

Dedicated to Professor Alexandru Şubă on the occasion of his 70th birthday

The method for solving the multi-criteria linear-fractional optimization problem in integers

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Abstract. In the paper we propose a method for solving the linear-fractional multi-criteria optimization model with identical denominators in whole numbers. Such models are in increasing demand, especially from an application point of view. The solving procedure of these models initially involves assigning utilities (weights) to each criterion [15] and building the optimization model with a single criterion, which is a synthetic function of all criteria weighted. It was found that the optimal solution of the model does not depend on the values optimum of the criteria obtained in R^+ or in Z^+ . So, the decision maker can combinatorially select the types of optimal values of criteria, a fact that represents the essential priority of the algorithm. By changing the utility values, at the decision maker's discretion, we will obtain a new optimal compromise solution of the model. Theoretical justification of the algorithm as well as a solved example are brought to work.

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Metodă de soluționare a problemei de optimizare multicriterială de tip liniar-fracționar în numere întregi

Rezumat. În actuala lucrare propunem o metodă de rezolvare a modelului de optimizare multicriterial de tip liniar-fracționar cu numitori identici în numere întregi. Acest tip de modele înregistrează o solicitare practică în creștere. Procedura de soluționare a modelului presupune atribuirea inițială a unor utilități (ponderi) fiecărui criteriu [15], apoi se construiește modelul de optimizare de tip liniar-fracționar în numere întregi cu un singur criteriu, care este o funcție sinteză a criteriilor ponderate. S-a dovedit că soluția de compromis optim a modelului nu depinde de tipul soluțiilor optime a fiecărui criteriu real sau întreg pentru funcția sinteză, astfel fiind posibilă selectarea combinatorială a acestora, iar modificând utilitățile, obținem o nouă soluție a modelului. Justificarea teoretică a algoritmului, cât și un exemplu rezolvat se aduc în lucrare.

Cuvinte-cheie: model multicriterial în numere întregi, soluție eficientă de bază, soluție de compromis optimal

1. INTRODUCTION

There is currently a growing demand for solving integer optimization problems. This happens because many decision situations require solving only in whole numbers [16]. Of course, this condition requires increased efforts to solve the optimization problem. Among the practical domains, where are needed optimal integer solutions, there is the problem of bi and three-dimensional cutting of materials [8], [9], [17]. A number of studies of this type have been done to solve the problem of dynamic memory allocation for multiprocessor and positioning systems. Several researchers have proposed various studies on this topic (Dowland and Dowland 1992, Sweeny and Paternoster 1992, Dyckhoff 1990, Coffman 1984, Golden 1976, Gilmore 1966). All approaches of these researchers can be divided into 3 categories: precise, heuristic and metaheuristic. The exact methods were investigated by Gilmore and Gomory (1961) and are considered the first methods actually applied in the tailoring industry. The fundamental drawback of these approaches is their inability to effectively solve the problems of large-dimensions. However, this effort increases significantly when the problem is multicriteria in nature, even if it is of linear type [5], [6], [7]. The requirement that the choice variables be of integer type increases the problem's complexity and the length of the solving time [1], [2]. That is why, the interest in this fertile field of scientific research remains opening further [10], [11], [12]. From a practical point of view, there is an increased interest for the multicriterial optimization models of linear-fractional type in whole numbers, a fact that intensified my research on these types of issues. Next, I will propose a study specifically dedicated to this type of models.

2. DEFINING THE PROBLEM WITH SPECIFIC REASONING

The integer multicriteria linear optimization problem is typically represented by a collection of linear constraints, including on the variables restrictions of non-negativity and integrity, such as equations and/or inequalities. The mathematical model of this type of problem [16] is as follows:

$$\begin{cases} \text{optimum}\{F_k(x)\}, k = \overline{1, r} \\ x \in D \\ D = \text{the field of the admissible solutions} \end{cases} \quad (1)$$

in which: $D = \{x = (x_1, x_2, \dots, x_n)^T \mid Ax \leq b, x \in Z^+\}$.

The explicit form of model (1), in which the objective functions have the same denominator, being of linear-fractional type, is the following:

$$\left\{ \begin{array}{l} \left\{ \begin{array}{l} \min \\ \max \end{array} \right\} F_k(x) = \frac{\sum_{j=1}^n c_{kj}x_j}{\sum_{j=1}^n d_jx_j} \\ Ax \leq b \\ x \in Z^+ \end{array} \right. \quad (2)$$

in which: $D = \{x = (x_1, x_2, \dots, x_n)^T \mid Ax \leq b, x \in Z^+\}$, $A = \|a_{ij}\|$ is an array of size $m \times n$ ($m < n$), $C = \|c_{ij}\|$ is an array of size $r \times n$ ($r < n$), $d-$ is a n -dimensional line vector, x is a n -dimensional column vector and b is a m -dimensional column vector.

