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Dedicated to Professor Alexandru Şubă on the occasion of his 70th birthday

On stability of some examples of ternary differential critical systems with quadratic nonlinearities

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Abstract. Starting with Example 1 of A.M. Lyaponov's thesis [1] (§32), which represents a ternary differential system with quadratic nonlinearities, examples of differential systems of the generalized Darboux type were constructed in the critical case.The stability conditions of the unperturbed motion described by these systems were determined.
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Despre stabilitatea unor exemple de sisteme critice diferenţiale ternare cu neliniarităţi pătratice

Rezumat. Pornind de la Exemplul 1, din teza lui A. M. Lyaponov [1] (§32), ce constă dintr-un sistem diferenţial ternar cu nelinearităţi pătratice, în cazul critic, au fost construite mai multe exemple de sisteme diferenţiale de tip generalizat Darboux. Pentru aceste sisteme au fost determinate condiţiile de stabilitate a mişcării neperturbate.
Cuvinte-cheie: sistem diferenţial, stabilitatea mişcării neperturbate, sistem diferenţial de tip generalizat Darboux, sistemul dinamicii răspândirii tuberculozei.

1. Introduction

Systems of autonomous differential equations of the first order are mathematical models of many processes in everyday life, for example the system of intrinsic transmission dynamics of tuberculosis (TB).

This mathematical model is described by ternary differential systems with quadratic nonlinearities, which are contained in ternary differential systems with quadratic nonlinearities generalized Darboux type. In the world, there are countless dedicated works to TB problems, both in medicine and in mathematics. For example, among the works devoted to the problem of dynamics of TB, in medicine and mathematics, we can mention the paper [2].

In the Institute of Mathematics and Computer Science of ASM, there were carried researches in the field of TB within a project [3], without examining the stability of the unperturbed motion governed by the system mentioned above.

Also, starting with Example 1 from A. M. Lyaponov's thesis [1] (§32), which consisted of a ternary differential system with quadratic nonlinearities, in the critical case, there were constructed some examples of differential systems of the generalized Darboux type.

In this work, there were determined the stability conditions of unperturbed motion in the critical case for the differential system aimed at the intrinsic transmission dynamics of tuberculosis TB in society and examples of differential systems of generalized Darboux type.

2. Intrinsic transmission dynamics of tuberculosis (TB)

The intrinsic transmission dynamics of tuberculosis [2, 4], represents a mathematical model, in which the entire population is divided into:

- the sensitive population (S);
- the population carrying latent infection (L);
- the population with active tuberculosis (T).

This dynamics is defined by the following system of differential equations:

$$\frac{dS}{dt} = \tau - \mu S - \beta ST \equiv P, \quad \frac{dL}{dt} = -\delta L - \mu L + (1-p)\beta ST \equiv Q,$$
$$\frac{dT}{dt} = \delta L - (\mu + \nu)T + p\beta ST \equiv R. \tag{1}$$

The variables and parameters of system (1) are described in the Table 1:

Table 1. The variables and parameters of the system (1)

Value	**Description**
$S(t)$	number of sensible persons in the moment t
$L(t)$	number of infected persons in the moment t
$T(t)$	number of infectious persons in the moment t
$\lambda(t)$	force of infection per capita in the moment t
τ	influx of young people
μ	average mortality from causes not related to TB
p	probability of rapid progression of the disease
δ	speed constant of reactivation of TB infection
ν	additional mortality caused by active TB
β	transfer coefficient of TB infection

The intrinsic transmission dynamics of tuberculosis [3] (1), through the affine transformation

$$x = \tau - \mu S; \quad y = L; \quad z = T, \tag{2}$$

and $\mu \neq 0$, according to the medical meaning of this variable, can be brought to the form

$$\begin{aligned} \frac{dx}{dt} &= ax + bz + 2gxz; \\ \frac{dy}{dt} &= cy + dz + 2hxz; \\ \frac{dz}{dt} &= ey + fz + 2kxz. \end{aligned} \tag{3}$$

where

$$\begin{gathered} a = -\mu \neq 0,\ b = \beta\tau,\ c = -\delta - \mu, d = \frac{(1-p)\beta\tau}{\mu},\ e = \delta,\ f = \frac{p\beta\tau}{\mu} - \mu - \mu_T, \\ g = -\frac{\beta}{2},\ h = -\frac{(1-p)\beta}{2\mu},\ k = -\frac{p\beta}{2\mu}. \end{gathered} \tag{4}$$

The characteristic equation of system (3) is

$$\rho^3 + (-a - c - f)\rho^2 + (ac + af + cf - de)\rho - a(cf - de) = 0. \tag{5}$$

Taking into account the medical meaning of the variables ($a \neq 0$), for equation (5) to have one zero root, we obtain the relation $cf - de = 0$.

