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This issue is dedicated to the 70th anniversary of Professor Mihail Popa

Chisinau 2018

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PROFESSOR MIHAIL POPA – 70TH ANNIVERSARY

Professor Mihail Popa is a Moldavian mathematician and a remarkable leader of the Moldavian school of differential equations, who contributed a lot to the qualitative theory of differential equations and to the education of new generations of highly-qualified specialists. Professor Mihail Popa is Habilitated Doctor in Mathematical and Physical Sciences and Full University Professor. On May 15th 2018, Professor Mihail Popa will celebrate his 70th anniversary.

Mihail Popa was born in the village Vălcineț of the Călărași District, Republic of Moldova. In 1963, he graduated from the elementary school of the village Temeleuți, Călărași District; in 1966 he finished the secondary school nr.1 of the city Călărași and in 1971 he graduated from the Faculty of Physics and Mathematics of the State University of Chișinău. In 1978, he started his Candidate's Degree (1st PhD equivalent) (at Institute of Mathematics and Computer Sciences of the Academy of Sciences of Moldova (specialty 01.01.02 – Differential Equations).

In 1979, Mihail Popa defended his Candidate's Degree thesis in Mathematical and Physical Sciences at Gorki State University. He did it under the supervision of the well-known mathematician Academician Constantin Sibirschi. In 1992, he defended his Habilitated Doctor's degree thesis (2nd PhD thesis) in Kiev at the Institute of Mathematics of the Ukrainian Academy of Sciences.

The professional activity of Professor Mihail Popa took place at the Institute of Mathematics and Informatics of the Academy of Sciences of Moldova and it evolved as follows: Collaborator of the Laboratory (1975 – 1977), Scientific Researcher (1977 – 1980), Scientific Secretary (1980 - 1999), Deputy Director (1999 – 2006), Director (2006-2010), Scientific Principal Researcher (2010 - present).

The scientific interests of Professor Popa involve the use of invariant processes in the qualitative study of differential equations. A new viewpoint on the qualitative theory of differential equations based on the method of algebraic invariants founded by the Academician C. Sibirschi was established. This new viewpoint consists in application of the Lie algebras of operators of representations of the linear groups in the space of coefficients of systems of polynomial differential equations and of the graduate algebras of invariants and comitants to the geometry of these systems. This new viewpoint extended the scientific domain where it was applied, thus, to comprise methods of group analysis. This brought forth the study of the graduate algebras of invariants of differential equations with the help of generating functions and of Hilbert series. A sequence of generating series and of Hilbert series for diverse graduate algebras of comitants and invariants of differential systems was obtained for which it is possible to evaluate their Krull dimension.

A substantial part of the results are about the study of the Lie algebra of operators L4 for the center-affine group and its representations in the space of coefficients of autonomous systems of polynomial ordinary differential equations (S.O.D.E) of first order. Another category of results is connected to the classification of the dimensions of orbits of polynomial S.O.D.E with respect to the admissible groups. A new direction in the use of Lie algebras and of algebras of invariants is the extension to autonomous multidimensional systems of first order differential equations with polynomial right-hand sides, which have constant coefficients.

In his works Professor Popa used Lie L4 algebra and the Sibirsky's graduate algebras of the invariants and thus, a numerical estimation of the maximum margin of the maximum number of algebraically independent focal lengths was obtained. Professor Popa solved the Problem of the Center and Focal Center formulated by Henri Poincare over 130 years ago with help of the above-mentioned results for any two-dimensional differential system with polynomial nonlinearities.

Professor Mihail Popa is the author of over 120 scientific publications, among them four monographs on applications of algebras to systems of differential equations, two text books for Master's Degree students on Lie algebras and systems of differential equations and three books to popularize science.

The scientific activity of Professor Mihail Popa was highly appreciated by the scholars from many Scientific Centers, as the Université de Limoges (France), the State University of Minsk (Belarus), the University of Pitești (România), the Center of Research in Mathematics of Montreal (Canada), the University of Lund (Sweden), the Institute of Mathematics of the Romanian Academy of Science (București), the State University M. Lomonosov of Moscow, etc.

At Scientific Symposium dedicated to 70-anniversary of Professor Mihail Popa, held on 16 May 2018, on this occasion, the following letter was received and signed by the scholars of the Department of Mathematics of University of Barcelona, Spain:

We are a group of four scholars, J.C. Artes (Universitat Autonoma de Barcelona), J. Llibre (Universitat Autonoma de Barcelona), D. Schlomiuk (University de Montreal) and N. Vulpe (Institute of Mathematics and Computer Science, Moldova) who know personally Professor Popa whom we met either in Chișinău or in Montreal and we all value his work in the development of the invariant theory of differential equations, founded in Moldova by academician C.S. Sibirschi.

