Acta et Commentationes, Exact and Natural Sciences ISSN: 2537-6284
Volume 2(16), 2023, Pages 78–88 E-ISSN: 2587-3644
CZU: 681.51:004.421:004.9 DOI: 10.36120/2587-3644.v16i2.78-88

Dedicated to Professor Alexandru Şubă on the occasion of his 70^{th} birthday

Tuning method of automatic controllers to object models with second order advance-delay and dead time

Dumitru Moraru

Abstract. This article explores the application of mathematical models in the design and analysis of automatic control system. By integrating mathematical concepts such as linear algebra, mathematical analysis, the performance and reliability of automatic control systems can be optimized. In the paper, an efficient procedure has been developed for tuning the standardized P, PI, PD, and PID control algorithms to mathematical models of second-order advance-delay with dead time control objects with known parameters, using the maximal stability degree method with iterations.The advantages of the maximum stability degree method with reduced calculations and minimal time are highlighted.

2010 Mathematics Subject Classification: 93A30, 93C85, 44A10.
Keywords: mathematical model, advance-delay control object, transfer function, automatic system, tuning methods, differential equation, system performances.

Metodă de acordare a regulatoarelor automate la modele de obiecte cu anticipaţie-întârziere de ordinul doi și timp mort

Rezumat. Acest articol explorează aplicarea modelelor matematice în proiectarea și analiza sistemelor automate. Prin integrarea conceptelor matematice, cum ar fi algebra liniară, analiza matematică, se pot optimiza performanțele și fiabilitatea sistemelor de conducere automată. În lucrare s-a elaborat o procedură eficientă de acordare a algoritmilor tipizați P, PI, PD şi PID la modele matematice ale obiectelor de reglare cu anticipație-întârziere de ordinul doi cu timp mort cu parametrii cunoscuţi după metoda gradului maximal de stabilitate cu iterații. Se evidențiază avantajele metodei gradul maximal de stabilitate cu iterații cu calcule reduse și timp minim.

Cuvinte-cheie: model matematic, obiect de reglare cu anicipație-întârziere, funcţie de transfer, sistem automat, metode de acordare, ecuație diferențială, performanţele sistemului.

1. Introduction

Automatic control systems are complex entities that can adapt their behavior based on external conditions or inputs. These can be mathematically modeled using differential equations, Laplace transforms, and transfer functions. Differential equations are used to describe the relationships between the input and output variables of a system as a function of time. In the context of automatic systems, these equations model the dynamic behavior of the system. The Laplace transform is used to convert differential equations into transfer functions, which represent algebraic equations of complex variables. This facilitates the analysis and solving of dynamic system problems. Control theory deals with the design and analysis of controllers that influence the behavior of a system. There are two main types of control: open-loop and closed-loop. In open-loop control, the input is set without considering the output, whereas in closed-loop control, the input is adjusted based on the output magnitude to achieve a desired behavior.

According to the concept of automatic control theory, the technological process presents the control object with the variables that interact in the process: the input flow is called the control variable, denoted by the vector $x(t)$, the characteristic variables $y_1, ..., y_n$, which represents the output flow known as the controlled variable, denoted by the vector $y(t)$ and disturbances denoted by the vector $p(t)$ (Figure 1), where FP is fixed part of control object [5].

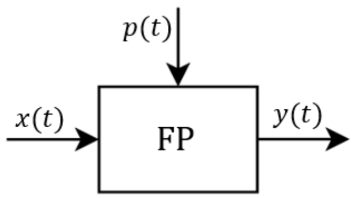

Figure 1. The block diagram of the control object.

The control object represents a technical, industrial, biological, economic, social, etc. process that requires control for optimal operation.

In paper is discussed the mathematical model of the control object, characterized as a advance-delay object with second-order inertia and dead time, described by the transfer function $H_{FP}(s)$ in form [2], [3]:

$$H_{FP}(s) = e^{-\tau s} \frac{k(T_1 s + 1)}{(T_2 s + 1)(T_3 s + 1)} = e^{-\tau s} \frac{b_0 s + b_1}{a_0 s^2 + a_1 s + a_2}, \tag{1}$$

where k is the transfer coefficient, T_1, T_2, and T_3 are the time constants of the process, τ is the dead time and the generic coefficients are $b_0 = kT_1$, $b_1 = k$, $a_0 = T_2 T_3$, $a_1 = T_2 + T_3$, $a_2 = 1$.