By a center-affine transformation

$$\bar{x} = -ey + cz; \quad \bar{y} = y; \quad \bar{z} = x + z\ (\Delta \equiv c \neq 0) \tag{6}$$

and $cf - de = 0$ or $f = \frac{de}{c}$, according to Hurwitz's theorem [5], the system (3) can be brought to the critical Lyapunov form

$$\begin{aligned} \frac{dx}{dt} &= \frac{2(ck - eh)}{c^2}(-x^2 - 2exy + cxz - e^2y^2 + ceyz); \\ \frac{dy}{dt} &= \frac{d}{c}x + (c + \frac{de}{c})y + \frac{2h}{c^2}(-x^2 - 2exy + cxz - e^2y^2 + ceyz); \\ \frac{dz}{dt} &= \frac{-a + f + b}{c}x + \frac{(-a + c + f + b)e}{c}y + az + \\ &+ \frac{2(g + k)}{c^2}(-x^2 - 2exy + cxz - e^2y^2 + ceyz), \end{aligned} \tag{7}$$

where

$$-a - c - f > 0, \quad a(c + f) > 0. \tag{8}$$

According to Lemma 4.2 [6], we have

$$C_1 = 0, \quad C_2 = \frac{2}{c^2}(-1 + cB_1 - eA_1)(1 + eA_1)(-eh + ck),$$
$$C_3 = \frac{2}{c^2}[cB_2 - 2eA_2 + ce(A_2B_1 + A_1B_2) - 2e^2A_1A_2](-eh + ck),$$
$$C_4 = \frac{2}{c^2}[cB_3 - 2eA_3 + ce(A_3B_1 + A_2B_2 + A_1B_3) - e^2(A_2^2 + 2A_1A_3)](-eh + ck),$$
$$C_5 = \frac{2}{c^2}[cB_4 - 2eA_4 + ce(A_4B_1 + A_3B_2 + A_2B_3 + A_1B_4)- -2e^2(A_2A_3 + A_1A_4)](-eh + ck),$$
$$C_6 = \frac{2}{c^2}[cB_5 - 2eA_5 + ce(A_5B_1 + A_4B_2 + A_3B_3 + A_2B_4 + A_1B_5)- -e^2(A_3^2 + 2A_2A_4 + 2A_1A_5)](-eh + ck),$$
$$C_7 = \frac{2}{c^2}[cB_6 - 2eA_6 + ce(A_6B_1 + A_5B_2 + A_4B_3 + A_3B_4 + A_2B_5 + A_1B_6)- -2e^2(A_3A_4 + A_2A_5 + A_1A_6)](-eh + ck), \tag{9}$$
$$C_8 = \frac{2}{c^2}[cB_7 - 2eA_7 + ce(A_7B_1 + A_6B_2 + A_5B_3 + A_4B_4 + A_3B_5+ +A_2B_6 + A_1B_7) - e^2(A_4^2 + 2A_3A_5 + 2A_2A_6 + 2A_1A_7)](-eh + ck),$$
$$C_9 = \frac{2}{c^2}[cB_8 - 2eA_8 + ce(A_8B_1 + A_7B_2 + A_6B_3 + A_5B_4 + A_4B_5+ +A_3B_6 + A_2B_7 + A_1B_8) - 2e^2(A_4A_5 + A_3A_6 + A_2A_7 + A_1A_8)](-eh + ck),$$
$$C_{10} = \frac{2}{c^2}[cB_9 - 2eA_9 + ce(A_9B_1 + A_8B_2 + A_7B_3 + A_6B_4 + A_5B_5 + A_4B_6+ +A_3B_7 + A_2B_8 + A_1B_9) - e^2(A_5^2 + 2A_4A_6 + 2A_3A_7+ +2A_2A_8 + 2A_1A_9)](-eh + ck), \dots,$$