The parameters c_{kj} , as well as d_j , may be the most different, according of their practical meanings such as unit costs or benefits, unit of damages or others close in this meaning. The type of related objective function, minimum or maximum, is determined by their relevance. Similar to how the elements of the vector b indicate the resources available by types, the elements of the matrix A , $\{a_{ij}\}$, represent the specific consumption of the resource j for the creation of a product unit of type i .

In order to solve the model (2) obviously, the value of the denominator function, which is the same for all criteria, must be nonzero on the domain D , that is the following condition must be satisfied:

$$\sum_{j=1}^n d_jx_j \neq 0, (\forall)x \in D.$$

It should be noted, that if in model (2) there are criteria of both minimum and maximum type, it is not complicated to homogenize them, if necessary.

Unfortunately, it is well known that the multicriteria optimization model rarely admits an optimal solution. That's why, in order to solve the multicriteria model, the notion of a solution that achieves the best compromise, solution of the optimal compromise, non-dominant solution, efficient solution, optimal solution in the Pareto sense, etc. is used. In [13] different ways of defining the vector solution x^* of the best compromise for the real-type multicriteria optimization model are proposed. We will adapt some of them to solve the integer multicriteria linear-fractional optimization model (2) as follows.

1. The solution $x^* \in Z^+$ for the model (1) is the vector that optimizes a synthesis function of all r criteria, ie: $h(F) = h[F_1, F_2, \dots, F_r]$. We mention that $h(\cdot)$ can be defined in various ways.

2. The solution $x^* \in Z^+$ is the vector which minimizes a single criterion such as: $\phi(x^*) = \min_{x \in D} h(\Psi_1(x-X_1), \dots, \Psi_1(x-X_r))$, in which $X_j = (x_{1j}, x_{2j}, \dots, x_{nj})^T, j = \overline{1, r}$

is the optimal solution of each criterion, F_j , and Ψ_k is a distance type function between the vector $x \in D$ and optimal solution X_k for each criterion F_k .

3. The solution $x^* \in Z^+$ is a vector which belongs to a set of efficient solutions of integer type. Because the model (1) is of multicriteria type, it is well known, that such of models in general rarely admit optimal solutions. Solving model (1) involves constructing a finite set of efficient integer solutions known as best compromise solutions [13], which we mentioned earlier. For the model (2) we will adapt the next definitions.

Definition 2.1. *The basic solution x^* of the model (2), where $x^* \in Z^+$, is called optimal overall if it is the optimal solution for each of criteria.*

Definition 2.2. *The basic solution \bar{X} , where $\bar{X} \in Z^+$ of the model (2) is a basic efficient one if and only if it doesn't exists any other basic solution $X \in Z^+$, where $X \neq \bar{X}$, which would improve the values of all criteria and at least one of criteria would be strictly improved.*

The more exact, mathematical version of the same definition is proposed below.

Definition 2.3. *The basic solution $\bar{X} \in Z^+$ of the model (2) is a basic efficient one if and only if for any other basic solution $X \in Z^+$, where $X \neq \bar{X}$, for which the relations $F_{j_1}(X) \geq F_{j_1}(\bar{X})$ are true, where $j_1 \in (1, \dots, j_2)$, indices corresponding to the maximum type of criteria immediately exists at least one index $\exists j_l \in (j_2 + 1, \dots, r)$, of the minimum type for which is true the relation: $F_{j_l}(X) > F_{j_l}(\bar{X})$ or, if the relation $F_{j_l}(X) \leq F_{j_l}(\bar{X})$ is true for all indices corresponding to the minimum type of criteria which are $j_l \in (j_2 + 1, \dots, r)$, immediately exists at least one index from the set of indices of the maximum type of criteria $\exists j_1 \in (1, \dots, j_2)$, for which the next relation $F_{j_2}(\bar{X}) < F_{j_2}(X)$ is true.*