For the control object model (1), it is necessary to synthesize the control algorithm. In the practice of automation various industrial processes, controllers with a fixed PID structure have a wide range of applications [1], [6], [7].

There are several methods for tuning the standard PID control algorithm to the model object (1): the frequency-domain method, the pole-zero allocation method, the polynomial method, the Ziegler-Nichols method, etc [2], [3], [4], [9].

The application of the frequency-domain method involves calculations in the frequency domain and graphical constructions, which can lead to difficulties in synthesizing control algorithms.

The pole-zero allocation method (or model-based method) is an analytical approach. Based on the model of the control object (1) and the performance requirements imposed on the designed system, PI and PID control algorithms are synthesized. This is done by solving a system of matrix equations to determine the control algorithm parameters that meet the stability, performance, and robustness requirements of the system. As a result, the control algorithm synthesis procedure involves iterations and can become challenging [3], [4].

The polynomial method is also an analytical approach that leads to solving the control algorithm synthesis problem. However, it can be challenging to determine the characteristic equation of the designed system [8].

The basic experimental method includes the Ziegler-Nichols (ZN) method, which is widely used in practice for tuning standard PID algorithms for the model (1), but it may lead to reduced system performance [4].

In the paper, a procedure for tuning the PID controller for the control object model (1) has been developed based on the maximum stability degree method with Iterations (MSDI) [1], [6], [7].

To verify and compare the obtained results, both the Ziegler-Nichols and parametric optimization (PO) methods are applied.

2. Tuning the controller using the Maximum Stability Degree Method with Iterations

The structural block diagram of the automatic control system, consisting of the object model with transfer function $H_{FP}(s)$ and the controller with transfer function $H_R(s)$, is shown in Figure 2. Here, $r(t) = 1(t)$ represents the unit reference, $e(t)$ is the system error, $u(t)$ is the command generated by the controller, and $y(t) = h(t)$ is the step response of the system.

The standardized control algorithms P, PI, PD and PID are represented by the transfer function:

$$H_P(s) = k_p, \tag{2}$$

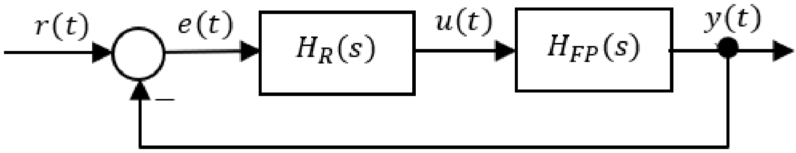

Figure 2. Structural block diagram of the automatic system.

$$H_{PI}(s) = k_p + \frac{k_i}{s} = \frac{k_p s + k_i}{s}, \tag{3}$$

$$H_{PD}(s) = k_p + k_d s, \tag{4}$$

$$H_{PID}(s) = k_p + \frac{k_i}{s} + k_d s = \frac{k_d s^2 + k_p s + k_i}{s}, \tag{5}$$

where k_p, k_i, and k_d are the tuning parameters of the proportional, integral and derivative components of the P, PI, PD and PID algorithms [1], [4], [8].

Tuning the P, PI, PD and PID control algorithms to the model (1) based on the maximum stability degree method of the designed system in the classical version becomes challenging when determining the algebraic equation for finding the maximum stability degree J.

The procedure for tuning the PID control algorithm according to the proposed method involves obtaining the characteristic equation of the closed-loop system. The notion of stability degree is introduced into the characteristic equation as a new unknown variable. Through operations of differentiation on this variable, relationships are derived that express the PID tuning parameters as nonlinear functions of the stability degree J and the known parameters of the object model parameters.

The transfer function of the closed-loop system with a P controller is given by:

$$H_0(s) = \frac{H_d(s)}{1 + H_d(s)} = \frac{k_p e^{-\tau s}(b_0 s + b_1)}{a_0 s^2 + a_1 s + a_2 + k_p e^{-\tau s}(b_0 s + b_1)} = \frac{C(s)}{D(s)}, \tag{6}$$

where $H_0(s)$is the transfer function of closed-loop system, $H_d(s)$ - transfer function of open-loop system, k_p - the parameter of the P controller, $C(s)$ and $D(s)$ - the system polynomials.

