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Dedicated to Professor Alexandru Şubă on the occasion of his 70th birthday

First integrals in a cubic differential system with one invariant straight line and one invariant cubic

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Abstract. In this paper we find conditions for a singular point $O(0,0)$ of a center or a focus type to be a center, in a cubic differential system with one invariant straight line and one invariant cubic. The presence of a center at $O(0,0)$ is proved by constructing Darboux first integrals.

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Integrale prime pentru un sistem diferenţial cubic cu o dreaptă invariantă şi o cubică invariantă

Rezumat. În lucrare se examinează sistemul diferenţial cubic cu punctul singular $O(0,0)$ de tip centru sau focar, care are o dreaptă invariantă şi o cubică invariantă. Pentru acest sistem sunt determinate condiţiile de existenţă a centrului în $O(0,0)$ prin construirea integralelor prime de forma Darboux.

Cuvinte-cheie: sistem diferenţial cubic, curbă algebrică invariantă, integrabilitatea Darboux, problema centrului şi focarului.

1. Introduction

We consider the cubic differential system

$$\begin{cases} \dot{x} = y + ax^2 + cxy + fy^2 + kx^3 + mx^2y + pxy^2 + ry^3 \equiv P(x,y), \\ \dot{y} = -(x + gx^2 + dxy + by^2 + sx^3 + qx^2y + nxy^2 + ly^3) \equiv Q(x,y), \end{cases} \tag{1}$$

where $P(x,y)$ and $Q(x,y)$ are real and coprime polynomials in the variables x and y, $\dot{x} = dx/dt, \dot{y} = dy/dt$. The origin $O(0,0)$ is a singular point of a center or a focus type for (1), i.e. a week focus. It arises the problem of distinguishing between a center and a focus (called *the problem of the center*), i.e. the problem of finding the coefficient conditions under which $O(0,0)$ is a center.

The problem of the center is equivalent to the problem of local integrability of a differential system in the neighboarhood of a singular point with pure imaginary eigenvalues.

It is known [1] that a singular point $O(0,0)$ is a center for (1) if and only if the system has a holomorphic first integral of the form $F(x,y) = C$ in some neighborhood of $O(0,0)$.

Although the problem of the center dates from the end of the 19th century, it is completely solved only for: quadratic systems $\dot{x} = y + p_2(x,y),\ \dot{y} = -x + q_2(x,y)$; cubic symmetric systems $\dot{x} = y + p_3(x,y),\ \dot{y} = -x + q_3(x,y)$; Kukles system $\dot{x} = y,\ \dot{y} = -x + q_2(x,y) + q_3(x,y)$ and a few particular cases in families of polynomial systems of higher degree, where $p_j(x,y)$ and $q_j(x,y)$ are homogeneous polynomials of degree j in the variables x and y.

If the cubic system (1) contains both quadratic and cubic nonlinearities, then the problem of the center is still open. For such systems the necessary and sufficient conditions for the origin to be a center were obtained in some particular cases (see, for example, [9], [20], [21], [22]), [25].

The problem of the center was solved for some families of cubic differential systems with algebraic solutions: four invariant straight lines [3], [4], [5], [9], [18]; three invariant straight lines [9], [24]; two parallel invariant straight lines [14], [23]; two invariant straight lines and invariant conic [6], [7], [9]; two invariant straight lines and invariant cubic [10], [11]. It was proved that every center in the cubic differential system (1) with two invariant straight lines and one invariant conic comes from a Darboux integrability.

The integrability conditions for some families of cubic differential systems having invariant algebraic curves were found in [2], [8], [9], [12]–[17], [19], [25].

The goal of this paper is to obtain the center conditions for cubic differential system (1) with one invariant straight line and one irreducible invariant cubic by using the method of Darboux integrability. Our main result is the following one.