3. SECTION PLANS METHOD

In order to iteratively improve the integer solution of the optimization model, the section plans approach involves a sectioning procedure for the domain of admissible solutions. Sections are executed in accordance with predetermined rules. The section plans algorithm is often referred to as the "Cyclic Algorithm". The algorithm iteratively modifies one of the components of the admissible solution of the optimization problem, cutting each time the admissible domain, so that the new obtained solution remains admissible. Of course, at each iteration the value of the objective function changes in the direction opposite to the criterion type. Since the algorithm is convergent and finite, after a finite number of steps it determines the optimal integer solution of the model, if it exists. Despite the fact that the convergence of this algorithm has not been proven, no examples have been found that contradict it. This algorithm is also known as Gomory's

algorithm [4], in honor of the American scientist, R. Gomory, who created it and published the method for the first time in 1958. We will describe further, applying mathematical formulas the sectioning process adapted for the linear-fractional optimization problem. Next we will consider the following couple of optimization problems:

$$(ILP) \left\{ \begin{array}{l} (\max) f = \frac{cx + c_0}{dx + d_0} \\ Ax = b \\ x \geq 0 \\ x \in Z^+ \end{array} \right. \quad (LP) \left\{ \begin{array}{l} (\max) f = \frac{cx + c_0}{dx + d_0} \\ Ax = b \\ x \geq 0 \end{array} \right.$$

where the elements of the matrix A and the components of the vector b, c, d , and the constants c_0, d_0 , all are of integer type.

We denote $D_0 = \{x | A \cdot x = b, x \in Z^+\}$ and $D = \{x | A \cdot x = b, x \geq 0\}$, where D_0 is the domain of admissible solutions of the problem (ILP) and D of the problem (LP), respectively. We will assume that the function at the denominator is different from zero in D , which appears like this: $dx + d_0 \neq 0, (\forall)x \in D$.

Algorithm stages with theoretical justifications

We'll start off assuming that x^* doesn't have all of the integer components. In this instance, a constraint of the fractional optimum x^* is constructed; nonetheless, it is satisfied by any admissible solution of whole type. It is added to the original problem noted with (LP_0) , after which the optimal solution will be re-optimized. Let x^{**} be the optimal solution to the new constructed problem, denoted by (LP_1) . Because of the way the additive constraint was defined, we will have the following true relationships between admissible domains: $D_{ILP} \subset D_{LP_1} \subset D_{LP_0} = D_{LP}$.

If x^{**} does not have all components of integer type, the described procedure is repeated: a new restriction is constructed, which is not satisfied by x^{**} , but is verified by the set of admissible solutions. This new restriction is added to (LP_1) , resulting a new linear optimization problem (LP_2) . The sectioning procedure is as follows: $D_{ILP} \subset D_{LP_2} \subset D_{LP_1} \subset D_{LP_0} = D_{LP}$.

After applying the reoptimization procedure of the new admissible solution, it is decided whether LP_2 admit or not optimal solution. The theory guarantees that, after a finite number of steps, we obtain a linear-fractional programming problem, (LP_{k-1}) , whose optimal solution is $x^{k(*)}$, which has all integer components, hence it is the optimal solution of our proposed problem (ILP).

Geometrically, each new added constraint removes some part of the set of admissible solutions, thus cutting off an intrusive section of the entire admissible domain. Next, we

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will describe the admissible domain partitioning algorithm proposed in [16], adapted for the linear-fractional optimization problem by imposing additional partitioning restrictions. We shall take into consideration the vector \bar{b} and matrix \bar{A} , which correspond to the optimal solution of model (LP). We'll assume that the vector \bar{b} doesn't have all integer components. Let the vector component \bar{b} with the largest fractional part be \bar{b}_r .

We can represent the constraint coefficients as follows:

$$\bar{b}_r = x_r + \sum_{j \in J} \bar{a}_{rj} x_j \quad (3)$$

which can be decomposed thus:

$$\lceil \bar{b}_r \rceil + \{ \bar{b}_r \} = x_r + \sum_{j \in J} (\lceil \bar{a}_{rj} \rceil + \{ \bar{a}_{rj} \}) x_j. \quad (4)$$

Because we have true the relationship $0 < \{ \bar{b}_r \} < 1$, the following equality is also true:

$$\lceil \bar{b}_r \rceil - \sum_{j \in J} \lceil \bar{a}_{rj} \rceil x_j - x_r = \sum_{j \in J} \{ \bar{a}_{rj} \} x_j - \{ \bar{b}_r \} \quad (5)$$

Let x be a whole admissible solution of the problem (ILP). Therefore, the left-hand member of the relation (5) is an integer, so we get the following relation true:

$$\lceil \bar{b}_r \rceil - \sum_{j \in J} \lceil \bar{a}_{rj} \rceil x_j - x_r \in Z \quad (6)$$