The characteristic equation of the automatic control system is the polynomial $D(s)$:

$$D(s) = a_0 s^2 + a_1 s + a_2 + k_p e^{-\tau s}(b_0 s + b_1) = 0. \tag{7}$$

According to the maximum stability degree method algorithm, it is substituted $s = -J$, and after some transformations, it is obtained the expression:

$$D(-J) = a_0J^2 - a_1J + a_2 + k_pe^{\tau J}(b_1 - b_0J) =$$

$$= \frac{e^{-\tau J}(a_0J^2 - a_1J + a_2)}{b_1 - b_0J} + k_p = 0. \quad (8)$$

In the case of a system with a P controller, expression (8) is differentiated once with respect to J and the resulting expression is:

$$\dot{D}(-J) = \frac{e^{-\tau J}(d_0J^3 - d_1J^2 + d_2J - d_3)}{b_0^2J^2 - 2b_0b_1J + b_1^2} = 0. \quad (9)$$

where $d_0 = a_0b_0\tau$, $d_1 = a_0b_0 + a_0b_1\tau + a_1b_0\tau$, $d_2 = 2a_0b_1 + a_1b_1\tau + a_2b_0\tau$, $d_3 = a_1b_1 - a_2b_0 + a_2b_1\tau$.

The optimal degree value J_{opt} is the smallest positive root of the expression:

$$e^{-\tau j}[a_0b_0J^3\tau - a_2b_1\tau + J^2(-a_0b_0 - a_0b_1\tau - a_1b_0\tau) +$$

$$+J(2a_0b_1 + a_1b_1\tau + a_2b_0\tau) - a_1b_1 + a_2b_0] = 0. \quad (10)$$

To determine the tuning parameter for the P controller from (8), the following relationship is used:

$$k_p = \frac{e^{-\tau J}(-a_0J^2 + a_1J - a_2)}{b_1 - b_0J} = f_p(J). \quad (11)$$

Further, the calculation mathematical expression for the tuning parameters k_p, k_i, k_d of the PI, PD, and PID control algorithms are presented using the MSDI to the object model (1) in a simplified form.

Mathematical expressions for determine of tuning parameters of PI controller are:

$$k_p = \frac{e^{-\tau J}(-d_0J^4 + d_1J^3 - d_2J^2 + d_3J - d_4)}{b_0^2J^2 - 2b_0b_1J + b_1^2} = f_p(J), \quad (12)$$

$$k_i = \frac{e^{-\tau J}(a_0J^3 - a_1J^2 + a_2J)}{b_1 - b_0J} + k_pJ = f_i(J), \quad (13)$$

where $d_0 = a_0b_0\tau$, $d_1 = 2a_0b_0 + a_0b_1\tau + a_1b_0\tau$, $d_2 = 3a_0b_1 + a_1b_0 + a_1b_1\tau + a_2b_0\tau$, $d_3 = 2a_1b_1 + a_2b_1\tau$, $d_4 = a_2b_1$.

Mathematical expressions for determine of tuning parameters of PD controller are:

$$k_p = \frac{e^{-\tau J}\left(a_0J^3 - a_1J^2 + a_2J\right)}{b_1 - b_0J} + k_dJ = f_p(J), \quad (14)$$

$$k_d = \frac{e^{-\tau J}(-d_0 J^4 + d_1 J^3 - d_2 J^2 + d_3 J - d_4)}{b_0^2 J^2 - 2b_0 b_1 J + b_1^2} = f_d(J), \tag{15}$$

where $d_0 = a_0 b_0 \tau$, $d_1 = 2a_0 b_0 + a_0 b_1 \tau + a_1 b_0 \tau$, $d_2 = 3a_0 b_1 + a_1 b_0 + a_1 b_1 \tau + a_2 b_0 \tau$, $d_3 = 2a_1 b_1 + a_2 b_1 \tau$.