where

$$A_1 = -\frac{d}{c^2 + de}, \quad B_1 = \frac{(a - b)c}{a(c^2 + de)};$$
$$A_2 = -\frac{2}{c(c^2 + de)}[(-1 + cB_1 - eA_1)(1 + eA_1)h],$$
$$B_2 = -\frac{2}{ac^3(c^2 + de)}(-1 + cB_1 - eA_1)(1 + eA_1)(c^3g + cdeg + aceh- -bceh - c^2eh - de^2h + c^3k + cdek),$$

$$A_3 = -\frac{2}{c(c^2+de)}(cB_2 - 2eA_2 + ce(A_2B_1 + A_1B_2) - 2e^2A_1A_2)h,$$

$$B_3 = -\frac{2}{ac^3(c^2+de)}(cB_2 - 2eA_2 + ce(A_2B_1 + A_1B_2) - $$
$$-2e^2A_1A_2)(c^3g + cdeg + aceh - bceh - c^2eh - de^2h + c^3k + cdek),$$

$$A_4 = -\frac{2}{c(c^2+de)}[cB_3 - 2eA_3 + ce(A_3B_1 + A_2B_2 + A_1B_3) - e^2(A_2^2 + 2A_1A_3)]h,$$

$$B_4 = -\frac{2}{ac^3(c^2+de)}[cB_3 - 2eA_3 + ce(A_3B_1 + A_2B_2 + A_1B_3) - $$
$$-e^2(A_2^2 + 2A_1A_3)](c^3g + cdeg + aceh - bceh - c^2eh - de^2h + c^3k + cdek),$$

$$A_5 = -\frac{2}{c(c^2+de)}[cB_4 - 2eA_4 + ce(A_4B_1 + A_3B_2 + A_2B_3 + A_1B_4) - $$
$$-2e^2(A_2A_3 + A_1A_4)]h,$$

$$B_5 = -\frac{2}{ac^3(c^2+de)}[cB_4 - 2eA_4 + ce(A_4B_1 + A_3B_2 + A_2B_3 + A_1B_4) - $$
$$-2e^2(A_2A_3 + A_1A_4)](c^3g + cdeg + aceh - bceh - c^2eh - de^2h + c^3k + cdek),$$

$$A_6 = -\frac{2}{c(c^2+de)}[cB_5 - 2eA_5 + ce(A_5B_1 + A_4B_2 + A_3B_3 + A_2B_4 + A_1B_5) - $$
$$-e^2(A_3^2 + 2A_2A_4 + 2A_1A_5)]h,$$

$$B_6 = -\frac{2}{ac^3(c^2+de)}[cB_5 - 2eA_5 + ce(A_5B_1 + A_4B_2 + A_3B_3 + A_2B_4 + A_1B_5) - $$
$$-e^2(A_3^2 + 2A_2A_4 + 2A_1A_5)](c^3g + cdeg + aceh - bceh - c^2eh - de^2h + c^3k + cdek),$$

$$A_7 = -\frac{2}{c(c^2+de)}[cB_6 - 2eA_6 + ce(A_6B_1 + A_5B_2 + A_4B_3 + A_3B_4 + $$
$$+A_2B_5 + A_1B_6) - 2e^2(A_3A_4 + A_2A_5 + A_1A_6)]h,$$

$$B_7 = -\frac{2}{ac^3(c^2+de)}[cB_6 - 2eA_6 + ce(A_6B_1 + A_5B_2 + A_4B_3 + A_3B_4 + $$
$$+A_2B_5 + A_1B_6) - 2e^2(A_3A_4 + A_2A_5 + A_1A_6)](c^3g + $$
$$+cdeg + aceh - bceh - c^2eh - de^2h + c^3k + cdek),$$

$$A_8 = -\frac{2}{c(c^2+de)}[cB_7 - 2eA_7 + ce(A_7B_1 + A_6B_2 + A_5B_3 + A_4B_4 + $$
$$+A_3B_5 + A_2B_6 + A_1B_7) - e^2(A_4^2 + 2A_3A_5 + 2A_2A_6 + 2A_1A_7)]h,$$

$$B_8 = -\frac{2}{ac^3(c^2+de)}[cB_7 - 2eA_7 + ce(A_7B_1 + A_6B_2 + A_5B_3 + A_4B_4 + A_3B_5+$$
$$+A_2B_6 + A_1B_7) - e^2(A_4^2 + 2A_3A_5 + 2A_2A_6 + 2A_1A_7)](c^3g+$$
$$+cdeg + aceh - bceh - c^2eh - de^2h + c^3k + cdek),$$