Theorem 1.1. *The origin is a center for cubic differential system* (1)*, with one invariant straight line and one irreducible invariant cubic, if one of the following conditions holds:*

(i) $a = d = k = r = 0,\ g = c + 1 - b,\ l = bf,\ m = -(c+1),\ n = [b(2c+3-b)]/2,$ $p = -f,\ q = -f(c+1+b),\ s = -b(c+1),\ b^2 - 2f^2 - b = 0;$

(ii) $d = 2a,\ k = -a,\ l = [f(2b-c-1)]/3,\ m = -(c+1),\ n = [(2b-c-2)(c+1)]/2,$ $p = -f,\ q = a(2b-c-3),\ r = 0.$

The paper is organized as follows. In Section 2 we present the known results concerning the relation between algebraic solutions and Darboux integrability. In Sections 3 and 4 we determine the integrability conditions for cubic differential system (1) with one invariant straight line and one invariant cubic by constructing Darboux first integrals. Finally in Section 5 we prove the Theorem 1.1.

2. Algebraic solutions and Darboux first integrals

An important problem for differential system (1) is whether the trajectories to (1) can be described by an algebraic formula, for example, $\Phi(x, y) = 0$, where Φ is a polynomial.

Definition 2.1. *An algebraic invariant curve of* (1) *is the solution set in* $\mathbb{C}^2$ *of an equation* $\Phi(x, y) = 0$*, where* Φ *is a polynomial in* x, y *with complex coefficients such that*

$$\frac{\partial \Phi}{\partial x} P(x, y) + \frac{\partial \Phi}{\partial y} Q(x, y) = K(x, y)\Phi(x, y)$$

for some polynomial in x, y*,* $K = K(x, y)$ *with complex coefficients, called the cofactor of the invariant algebraic curve* $\Phi = 0$.

We say that the invariant algebraic curve $\Phi(x, y) = 0$ is an *algebraic solution* of (1) if and only if $\Phi(x, y)$ is an irreducible polynomial in $\mathbb{C}[x, y]$. We shall study the problem of the center for cubic differential system (1) assuming that (1) has algebraic solutions: one invariant straight line and one invariant cubic.

By Definition 2.1 a straight line

$$1 + Ax + By = 0, \ (A, B) \neq 0, \ A, B \in \mathbb{C} \tag{2}$$

is said to be invariant for (1), if there exists a polynomial with complex coefficients $K(x, y)$ such that the following identity holds

$$AP(x, y) + BQ(x, y) \equiv (1 + Ax + By)K(x, y).$$

Let the cubic system (1) have a real invariant straight line of the form (2). Then by rotating the system of coordinates ($x \to x\cos\varphi - y\sin\varphi, y \to x\sin\varphi + y\cos\varphi$) and rescaling the axes of coordinates ($x \to \alpha x, \ y \to \alpha y$), we can make the line to be $1-x = 0$.

In [9] it was proved the following Lemma

Lemma 2.1. *The cubic system (1) has the invariant straight line* $1 - x = 0$ *if and only if the following set of conditions holds*

$$k = -a, \ m = -c - 1, \ p = -f, \ r = 0. \tag{3}$$

Suppose the set of conditions (3) is realized, then the cubic system (1) can be written as follows

$$\begin{cases} \dot{x} = (1 - x)(y + xy + ax^2 + cxy + fy^2) \equiv P(x, y), \\ \dot{y} = -(x + gx^2 + dxy + by^2 + sx^3 + qx^2y + nxy^2 + ly^3) \equiv Q(x, y), \end{cases} \tag{4}$$

We are interested in finding the conditions under which the cubic system (4) has one real irreducible invariant cubic curve. According to [9], a real irreducible invariant cubic curve of (1) can have one of the following two forms

$$a_{30}x^3 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3 + a_{20}x^2 + a_{11}xy + a_{02}y^2 + a_{10}x + a_{01}y + 1 = 0 \quad (5)$$

or

$$a_{30}x^3 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3 + x^2 + y^2 = 0, \quad (6)$$

where $(a_{30}, a_{21}, a_{12}, a_{03}) \neq 0,\ \ a_{ij} \in \mathbb{R}$.