Mathematical expressions for determine of tuning parameters of PID controller are:

$$k_p = \frac{e^{-\tau J}\left(-d_0 J^4 + d_1 J^3 - d_2 J^2 + d_3 J - d_4\right)}{b_0^2 J^2 - 2b_0 b_1 J + b_1^2} + 2k_d J = f_p(J), \tag{16}$$

$$k_i = \frac{e^{-\tau J}(a_0 J^3 - a_1 J^2 + a_2 J)}{b_1 - b_0 J} - k_d J^2 + k_p J = f_i(J), \tag{17}$$

$$k_d = \frac{e^{-\tau J}(-d_5 J^6 + d_6 J^5 - d_7 J^4 + d_8 J^3 - d_9 J^2 + d_{10} J - d_{11})}{2(b_0^4 J^4 - 4b_0^3 b_1 J^3 + 6b_0^2 b_1^2 J^2 - 4b_0 b_1^3 J + b_1^4)} = f_d(J), \tag{18}$$

where $d_0 = a_0 b_0 \tau$, $d_1 = 2a_0 b_0 + a_0 b_1 \tau + a_1 b_0 \tau$, $d_2 = 3a_0 b_1 + a_1 b_0 + a_1 b_1 \tau + a_2 b_0 \tau$, $d_3 = 2a_1 b_1 +$
$+ a_2 b_1 \tau$, $d_4 = a_2 b_1$, $d_5 = a_0 b_0^3 \tau^2$, $d_6 = 4a_0 b_0^3 \tau + 3a_0 b_0^2 b_1 \tau^2 + a_1 b_0^3 \tau^2$, $d_7 = 2a_0 b_0^3 + 14a_0 b_0^2 b_1 \tau +$
$+ 3a_0 b_0 b_1^2 \tau^2 + 2a_1 b_0^3 \tau + 3a_1 b_0^2 b_1 \tau^2 + a_2 b_0^3 \tau^2$, $d_8 = 8a_0 b_0^2 b_1 + 16a_0 b_0 b_1^2 \tau + a_0 b_1^3 \tau^2 + 8a_1 b_0^2 b_1 \tau +$
$+ 3a_1 b_0 b_1^2 \tau^2 + 3a_2 b_0^2 b_1 \tau^2$, $d_9 = 12a_0 b_0 b_1^2 + 6a_0 b_1^3 \tau + 10a_1 b_0 b_1^2 \tau + a_1 b_1^3 \tau^2 + 2a_2 b_0^2 b_1 \tau + 3a_2 b_0 b_1^2 \tau^2$, $d_{10} = 6a_0 b_1^3 + 2a_1 b_0 b_1^2 + 4a_1 b_1^3 \tau + 2a_2 b_0^2 b_1 + 4a_2 b_0 b_1^2 \tau + a_2 b_1^3 \tau^2$, $d_{11} = 2a_1 b_1^3 - 2a_2 b_0 b_1^2 + 2a_2 b_1^3 \tau$.

3. Applications and computer simulation

The mathematical model of object described by the transfer function (1) is considered with the following numerical values: $\tau = 2$, $b_0 = 0.35$, $b_1 = 0.2313$, $a_0 = 1$, $a_1 = 0.3872$, and $a_2 = 0.04851$.

$$H_{PF}(s) = e^{-\tau s} \frac{b_0 s + b_1}{a_0 s^3 + a_1 s^2 + a_2 s} = e^{-2s} \frac{0,35s + 0,2313}{s^3 + 0,3872s^2 + 0,04851s}. \tag{19}$$

It is required: to tune the P, PI, PD and PID controllers.

Solution. The parameters of the control algorithms P, PI, PD and PID of the automatic system with the model of object in (1) with the given parameters and the respective controller according to relations (12)-(18) are calculated.

Substitute the numerical data in (11) and it is obtained the mathematical calculation expression for the P controller:

$$k_p = \frac{e^{-2J}(-J^2 + 0.3872J - 0.04851)}{0.2313 - 0.35J}. \tag{20}$$

The value of stability degree J is varied from 0.01 to 4.8, and the dependence $k_p = f(J)$ is plotted (Figure 3).