$$A_9 = -\frac{2}{c(c^2+de)}[cB_8 - 2eA_8 + ce(A_8B_1 + A_7B_2 + A_6B_3 + A_5B_4 + A_4B_5+$$
$$+A_3B_6 + A_2B_7 + A_1B_8) - 2e^2(A_4A_5 + A_3A_6 + A_2A_7 + A_1A_8)]h,$$

$$B_9 = -\frac{2}{ac^3(c^2+de)}[cB_8 - 2eA_8 + ce(A_8B_1 + A_7B_2 + A_6B_3 + A_5B_4 + A_4B_5+$$
$$+A_3B_6 + A_2B_7 + A_1B_8) - 2e^2(A_4A_5 + A_3A_6 + A_2A_7 + A_1A_8)](c^3g+ \qquad (10)$$
$$+cdeg + aceh - bceh - c^2eh - de^2h + c^3k + cdek),$$

$$A_{10} = -\frac{2}{c(c^2+de)}[cB_9 - 2eA_9 + ce(A_9B_1 + A_8B_2 + A_7B_3 + A_6B_4 + A_5B_5+$$
$$+A_4B_6 + A_3B_7 + A_2B_8 + A_1B_9) - e^2(A_5^2 + 2A_4A_6 + 2A_3A_7 + 2A_2A_8 + 2A_1A_9)]h,$$

$$B_{10} = -\frac{2}{ac^3(c^2+de)}[cB_9 - 2eA_9 + ce(A_9B_1 + A_8B_2 + A_7B_3 + A_6B_4 + A_5B_5+$$
$$+A_4B_6 + A_3B_7 + A_2B_8 + A_1B_9) - e^2(A_5^2 + 2A_4A_6 + 2A_3A_7 + 2A_2A_8+$$
$$+2A_1A_9)](c^3g + cdeg + aceh - bceh - c^2eh - de^2h + c^3k + cdek), \ldots$$

Taking into account the medical meaning of the parameters from system (3) – (4), we mention that the denominators in (9) and (10) are different from zero. We get

Lemma 2.1. *Let $a+c+f < 0$ and $a(c+f) > 0$. Then the stability of unperturbed motion governed by system* (7) *includes all possible cases in the following two:*

I. *When $(-1 + cB_1 - eA_1)(1 + eA_1)(ck - eh) \neq 0$ the unperturbed motion is* ***unstable****;*

II. *When $(-1 + cB_1 - eA_1)(1 + eA_1)(ck - eh) = 0$ the unperturbed motion is* ***stable****.*

In the last case, the unperturbed motion belongs to some continuous series of stabilized motion. For sufficiently small perturbations, any perturbed motion will asymptotically approach to one of the stabilized motions of the mentioned series. Moreover, this motion is also asymptotic stable [5].

Proof. To prove Lemma we use the Lyapunov's theorem [1] (§32) in the ternary case. Next we analyze the coefficients of the series C_i from (9).

If $C_2 \neq 0$, then we obtain the Case I of Lemma 2.1.

If $-eh + ck = 0$, then $C_i = 0$ $(\forall i)$ and if $(-1 + cB_1 - eA_1)(1 + eA_1) = 0$, then $A_i = B_i = 0$ $(i \geq 2)$ from (10). This implies $C_i = 0$ $(i \geq 3)$. Lemma 2.1 is proved. □

3. Stability conditions of unperturbed motion for some differential systems of generalized Darboux type with quadratic nonlinearities

Following Example 1 from [1] (§32), which in the critical equation has 2 parameters, we will examine a few cases of ternary systems of Lyapunov critical canonical form:

Example 3.1. *We will examine the ternary differential system with three parameters in the critical equation of the form*

$$\begin{aligned} \frac{dx}{dt} &= (ax + by + cz)(-x + y + z), \\ \frac{dy}{dt} &= x - y + (x - y + 2z)(-x + y + z), \\ \frac{dz}{dt} &= x - z + (x + 2y - z)(-x + y + z), \end{aligned} \tag{11}$$

where a, b, c are real arbitrary coefficients.

