Acta et Commentationes, Exact and Natural Sciences ISSN: 2537-6284
Volume 2(16), 2023, Pages 89–96 E-ISSN: 2587-3644
CZU: 515.12 (043.3) DOI: 10.36120/2587-3644.v16i2.89-96

Dedicated to Professor Alexandru Șubă on the occasion of his 70th birthday

Generalized Hausdorff compactifications

Laurenţiu Calmuţchi

Abstract. This article investigates some properties of generalized Hausdorff compactifications of topological T_0-spaces. In particular, it is show that the totality of these compactifications forms a lattice of g-extensions in which there is the maximum element.

2010 Mathematics Subject Classification: 54D30, 54D40.

Keywords: continuous application, extension, g-extension, compactification, lattice, space.

Compactificări generalizate Hausdorff

Rezumat. În acest articol se studiază unele proprietăţi ale compactificărilor generalizate Hausdorff ale T_0-spaţiilor topologice. În particular, se demonstrează că totalitatea compactificărilor formează o latice de g-extensii în care există elementul maximal.

Cuvinte-cheie: aplicaţie continuă, extensie, g-extensie, compactificare, latice, spaţiu.

1. Extensions

Let us mention, that in case there are no concrete indications, then the topological space is considered T_0–space.

Definition 1.1. *A pair (Y, f) it is called a generalized extension or g–extension of space X, where Y is a space, $f : X \to Y$ is a continuous mapping and the set $f(X)$ is dense in Y. If f is an embadding of space X in Y, i.e. an omeomorphism of space X on subspace $f(X)$ of Y, then the pair (Y, f) is called an extension of space X.*

If $(Y,\ f)$ is an extension of space X, then, as a rule, the point $x \in X$ is identified with $f(x) \in Y$ and it is considered to be $X \subseteq Y$. In this case $f(x) = x$ for any $x \in X$.

Let $GE(X)$ be the set of all g–extensions of the space X and $E(X)$ be the set of all extensions of X. Obviously, $E(X) \subseteq GE(X)$.

In class $GE(X)$ the binary increased relationship is introduced. If (Y, f) and (Z, g) are two g–extensions of X space, then it is considered $(Z, g) \leq (Y, f)$. If there is a continuous mapping $\varphi : Y \to Z$, for which $g(x) = \varphi(f(x))$ for any $x \in X$, i.e. $g = \varphi \circ f$ and Diagram 1 is commutative.

Figure 1. Diagrams 1 and 2.

If $(Y, f) \leq (Z, g)$ and $(Z, g) \leq (Y, f)$, then these g–extensions (Y, f) and (Z, g) are called *equivalent* and we denote this by $(Y, f) \sim (Z, g)$.

Proposition 1.1. *If (Y, f) and (Z, g) are two g–extensions of space X, $(Z, g) \leq (Y, f)$ and $(Z, g) \in E(X)$, then $(Y, f) \in E(X)$.*

Proof. Let $\varphi : Y \to Z$ be a continuous mapping and $g = \varphi \circ f$. According to the definition of relationship $\leq$, g is a dive. Let us denote $h = \varphi \mid f(X) \ : f(X) \to g(X)$. Then we get Diagram 2. As g is a bijection and f, h are surjections, it turns out that f and h are bijections. We have $f(A) = h^{-1}(g(A))$. Therefore, for any open set U of X the set $f(U)$ is open in $f(X)$, and the mapping $f^{-1} : f(X) \to X$ is continuous. So, f is a dive. Obviously, h is a homomorphism. Proposition 1.1 is proved. □

Corollary 1.1. *If (Y, f) and (Z, g) are two g–extensions equivalent of space X and one of them is extension, then the other one is extension.*

The pair (X, f), where $f(x) = x$ for any $x \in X$ is an extension of space X. This is the trivial extension or maximum extension. Let us denote this extension by (X, e_X).

Let S be a space consisting of a single point and let $s_X(x) = S$ for any $x \in X$. Then $(S,\ s_X)$ is called g–extension minimal or g–zero extension of space X.

Let P be a property of topological spaces. The property P is called *multiplicative* if the product of a set of spaces with the property P is a space with the property P.

The property P is called *hereditary closed* if any closed subspace of a space with the property P is a space with the property P.