By Definition 2.1, the cubic curve (5) ((6)) is said to be an invariant cubic for (1), if there exists a polynomial with real coefficients $K(x, y) = c_{10}x+c_{01}y+c_{20}x^2+c_{11}xy+c_{02}y^2$ such that the following identity holds

$$\frac{\partial \Phi}{\partial x}P(x, y) + \frac{\partial \Phi}{\partial y}Q(x, y) \equiv \Phi(x, y)K(x, y).$$

Definition 2.2. *System (1) is integrable on an open set D of R^2 if there exists a nonconstant analytic function $F : D \to R$ which is constant on all solution curves $(x(t), y(t))$ in D, i.e. $F(x(t), y(t)) = C$ for all values of t where the solution is defined. Such an F is called a first integral of the system on D.*

When F exists in D all the solutions of the differential system in D are known [1], since every solution is given by $F(x, y) = C$, for some $C \in \mathbb{R}$. Clearly F is a first integral of (1) on D if and only if

$$P\frac{\partial F}{\partial x} + Q\frac{\partial F}{\partial y} \equiv 0. \quad (7)$$

A first integral constructed from invariant algebraic curves $f_j(x, y) = 0,\ \ j = \overline{1, q}$

$$F(x, y) \equiv f_1^{\alpha_1} f_2^{\alpha_2} \cdots f_q^{\alpha_q} = C \quad (8)$$

with $\alpha_j \in \mathbb{C}$ not all zero is called *a Darboux first integral* [21], [25].

By constracting Darboux first integrals (8), the center conditions where obtained for cubic system (1) with two invariant straight lines and one invariant conic in [9], with two invariant straight lines and one invariant cubic of the form (6) in [10] and [11].

The qualitative investigation in 2-dimensional parameter space of cubic systems (1) with a center and having the Darboux first integral of the form

$$(1 + Ax + By)^2\Phi = 0,$$

where $\Phi = 0$ is an irreducible invariant cubic curve of the form (6), was done in [26].

In [13] it was found the center conditions for cubic differential system (1) by constructing Darboux integrating factors of the form

$$\mu(x, y) = (1 - x)^{\alpha}\Phi^{\beta},$$

where $\Phi = 0$ is an irreducible invariant cubic of the form (6) and $\alpha, \beta \in \mathbb{R}$.

In this paper, using the equation (7), we find the conditions under which the cubic differential system (1) has Darboux first integrals of the form

$$F(x, y) \equiv (1 - x)^{\alpha} \Phi^{\beta} = C \tag{9}$$

composed of one invariant straight line $1 - x = 0$ and one irreducible invariant cubic $\Phi = 0$ of the form (5) ((6)), where $\alpha, \beta \in \mathbb{R}$.

3. One invariant straight line and one invariant cubic of the form (5)

In this section we find Darboux integrability conditions for cubic differential system (1) with one invariant straight line and one invariant cubic of the form (5).

Lemma 3.1. *The cubic differential system (1) with one invariant straight line* $1 - x = 0$ *and one invariant cubic (5) has a Darboux first integral of the form (9) if and only if one of the following two sets of conditions is satisfied:*

(i) $a = d = k = r = 0$, $g = c + 1 - b$, $l = bf$, $m = -(c + 1)$, $n = [b(2c + 3 - b)]/2$, $p = -f$, $q = -f(c + 1 + b)$, $s = -b(c + 1)$, $b^2 - 2f^2 - b = 0$;

(ii) $d = 2a$, $k = -a$, $l = [f(2b - c - 1)]/3$, $m = -(c + 1)$, $n = [(2b - c - 2)(c + 1)]/2$, $p = -f$, $q = a(2b - c - 3)$, $r = 0$.