The four of us work on a long term project based on the results obtained by the Moldavian school in the invariant theory of differential equations.

Professor M. Popa is a brilliant disciple of C.S. Sibirschi and his work introduced a new viewpoint in the method of invariant theory, by using Lie algebras and differential operators for constructing new invariant polynomials and applying them in the qualitative study of differential equations. During the period 1998 - 2014 Professor M. Popa has been the scientific advisor of nine young mathematicians who obtained their doctorate under his supervision and he continues to form other young mathematicians. Thus, he is a leading member of the Moldavian school in mathematics.

On the occasion of his 70th birthday we congratulate Professor M. Popa on his achievements and we wish him good health and many more contributions in mathematics.

The contribution of Professor Mihail Popa to the education of new generations of highly-qualified mathematicians is enormous. From the year 1996 he works fruitfully at Tiraspol State University, where he won by competition the position of Full Professor and holds lectures for students, master students and PhD students. He was appointed as a scientific adviser of thesis for several university's graduates and master degree students. He is an exemplary figure and exceptional teacher who is inspiring his colleagues and former students in the best possible way in math and in real life.

It is one of the founders of the Seminar on Differential Equations and Algebras at Tiraspol State University, which works on regular basis since 2002 and it is designed for students, Master degree students, PhD students and scientific researches. Professor Mihail Popa has been a supervisor for ten defended PhD thesis; eight PhD graduates studied at Tiraspol State University (in Chișinău).

From February to June 2001, Mihail Popa was Invited Professor at the Université de Limoges (France), where he gave courses and seminars for students and professors.

Professor Mihail Popa was a director for many scientific projects, *in particular: the Workshop*" Qualitative Study of Differential Equations" (Chișinău, February 14-15, 2003), Second Conference of the Mathematical Society of Moldova, (Chișinău, August 17-19, 2004), International Conference „Algebraic Systems and their Applications to Differential Equations and to other mathematical domains" (Chișinău, August 21-23, 2007). He is a member of the Scientific Committee of the Institute of Mathematics and Informatics, a member of the Commission of Experts of the National Council of Accreditations and Attestation of the Republic of Moldova, a member of the Editorial

Boards of the Bulletin of the Academy of Sciences in Mathematics (Moldova) and of ROMAI Journal (Romania).

Professor Mihail Popa was awarded of Doctor Honorius Causa Degree of Tiraspol State University (2013), the Academy of Sciences of Moldova Award (2003), the ”Academician Constantin Sibirschi” Award (2004).

At the age of 70, full of vigor and optimism, Professor Mihail Popa is very active in the academic community of the Republic of Moldova. We wish him a good health, prosperity and new accomplishments in his scientific and didactic activities:

”Happy Birthday to You, Many Happy returns of the Day”.

The present volume is dedicated to Professor Mihail Popa and contains a part of communications presented at the Scientific Symposium dedicated to 70-anniversary of Professor Mihail Popa, held on 16 May 2018.

The more complete description of the life of Professor Mihail Popa and his scientific works can be found in the following publications:

1. M. Ciobanu, T. Rotaru. *130 years of the effort in the solving of the Poincaré center-focus problem*. Akademos 2013, no. 3, 13-21. (in Romanian)
2. M. Popa. *My way in mathematics*. Academy of Scenice of Moldova. Chișinău, 2018, 343 p. (in Romanian)
3. M.N. Popa, V.V. Pricop. *The center-focus problem: algebraic solutions and hypotheses*. Academy of Scenice of Moldova. Chișinău, 2018, 240 p. (in Russian)
4. M. Popa, V. Repeşco. *Lie algebras and dynamical systems in the plane*. Tiraspol State University. Chișinău, 2016, 237 p. (in Romanian)
5. M.N. Popa. *Invariant processes to differential systems and their applications in the qualitative theory.* Academy of Scenice of Moldova, 2014, 223 p. (in Russian)
6. M. Popa and T. Rotaru editors. *Academician Vladimir Andrunachievici.* Academy of Scenice of Moldova, 2009, 269 p. (in Romanian)
7. M.N. Popa. *Lie algebras and differential systems*. Academy of Scenice of Moldova, 2008, 163 p. (in Romanian)
8. M. Popa and T. Rotaru editors. *Institute of Mathematics and Informatics.* Academy of Scenice of Moldova, 2004, 454 p. (in Romanian)
9. M.N. Popa. *Algebraic methods for differential syst*ems. Flower Power edition. University of Pitești, Applied and Industrial Mathematical series, no. 15, 2004, 340 p. (in Romanian)
10. M.N. Popa. *Applications of algebras to differential systems*. Academy of Science of Moldova, Chișinău, 2001, 224 p. (in Russian)

