Acta et Commentationes, Exact and Natural Sciences ISSN: 2537-6284
Volume 2(16), 2023, Pages 106–110 E-ISSN: 2587-3644
CZU: 512.548 (043.3) DOI: 10.36120/2587-3644.v16i2.106-110

Dedicated to Professor Alexandru Șubă on the occasion of his 70th birthday

T-quasigroups with Stein 2-nd and 3-rd identity

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Abstract. In this paper we prolong research of T-quasigroups with Stein 2-rd ($xy \cdot x = y \cdot xy$) and Stein 3-rd ($xy \cdot yx = y$) identities [9].
2010 Mathematics Subject Classification: 20N05.
Keywords: quasigroup, loop, groupoid, Schröder quasigroups, Stein identity.

T-cvasigrupuri cu a doua şi a treia identitate Stein

Rezumat. În această lucrare sunt prelungite cercetările T-cvasigrupurilor cu a 2-a identitate Stein ($xy \cdot x = y \cdot xy$) şi a 3-a identitate Stein ($xy \cdot yx = y$) [9].
Cuvinte-cheie: cvasigrup, buclă, grupoid, cvasigrupuri Schröder, identitate Stein.

1. Introduction

Necessary definitions can be found in [2, 3, 4, 5, 7, 10, 14].

Definition 1.1. *Binary groupoid $(Q, \circ)$ is called a left quasigroup if for any ordered pair $(a, b) \in Q^2$ there exist the unique solution $x \in Q$ to the equation $a \circ x = b$* [2].

Definition 1.2. *Binary groupoid $(Q, \circ)$ is called a right quasigroup if for any ordered pair $(a, b) \in Q^2$ there exist the unique solution $y \in Q$ to the equation $y \circ a = b$* [2].

Definition 1.3. *Binary groupoid $(Q, \cdot)$ is called medial if this groupoid satisfies the following medial identity:*

$$xy \cdot uv = xu \cdot yv \tag{1}$$

for all $x, y, u, v \in Q$ [2].

We recall

Definition 1.4. *Quasigroup $(Q, \cdot)$ is a T-quasigroup if and only if there exists an abelian group $(Q, +)$, its automorphisms φ and ψ and a fixed element $a \in Q$ such that $x \cdot y = \varphi x + \psi y + a$ for all $x, y \in Q$* [6].

We mantion that a T-quasigroup with the additional condition $\varphi\psi = \psi\varphi$ is medial.

2. T-quasigroups with Stein 2-rd ($xy \cdot x = y \cdot xy$) identity

Theorem 2.1. *In T-quasigroup $(Q, \cdot)$ of the form $x \cdot y = \varphi x + \psi y$ Stein 2-nd identity ($xy \cdot x = y \cdot xy$) is true if $\varphi y + \psi^2 y = \varphi\psi y$, $\varphi^2 x + \psi x = \psi\varphi x$.*

Proof. From identity

$$xy \cdot x = y \cdot xy \tag{2}$$

we obtain

$$\varphi(\varphi x + \psi y) + \psi x = \varphi y + \psi(\varphi x + \psi y), \tag{3}$$

$$\varphi^2 x + \varphi\psi y + \psi x = \varphi y + \psi\varphi x + \psi^2 y, \tag{4}$$

If in (4) $x = 0$, then

$$\varphi y + \psi^2 y = \varphi\psi y. \tag{5}$$

If in (4) $y = 0$, then

$$\varphi^2 x + \psi x = \psi\varphi x. \tag{6}$$

□

Corollary 2.1. *In medial quasigroup $(Q, \cdot)$ of the form $x \cdot y = \varphi x + \psi y$, Stein 2-nd identity ($xy \cdot x = y \cdot xy$) is true if $(\varphi + \psi - 1)(\varphi - \psi) = 0$.*

Proof. From the mediality of quasigroup $(Q, \cdot)$, equalities (5), (6) we have the following

$$(\varphi + \psi - 1)(\varphi - \psi) = 0. \tag{7}$$

Indeed, $\varphi^2 x + \psi x = \varphi x + \psi^2 x$, $\varphi^2 x - \psi^2 x = \varphi x - \psi x$, $(\varphi x - \psi x)(\varphi x + \psi x) = \varphi x - \psi x$, $(\varphi x - \psi x)(\varphi x + \psi x - 1) = 0$ □

Example 2.1. *Suppose we have the group Z_n of residues modulo n. We define quasigroup $(Q, \circ)$ in the following way: $x \circ y = 4 \cdot x + 2 \cdot y \pmod 5$.*

Check: $(x \circ y) \circ x = y \circ (x \circ y)$, $16x + 8y + 2x = 4y + 8x + 4y \pmod 5$, $18x + 8y = 8x + 8y \pmod 5$, $10x = 0 \pmod 5$.