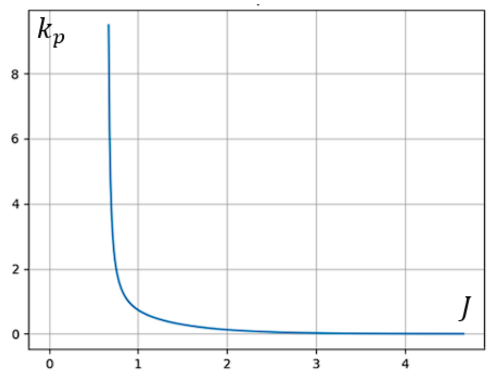

Figure 3. Dependence of $k_p = f(J)$.

Substitute the numerical data in (12), (13) and obtain the mathematical calculation expressions for the PI controller:

$$k_p = \frac{e^{-2J}(-0.7J^4 + 1.433J^3 - 1.042J^2 + 0.201J - 0.011)}{0.122J^2 - 0.162J + 0.053}, \tag{21}$$

$$k_i = \frac{e^{-2J}(J^3 - 0.3872J^2 + 0.04851J)}{0.2313 - 0.35J} + k_pJ. \tag{22}$$

The value of stability degree J is varied from 0.76 to 1.9, and the dependencies $k_p = f(J)$, $k_i = f(J)$ are plotted (Figure 4).

Substitute the numerical data in (14), (15) and obtain the mathematical calculation expressions for the PD controller:

$$k_p = \frac{e^{-2J}(-J^2 + 0.3872J - 0.04851)}{0.2313 - 0.35J} + k_dJ, \tag{23}$$

$$k_d = \frac{e^{-2J}(0.7J^3 - 1.083J^2 + 0.6756J - 0.095)}{0.122J^2 - 0.162J + 0.053}. \tag{24}$$

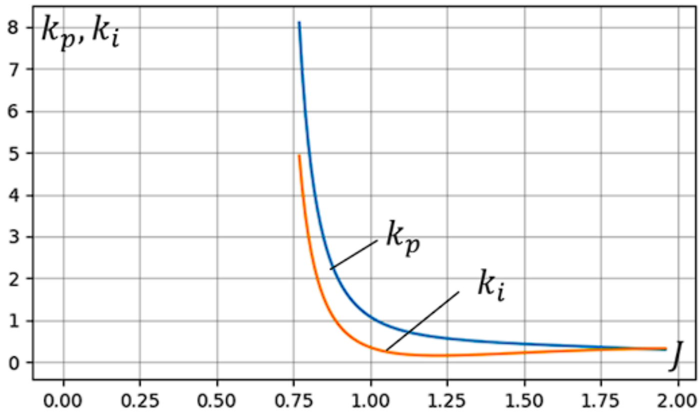

Figure 4. Dependencies of $k_p = f(J)$, $k_i = f(J)$.

The value of stability degree J is varied from 0.76 to 3, and the dependencies $k_p = f(J)$, $k_d = f(J)$ are plotted (Figure 5).

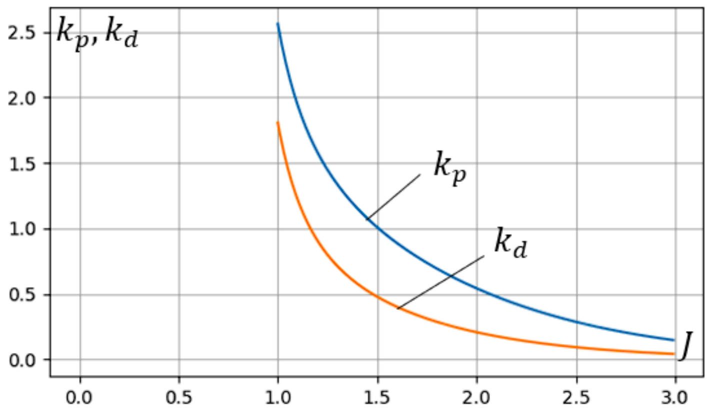

Figure 5. Dependencies of $k_p = f(J)$, $k_d = f(J)$.