Property P is called *additive* if the reunion space of a finite number of subspaces with the property P is a space with the property P.

Example 1.1. *The property of being compact space is multiplicative, hereditary closed and additive.*

Example 1.2. *The property of being countable compact space is hereditary, additive but not multiplicative. The product of two countable compact spaces can not be a countable compact space ([4], Example 3.10.19).*

Example 1.3. *The property of being pseudocompact is additive, but it is neither multiplicative and not hereditary closed [4].*

Example 1.4. *The property of being space is multiplicative, hereditary and additive. This property is called trivial property.*

2. Lattice of extensions

Let us fix a property P of topological spaces. We denote by $PGE(X)$ the totality of g–extensions (Y, f) with the property P, i.e. Y possesses the property P and denote $Y \in P$.

Let $PE(X) = E(X) \cap PGE(X)$. If P is a trivial property, then $PE(X) = E(X)$ and $PGE(X) = GE(X)$.

Definition 2.1. *If L is a nonempty set of $PGE(X)$ and $(Y, f) \in PGE(X)$, then:*

(1) *the extension (Y, f) is called the upper bound of a set L in $PGE(X)$ and denote $(Y, f) \in \vee L$, if $(Z, g) \leq (Y, f)$ for any $(Z, g) \in L$. If $(Y_1, f_1) \in PGE(X)$ and $(Z, g) \leq (Y, f)$ for any $(Z, g) \in L$, then $(Y, f) \leq (Y_1, f_1)$;*

(2) *the extension (Y, f) is called the lower bound of a set L in $PGE(X)$ and denote $(Y, f) \in \wedge L$, if $(Y, f) \leq (Z, g)$ for anything $(Z, g) \in L$. If $(Y_1, f_1) \in PGE(X)$ and $(Y, f) \leq (Z, g)$ for any $(Z, g) \in L$, then $(Y, f) \leq (Y_1, f_1)$.*

Proposition 2.1. *Let P be a multiplicative and hereditary closed property. Then for any nonempty set $L \subseteq PGE(X)$ there are extensions $(Y, f) \in \vee L$.*

Proof. Let $L = \{(Y_\mu, f_\mu) : \mu \in M\}$, $f(x) = (f_\mu(x) : \mu \in M) \in \prod \{Y_\mu : \mu \in M\}$ for any $x \in X$ and let Y be the adherence of a set $f(x)$ in $\prod \{Y_\mu : \mu \in M\}$. Then $(Y, f) \in \vee L$. Proposition 2.1 is proved. □

Definition 2.2. *The set $L \subseteq PGE(X)$ is called:*

(1) *the upper semilattice of extensions, if L is nonempty and for any nonempty subset $M \subseteq L$ there exists $(Y, f) \in \vee M$.*

(2) *the lower semilattice of extensions, if L is nonempty and for any nonempty subset $M \subseteq L$ there exists $(Y, f) \in \wedge M$;*

(3) *the lattice of extensions, if it is an upper semilattice and a lower semilattice of extensions.*

Proposition 2.2. *Let P be a multiplicative and closed hereditary property. Then for any nonempty set $H \subseteq PGE(X)$ there exists an upper semilattice of extensions $L^*(H)$ with properties:*

(1) *$H \subseteq L^*(H)$;*
(2) *if L is an upper semilattice of extensions and if $H \subseteq L \subseteq L^*(H)$, then $L = L^*(H)$.*

Proof. Let us fix $(Y_M,\ f_M) \in \vee M$ for any nonempty subset $M \subseteq H$. If $M = \{(Y, f)\}$, then $Y_M = Y$ and $f_M = f$. They can be obtained by constructing (Y_M, f_M) as in the proof of Proposition 2.1.