Proof. Let the cubic system (1) have the invariant straight line $1 - x = 0$ and an invariant cubic $\Phi = 0$ of the form (5). In this case the system (1) will have a Darboux first integral of the form (9) if and only if the identity (7) holds. Identifying the coefficients of the monomials $x^i y^j$ in (7), we obtain a system of twenty equations

$$\{U_{ij} = 0, \quad i + j = 1, \ldots, 5\} \tag{10}$$

for the unknowns a_{30}, a_{21}, a_{12}, a_{03}, a_{20}, a_{11}, a_{02}, a_{10}, a_{01}, α, β and the coefficients of system (1).

When $i + j = 1$, the equations $U_{10} = 0$ and $U_{01}=0$ of (10) yield $a_{01} = 0$ and $\alpha = a_{10}\beta$. If $i + j = 2$, the equations $U_{20} = 0$, $U_{11} = 0$ and $U_{02} = 0$ of (10) imply $a_{11} = 0$ and $a_{20} = (2a_{02} + a_{10}^2 + a_{10})/2$.

When $i + j = 3$, the equations $U_{ij} = 0$ of (10) give $a_{21} = 2aa_{02}$, $a_{12} = a_{02}(a_{10} + 2b)$, $a_{30} = (6a_{02}a_{10}+8ba_{02}-4ca_{02}+4ga_{02}+a_{10}^3+3a_{10}^2+2a_{10})/6$ and $a_{03} = [2a_{02}(2a-d+f)]/3$. We express l, n, q, s from the equations $U_{04} = 0$, $U_{13} = 0$, $U_{22} = 0$, $U_{31} = 0$ of (10). Then $U_{40} \equiv 2a\beta a_{02}(a_{10} + 2b - c - 1) = 0$, where $\beta a_{02} \neq 0$. We divide the investigation into two cases: $\{a = 0\}$; $\{a_{10} = c - 2b + 1, a \neq 0\}$.

1. Let $a = 0$. Then $U_{05} \equiv d(d - f)(a_{10} + 3b) = 0$.

1.1. Assume that $d = f$. In this case $U_{14} \equiv f(a_{10} + 2b + 1)(a_{10} + 2b) = 0$.

1.1.1. Suppose $a_{10} = -2b$. Then $U_{32} \equiv f_1 f_2 f_3 = 0$, where

$$f_1 = f,\ f_2 = 2b - 3,\ f_3 = (2b - 1)(b - 1)b + (b + c - g)a_{02}.$$

If $f_2 = 0$ or $f_3 = 0$, then the right-hand sides of (1) have a common factor $cx+x+fy+1$. The case $f_1 = 0$ is contained in (ii) $(a = f = 0, c = -1)$.

1.1.2. Suppose $a_{10} = -2b - 1$. Then $U_{32} \equiv g_1 g_2 g_3 = 0$, where

$$g_1 = f,\ g_2 = 2b + 1,\ g_3 = (2b - 1)(b - 1)b + (b + c - g)a_{02}.$$

If $g_2 = 0$ or $g_3 = 0$, then the right-hand sides of (1) have a common factor $cx+x+fy+1$. The case $g_1 = 0$ is contained in (ii) $(a = f = 0, c = -2)$.

1.1.3. Suppose $(a_{10} + 2b + 1)(a_{10} + 2b) \neq 0$ and let $f = 0$. Then $U_{32} \equiv 0$ and $U_{23} = 0$ yields $a_{10} = c - 2b + 1$. This case is contained in (ii) $(a = f = 0)$.

1.2. Assume that $d \neq f$ and let $d = 0$. Then $U_{14} \equiv (a_{10} + 2b - c - 1)(a_{10} + 3b) = 0$.

1.2.1. The case $a_{10} = c + 1 - 2b$ is contained in (ii) $(a = 0)$.

1.2.2. If $a_{10} \neq c + 1 - 2b$ and $a_{10} = -3b$, then from equations $\{U_{23} = 0, U_{32} = 0, U_{41} = 0\}$ of (10) we get $2f^2 - b^2 + b = 0$, $g = c - b + 1$ and $a_{02} = (3b - 3b^2)/2$. In this case we obtain the set of conditions (i) for the existence of the first integral (9) with $\alpha = -3b,\ \beta = 1$ and