The characteristic equation of the linear part is

$$\rho^3 + 2\rho^2 + \rho = 0, \tag{12}$$

where

$$\rho_1 = 0, \quad \rho_2 = \rho_3 = -1. \tag{13}$$

According to Lemma 4.2 [6], we have

$$\begin{gathered} C_1 = 0, \quad C_2 = a + b + c, \\ C_3 = 4a + 6b + 6c = 2[2a + 3(b + c)], \\ C_4 = 20a + 38b + 38c = 2[10a + 19(b + c)], \\ C_5 = 116a + 254b + 254c = 2[58a + 127(b + c)], \\ C_6 = 740a + 1774b + 1774c = 2[370a + 887(b + c)], \\ C_7 = 5028a + 12822b + 12822c = 2[2514a + 6411(b + c)], \\ C_8 = 35700a + 95190b + 95190c = 2[17850a + 47595(b + c)], \\ C_9 = 261780a + 721870b + 721870c = 2[130890a + 360935(b + c)], \\ C_{10} = 1967300a + 5569118b + 5569118c = 2[982650a + 2784559(b + c)], \ldots, \end{gathered} \tag{14}$$

where

$$A_1 = B_1 = 1; \quad A_2 = B_2 = 2, \quad A_3 = B_3 = 10, \quad A_4 = B_4 = 58, \quad A_5 = B_5 = 370,$$
$$A_6 = B_6 = 2514, \quad A_7 = B_7 = 17850, \quad A_8 = B_8 = 130890, \quad A_9 = B_9 = 983650, \tag{15}$$
$$A_{10} = B_{10} = 7536418, \ldots$$

As the characteristic equation (12) of system (11) has the roots (13), then according to Lyapunov's Theorem [1] (§32), in the ternary case, we obtain

Lemma 3.1. *The stability of the unperturbed motion, governed by system* (11)*, includes all possible cases in the following four:*

I. $a + b + c \neq 0$*, then the unperturbed motion is* ***unstable****;*

II. $a > 0$*, then the unperturbed motion is* ***stable****;*

III. $a < 0$*, then the unperturbed motion is* ***unstable****;*

IV. $b + c = -a = 0$*, then the unperturbed motion is* ***stable****.*

In the last case, the unperturbed motion belongs to some continuous series of stabilized motion. For sufficiently small perturbations, any perturbed motion will asymptotically approach to one of the stabilized motions of the mentioned series. Moreover, this motion is also asymptotic stable [5].

Proof. According to Lyapunov's Theorem [1] (§32), we analyze the coefficients of the series C_i, from (14). If $C_2 \neq 0$, then we get the case I of Lemma 3.1.

If $C_2 = 0$, then $b + c = -a$. Substituting in C_3 we obtain $C_3 = -2a$. Depending on the sign of this expression, we get the cases II and III of Lemma 3.1.

If $C_3 = 0$, then $b + c = -a = 0$. In this case, $C_i = 0$ $(i \geq 4)$. Lemma 3.1 is proved. □

Example 3.2. *We examine the ternary differential system with six parameters in the critical equation of the form*

$$\begin{aligned}
\frac{dx}{dt} &= a_1x^2 + a_2y^2 + a_3z^2 + 2a_4xy + 2a_5xz + 2a_6yz,\\
\frac{dy}{dt} &= x - y + (x - y + 2z)(-x + y + z),\\
\frac{dz}{dt} &= x - z + (x + 2y - z)(-x + y + z),
\end{aligned} \tag{16}$$

where a_i $(i = \overline{1, 6})$ *are real arbitrary coefficients.*

According to Lemma 4.2 [6], we have

$$C_1 = 0, \quad C_2 = a_1 + (a_2 + a_3 + 2a_6) + 2(a_4 + a_5),$$
$$C_3 = 4[(a_2 + a_3 + 2a_6) + (a_4 + a_5))],$$
$$C_4 = 4[6(a_2 + a_3 + 2a_6) + 5(a_4 + a_5)],$$

$$
\begin{aligned}
C_5 &= 4[39(a_2 + a_3 + 2a_6) + 29(a_4 + a_5)],\\
C_6 &= 4[268(a_2 + a_3 + 2a_6) + 185(a_4 + a_5)],\\
C_7 &= 12[639(a_2 + a_3 + 2a_6) + 419(a_4 + a_5)],\\
C_8 &= 60[942(a_2 + a_3 + 2a_6) + 595(a_4 + a_5)],\\
C_9 &= 20[21319(a_2 + a_3 + 2a_6) + 13089(a_4 + a_5)],\\
C_{10} &= 4[819096(a_2 + a_3 + 2a_6) + 491825(a_4 + a_5)], \dots,
\end{aligned}
\tag{17}
$$

where

$$
\begin{gathered}
A_1 = B_1 = 1;\ \ A_2 = B_2 = 2,\ \ A_3 = B_3 = 10,\ \ A_4 = B_4 = 58,\\
A_5 = B_5 = 370,\ \ A_6 = B_6 = 2514,\ \ A_7 = B_7 = 17850,\\
A_8 = B_8 = 130890,\ \ A_9 = B_9 = 983650,\\
A_{10} = B_{10} = 7536418, \dots
\end{gathered}
\tag{18}
$$