Mitrofan CIOBAN

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AROUND THE POINCARÉ CENTER-FOCUS PROBLEM

Mitrofan M. CIOBAN, Academician
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Summary . It is well known that many mathematical models use differential equation systems and apply the qualitative theory of differential equations, introduced by Poincaré and Liapunoff. One of the problems that persists in order to control the behavior of systems of this type, is to distinguish between a focus or a center (the Center-Focus Problem). The solving of this problem goes through the computation of the Poincaré–Liapunoff quantities. The problem of estimating the maximal number of algebraically independent essential constants is called the Generalized Center-Focus Problem. The present article contains: some moments related to the history of the Center-Focus Problem; the contribution of the Academician C. Sibirschi's school in the solving of the Center-Focus Problem; methodological aspects of the M. N. Popa and V. V. Pricop solution of the Generalized Center-Focus Problem.

Key words: Poincaré-Liapunoff quantities, center-focus problem, generalized center-focus problem, Krull dimension, sober spaces.
2010 Mathematics Subject Classification: 34C05, 34C07

REFLECŢII ASUPRA PROBLEMEI LUI POINCARÉ DESPRE CENTRU ŞI FOCAR

Rezumat. Multe modele matematice folosesc sisteme de ecuaţii diferenţiale şi aplică teoria calitativă a ecuaţiilor diferenţiale, elaborată de Poincaré şi Liapunoff. Una din probleme ce persistă în studiul acestor sisteme constă în determinarea condiţiilor care asigură că punctul singular este un centru (Problema Centrului şi Focarului). Problema Generalizată a Centrului şi Focarului constă în estimarea de sus a numarului de elemete algebric independente din careva sistem complet de condiţii esenţiale. Problema Generalizată a Centrului şi Focarului a fost rezolvată de M. N.Popa şi V. V. Pricop. În articolul prezent: se expun unele momente din istoria rezolvării Problemei Centrului si Focarului; se mentionează contribuţia şcolii acad. C. Sibirschi la rezolvarea Problemei Centrului şi Focarului; se analizează aspectele metodologice ale soluţiei propusă de M. N.Popa şi V. V. Pricop.

Cuvinte-cheie: constantele Poincaré-Liapunoff, Problema Centrului şi Focarului, Problema Generalizată a Centrului şi Focarului, dimensiunea Krull, spaţiu sobru.

1. Introduction

Mathematical research has helped to solve a number of problems that have sprouted the scientists' minds for almost 2500 years, starting with Plato, Aristotle, Euclid, Archimedes. The nineteenth century brought to human civilization several surprising discoveries. Much of them is the result of the logical analysis and, in general, of the mathematical analysis of phenomena: Gauss discovered through calculus the asteroids Ceres, Palass, Vesta, Iunona; Galle also, based on the calculations, identified the planet Neptune; Mendeleev, starting from the atomic table, systematized the chemical elements and anticipated the existence of many new ones; Schliemann, based on Homer's descriptions, determined the place of Troy's placement, etc. At the end of the nineteenth century, the genius French mathematician Jules Henri Poincaré (1854 – 1912) created new areas of research such as topology, qualitative theory of dynamic systems, etc. We mention that by the quantitative methods, the Romanian mathematician Spiru Haret (1851 – 1912) demonstrated in 1878 the instability of the Solar System. He made a fundamental contribution to the n-body problem in celestial mechanics. Haret's major scientific contribution was made in 1878, in his Ph.D. thesis "Sur linvariabilité des grandes axes des orbites planétaires". At the time it was known that planets disturb each others orbits, thus deviating from the elliptic motion described by Johannes Kepler's First

Law. Pierre Laplace (in 1773) and Joseph Louis Lagrange (in 1776) had already studied the problem, both of them showing that the major axes of the orbits are stable, by using a first degree approximation of the perturbing forces. In 1808 Siméon Denis Poisson had proved that the stability also holds when using second degree approximations. In his thesis, Haret proved by using third degree approximations that the axes are not stable as previously believed, but instead feature a time variability, which he called secular perturbations. This result implies that planetary motion is not absolutely stable. Henri Poincaré considered this result a great surprise and continued Haret's research, which eventually led him to the creation of chaos theory and qualitative theory of dynamic systems [10, 19].