It follows that the right-hand side of equality (5), which is calculated in the same solution, is an integer, so we have true the next relationship:

$$\sum_{j \in J} \{ \bar{a}_{rj} \} x_j - \{ \bar{b}_r \} \in Z. \quad (7)$$

Obviously, the next real relation is true:

$$\sum_{j \in J} \{ \bar{a}_{rj} \} x_j - \{ \bar{b}_r \} \geq 0. \quad (8)$$

If, however, by absurdity, we assume that:

$$\sum_{j \in J} \{ \bar{a}_{rj} \} x_j - \{ \bar{b}_r \} < 0, \quad (9)$$

then from the equality (5) yields the following true inequality:

$$\lceil \bar{b}_r \rceil - \sum_{j \in J} \lceil \bar{a}_{rj} \rceil x_j - x_r < 0 \quad (10)$$

and from (6) results the true expression:

$$\lceil \bar{b}_r \rceil - \sum_{j \in J} \lceil \bar{a}_{rj} \rceil x_j - x_r \leq -1. \quad (11)$$

From (5), we will obtain the next relations: $\sum_{j \in J} \{\overline{a_{rj}}\} x_j - \{\overline{b_r}\} \leq -1$, whence it follows:

$$\sum_{j \in J} \{\overline{a_{rj}}\} x_j \leq \{\overline{b_r}\} - 1 \quad (12)$$

Since the relationship is true: $\{\overline{a_{rj}}\} \geq 0, (\forall) j \in J$, we get that the left member of the relationship (5) is also positive, while the right limb is negative, since $\{\overline{b_r}\} < 1$. The obtained contradiction proves that the inequality (8) is true. Because x has been chosen arbitrarily, we conclude that the next restriction is also true:

$$\sum_{j \in J} \{\overline{a_{rj}}\} x_j \geq \{\overline{b_r}\} \quad (13)$$

and that this is verified by any admissible solution of integer type.

But in the optimal solution, which is not of integer type x^* , we will get: $x_j^* = 0, j \in J$. Inserting these values into (8), we obtain the inequality: $\{\overline{b_r}\} \leq 0$, which contradicts the hypothesis (4), according to which we had: $\{\overline{b_r}\} > 0$. So, it turns out that the fractional optimum x^* does not verify the inequality (10). Adding this constraint to model (10), we obtain a new linear optimization problem with $(m + 1)$ constraints, which we denote by (LP_1) . By introducing a new deviation variable x_{n+1} , we will transform the added constraint into equality, after which we will apply the re-optimization procedure of the model solution. The new restriction introduced in (10): $-\sum_{j \in J} \{\overline{a_{rj}}\} x_j + x_{n+1} = -\{\overline{b_r}\}$, is considered a *plane sectioning restriction*.

4. METHOD OF MAXIMIZING GLOBAL UTILITY

The global utility maximization method is known as the method of Boldur Latescu, a Romanian researcher, who developed it, as mentioned in [13]. It is based on the idea of transforming of the objective functions of a multicriteria problem into utility functions in the sense of von Neumann—Morgenstern [3], which are to be summed to obtain a synthesis function. In the hypothesis of the existence of a multicriteria linear programming problem, this method can be used quite effectively even in the case of an infinite number of decision variants. We will extend the method for the case when the objective functions are of linear fractional type with the same denominator.

Definition 4.1. *Utility is a subjective amount of appreciation of the event by the decision maker on a certain scale of values depending on the specifics of the event [3].*

Definition 4.2. Given n criteria C_1, C_2, \dots, C_n , they are called mutually independent in the sense of the theory of utility, if and only if we have the true relation: $\omega_i \sim \omega_j$ for anything $(\omega_i, \omega_j) \in G$, where G is the events space.

Since the additivity of utilities is obviously possible, we will have true the following relationship:

$$U(a_{i1}, a_{i2}, \dots, a_{in}) = u_1(a_{i1}) + u_2(a_{i2}) + \dots + u_n(a_{in}).$$

The independence of the criteria in the sense of the utility theory specifies that any consequence of the possible decision variant of a criterion always corresponds to the same a priori assigned utility.

Global utility maximization algorithm

We will present the algorithm of the global utility maximization method [13], considering the case of the linear multicriteria optimization problem (1).

Step 1. We will consider for each objective function its optimal value X_j , which is determined, where $F_j = \text{optim}_{x \in D} F_j(x)$ and Y_j is its pessimistic value, where $F_j^P = \text{pessim}_{x \in D} F_j(x)$. We note that in these cases we will solve fractional linear models, applying the adapted simplex algorithm [14].

Step 2. For all sets of optimal and worst values of the criteria, the corresponding values of utilities in the sense Neumann–Morgenstern [13] are associated as follows:

$$\{F_1, F_2, \dots, F_r; F_1^P, F_2^P, \dots, F_r^P\} \longrightarrow \{U_1, U_2, \dots, U_r; U_{r+1}, U_{r+2}, \dots, U_{2r}\}.$$

Step 3. The objective functions F_j are presented as utility functions FU_j , by solving r linear systems with $2r$ variables. The unknowns in these equations are the coefficients of the type: $\{(\alpha_j, \beta_j)\}_{j=\overline{1, r}}$.