Let us denote $L^*(H) = \{(Y_M, f_M) : M \subseteq H, M \neq \emptyset\}$. Obviously, $H \subseteq L^*(H)$. If $M \subseteq K \subseteq H$, then $(Y_M, f_M) \leq (Y_K, f_K)$. According to construction $L^*(H)$ is an upper semilattice. If $K = \left\{(Y_{M_\alpha}, f_{M_\alpha}) : \alpha \in A\right\}$ and $M = \cup\{M_\alpha : \alpha \in A\}$, then $(Y_M, f_M) \in \vee K$. The proof is complete. □

Definition 2.3. *The upper semilattice $L^*(H)$ built in the proof of Proposition 2.2 is called the upper semilattice generated by set H.*

Corollary 2.1. *Let P be a multiplicative and closed hereditary property. Suppose that the continuous image of a space with property P is a space with property P. Then any nonempty set $H \subseteq PGE(X)$ is contained in a lattice of extensions of $PGE(X)$.*

Proof. Let (Z_0, g_0) be the extension, where Z_0 is a space consisting of a single point, and let $g_0 : X \to Z_0$ be the only possible application. It is clear that $(Z_0, g_0) \leq (Y, f)$ for any $(Y, f) \in GE(X)$. Let us denote $L(H) = L^*(H \cup \{(Z_0, g_0)\})$. Obviously, $L(H)$ is an upper semilattice. As the upper lattice $L(H)$ contains an element of $\wedge L(H)$, it is a lattice. But $(Z_0, g_0) \in \wedge L(H)$. The proof is complete. □

Definition 2.4. *A g–extension (Y, f) of the space X is called correct, if the family $\{cl_Y f(A) : A \subseteq X\}$ forms a closed base of the space Y.*

Let us denote by $KGE(X)$ the totality of correct g–extensions of the space X and let $KE(X) = E(X) \cap KGE(X)$.

Proposition 2.3. *If (Y, f), (Z, g) are two correct and equivalent g–compactifications of the space X, then $(Y, f) = (Z, g)$, i.e. the continuous application $\varphi : Y \to Z$ for any $g = \varphi \circ f$ is a homeomorphism of the space Y onto the space Z.*

Proof. Let $\varphi : Y \to Z$ and $\psi : Z \to Y$ be two continuous applications, for which $g = \varphi \circ f$ and $f = \psi \circ g$. If $A \subseteq X$, then $\varphi(cl_Y f(A)) \subseteq cl_Z g(A)$ and $\psi(cl_Z g(A)) \subseteq (cl_Y f(A))$. Hence, $\varphi(cl_Y f(A)) = cl_Z g(A)$ and $\psi(cl_z g(A)) = cl_Y f(A)$. From these two equalities we conclude that φ, ψ are reciprocal bijective applications and $\varphi^{-1} = \psi$. Proposition 2.3 is proved. □

3. Compacts

For topological spaces the notion of compact space was introduced by P.S. Alexandroff and P.S. Urysohn (see [1]).

Definition 3.1. *The class P of topological spaces is called strict compactness if it satisfies the conditions:*

$(C1)$ *class P is not empty;*

$(C2)$ *in P there is a space X containing at least two different points;*

$(C3)$ *class P is multiplicative;*

$(C4)$ *class P is closed hereditary;*

$(C5)$ *if Y is a dense subspace of the space $X \in P$, then $\{cl_X A : A \subseteq Y\}$ is a closed basis of the space X.*

Definition 3.2. *The class P of spaces with properties $(C1)$–$(C4)$ is called quasi-compactness.*

Definition 3.3. *A quasi-compactness P of Hausdorff spaces is called compactness.*

Proposition 3.1. *If P is a strict compactness, then:*

(1) *$PGE(X) = KPGE(X)$ for any space X;*

(2) *$PGE(X)$ is a set for any space X;*

(3) *$PGE(X)$ is a lattice of extensions for any space X.*

Proof. Equality (1) is a consequence of condition (C5) in Definition 3.1. It follows from Proposition 2.3 that $PGE(X)$ is a set. Since $L^*PGE(X)) = L(PGE(X)) = PGE(X)$, from Corollary 2.1 we obtain that $PGE(X)$ is a lattice of extensions. The proof is complete. □

Proposition 3.2. *If P is a compactness, then $PGE(X)$ is a lattice of extensions for any space X.*

Proof. If (Y, f) is a g–extension of the space X, Y is a Hausdorff space, and τ is the power of the set X, then the weight $\omega(Y) \leq 2^\tau$. Therefore, $PGE(X)$ is a set. Then, based on Corollary 2.1, we obtain that $L^*(PGE(X)) = L(PGE(X)) = PGE(X)$ is a lattice of extensions. The proof is complete. □