Henri Poincaré formulated a series of important problems, the solution of which determines the further development of mathematical sciences. One of them is the the Poincaré conjecture about the characterization of the 3-sphere, which is the hypersphere that bounds the unit ball in four-dimensional space. In 2000, it was named one of the seven Millennium Prize Problems, for which the Clay Mathematics Institute offered one million dollars prize for the first correct solution. The enigmatic Russian mathematician Grigori Perelman presented a proof of the conjecture in three papers made available in 2002 and 2003 on arXiv. On 22 December 2006, the scientific journal Science recognized Perelman's proof of the Poincaré conjecture as the scientific "Breakthrough of the Year", the first such recognition in the area of mathematics.

One of the famous problems of the qualitative theory of differential equations is the Center-Focus Problem, formulated by Poincaré about 135 years ago, in period 1881-1885 [10]. The Center-Focus Problem consists in distinguishing when a monodromic singular point is either a center or a focus. The Center-Focus Problem arises many open questions and it has deep links with Hilbert's 16th Problem.

Hilbert's 16th problem was posed by David Hilbert (1862 – 1943) at the Paris International Congress of Mathematicians in 1900, as part of his list of 23 problems in mathematics (see [4, 1, 6]). The original problem was posed as the Problem of the topology of algebraic curves and surfaces. Actually the problem consists of two similar problems in different fields of mathematics:

1. An investigation of the relative positions of the branches of real algebraic curves of degree n.

2. The determination of the upper bound for the number of limit cycles in two-dimensional polynomial vector fields of degree n and an investigation of their relative positions.

In 1976, Academician Constantin Sibirschi (Sibirsky) (1928 – 1990), Head of Laboratory at the Institute of Mathematics and Computer Science of the Academy of Sciences of Moldova, founder of the scientific school of differential equations in the Republic of Moldova, published the monograph "Algebraic Invariants of Differential Equations and Matrices" (see [16, 15]), which had a great resonance in the world of mathematicians. Over three years in 1979, Professor C.S. Coleman has published a review of this scientific paper, in which he stated that it is written in the spirit of the research of Norwegian mathematician Marius Sophus Lie (1842 – 1899). Marius Sophus Lie obtained his PhD at the University of

Christiania (present day Oslo) in 1871 with the thesis entitled "Over en Classe Geometriske Transformationer". He created the theory of continuous symmetry, introducing the concept of algebra, those bearing his name today, and applied it to the study of geometry and differential equations. It would be described by Darboux as "one of the most handsome discoveries of Modern Geometry".

The mathematician Mihail Popa, who was a student of the Professor C. Sibirschi, went his own way, starting from establishing the link between the Lie algebras and the graduated algebra of Sibirschi invariants – a tool for further researches. M. Popa took as a basis the Generalized Center-Focus Problem for Polynomial Differential Systems, avoiding calculating Poincaré–Lyapunoff quantities for each system. Poincaré–Lyapunoff's quantities was substituted by a sequence of Lie algebras and a sequence of linear subspaces of the graduate algebra of Sibirsky's invariants (see [17, 13, 14]). When estimating the maximum number of algebraically independent focal constants, he applied these algebras. As a result, a finite numerical estimation was obtained for independent algebraic focal quantities, participating in the solving of the generalized Center–Focus Problem for any polynomial differential system (see Theorem 1). Currently, Professor Mihail Popa, along with his disciples, continues his research in the theory of polynomial differential systems, successfully using Lie algebras. An analysis of the activity of Professor Mihail Popa is contained in article [2].

2. The Center-Focus Problem

Consider the differential system

$$dx/dt = P(x, y),\ dy/dt = Q(x, y), \tag{1}$$

where $P(x, y)$ and $Q(x, y)$ are polynomials that contain the linear part and satisfy the conditions $P(0, 0) = Q(0, 0) = 0$. The coefficients of polynomials $P(x, y)$, $Q(x, y)$ and variables from the system (1) takes values from the field of the real numbers $\mathbb{R}$. It is known [7, 10] that the conditions which distinguish center from focus for the system (1) consist in study of an infinite sequence of polynomials (focal quantities, Lyapunoff constants, Poincaré–Lyapunoff quantities (constants))

$$L_1,\ L_2,\ ...,\ L_k,\ ... \tag{2}$$

in the coefficients of the polynomials from the right side of the system (1).