Substitute the numerical data in (16), (17), (18) and obtain the mathematical calculation expressions for the PID controller:

$$k_p = \frac{e^{-2J}(-0.7J^4 + 1.433J^3 - 1.042J^2 + 0.201J - 0.011)}{0.122J^2 - 0.162J + 0.053} + 2k_dJ, \tag{25}$$

$$k_i = \frac{e^{-2J}(J^3 - 0.3872J^2 + 0.04851J)}{0.2313 - 0.35J} - k_dJ^2 + k_pJ, \tag{26}$$

$$k_d = \frac{e^{-2J}(-0.1715J^6 + 0.343J^5 - 1.31J^4 + 1.182J^3 - 0.553J^2 + 0.139J - 0.01)}{2(0.015J^4 - 0.0396J^3 + 0.0393J^2 - 0.0173J + 0.0028)}. \tag{27}$$

The value of stability degree J is varied from 0.01 to 0.57, and the dependencies $k_p = f(J)$, $k_i = f(J)$, $k_d = f(J)$ is plotted (Figure 6).

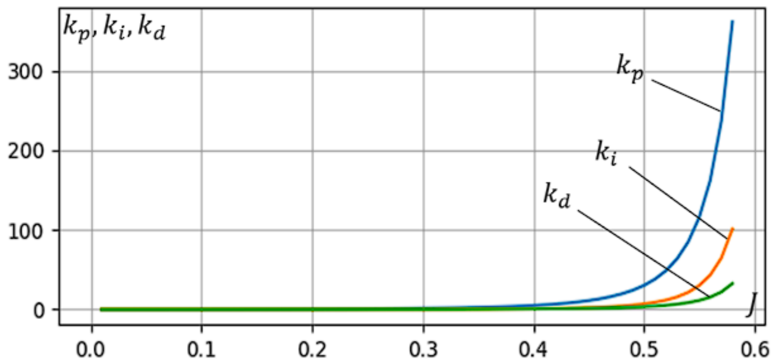

Figure 6. Dependencies of $k_p = f(J)$, $k_i = f(J)$ and $k_d = f(J)$.

To verify the tuning results of the controller, the system is simulated in the MATLAB software package, and the step responses (set point = 80) of the system with the respective P, PI, PD and PID controllers are illustrated in Figure 7.

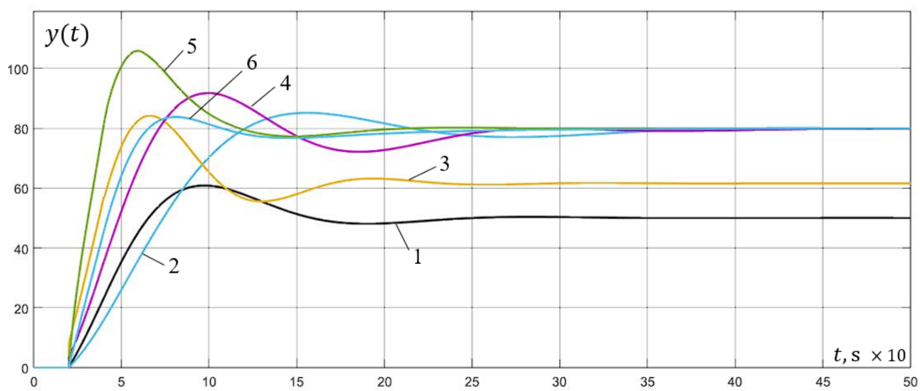

Figure 7. The step responses of the system with different controller types: P, PI, PD and PID: curve 1 is with P controller, 2 - PI, 3 - PD, 4 - PID tuned with MSDI, 5 - PID with Ziegler-Nichlos method, 6 - PID with parametric optimization method.

In Table 1, the performance of the simulated automatic control system in the MATLAB software package is presented with different P, PI, PD, and PID controllers tuned using the MSDI, Ziegler-Nichols, and parametric optimization methods.