We introduce the notation

$$
\begin{gathered}
L_1 = a_2 + a_3 + 2a_6;\ \ L_2 = -a_1 - 2(a_4 + a_5);\\
L_3 = -a_1 - (a_4 + a_5);\ \ L_4 = -(a_4 + a_5).
\end{gathered}
\tag{19}
$$

As the characteristic equation (12) of the system (16) has the roots (13), then according to Lyapunov's Theorem [1] (§32), in the ternary case, we obtain

Lemma 3.2. *The stability of the unperturbed motion, governed by system* (16)*, includes all possible cases in the following five:*

I. $L_1 \neq L_2$*, then the unperturbed motion is* ***unstable****;*

II. $L_1 = L_2,\ L_3 < 0$*, then the unperturbed motion is* ***stable****;*

III. $L_1 = L_2,\ L_3 > 0$*, then the unperturbed motion is* ***unstable****;*

IV. $L_1 = L_2,\ L_3 = 0,\ L_4 \neq 0$*, then the unperturbed motion is* ***unstable****;*

V. $a_2 + a_3 + 2a_6 = a_1 = 0,\ a_4 = -a_5$*, then the unperturbed motion is* ***stable****.*

In the last case, the unperturbed motion belongs to some continuous series of stabilized motion. For sufficiently small perturbations, any perturbed motion will asymptotically approach to one of the stabilized motions of the mentioned series. Moreover, this motion is also asymptotic stable [5]*. The expressions* L_i $(i = \overline{1,4})$ *are given in* (19).

Proof. According to Lyapunov's Theorem [1] (§32), we analyze the coefficients of the series C_i from (17). If $C_2 \neq 0$, then $L_1 - L_2 \neq 0$. We obtain the Case I of Lemma 3.2.

If $C_2 = 0$, then $L_1 = L_2$, and

$$C_3 = 4[L_1 + (a_4 + a_5)] = 4[L_2 + (a_4 + a_5)] = 4[-a_1 - (a_4 + a_5)] = 4L_3.$$

Depending on the sign of this expression, we get the Cases II and III of Lemma 3.2.

If $C_2 = C_3 = 0$, then $L_1 = L_2$ and $L_3 = 0$ implies $a_1 = -(a_4 + a_5)$ and

$$C_4 = 4[6L_1 + 5(a_4 + a_5)] = 4[6L_2 + 5(a_4 + a_5)] = 4[-6a_1 - 7(a_4 + a_5)] = 4L_4.$$

Assume that $L_4 \neq 0$. In this case we get the Case IV of Lemma 3.2.

If $C_2 = C_3 = C_4 = 0$, then $L_1 = L_2, L_3 = L_4 = 0$ or $a_2 + a_3 + 2a_6 = a_1 = 0$, $a_4 = -a_5$. In this case, all $C_i = 0$ $(i \geq 5)$. Lemma 3.2 is proved. □

Example 3.3. *We examine the ternary differential system with 6 parameters in the critical equation, of which three form the common factor of the quadratic part, of the form*

$$\begin{aligned} \frac{dx}{dt} &= (a_1x + b_1y + c_1z)(ax + by + cz), \\ \frac{dy}{dt} &= x - y + (x - y + 2z)(ax + by + cz), \\ \frac{dz}{dt} &= x - z + (x + 2y - z)(ax + by + cz), \end{aligned} \tag{20}$$

where a, b, c, a_1, b_1, c_1 are real arbitrary coefficients.