Example 2.2. *Suppose we have the group Z_n of residues modulo n. We define quasigroup $(Q, \circ)$ in the following way: $x \circ y = 2 \cdot x + 4 \cdot y \pmod{10}$.*

Verify: $(x \circ y) \circ x = y \circ (x \circ y)$, $4x + 8y + 4x = 2y + 8x + 16y \pmod{10}$, $8x + 8y = 8x + 18y \pmod{10}$, $0 = 10y \pmod{10}$.

Example 2.3. *Let us consider the group Z_n of residues modulo n. We define quasigroup $(Q, \circ)$ in the following way: $x \circ y = 11 \cdot x + 3 \cdot y \pmod{13}$.*

Check: $(x \circ y) \circ x = y \circ (x \circ y)$, $121x + 33y + 3x = 11y + 33x + 9y \pmod{13}$, $0 = 91x + 13y \pmod{13}$, $0 = 0 \pmod{13}$.

Example 2.4. *Suppose we have the group Z_n of residues modulo n. We define quasigroup* $(Q, \circ)$ *in the following way:* $x \circ y = 7 \cdot x + 11 \cdot y \pmod{17}$.

Verify: $(x \circ y) \circ x = y \circ (x \circ y)$, $49x + 77y + 11x = 7y + 77x + 121y \pmod{17}$, $60x + 77y = 128y + 77x \pmod{17}$, $0 = 0 \pmod{17}$.

Example 2.5. *Suppose we have the group Z_n of residues modulo n. We define quasigroup* $(Q, \circ)$ *in the following way:* $x \circ y = 21 \cdot x + 9 \cdot y \pmod{29}$.

Check: $(x \circ y) \circ x = y \circ (x \circ y)$, $441x + 189y + 9x = 21y + 189x + 81y \pmod{29}$, $450x + 189y = 189x + 102y \pmod{29}$, $261x + 87y = 0 \pmod{29}$, $0 = 0 \pmod{29}$.

3. T-quasigroups with Stein 3-rd identity $xy \cdot yx = y$

T-quasigroups with Stein 3-rd identity are researched in [13]. Sufficiently big number of simple medial qusigroups with 3-rd Stein identity is constructed in [12].

Theorem 3.1. *In T-quasigroup* $(Q, \cdot)$ *of the form* $x \cdot y = \varphi x + \psi y$ *Stein 3-rd identity is true if and only if* $\varphi^2 + \psi^2 = 0$, $\varphi\psi y + \psi\varphi y = \varepsilon$ [13].

Corollary 3.1. *In medial quasigroup* $(Q, \cdot)$ *of the form* $x \cdot y = \varphi x + \psi y$ *Stein 3-rd identity is true if and only if* $\varphi^2 + \psi^2 = 0$, $2\varphi\psi = \varepsilon$ [13].

Example 3.1. *Let Z_n be the group of residues modulo n. Assume that* $\varphi = 8$, $\psi = 20$. *Then* $\varphi^2 + \psi^2 = 64 + 400 = 464 = 0 \pmod{29}, n = 29$. *Next,* $2\varphi\psi = 2 \cdot 8 \cdot 20 = 320 = \varepsilon = 1 \pmod{29}$, $x \cdot y = 8x + 20y \pmod{29}$.

Verify: $8(8x + 20y) + 20(8y + 20x) = y \pmod{29}$, $64x + 160y + 160y + 400x = y \pmod{29}$, $y = y \pmod{29}$.

