t \mp 0) \neq 0 \ (\nabla t \in \Gamma)$.

Along with the operator A of the form (16), we also consider the operator $\tilde{A}$ defined on the space $L_p^2(\Gamma) = L_p(\Gamma) \times L_p(\Gamma)$ by the equality

$$\tilde{A} = \begin{Vmatrix} aI + bS & cI + dS \\ \tilde{c}I + \tilde{d}S & \tilde{a}I + \varepsilon\tilde{b}S \end{Vmatrix} + \begin{Vmatrix} 0 & 0 \\ \tilde{d}(VSV - \varepsilon S) & \tilde{b}(VSV - \varepsilon S) \end{Vmatrix} = \tilde{A}_0 + R, \tag{17}$$

where $\tilde{f} = f(\nu(t))$ and $\varepsilon = 1$ ($\varepsilon = -1$), if the mapping ν preserves (changes) its orientation on the contour Γ. As it is known (see [4], [2] and the bibliography given in these papers), if a, b, c and d are continuous functions and $\nu'(t) \in H(\Gamma)$, then the operator is R completely continuous in $L_p(\Gamma)$ and

Theorem 5.1. *The operator A defined by equality (16) is Noetherian on the space $L_p(\Gamma)$ if and only if the operator $\tilde{A}_0$ is Noetherian in the space $L_p^2(\Gamma)$. When these conditions are fulfilled, the index of the operator A is calculated by formula*

$$IndA = \frac{1}{2}Ind\tilde{A}_0.$$

In this section, we prove that this assertion ceases to be true if Γ has corner points. In this case, as a rule, the derivative $\nu'(t)$ has discontinuity points on Γ, and it turns out that if the operator A is Noetherian, then the operator $\tilde{A}_0$ is also Noetherian. However, the converse assertion does not hold.

Theorem 5.2. *If operator $A = a(t)I + b(t)S + (c(t)I + d(t)S)V$ $(a, b, c, d \in C(\Gamma))$ is Noetherian on the space $L_p(\Gamma)$, then operator $\tilde{A}_0$ is also Noetherian on the space $L_p^2(\Gamma)$.*

Proof. Indeed, operator $\tilde{A}_0$ is Noetherian if and only if the conditions

$$\Delta_1(t) = (a(t) - b(t))(\tilde{a}(t) - \varepsilon\tilde{b}(t)) - (c(t) - d(t))(\tilde{c}(t) - \varepsilon d(t)) \neq 0,$$

$$\Delta_2(t) = (a(t) + b(t))(\tilde{a}(t) + \varepsilon\tilde{b}(t)) - (c(t) + d(t))(\tilde{c}(t) + \varepsilon d(t)) \neq 0,$$

hold for all $t \in \Gamma$. Let the operator A be Noetherian. Then the determinant of its symbol [1] does not vanish: $detA(t, \xi)$ $(t \in \Gamma, -\infty \leq \xi \leq +\infty)$. It is directly verified that

$$detA(t, -\infty) \cdot detA(t, -\infty) = \Delta_1(t) \cdot \Delta_2(t).$$

Hence the operator $\tilde{A}_0$ is Noetherian in $L_p^2(\Gamma)$. Theorem is proved. □

The following example shows that Theorem 5.1 cannot be inverted. Let us change the orientation of the contour Γ and the corner point $t_0 \in \Gamma$ with the angle θ $(0 < \theta < \pi)$ be a fixed point of the mapping $\nu : \nu(t_0) = t_0$. In this case, it is easy to verify that the derivative $\nu'(t)$ is discontinuous at the point t_0, and $\nu'(t_0 - 0) = \exp(-i\theta - \sigma)$ and $\nu'(t_0 + 0) = \exp(i\theta + \sigma)$, where σ is some real number. Consider the operator

$$A = I + \delta SV,$$

where δ is a complex number. The corresponding operator $\tilde{A}$ has the form

$$\tilde{A} = \begin{Vmatrix} I & \delta S \\ -\delta S & I \end{Vmatrix} + \begin{Vmatrix} 0 & 0 \\ \delta(VSV - S) & 0 \end{Vmatrix} = \tilde{A}_0 + R.$$

If $\delta \neq \pm i$, then the operator $\tilde{A}_0$ is Noetherian. Let $A(t_0, \xi)$ $(-\infty \leq \xi \leq +\infty)$ be the symbol of the operator A at the point t_0. It is directly verified that

$$detA(t_0, \xi) = \delta^2 + 2(\gamma + \beta)\delta + 1,$$

where

$$\gamma = \frac{\exp[(2\pi - \theta - i\sigma)(\xi + i/p)]}{\exp(\xi + i/p) - 1} \text{ and } \beta = \frac{\exp[(\theta + i\sigma)(\xi + i/p)]}{\exp(\xi + i/p) - 1}.$$

Hence, by virtue of Theorem 1.1 from [5], it follows that for all $\delta = -(\gamma + \beta)^{-1} \pm \sqrt{(\gamma + \beta)^2 - 1}$ the operator A is not Noetherian on the space $L_p(\Gamma)$. Thus, the condition for the operator A to be Noetherian depends on the angle θ.