It was shown that if the focal quantities (2) are equal to zero then the origin of coordinates for the system (1) is a center, i.e. the trajectories near this point are closed. On the contrary the origin of coordinates is a focus and the trajectories are spirals.

We can assume that $P(x, y) = \Sigma\{P_{m_i} : i \in \{0, 1, 2, ...l\}\}$ and $Q(x, y) = \Sigma\{Q_{m_i} : i \in \{0, 1, 2, ...l\}\}$, where P_{m_i} and Q_{m_i} are homogeneous polynomials of degree $m_i \geq 1$ in x and y, $m_0 = 1$. In this case we denote the system (1) by $s(1, m_1, m_2, ..., m_l)$

It is known that if the roots of characteristic equation of the singular point $O(0, 0)$ of the system (1) are imaginary, then the singular point O is a center or a focus. In this case the origin of coordinates is a singular point of the second type.

The Center-Focus Problem can be formulated as follows: *Let for the system $s(1, m_1, m_2, ..., m_l)$ the origin of coordinates be a singular point of the second type (center or focus). Find the conditions which distinguish center from focus.* This problem was posed by H. Poincaré [10]. The basic results were obtained by A. M. Lyapunoff (1857 – 1918) [7].

It is well known that, from the Hilbert's theorem on the finiteness of basis of polynomial ideals, for any concrete system $s(1, m_1, m_2, ..., m_l)$ the set

$$PL(1, m_1, m_2, ..., m_l) = \{i \in \mathbb{N} = \{1, 2, ...\} : L_i \neq 0\} \tag{3}$$

is finite. Assume that

$$PL(1, m_1, m_2, ..., m_l) = \{n_1, n_2, ..., n_\beta\} \tag{4}$$

and

$$n_\alpha = n_1 < n_2 < ... < n_\beta. \tag{5}$$

The Poincaré's Center-Focus Problem determines the following problems:

P1. *The problem of finding the number n_α or obtaining for it an argued numerical upper bound.*

P2. *The problem of finding the number n_β or obtaining for it an argued numerical upper bound.*

P3. *The problem of finding the number β or obtaining for it an argued numerical upper bound.*

Problems P1 and P2 are open. Solution of the Problem P1 contains a solution of the Center-Focus Problem. Positive solution of Problem P2 contains the solution of Problem P1. Hence Problem P2 is the strong Center-Focus Problem. Problem P3 is the weakly Center-Focus Problem.

Denote by $\mathcal{D}$ the set of all systems (1). Since the Center-Focus Problem is very complicated, it presents interest the following problem: *Finding the subsets $\mathcal{H}$ of the set $\mathcal{D}$ for which Problems P1–P3 (or some of them) are positive solutions.* Monographs [16, 18, 11, 14, 12, 3] contain some results of that kind. The Center-Focus Problem is solved for the class QS of all quadratic systems (see [16, 18, 17, 15]). Using global geometric concepts, was completely studied the class $QW3$ of quadratic systems with a third order weak focus (see [15]). The class $QW2$ of all quadratic differential systems with a weak focus of second order is important for Hilbert's 16th problem (see [15, 1, 6]). Are important (see [3, 15]) the classes:

- the class of dynamical systems with special invariant algebraic curves;

- the class of dynamical systems with a Darboux first integral or a Darboux integrating factor.

3. Sibirschi graded algebras

C. S. Sibirschi (see [13, 11, 14]), for any system $s(1, m_1, m_2, ..., m_l)$, were introduced the graded algebra $SI = SI(1, m_1, m_2, ..., m_l)$ of unimodular invariants and the graded algebra $S = S(1, m_1, m_2, ..., m_l)$ of comitants of the system $s(1, m_1, m_2, ..., m_l)$. Obviously $SI(1, m_1, m_2, ..., m_l) \subset S(1, m_1, m_2, ..., m_l)$.

The maximal number of algebraically independent elements of the Sibirsky graded algebra S is denoted by $\rho(S)$.

Let R be a finitely generated algebra over a field K. By the virtue of Krull's theorem the maximum number of elements of R that are algebraically independent over K is the same as the Krull dimension of R. Hence $\rho(S)$ is the Krull dimension of the Sibirschi algebra S.