Example 3.2. *Let Z_n be the group of residues modulo n. We consider* $\varphi = 9$, $\psi = 21$. *Then* $\varphi^2 + \psi^2 = 81 + 441 = 522 = 0 \pmod{29}, n = 29$. *Next,* $2\varphi\psi = 2 \cdot 9 \cdot 21 = 378 = \varepsilon = 1 \pmod{29}$, $x \cdot y = 9x + 21y \pmod{29}$.

Verify: $9(9x + 21y) + 21(9y + 21x) = y \pmod{29}$, $81x + 189y + 189xy + 441x = y \pmod{29}$, $y = y \pmod{29}$.

Example 3.3. *Let Z_n be the group of residues modulo n. Let* $\varphi = 3$, $\psi = 11$. *In this case* $\varphi^2 + \psi^2 = 9 + 121 = 130 = 0 \pmod{65}, n = 65$. *Next,* $2\varphi\psi = 2 \cdot 3 \cdot 11 = 66 = \varepsilon = 1 \pmod{65}$, $x \cdot y = 3x + 11y \pmod{65}$.

Verify: $3(3x+11y)+11(3y+11x) = y \pmod{65}$, $9x+33y+33y+121x = y \pmod{65}$, $y = y \pmod{65}$.

Example 3.4. *Let Z_n be the group of residues modulo n. Let $\varphi = 11$, $\psi = 41$. Then we get* $\varphi^2+\psi^2 = 121+1681 = 1802 = 0 \pmod{53}, n = 53$. *Next,* $2\varphi\psi = 2\cdot 41\cdot 11 = 902 = \varepsilon = 1 \pmod{53}$, $x \cdot y = 11x + 41y \pmod{53}$.

Check: $11(11x+41y)+41(11y+41x) = y \pmod{53}$, $121x+451y+451y+1681x = y \pmod{53}$, $y = y \pmod{53}$.

Example 3.5. *Let Z_n be the group of residues modulo n. We consider $\varphi = 12$, $\psi = 42$. Then* $\varphi^2 + \psi^2 = 144 + 1764 = 1908 = 0 \pmod{53}, n = 53$. *Next,* $2\varphi\psi = 2 \cdot 42 \cdot 12 = 1008 = \varepsilon = 1 \pmod{53}$, $x \cdot y = 12x + 42y \pmod{53}$.

Verify: $12(12x+42y)+42(12y+42x) = y \pmod{53}$, $144x+504y+504y+1764x = y \pmod{53}$, $y = y \pmod{53}$.

Example 3.6. *Let Z_n be the group of residues modulo n. Let $\varphi = 55$, $\psi = 5$. Then* $\varphi^2 + \psi^2 = 3050 \pmod{61}, n = 61$. *Next,* $2\varphi\psi = 2 \cdot 5 \cdot 55 = 550 = \varepsilon = 1 \pmod{61}$, $x \cdot y = 55x + 5y \pmod{61}$.

Check. $55(55x + 5y) + 5(55y + 5x) = y \pmod{61}$, $3025x + 275y + 275y + 25x = y \pmod{61}$, $y = y \pmod{61}$.

Similar example gives us numbers $x\cdot y = 56x+6y \pmod{61}$. *Indeed, we have to check.* $56(56x + 6y) + 6(56y + 6x) = y \pmod{61}$, $3136x + 336y + 336y + 36x = y \pmod{61}$, $0 = 0 \pmod{61}$. *Notice, the last quasigroup is idempotent.*

Example 3.7. *Let Z_n be the group of residues modulo n. Assume that $\varphi = 59$, $\psi = 13$. Then we have* $\varphi^2+\psi^2 = 59^2+13^2 = 3650 = 0 \pmod{73}, n = 73$. *Next,* $2\varphi\psi = 2\cdot 59\cdot 13 = 1534 = \varepsilon = 1 \pmod{73}$, $x \cdot y = 59x + 13y \pmod{73}$.

Verify: $59(59x+13y)+13(59x+13y) = y \pmod{73}$, $3481x+767y+767y+169x = y \pmod{73}$, $y = y \pmod{73}$.

Idempotent example gives us numbers φ =60, ψ=14. Check: $60(60x+14y)+14(60x+14y) = y \pmod{73}$, $3600x + 840y + 840y + 196x = y \pmod{73}$, $y = y \pmod{73}$.

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Received: October 09, 2023 *Accepted: December 15, 2023*

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