Using the solutions of the r systems of linear equations of this type:

$$\begin{cases} \alpha_j F_j + \beta_j = u_j \\ \alpha_j F_j^P + \beta_j = u_{j+r} \end{cases}, \quad j = \overline{1, r},$$

we will build the following r utility functions such as:

$$FU_j = \alpha_j F_j(X) + \beta_j, \quad j = \overline{1, r}.$$

Step 4. At the final stage, we will solve a single problem of linear programming whose objective is to maximize the global utility function UG , which is as follows:

$$\max_{x \in D} UG = \max_{x \in D} \sum_{j=1}^r \pi_j FU_j,$$

where π_j is the weight coefficient of the criterion C_j , which, obviously, can be changed by the decision maker, thus obtaining another, a new linear optimization problem.

5. THE GENERALIZED SYNTHESIS ALGORITHM

A rather important problem, which we obviously face when solving the multicriteria optimization problem in integers, using the methods of synthesis functions, as mentioned in [16], is formulated as follow: what type of optimal solution of each criterion must be used to construct the synthesis function of all criteria, in R^+ or in Z^+ , so that the final model efficiently solves the problem in Z^+ ?

In this paragraph we will answer and justify the answer to this question. We will adjust the global utility maximization method for the objective functions of the linear-fractional criteria, in order to use it in solving the proposed model (2). The algorithm will be performed in two stages.

Stage I:

1. At this stage it is necessary to solve $2r$ unicriteria linear fractional programming problem from model (2), of which r are of the type: $F_j = \text{optim}_{x \in D} F_j(x)$ and the other r of the type: $F_j^P = \text{pessim}_{x \in D} F_j(x)$ on the same admissible domain:

$$D = \{x \in R | Ax \leq b, x \geq 0\};$$

2. Next, we will analogically solve $2r$ linear fractional programming problems of integer type as follows, the first r of the type: $F_j = \text{optim}_{x \in D} F_j(x)$ and the others r of the type: $F_j^P = \text{pessim}_{x \in D} F_j(x)$ all on the domain: $D = \{x \in Z^+ | Ax \leq b, x \geq 0\}$;

3. We will build the vectors of records of the optimal values of the objective functions, using in each combinatorial vector both values of some criteria in Z^+ and of others in R^+ . Analogously, we will build the vectors of the worst records of the objective functions in R^+ and Z^+ . Since the size of the optimization problem is finite, it follows that the number of such combinations is also finite. These combinations can be described as follows:

$$\left\{ \left(\begin{array}{c} F_1(R^+) \\ F_2(R^+) \\ \dots \\ F_r(R^+) \end{array} \right) \vee \left(\begin{array}{c} F_1(Z^+) \\ F_2(Z^+) \\ \dots \\ F_r(Z^+) \end{array} \right) \vee \left(\begin{array}{c} F_1(R^+) \\ F_2(R^+) \\ \dots \\ F_r(Z^+) \end{array} \right) \vee \dots \vee \left(\begin{array}{c} F_1(Z^+) \\ F_2(Z^+) \\ \dots \\ F_r(Z^+) \end{array} \right) \right\},$$

$$\left\{ \left(\begin{array}{c} F_1^P(R^+) \\ F_2^P(R^+) \\ \dots \\ F_r^P(R^+) \end{array} \right) \vee \left(\begin{array}{c} F_1^P(Z^+) \\ F_2^P(Z^+) \\ \dots \\ F_r^P(Z^+) \end{array} \right) \vee \left(\begin{array}{c} F_1^P(R^+) \\ F_2^P(R^+) \\ \dots \\ F_r^P(Z^+) \end{array} \right) \vee \dots \vee \left(\begin{array}{c} F_1^P(Z^+) \\ F_2^P(Z^+) \\ \dots \\ F_r^P(Z^+) \end{array} \right) \right\}.$$

The total number of such vectors is: $N(V) = C_r^1 + C_r^2 + \dots + C_r^r$, analogously, the same number is for pessim type of criteria.