Theorems 5.1 and 5.2, and the above example imply the following assertions

Corollary 5.1. *Let $\nu'(t) \notin H(\Gamma)$. Then operator $VSV - \varepsilon S$ is not compact on the space $L_p(\Gamma)$.*

Corollary 5.2. *If the operator A, defined by equality (16), is Noetherian, then the operators $\tilde{A}$ and $\tilde{A}_0$ defined by equality (17) are also Noetherian.*

Corollary 5.3. *If the operator $\tilde{A}$ is Noetherian, then $\tilde{A}_0$ is also Noetherian. The converse is generally not true.*

In conclusion, we note that the corresponding example of a non-Noetherian operator A for which $\tilde{A}_0$ is Noetherian can also be given in the case when the mapping ν preserves the orientation of the contour Γ.

6. Examples

The symbol of the singular integral operators and Theorem 2.1 can be used in studying different classes of composite singular operators. The difficulties which can arise in this context are the following: to show that the operator under consideration belongs to an algebra of operators with symbol; to write in the explicit form the symbol of this operator; to show that the symbol can be expressed as a singular perturbed operator that satisfies the conditions of Theorem 2.1.

We will consider an example where these difficulties arise and are overcome. Studying singular integral operators with homographic translations on the real axis in the space

$$L_p^\gamma = \left\{\varphi : \int_{-\infty}^{+\infty} |\varphi(x)|^p |x-\delta|^\gamma dx < \infty\right\} \;\; (-1 < \gamma < p-1,\; \delta \in R),$$

operators of the following form are investigated (see [12])

$$H\varphi = a(x) + \frac{b(x)}{\pi i}\int_{-\infty}^{+\infty} \frac{\varphi(t)}{t-x}dt + c(x)\frac{(x-\delta)^\lambda}{\pi i}\int_{-\infty}^{+\infty} \frac{(t-\delta)^{-\lambda}\varphi(t)}{t-x}dt,$$
$$-\frac{1+\gamma}{p} < \lambda < 1 - \frac{1+\gamma}{p}. \tag{18}$$

To apply the conditions of Theorem 2.1, we will express the operator H as follows

$$H = aI + bS + cM, \tag{19}$$

where

$$S\varphi = \int_{-\infty}^{+\infty} \frac{\varphi(t)}{t-x}dt$$

and

$$M\varphi = \frac{(x-\delta)^\lambda}{\pi i}\int_{-\infty}^{+\infty} \frac{(t-\delta)^{-\lambda}\varphi(t)}{t-x}dt.$$

The expression (19) implies the operator H to be a singular integral operator perturbed with the operator cM.

Theorem 6.1. *Let $a, b, c \in C(\bar{R})$. The operator $H = aI + bS + cM$ is Noetherian in the space L_p^γ if and only if the operator $H_0 = aI + bS$ is Noetherian. Moreover, $Ind\, H = Ind\, H_0$.*

Thus, the operator cM is a permissible perturbation to the operator H_0 and as a result of this perturbation his index remains the same.

Proof. Denote by $H(x,\mu)$, respectively by $H_0(x,\mu)$ the symbols of operators H and H_0. The symbol of the operator M at the point $x = \delta$ has the form

$$M(\delta,\mu) = \left\| \begin{matrix} 0 & u(\mu) \\ 0 & 0 \end{matrix} \right\|, \tag{20}$$

where

$$u(\mu) = \frac{4ih(\mu)\sin\pi\lambda \cdot \exp(\pi i\lambda)}{2if(\mu)\sin\pi\lambda \cdot \exp(\pi i\lambda) + 1},$$

$$f(\mu) = \begin{cases} \frac{\sin\theta\mu\cdot\exp(i\theta\mu)}{\sin\theta\cdot\exp(i\theta)}, & \text{for } \theta \neq 0, \\ \mu, & \text{for } \theta = 0, \end{cases} \qquad \theta = \pi - \frac{2\pi(1+\gamma)}{p}$$

and

$$h(\mu) = \sqrt{f(\mu)(1 - f(\mu))},\ 0 \le \mu \le 1.$$

Obviously, the operator M has singularities only at the points $x = \delta$ and $x = \infty$, therefore it is equivalent to the null operator at the points $x \in \bar{R} \setminus \{\delta, \infty\}$. Thus

$$M(x,\mu) = \left\| \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \right\|,\ x \neq \delta,\ x \neq \infty.$$

To calculate the symbol of M at the point $x = \infty$, we proceed as follows. We consider the linear and bounded operator $A : L_p(R, |x-\delta|^\gamma) \to L_p(R, |x-\delta|^{p-2-\gamma})$ defined by the relation

$$(A\varphi)(t) = \frac{1}{t}\varphi\left(\frac{\delta t - 1}{t}\right)\left((A^{-1}\psi)(x) = \frac{1}{x-\delta}\varphi\left(-\frac{1}{x-\delta}\right)\right).$$