Table 1. Controller parameters and simulated automated system performances

Iter. Nr.	**Tune method**	**Contr. type**	**Controller parameters**					**System performances**			
			J	k_p	k_i	T_i, s	k_d	t_c, s	σ, %	t_r, s	n
1	MSDI	P	1.38	0.35	-	-	-	29.3	31.97	187.2	2
2	MSDI	PI	1.35	0.209	0.032	31.25	-	57.5	12.02	263.4	2
3	MSDI	PD	1.80	0.701	-	-	0.29	18.7	57.49	245.3	4
4	MSDI	PID	0.22	0.424	0.046	21.73	0.119	28.1	8.75	82.8	1
5	ZN	PID	-	0.8922	0.1965	5.08	1.029	18.12	32.12	102.2	1
6	PO	PID	-	0.404	0.0501	19.96	0.572	36.88	-	60.94	-

4. Conclusions

Based on the conducted study, the following conclusions are formulated:

1. Good performances of the automatic control system were obtained for the version with PID controller tuned by the MSDI (Figure 7, curve 4, iteration 4, Table 1), having the settling time t_r = 82.8 s, the overshoot σ = 8.75 % and a deviation n = 1.
2. The best performance of the automatic control system was obtained for the system with PID controller tuned according to the parametric optimization method (Figure 7, curve 6, iteration 6, Table 1), having the lowest settling time t_r = 60.94 s, no overshoot $\sigma = 0$ and no oscillation $n = 0$.
3. The MSDI tuning method is the least computationally intensive and performs satisfactorily compared to the ZN and PO.
4. It is not recommended to use the P and PD controllers for the system with the given mathematical model of object (1) because they have a high stationary error (Figure 7, curve 1 and 3, Table 7, iteration 1 and 3).

Acknowledgement *This work was supported by the project 20.80009.5007.26 "Models, algorithms and technologies for the control, optimization and security of the Cyber-Physical systems".*

References

[1] Cojuhari, I., Fiodorov, I., Izvoreanu, B., Moraru, D. Synthesis of PID Controller for the Automatic Control System with Imposed Performance based on the Multi-Objective Genetic Algorithm. *Proceedings of the 11th International Conference and Exposition on Electrical and Power Engineering (EPE), 22-23 October, 2020*, Iași, Romania, 598–603.

[2] Dib, F., Benaya, N., Ben Meziane, K., Boumhidi, I. Comparative Study of Optimal Tuning PID Controller for Manipulator Robot. *The Proceedings of the International Conference on Smart City Applications SCA 2022: Innovations in Smart Cities Applications*, vol. 6, 252-–261.

[3] Dorf, R., Bishop, R. *Sovremennye sistemy upravlenia (Modern Control Systems)*. Moskva: Laboratoria Bazovyh Znanii, 2004. 832 p. (in Russian).

[4] Dumitrache, I. *Ingineria reglării automate*. București: Politehnica Press. V.1, 2016.

[5] Izvoreanu, B. *Teoria sistemelor automate.* Chișinău: Tehnica-UTM, 2022. 343 p.

[6] Izvoreanu, B., Cojuhari, I., Fiodorov, I., Moraru, D. Metodă de identificare a modelelor aproximate ale obiectului de reglare cu elemente identice cu întârziere şi timp mort după răspunsul experimental al procesului. *Materialele Conferinţei Tehnico–Ştiinţifice a Colaboratorilor, Doctoranzilor şi Studenţilor a UTM, 17 noiembrie 2017*. Chişinău: Editura ”Tehnica-UTM”, 2017, 13–17.

[7] Izvoreanu, B., Cojuhari, I., Fiodorov, I., Moraru, D., Secrieru, A. Tuning the PID Controller to the Model of Object with Inertia Second Order According to the Maximum Stability Degree Method with Iteration. *Annals of the University of Craiova. Electrical Engineering series*, 2019, vol. 43, no. 1, 79–85.

[8] Kim, D.P. *Teoria avtomaticheskogo upravlenia. Lineinye sistemy*. Moskva: FIZMATLIT, 2003 (in Russian).

[9] Sule, A.H. Studies of PID Controller Tuning using Metaheuristic Techniques: A Review. *International Journal of Innovative Scientific & Engineering Technologies Research*, 2022, vol. 10, no. 4, 44–63.

Received: September 18, 2023 *Accepted: December 06, 2023*

(Moraru Dumitru) Technical University of Moldova, 9/7 Studenţilor st., Chişinau, MD-2045, Republic of Moldova
E-mail address: dumitru.moraru@ati.utm.md