Stage II:

1. Randomly considering one of the vector records of the optimal values of the objective functions and correspondingly the vector of records of the worst values, we construct the synthesis function of the model, which expresses the summary utility of all criteria thus: $G = \sum_{j=1}^r (\alpha_j F_j + \beta_j)$, which is obviously to be maximized. The utility coefficients $\{(\alpha_j, \beta_j)\}_{j=1, \dots, r}$ are determined by solving the systems of equations for each criterion, as previously described. Finally, we will solve the following model on linear programming:

$$\max_{x \in D} G = \sum_{j=1}^r (\alpha_j F_j (X) + \beta_j),$$

where $D = \{x | Ax = b, x \in Z^+\}$. The optimal solution of this problem is the optimal compromise solution for model (2). By calculating the values of each objective function of the model (2) in the obtained optimal solution we will construct the following vector of records of all objective functions:

$$\begin{pmatrix} F_1(X^*) \\ F_2(X^*) \\ \dots \\ F_r(X^*) \end{pmatrix}.$$

Theorem 5.1. *For any utility values assigned a priori to the objective functions in model (2), where the identical denominator is nonzero over the admissible domain, the optimal compromise solution corresponding to them remains the same for any combinatorial selection of the optimal values of the criteria and the corresponding pessimistic ones from in R^+ or in Z^+ .*

Proof. Let X_{eff}^1 be a solution of the optimal compromise for the model (2) of integer type, for a given a priori set of utilities, obtained by applying the global utility maximization method. Because the solution is of the optimal compromise, it turns out that it is the closest located to the optimal solutions of the whole type of each criterion. We will assume that the synthesis function of the model, which generated the given solution, was obtained using a certain combination of optimal and pessimistic values of the objective functions of the model (2), some being solved in R^+ , others in Z^+ .

Let $\begin{pmatrix} F_1(R^+) \\ F_2(R^+) \\ \dots \\ F_r(Z^+) \end{pmatrix}$ be the vector of optimal and correspondingly pessimistic $\begin{pmatrix} F_1^P(R^+) \\ F_2^P(R^+) \\ \dots \\ F_r^P(Z^+) \end{pmatrix}$ objective functions values. We will admit, analogously to the demonstration in [16], that

for another record values, different from the previous one, of the values of the objective functions of model (2), let it be

$$\begin{pmatrix} F_1(Z^+) \\ F_2(R^+) \\ \dots \\ F_r(Z^+) \end{pmatrix} \text{ and corresponding vector of the pessimistic values } \begin{pmatrix} F_1^p(Z^+) \\ F_2^p(R^+) \\ \dots \\ F_r^p(Z^+) \end{pmatrix}, \text{ the ob-}$$

jective synthesis function admits another integer-optimal compromise solution that is different from the first one, either it is: X_{eff}^2 . If $X_{eff}^1 \neq X_{eff}^2$, there is at least one coordinate by which these solution vectors differ from each other. So, we will have at least one criterion of model (2), be it having indices i_1 , for which the distance between its optimal solution in integers and the new solution is smaller than the previously received one. Therefore, we will have the following true relation: $\rho(X_{eff}^1, X_{i_1}^*) > \rho(X_{eff}^2, X_{i_1}^*)$, where $X_{i_1}^*$ is optimal solution in integer of criterion i_1 , fact that contradicts the assumption that X_{eff}^1 is a optimal compromise solution of integer type for the model (2). Therefore, our assumption is wrong. So, in conclusion, we obtained, that model (2) admits a single optimal compromise solution in integers, regardless of the type of optimal values of the objective functions of the model solved in R^+ or Z^+ , which is used to build the synthesis function of the proposed model.

Remark 5.1. *Obviously, for any values of the a priori utilities assigned to the criteria in the multicriteria optimization model (2), applying the global utility maximization method, we will obtain the optimal compromise solution of integer type corresponding to them.*

6. CONCLUSIONS

Multicriteria optimization models have always enjoyed increased interest. This trend is maintained even today, especially due to the fact that they more adequately describe the decision-making situations in the most diverse socio-economic fields, and the optimal compromise solution of such a model effectively solves the real situation described. In the current paper, an efficient algorithm is proposed for solving the multicriteria optimization model in integers, where the objective functions are of the linear-fractional type with identical denominators. Obviously, the complexity of such a problem is increased, but the practical necessity of its solution is certainly imposed. For this purpose, we focused on using the synthesis function methods, namely the global utility maximization method, adapted for solving the proposed multicriteria optimization model. This method leads to the determination of an optimal compromise integer solution for all criteria, which are of linear-fractional type and which is closest to the optimal integer solutions of each criterion.

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As a result of the algorithm investigation, we obtained a significant result for determining the optimal compromise solution in whole numbers of the proposed model. Thus, for its determination, the decision-maker can use combinatorially both the optimal values of some criteria in whole numbers, as well as others calculated on the set of real numbers, all, of course, positive when constructing the synthesis function. Regardless of the configuration used to construct the synthetic function, its optimal integer solution does not change. Therefore, the decision-maker has the free choice to select more advantageous values of the criteria from his point of view for building the synthesis function of the model. This fact is quite important, making it possible to solve the model interactively, obviously increasing both the efficiency and the attractiveness of the algorithm.