$$\Phi \equiv 2(1 - bx)^3 + b(b - 1)(3bx - 2fy - 3)y^2 = 0.$$

1.3. Assume that $d(d - f) \neq 0$ and let $a_{10} = -3b$. In this case $U_{05} \equiv 0$ and

$$U_{32} \equiv (3b - 1)(3b - 2)(3b(b - 1) + 2a_{02}) = 0.$$

1.3.1. If $b = 1/3$ or $b = 2/3$, then $U_{14} \equiv d(9(d - f)^2 + 1) \neq 0$.

1.3.2. If $a_{02} = (3b(1 - b))/2$ and $(3b - 1)(3b - 2) \neq 0$, then $U_{41} \equiv U_{32} \equiv 0$. The equations $U_{23} = 0, U_{14} = 0$ yield $b^2 - b - 2(d - f)^2 = 0$. In this case, the right-hand sides of (1) have a common factor $cx + x + fy + 1$.

2. Assume that $a_{10} = c - 2b + 1$ and let $a \neq 0$. Then the equations of (10) imply $d = 2a$. In this case we obtain the set of conditions (ii) for the existence of the first integral (9) with $\alpha = c - 2b + 1,\ \beta = 1$ and

$$\Phi \equiv a_{30}x^3 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3 + a_{20}x^2 + a_{11}xy + a_{02}y^2 + a_{10}x + a_{01}y + 1 = 0,$$

where $a_{01} = a_{11} = 0$, $a_{10} = c+1-2b$, $a_{20} = (2a_{02}+a_{10}^2+a_{10})/2$, $a_{02} = [(c+1-2b)(2b-c-2)(2b-c-3)(2b-c-4)]/[2(2b-c-2g-3)(2b-c-4)+12s]$, $a_{21} = 2aa_{02}$, $a_{12} = a_{02}(a_{10} + 2b)$, $a_{03} = [2a_{02}(2a - d + f)]/3$, $a_{30} = (6a_{02}a_{10} + 8ba_{02} - 4ca_{02} + 4ga_{02} + a_{10}^3 + 3a_{10}^2 + 2a_{10})/6$.

In each of the cases (i) and (ii), the system (1) has a Darboux first integral of the form (9) and therefore the origin is a center for (1). □

4. One invariant straight line and one invariant cubic of the form (6)

In this section we find Darboux integrability conditions for cubic differential system (1) with one invariant straight line and one invariant cubic of the form (6).

Lemma 4.1. *The cubic differential system (1) with one invariant straight line* $1 - x = 0$ *and one invariant cubic (6) has a Darboux first integral of the form (9) if and only if the following set of conditions is satisfied:*

(iii) $d = 2a,\ k = -a,\ l = [f(2b-c-1)]/3,\ m = -(c+1),\ n = [(2b-c-2)(c+1)]/2,$
$p = -f,\ q = a(2b-c-3),\ r = 0,\ s = [(c-2b+2g+3)(2b-c-4)]/6.$

Proof. Let the cubic system (1) have the invariant straight line $1 - x = 0$ and an invariant cubic $\Phi = 0$ of the form (6). In this case the system (1) will have a Darboux first integral of the form (9) if and only if the identity (7) holds. Identifying the coefficients of the monomials $x^i y^j$ in (7), we obtain a system of fifteen equations

$$\{V_{ij} = 0, \quad i + j = 3, 4, 5\} \tag{11}$$

for the unknowns a_{30}, a_{21}, a_{12}, a_{03}, α, β and the coefficients of system (1).

When $i + j = 3$, the equations of (11) yield $a_{21} = 2a$, $\alpha = \beta(a_{12} - 2b)$, $g = (3a_{30} - 3a_{12} + 2b + 2c)/2$, $d = (4a + 2f - 3a_{03})/2$, where $\beta \neq 0$.