We introduce the notation

$$\begin{gathered} M_1 = a + b + c; \quad M_2 = a_1 + b_1 + c_1; \\ M_3 = -aa_1 + (b + c)(b_1 + c_1); \quad M_4 = (b + c)(b_1 + c_1). \end{gathered} \tag{21}$$

According to Lemma 4.2 [6], we have

$$\begin{gathered} C_1 = 0, \quad C_2 = M_1M_2, \quad C_3 = (M_1M_2 + M_3)A_2, \\ C_4 = M_4A_2^2 + (M_1M_2 + M_3)A_3, \\ C_5 = 2M_4A_2A_3 + (M_1M_2 + M_3)A_4, \\ C_6 = M_4(2A_2A_4 + A_3^2) + (M_1M_2 + M_3)A_5, \\ C_7 = 2M_4(A_2A_5 + A_3A_4) + (M_1M_2 + M_3)A_6, \\ C_8 = M_4(2A_2A_6 + 2A_3A_5 + A_4^2) + (M_1M_2 + M_3)A_7, \\ C_9 = 2M_4(A_2A_7 + A_3A_6 + A_4A_5) + (M_1M_2 + M_3)A_8, \\ C_{10} = M_4(2A_2A_8 + 2A_3A_7 + 2A_4A_6 + A_5^2) + (M_1M_2 + M_3)A_9, \dots, \end{gathered} \tag{22}$$

where

$$A_1 = B_1 = 1; \quad A_2 = B_2 = 2(a + b + c),$$

$$A_3 = B_3 = (a + 3b + 3c)A_2,$$

$$
\begin{aligned}
A_4 = B_4 &= (b+c)A_2^2 + (a+3b+3c)A_3, \\
A_5 = B_5 &= 2(b+c)A_2A_3 + (a+3b+3c)A_4, \\
A_6 = B_6 &= (b+c)(2A_2A_4 + A_3^2) + (a+3b+3c)A_5, \\
A_7 = B_7 &= 2(b+c)(A_2A_5 + A_3A_4) + (a+3b+3c)A_6, \\
A_8 = B_8 &= (b+c)(2A_2A_6 + 2A_3A_5 + A_4^2) + (a+3b+3c)A_7, \\
A_9 = B_9 &= 2(b+c)(A_2A_7 + A_3A_6 + A_4A_5) + (a+3b+3c)A_8, \\
A_{10} = B_{10} &= (b+c)(2A_2A_8 + 2A_3A_7 + 2A_4A_6 + A_5^2) + (a+3b+3c)A_9, \dots
\end{aligned}
\tag{23}
$$

As the characteristic equation (12) of system (20) has the roots (13), then according to Lyapunov's Theorem [1] (§32), in the ternary case, we obtain

Lemma 3.3. *The stability of the unperturbed motion governed by the system* (20) *includes all possible cases in the following six:*

I. $M_1M_2 \neq 0$*, then the unperturbed motion is* ***unstable****;*

II. $M_2 = 0,\ M_1M_3 < 0$*, then the unperturbed motion is* ***stable****;*

III. $M_2 = 0,\ M_1M_3 > 0$*, then the unperturbed motion is* ***unstable****;*

IV. $M_1M_4 \neq 0$*, then the unperturbed motion is* ***unstable****;*

V. $M_4 = 0$*, then the unperturbed motion is* ***stable****;*

VI. $M_1 = 0$*, then the unperturbed motion is* ***stable****;.*

In the last case, the unperturbed motion belongs to some continuous series of stabilized motion. For sufficiently small perturbations, any perturbed motion will asymptotically approach to one of the stabilized motions of the mentioned series. Moreover, this motion is also asymptotic stable [5]*. The expressions* M_i $(i = \overline{1,4})$ *are given in* (21).

Proof. According to Lyapunov's Theorem [1] (§32), we analyze the coefficients of the series C_i from (22). Suppose that $M_1 \neq 0$.

If $C_2 \neq 0$, then $M_1M_2 \neq 0$ and we get the Case I of Lemma 3.3.

If $C_2 = 0$, then $M_2 = 0$, and $C_3 = (M_1M_2 + M_3)A_2 = 2M_1M_3$. Depending on the sign of this expression, we obtain the Cases II and III of Lemma 3.3.

If $C_2 = C_3 = 0$, then $M_2 = M_3 = 0$ and $C_4 = M_4A_2^2 + (M_1M_2 + M_3)A_3 = 4M_1^2M_4$. If $M_1M_4 \neq 0$, we get the Case IV of Lemma 3.3.

If $C_2 = C_3 = C_4 = 0$, then $M_2 = M_3 = M_4 = 0$, and all $C_i = 0$ $(i \geq 5)$. We have the Case V of Lemma 3.3.

If $M_1 = 0$, then all $C_i = 0$ $(\forall i)$ and we obtain the Case VI. Lemma 3.3 is proved. □

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