Example 5.1. For the following linear-fractional multicriteria optimization model in integers and for the proposed values of the criteria's utilities, the optimal compromise solution is to be determined, using the global utility maximization method:

$$\min \left\{ F_1(X) = \frac{x_1 + 2x_2 + x_3}{x_1 + x_2 + x_3 + 1} \right\}, \quad \max \left\{ F_2(X) = \frac{2x_1 + x_2 + 2x_3}{x_1 + x_2 + x_3 + 1} \right\},$$

$$\max \left\{ F_3(X) = \frac{2x_1 + 3x_2 + x_3}{x_1 + x_2 + x_3 + 1} \right\},$$

$$\begin{cases} 3x_1 + 5x_2 + x_3 \leq 18 \\ 5x_1 + 3x_2 + 2x_3 \leq 20 \\ 2x_1 + x_2 + 2x_3 \geq 5 \\ x_j \in Z^+. \end{cases}$$

F_1	F_2	F_3	F_1^P	F_2^P	F_3^P
$U_1 = 4$	$U_2 = 8$	$U_3 = 9$	$U_4 = 1$	$U_5 = 2$	$U_6 = 2$

Solving procedure: For solving the proposed model, we will apply the global utility maximization method, being one of synthesis type. Initially, we can observe, that in the model the value of the denominator will be non-zero in D . We will go through stage I of the algorithm. For this purpose we will solve six unicriteria linear-fractional programming problems in R^+ , recording the optimal and worst values for each objective function. Next, in an analogous way, we will solve the same six unicriteria linear-fractional programming problems on the set Z^+ , keeping the optimal and pessimistic values of each criterion. For the construction of the synthesis function, using the global utility maximization method, we will randomly select any combination of records of the corresponding optimal and worst criteria values, some criteria being solved in R^+ and others in Z^+ .

These are as follows:

$$\begin{aligned}
 & 1) \left\{ \begin{array}{l} F_1(R^+) \\ F_2(R^+) \\ F_3(R^+) \end{array} \right\}, \left\{ \begin{array}{l} F_1^P(R^+) \\ F_2^P(R^+) \\ F_3^P(R^+) \end{array} \right\}; 2) \left\{ \begin{array}{l} F_1(Z^+) \\ F_2(Z^+) \\ F_3(Z^+) \end{array} \right\}, \left\{ \begin{array}{l} F_1^P(Z^+) \\ F_2^P(Z^+) \\ F_3^P(Z^+) \end{array} \right\}; \\
 & 3) \left\{ \begin{array}{l} F_1(R^+) \\ F_2(Z^+) \\ F_3(Z^+) \end{array} \right\}, \left\{ \begin{array}{l} F_1^P(R^+) \\ F_2^P(Z^+) \\ F_3^P(Z^+) \end{array} \right\}; 4) \left\{ \begin{array}{l} F_1(R^+) \\ F_2(R^+) \\ F_3(Z^+) \end{array} \right\}, \left\{ \begin{array}{l} F_1^P(R^+) \\ F_2^P(R^+) \\ F_3^P(Z^+) \end{array} \right\}; \\
 & 5) \left\{ \begin{array}{l} F_1(R^+) \\ F_2(Z^+) \\ F_3(R^+) \end{array} \right\}, \left\{ \begin{array}{l} F_1^P(R^+) \\ F_2^P(Z^+) \\ F_3^P(R^+) \end{array} \right\}; 6) \left\{ \begin{array}{l} F_1(Z^+) \\ F_2(Z^+) \\ F_3(R^+) \end{array} \right\}, \left\{ \begin{array}{l} F_1^P(Z^+) \\ F_2^P(Z^+) \\ F_3^P(R^+) \end{array} \right\}; \\
 & 7) \left\{ \begin{array}{l} F_1(Z^+) \\ F_2(R^+) \\ F_3(Z^+) \end{array} \right\}, \left\{ \begin{array}{l} F_1^P(Z^+) \\ F_2^P(R^+) \\ F_3^P(Z^+) \end{array} \right\}; 8) \left\{ \begin{array}{l} F_1(Z^+) \\ F_2(R^+) \\ F_3(R^+) \end{array} \right\}, \left\{ \begin{array}{l} F_1^P(Z^+) \\ F_2^P(R^+) \\ F_3^P(R^+) \end{array} \right\}.
 \end{aligned}$$