A natural question is of course: *Which properties of $s(1, m_1, m_2, ..., m_l)$ are described in $SI(1, m_1, m_2, ..., m_l)$ and $S(1, m_1, m_2, ..., m_l)$?* In particular, the following problem may be considered as the generalized Poincaré Center-Focus Problem (see [13, 2, 14]):

P4. *The problem of finding the number* $\rho(S(1, m_1, m_2, ..., m_l))$.

In [13, 14] was proved the following unexpected assertion.

Theorem 1. $\rho(S(1, m_1, m_2, ..., m_l)) = 2(\Sigma\{m_i : 1 \leq i \leq l\} + l) + 3$ *for any system* $s(1, m_1, m_2, ..., m_l)$.

In this context, in [13] was formulated the following

Conjecture. $\beta \leq 2(\Sigma\{m_i : 1 \leq i \leq l\} + l) + 3$ *for any system* $s(1, m_1, m_2, ..., m_l)$.

Present interest the following open question

P5. *For which* n *there exist two polynomials* $P(x, y) = \Sigma\{P_{m_i} : i \in \{0, 1, 2, ...l\}\}$ *and* $Q(x, y) = \Sigma\{Q_{m_i} : i \in \{0, 1, 2, ...l\}\}$ *for which:*

- $P(x, y)$ *is a polynomial of degree* n_1, $Q(x, y)$ *is a polynomial of degree* n_2 *and* $n = maximum\{n_1, n_2\}$;

- $\beta = 2(\Sigma\{m_i : 1 \leq i \leq l\} + l) + 3$.

4. Krull's dimension of spaces

Any space X is considered to be a Kolmogorov space, i.e. for any two distinct points $x, y \in X$ there exists an open subset U of X for which the intersection $U \cap \{x, y\}$ is a singleton set.

A subset F of a space X is called an irreducible subset if for any two closed subsets F_1, F_2 of X for which $F \subset F_1 \cup F_2$ we have $F \subset F_i$ for some $i \in \{1, 2\}$. The closure $cl_X\{x\}$ of the singleton set $\{x\}$ is irreducible. A sober space is a topological space X such that every non-empty irreducible closed subset of X is the closure of one point of X. If $F = cl_X\{x\}$, then x is a generic point of the set F. A non-empty irreducible subset has a unique generic point.

Denote by $|L|$ the cardinality of a set L.

The following assertion is obvious.

Proposition 1. *A subset* L *of a space* X *is irreducible if and only if the its closure* $cl_X L$ *is irreducible.*

Example 1. Let $X = \{1, 2, 3\}$ with the topology $\{\varnothing, X, \{2\}, \{1, 2\}, \{2, 3\}\}$. Then X is a sober irreducible space and the closed subspace $Y = \{1, 3\}$ is discrete and not irreducible.

A closed subspace of a sober space is a sober space.

Example 2. Let $\omega = \{0, 1, 2, ..., n, ...\}$ and $X = \{0, 1, 2, ..., n, ..., \omega\}$ with the topology $\{\varnothing, X\} \cup \{X \setminus F : F$ *is a finite subset of* $\omega\}$. Then X is a sobre irreducible space and the subspace $Y = \{0, 1, 2, ..., n, ...\}$ is irreducible and not sober.

Define the Krull dimension $dk(X)$ of a space X to be the maximum n such that there exists a chain of pairwise distinct non-empty irreducible closed sets F_0, F_1, F_2 ..., F_n such that $F_0 \subset F_1 \subset F_2 \subset ... \subset F_n$. If Y is an irreducible closed subset of X the Krull co-dimension *co*-$dk_X(Y)$ of Y in X is the supremum over all n such that there is a chain of pairwise distinct non-empty irreducible closed sets F_0, F_1, F_2 ..., F_n such that $Y \subset F_0 \subset F_1 \subset F_2 \subset ... \subset F_n$. We observe that $dk(X) =$ *co*-$dk_X(\varnothing)$. We can assume that $dk(X) = -1$ for $X = \varnothing$.

From Proposition 1 it follows that $dk(Y) \leq dk(X)$ for any subspace Y of a space X.

If $\{X_i : i \in \mathbb{N}_n = \{1, 2, ..., n\}\}$ is a finite family of closed subspaces of a space X, $n \geq 2$ and $X = \cup\{X_i : i \in \mathbb{N}_n\}$, then $dk(X) = supremum\{dk(X_i) : i \in \mathbb{N}_n\}$. This fact follows from Claim 1 in the proof of the following proposition.