Corollary 3.1. *Let P be a compactness or a strict compactness, X be a space and suppose that $PE(X) \neq \emptyset$. Then $PE(X)$ is a upper semilattice of extensions.*

Corollary 3.2. *Let P be a compactness or a strict compactness. Then:*

(1) *for any space X a unique maximal g–extension $(\beta_P X,\ \beta_X) \in PGE\,(X)$ is determined;*
(2) *for any continuous mapping $\varphi : X \to Y$ there is a unique continuous mapping $\beta_{P\varphi} : \beta_P X \to \beta_P Y$, for which $\beta_Y \circ \varphi = \beta_{P\varphi} \circ \beta_X$, i.e. Diagram 3 is commutative.*

Proof. The statement (1) follows from Propositions 3.1 and 3.2. If $\varphi : X \to Y$ is continuous mapping and $(Z,\ g) \in PGE\,(X)$, then $(Z,\ g \circ \varphi) \in PGE(X)$. This fact proves the presence of $\beta_{P\varphi}$. The proof is complete. □

4. Generalized Hausdorff Compactifications

Let us denote by $HGC(X)$ the totality of g–compactifications $(bX,\ b_X)$ of the space X for which bX is a Hausdorff space.

Theorem 4.1. *The totality of $HGC\,(X)$ is a complete lattice of g–extensions.*

Proof. We will prove this theorem after the following steps:

(1) Let us note that the totality of $HGC(X)$ is not empty, since it contains the minimal extension $(mX,\ m_X)$ of a point.
(2) If Y is a Hausdorff space, then for the power (cardinality) of the set Y we have $|Y| \leq \exp\,(\exp d\,(Y)\,)$, where $d(Y)$ is the density of the space Y (see [4], Theorem 1.5.3). If $(Y,\ f) \in GE(X)$, then $d\,(Y) \leq |X|$. Hence, $|Y| \leq \exp\,(\exp|X|\,)$ for any Hausdorff g–compactification $(Y,\ f) \in HGC(X)$. But all topological spaces of power $\leq \exp\,(\exp|X|\,)$ form a set, which contains the entirety of $HGC(X)$. So the totality of $HGC(X)$ is a set.
(3) If (Y, f) and (Z, g) are two equivalent Hausdorff g–compactifications, then they coincide. Let $\varphi : Y \to Z$ and $\psi : Z \to Y$ be two continuous maps for which $g = \varphi \circ f$ and $f = \psi \circ g$. Let us prove that $\psi = \varphi^{-1}$. We examine the application $h = \psi \circ \varphi : Y \to Y$. This mapping is continuous and $h\,(y) = y$ for any $y \in f(X)$. Indeed, let $y = f(x)$ and $x \in X$. Then $\varphi(y) = \varphi\,(f\,(x)) = g(x)$ and $\psi\,(\varphi\,(y)) = \psi\,(g\,(x)) = f(x) = y$. Therefore, $h(y) = y$. The space Y is Hausdorff and $Y_1 = \{y \in Y : h(y) = y\}$ contains the set $f(X)$. So the set Y_1 is dense in Y. Now let us prove that $Y_1 = Y$. Assume that $y_0 \in Y \backslash Y_1$. Then $y_1 = h\,(y_0) \neq y_0$ and there are two open sets U, V in Y for which $y_1 \in U$, $y_0 \in V$ and $U \cap V{=}\emptyset$. The set $W = U \cap h^{-1}\,(V)$ is open in Y and $y_1 \in W$. If $Y \in W$, then $h\,(y) \in V$ and $h\,(y) \neq y$. Hence, $W \cap Y_1 = \emptyset$. Therefore, the set Y_1 is closed in Y. But Y_1 is dense in Y, and a dense and closed set in Y coincides with Y. So, $Y_1 = Y$. We proved that $h(y) = y$ for any $y \in Y$. Therefore, $\psi = \varphi^{-1}$.

(4) The property H to be compact and Hausdorff space is multiplicative and hereditary over closed subspaces. Applying Proposition 3.2 we obtain that $HGC(X)$ is a complete lattice. Theorem 4.1 is proved.