The optimal solutions of the unicriteria models as well as the weight criteria, we placed them directly in the vectors of value combinations of the objective functions proposed above. Next, we solved 24 systems of linear equations in order to determine the weight coefficients of each criterion in the synthesis function: $\{(\alpha_j, \beta_j)\}_{j=1, \overline{r}}$. For each of the selected combinations of objective function values we have built the corresponding synthesis functions using the same criterion utility table for the proposed model. We obtained the following utility functions:

$$\begin{aligned}
 F_1(U) &= \frac{1,73x_1 + 1,63x_2 + 1,09x_3}{x_1 + x_2 + x_3 + 1} \longrightarrow \max \\
 F_2(U) &= \frac{1,83x_1 + 1,75x_2 + 1,13x_3}{x_1 + x_2 + x_3 + 1} \longrightarrow \max \\
 F_3(U) &= \frac{1,85x_1 + 1,8x_2 + 1,15x_3}{x_1 + x_2 + x_3 + 1} \longrightarrow \max \\
 F_4(U) &= \frac{1,85x_1 + 1,8x_2 + 1,15x_3}{x_1 + x_2 + x_3 + 1} \longrightarrow \max \\
 F_5(U) &= \frac{1,73x_1 + 1,63x_2 + 1,09x_3}{x_1 + x_2 + x_3 + 1} \longrightarrow \max \\
 F_6(U) &= \frac{1,7x_1 + 1,57x_2 + 1,07x_3}{x_1 + x_2 + x_3 + 1} \longrightarrow \max \\
 F_7(U) &= \frac{1,83x_1 + 1,75x_2 + 1,13x_3}{x_1 + x_2 + x_3 + 1} \longrightarrow \max \\
 F_8(U) &= \frac{1,7x_1 + 1,57x_2 + 1,07x_3}{x_1 + x_2 + x_3 + 1} \longrightarrow \max
 \end{aligned}$$

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OPTIMIZATION PROBLEM IN INTEGERS

The above expressions express the summary utility of all criteria for the corresponding weights and are to be maximized over the admissible domain of the model, given by the same restrictions:

$$\begin{cases} 3x_1 + 5x_2 + x_3 \leq 18 \\ 5x_1 + 3x_2 + 2x_3 \leq 20 \\ 2x_1 + x_2 + 2x_3 \geq 5 \\ x_j \in Z^+ \end{cases}$$

By solving these 8 constructed problems, which are of integer linear programming type, we obtained the same optimal compromise solution:

$$X_{eff}^1 = X_{eff}^2 = X_{eff}^3 = X_{eff}^4 = X_{eff}^5 = X_{eff}^6 = X_{eff}^7 = X_{eff}^8 = \{x_1 = 1, x_2 = 3, x_3 = 0\}.$$

Further we calculated the values of utility functions, which are as follows:

$$F_1(U) \approx 2,38; F_2(U) \approx 2,41; F_3(U) \approx 2,4; F_4(U) \approx 2,4;$$

$$F_5(U) \approx 2,38; F_6(U) \approx 2,39; F_7(U) \approx 2,41; F_8(U) \approx 2,39;$$

$$\begin{aligned} F(X_{eff}^1) &= \begin{Bmatrix} F_1(X_{eff}^1) \\ F_2(X_{eff}^1) \\ F_3(X_{eff}^1) \end{Bmatrix} = F(X_{eff}^2) = \begin{Bmatrix} F_1(X_{eff}^2) \\ F_2(X_{eff}^2) \\ F_3(X_{eff}^2) \end{Bmatrix} = \\ &= F(X_{eff}^3) = \begin{Bmatrix} F_1(X_{eff}^3) \\ F_2(X_{eff}^3) \\ F_3(X_{eff}^3) \end{Bmatrix} = F(X_{eff}^4) = \begin{Bmatrix} F_1(X_{eff}^4) \\ F_2(X_{eff}^4) \\ F_3(X_{eff}^4) \end{Bmatrix} = \\ &= F(X_{eff}^5) = \begin{Bmatrix} F_1(X_{eff}^5) \\ F_2(X_{eff}^5) \\ F_3(X_{eff}^5) \end{Bmatrix} = F(X_{eff}^6) = \begin{Bmatrix} F_1(X_{eff}^6) \\ F_2(X_{eff}^6) \\ F_3(X_{eff}^6) \end{Bmatrix} = \\ &= F(X_{eff}^7) = \begin{Bmatrix} F_1(X_{eff}^7) \\ F_2(X_{eff}^7) \\ F_3(X_{eff}^7) \end{Bmatrix} = F(X_{eff}^8) = \begin{Bmatrix} F_1(X_{eff}^8) \\ F_2(X_{eff}^8) \\ F_3(X_{eff}^8) \end{Bmatrix} = \begin{Bmatrix} 7 \\ 5 \\ 11 \end{Bmatrix}. \end{aligned}$$

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