Proposition 2. *Let $\{X_i : i \in \mathbb{N}_n\}$ be a finite family of subspaces of a space X, $n \geq 2$ and $X = \cup\{X_i : i \in \mathbb{N}_n\}$. Then:*

1. If F is a closed irreducible subset of X, then there exists $i \in \mathbb{N}_n$ such that $F_i = F \cap X_i$ is an irreducible subset of the spaces X_i and X, and $F = cl_X F_i$.

2. If F is a closed irreducible subset of X, $i \in \mathbb{N}_n$, $F_i = F \cap X_i$ and $F = cl_X F_i$, then F_i is an irreducible subset of the spaces X_i and X.

3. $dk(X) = \Sigma\{dk(X_i) : i \in \mathbb{N}_n\}$.

Proof. In the first we prove the following assertion.

Clam 1. *Let F be an irreducible subset of the space X, γ is a finite family of closed subsets of X and $F \subset \cup\gamma$. Then $F \subset Y$ for some $Y \in \gamma$.*

The assertion follows from the definition for $|\gamma| \leq 2$. Assume that $k > 2$ and the assertion is true provided $|\gamma| < k$. Fix a collection γ of closed subsets of X for which $|\gamma| = k$ and $F \subset \cup\gamma$. Now fix $Y \in \gamma$ and put $\gamma_1 = \gamma \setminus \{Y\}$. We have two possible cases.

Case 1. $F \subset \cup\gamma_1$.

Since $\gamma_1| = k - 1 < k$, there exists $Z \in \gamma_1$ such that $F \subset Z$.

Case 2. $F \not\subset \cup\gamma_1$.

We put $Z = \cup\gamma_1$. Then $F \subset Z \cup Y$ and $F \not\subset Z$. Hence $F \subset Y \in \gamma$. The proof of Claim 1 is complete.

Clam 2. *If F is a closed irreducible subset of X, then there exists $i \in \mathbb{N}_n$ such that $F = cl_X(F \cap X_i)$.*

We put $F_i = F \cap X_i$ and $\Phi_i = cl_X F_i$. Then $\gamma = \{\Phi_i : i \in \mathbb{N}_n\}$ is a finite family of closed subsets of X and $F \subset \cup\gamma$. Thus $F \subset \Phi_i$ for some $i \in \mathbb{N}_n$. Claim is proved.

Assertion 2 follows from Proposition 1.

Fix a chain of pairwise distinct non-empty irreducible closed sets F_0, F_1, F_2 ..., F_m of the space X such that $F_0 \subset F_1 \subset F_2 \subset ... \subset F_n$. We put $F_{ij} = F_j \cap X_i$. Let $A_i = \{j : 0 \leq j \leq m, F = cl_X F_{ij}\}$ and $m_i = |A_i|$. Then $m_i \leq dk(X_i)$ and, by virtue of assertions 1 and 2, we have $m \leq \Sigma\{m_i : i \in \mathbb{N}_n\} \leq \Sigma\{dk(X_i) : i \in \mathbb{N}_n\}$. Assertion 3 is proved. The proof is complete.

Proposition 3. *Let $\{X_i : i \in \mathbb{N}_n\}$ be a finite family of sober subspaces of a space X, $n \geq 2$ and $X = \cup\{X_i : i \in \mathbb{N}_n\}$. Then:*

1. If F is a closed irreducible subset of X, $i \in \mathbb{N}_n$, $F_i = F \cap X_i$ and $F = cl_X F_i$, then F_i is an irreducible subset of X_i and the generic point $x \in X_i$ of F_i in X_i is a generic point of F in X.

2. X is a sober space.

Proof. Assertion 1 follows from Proposition 1. Assertion 2 follows from assertions 1 and Proposition 2.

5. Spectrum of a ring

Let R be a commutative ring [9, 5]. A subset I of R is called an ideal of R if:

1. $(I, +)$ is a subgroup of the group $(R, +)$.
2. $R \cdot I \subset I$.

3. If R is an algebra over field K, then $K \cdot I \subset I$ for any ideal I of R.

An ideal I of R is said to be prime ideal if $x, y \in R$ and $x \cdot y \in I$ implies $I \cap \{x, y\} \neq \varnothing$.