The symbol of the operator M at the point $x = \infty$ is defined as the symbol of the operator AMA^{-1} at the point $x = 0$. We calculate AMA^{-1}. Let $f(x) = (x-\delta)^\lambda$, then

$$AfA^{-1} = \left(-\frac{1}{t}\right)^\lambda = t^{-\lambda}e^{\pi i\lambda}I,$$

$$(AfA^{-1}\varphi)(t) = -\frac{1}{t}\cdot\frac{1}{\pi i}\int_{-\infty}^{+\infty}\frac{\frac{1}{x-\delta}\varphi\left(\frac{1}{x-\delta}\right)}{x - \frac{\delta t-1}{t}}dx = -\frac{1}{t}\cdot\frac{1}{\pi i}\int_{-\infty}^{+\infty}\frac{\tau\varphi(\tau)}{\frac{\delta\tau-1}{\tau}\,\frac{\delta t-1}{\tau}}\frac{d\tau}{\tau^2}.$$

$$(AfA^{-1}\varphi)(t) = (S\varphi)(t).$$

Thus, $AMA^{-1} = t^{-\lambda}St^{\lambda}I$ and λ verifies the condition

$$\frac{1+p-\gamma-2}{p} < \lambda < 1 - \frac{1+p-\gamma-2}{p}.$$

Therefore, the symbol of the operator M at the point $x = \infty$ has the form (20), where λ must be replaced by $-\lambda$, and θ by $-\theta$. This is equivalent to the fact that the function $u(\mu)$ from (20) is replaced by $\overline{u(\mu)}$. Thus, the symbol of the operator (18) has the form

$$H(x,\mu) = \left\| \begin{array}{cc} a(x)+b(x) & c(x)\sigma(x,\mu) \\ 0 & a(x)-b(x) \end{array} \right\|,$$

where $\sigma(x,\mu) = 0$ for any $x \in \bar{R} \setminus \{\delta, \infty\}$, $\sigma(\delta,\mu) = u(x)$ and $\sigma(\infty,\mu) = \overline{u(\mu)}$. It follows from this that $detH(x,\mu) = detH_0(x,\mu)$ which means that both operators H and H_0 are or are not Noetherians and $Ind\, H = Ind\, H_0$. □

Remark 6.1. *The statements of Theorem 6.1 remain to be true even if the functions are replaced by matrices functions with elements from $C(\bar{R})$.*

Conclusion 6.1. *The results presented in this paper show us that the property of singular integral operators to be Noetherian is stable with respect to their perturbation with certain noncompact operators. This property was established due to the symbol of some operators with singularities was determined. It was shown that the determinant of the symbol of the original operator coincides with the determinant of the symbol of the perturbed operator. Moreover, the indices of these operators are also equal.*

References

[1] Dovgiy, S., Lifanov, D., Cherniy, D. *Method of singular integral equations and computational technologies*. "Iustin" publishing house, Kyiv, 2016.

[2] Gokhberg, I., Krupnik, N. On one-dimensional singular integral operators with a shift. *Izv. Akad. Nauk Arm. SSR, Mat.* 1973, no. 1, 3–12 (in Russian).

[3] Karapetiants, N., Samko, S. *Equations with involutive operators.* Birkhäuser, Boston, 2012.

[4] Kravchenko, V., Litvinchiuk, G. *Introduction to the Theory of Singular Integral Operators with Shift*. Kliuwer, 2012.

[5] Krupnik, N. *Banach Algebras with Symbol and Singular integral Operators*. Operator Theory, 26, Birkhäuser Verlag, Basel, 1992.

[6] Krupnik, N., Neagu, V. On singular operators with shift in the case of piecewise Lyapunov contour. *Soobsch. Akad. Nauk Gruz. SSR*, 1974, vol. 76, 25–28 (in Russian).

[7] Krupnik, N., Neagu, V. Singular integral operators with shift along piecewise Lyapunov contour. *Izv. Vyssh. Ucebn. Zaved., Mat.*, 1975, no. 6, 60–72 (in Russian).

[8] Kvedelidze, B. The merthod of Cauchy typ integrals in discontinuous boundary value problems of theory of holomorphie functions of a complex variable. *Itogi Nauki I Tekhniki, Ser. Sovrem. Probl. Mat.*, 1975, vol. 7, 5–162 (in Russian).

[9] Neagu, V. Some general question of the theory of singular operators in the case of piece-wise Lyapunov contour. *Revue analyse numérique de approximation*, 2000, vol. XXIX, no. 1, 57–73.

[10] Neagu, V. *Noetherian operators with applications*. Editura UST, Chișinău, 2015.

[11] Soldatov, A. Singular integral operators and elliptic boundary value problems. *Modern mathematics. Fundamental Directions*, 2017, vol. 63, no. 1, 1–189 (in Russian).

[12] Simonenko, I. A new general method of investigation linear operator equation of singular integral equation type. *Izv. Akad. Nauk. SSSR*, 1968, vol. 32, no. 6, 1138–1145 (in Russian).

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