We express l, n, q, s from the equations $V_{04} = 0$, $V_{13} = 0$, $V_{22} = 0$, $V_{31} = 0$ of (11). Then $V_{05} \equiv (3a_{03} - 2f)(a_{12} + b)a_{03} = 0$. We divide the investigation into three cases:

$\{a_{03} = (2f)/3\}$; $\{a_{03} = 0, f \neq 0\}$; $\{a_{12} = -b, a_{03}(3a_{03} - 2f) \neq 0\}$.

1. Let $a_{03} = (2f)/3$. Then $V_{40} \equiv a(a_{12} - c - 1) = 0$.

1.1. Assume that $a_{12} = c+1$. Then we obtain the set of conditions (iii) for the existence of the first integral (9) with $\alpha = c - 2b + 1,\ \beta = 1$ and

$$\Phi \equiv 3(x^2 + y^2) + (c - 2b + 2g + 3)x^3 + 6ax^2y + 3(c + 1)xy^2 + 2fy^3 = 0.$$

1.2. Assume that $a_{12} \neq c + 1$ and let $a = 0$. Then $V_{14} \equiv f(a_{12} + b) = 0$.

If $f = 0$, then $V_{23} \equiv a_{12}(a_{12} + 1) = 0$. When $a_{12} = 0$ or $a_{12} = -1$ the right-hand sides of (1) have a common factor $cx + x + 1$.

If $f \neq 0$ and $a_{12} = -b$, then $a_{30} = (2-7b)/3$. In this case $V_{41} \equiv (3b-1)(3b-2) = 0$. When $b = 1/3$ or $b = 2/3$, we have that $V_{23} \neq 0$.

2. Let $a_{03} = 0$ and $f \neq 0$. Then $V_{40} \equiv a(a_{12} - c - 1) = 0$.

2.1. Assume that $a = 0$. Then $V_{14} \equiv a_{12}(a_{12} + 1) = 0$. If $a_{12} = 0$, then $b = 3/2$ and the right-hand sides of (1) have a common factor $cx + x + fy + 1$.

If $a_{12} = -1$, then $V_{32} \equiv (a_{30} + 1)(2b + 1) = 0$. When $a_{30} = -1$, the cubic curve (6) is reducible and when $b = (-1)/2$, the right-hand sides of (1) have a common factor $cx + x + fy + 1$.

2.2. Assume that $a_{12} = c + 1$ and let $a \neq 0$. Then $V_{50} \neq 0$.

3. Let $a_{12} = -b$ and $(3a_{03} - 2f)a_{03} \neq 0$. Then $V_{40} \equiv a(b + c + 1) = 0$.

3.1. Assume that $a = 0$. Then $V_{41} \equiv f_1 f_2 = 0$, where

$$f_1 = b + c + 1,\ f_2 = (6b - 3)a_{30} + 5b^2 - 2b.$$

If $f_1 = 0$, then $c = -b - 1$ and $V_{23} = 0$ yields $a_{30} = (2 - 7b)/3$. In this case $V_{32} \equiv (3b - 1)(3b - 2) = 0$. If $b = 1/3$ or $b = 2/3$, then $V_{14} \neq 0$.

Suppose that $f_2 = 0$ and $f_1 \neq 0$. Then $a_{30} = (b(5b - 2))/(3(1 - 2b))$ and $V_{32} \equiv (3b - 1)(3b - 2) = 0$. If $b = 1/3$ or $b = 2/3$, then $V_{14} \neq 0$.

3.2. Assume that $c = -b - 1$ and $a \neq 0$. Then $V_{50} \neq 0$.

In the case (iii), the system (1) has a Darboux first integral of the form (9) and therefore the origin is a center for (1). □

5. Proof of the Main Theorem

The proof of the main result, Theorem 1.1, follows directly from Lemmas 3.1 and 4.1. The existence of a center for system (1), in Cases (i), (ii) and (iii), is equivalent to the existence of the Darboux first integrals of the form (9) defined in a neighborhood of the origin [25]. It is easy to verify that the Case (iii) is contained in the Case (ii) ($s = [(c - 2b + 2g + 3)(2b - c - 4)]/6$).

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