□

Definition 4.1. *The maximal element of the lattice $HGC(X)$ is denoted by $(\beta X, \beta_X)$ and is called the Stone-Čech g–compactification.*

Corollary 4.1. *If f: $X \to Y$ is a continuous mapping, then there is a unique continuous mapping $\beta f : \beta X \to \beta Y$ for which $\beta f \circ \beta_X$=$\beta_Y \circ f$, i.e. Diagram 4 is commutative.*

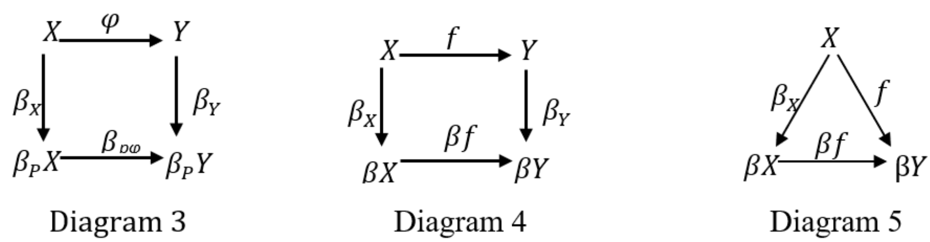

Figure 2. Diagrams 3, 4 and 5.

Corollary 4.2. *If f: $X \to Y$ is a continuous application of X space in the Hausdorff and compact space Y, then there is a unique continuous mapping $\beta f : \beta X \to Y$ for which $f = \beta f \circ \beta_X$, i.e. Diagram 5 is commutative.*

Corollary 4.3. *(See [4], Chapter 3). Let $HC(X)$ =$E(X) \cap HGC(X)$ be the set of Hausdorff compactifications of the space X. Then:*

(1) *if $HC\ (X) \neq \emptyset$, then $HC(X)$ is a complete upper semilattice with maximal element βX;*

(2) *the following statements are equivalent:*

(2.1) *$HC(X) \neq \emptyset$;*

(2.2) *X is a T_0–completely regular space;*

(2.3) *βX is an extension of the space X.*

Theorem 4.2. *(see [4], for T_1 spaces). For any continuous application $f : X \to Y$ in a compact Hausdorff space Y there is a unique continuous application $\omega f : \omega X \to Y$ for which $f = \omega X \mid X$. The mapping ωf is always perfect.*

Proof. Denote $\varphi(x) = f(x)$ for any $x \in X$ and let $\varphi(\xi) = \cap \{cl_Y f\ (H) : H \in \xi\}$ for any ultrafilter $\xi \in \omega X \backslash X$. Let $y, z \in cl_Y f\ (X)$ be two different points. There are two open sets

U and V in Y for which $x \in U, y \in V$, and the sets $F = cl_Y U, \Phi = cl_Y V$ do not intersect. Then $cl_{\omega X} f^{-1}(F) \cap cl_{\omega X} f^{-1}(\Phi) = \emptyset$. If $X \backslash f^{-1}(U) \in \xi$, then $y \notin \varphi(\xi)$. If $X \backslash V \in \xi$, then $z \notin \varphi(\xi)$. But $\xi \cap \left\{X \backslash f^{-1}(U), X \backslash f^{-1}(V)\right\} = \emptyset$. So the mapping $\varphi \colon \omega X \to Y$ is unique and $f = \varphi \mid X$. The set $Z = \{cl_Y A : A \subseteq f(x)\}$ forms a closed basis of the space $Z = cl_Y f(X)$. If A is closed in $f(X)$ and $y \in cl_Y A$, then there exists an ultrafilter η of closed sets in $f(X)$ for which $\{y\} = \cap \{cl_Y H : H \in \eta\}$. There exists at least one ultrafilter $\xi \in \omega X$ for which $f^{-1}(\eta) \subseteq \xi$. Then $\varphi(\xi) = y$. Therefore, $\varphi^{-1}(A) = cl_{\omega X} f^{-1}(A)$ is a closed set in ωX. So, φ is a continuous mapping. From the construction and continuity of the mapping φ we obtain its uniqueness. If the set F is closed in ωX, then $\varphi(F)$ is a compact set. The compact set in a Hausdorff space is closed. So, φ is a closed mapping. Theorem 4.2 is proved. □

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Received: January 23, 2023 *Accepted: October 11, 2023*

(Laurenţiu Calmuţchi) "Ion Creangă" State Pedagogical University, 5 Gh. Iablocikin st., Chişinău, MD-2069, Republic of Moldova