The set of all prime ideal of R, is denoted by $Spec(R)$, is called spectrum of the ring R. Let A be an ideal of R and let $V(A)$ be the collection of all prime ideal contains A. The collection of all $V(A)$ satisfies the axioms of closed subsets of a topology for $Spec(R)$, called the Zariski topology for $Spec(R)$. The space $Spec(R)$ is a compact Kolmogorov space.

For any commutative ring R and $m \in \mathbb{N}$ the following assertions are equivalent:

1. $dk(Spec(R) = m$.

2. If $I_0 \subset I_1 \subset ... \subset I_n$ is a chain of distinct prime ideals of R, then $n \leq m$.

From Theorem 1 it follows that $dk(S(1, m_1, m_2, ..., m_l)) = 2(\Sigma\{m_i : 1 \leq i \leq l\} + l) + 3$ for any system $s(1, m_1, m_2, ..., m_l)$.

6. Representation of a class of problems

The problem of determining the finite numbers n_α, n_β and β (see (5) in Section 2), or obtaining for them some numerical boundaries from the top, is important for the complete solution of the Center-Focus Problem. Obviously the Center-Focus Problem is a difficult one. So far, no general methods have been found for studying the Poincaré-Liapunoff quantities (2). In particular, there is no a general strategy to solve. Another impediment is the enormous calculations that can not be overcome by the modern supercomputers, even for the system $s(1, 2, 3)$, not to mention more complicated systems. From a psychological point of view, there are also impediments to the human conservatism to explore the problems traditionally, classically. History confirms that new, unusual methods with great difficulty are approved and valued at their fair value. However, according to Kurt Gödel's incompleteness theorem, as a rule, the resources created up to now are not sufficient for further studies. Therefore, it is undeniable that the successes of the future depend to a large extent on the newly created tools.

The study of a new problem or an unsolved problem, applying the methods of solving the known problem is done by various methods: the method of substitution of the variables; the method of crossing on limit, etc. Some of them have been well-known since ancient times and have generated new methods, appropriate to the mathematical concepts of the respective period. For example, with the method of crossing on limit, Hopf has solved the quasi-linear equations. In [8] the method of substitution of algebraic operations was successfully used in the solving of some problems of the theory of differential equations.

The principle of contrast revealed in "matter and anti-matter", "parallel spaces", "world and anti-world" penetrates into the essence of the universe, thus constituting amazing "symmetries" in the world of known phenomena. From a mathematical point of view such "symmetries" are built based on the duality principle. To build a duality means to determine a correspondence between certain types of objects, where each property of the original object corresponds to a particular property of that object in that correspondence. In any duality, their "objects" and "properties" have dual "objects" and "properties". Any concrete duality is a valuable event for these theories. The dualities in the projective geometry, the duality of Pontryagin in the theory of the local compact Abelian groups, the

Kolmogorov–Gelfand duality of compact spaces and functional Banach algebras, the dualities of Serre and Alexander in the topology, the duality of Radu Miron of the Cartan spaces and the Finsler spaces, the duality of De Morgan in the theory of sets, the Stone duality between zero-dimensional compact spaces and Boolean rings, wave-particle duality in quantum mechanics, Kramers–Wannier dualism in statistical physics, etc. This method, which is also an "anti-analogy reasoning", determines from the point of view of formal logic that many objects different in form and content are built in a similar way.

From this point of view, represent interest some correspondences of concrete class of objects of one theory into other theory. Let $\mathcal{A}$ and $\mathcal{B}$ be two theories, $\mathcal{P}$ be a class of problems of the theory $\mathcal{A}$ and $\mathcal{S}(P)$ be a set of solution of the problem $P \in \mathcal{P}$. A correspondence $\Psi : \mathcal{P} \longrightarrow \mathcal{B}$ is a representation of the class of problems $\mathcal{P}$ in the theory $\mathcal{B}$ if:

- $\Psi(P)$ is a problem of the theory $\mathcal{B}$ for any problem $P \in \mathcal{P}$;

- if $P \in \mathcal{P}$ and $\Omega \in \mathcal{S}(P)$ is a given solution of the problem P, then $\Psi(\Omega)$ is a solution of the problem $\Psi(P)$.

In this case the problem $\Psi(P)$ is a generalized form of the initial problem $P \in \mathcal{P}$. Solving generalized forms is important if for a long time there is no solution for the initial problem. Moreover, the solutions of the generalized problem propose strategies and hypotheses to solve the initial problem. Some estimates in the generalized problem solution can serve as working hypotheses for the initial problem. Furthermore, the solution to the generalized problem reflects possible ways of examining some particular cases.

Denote by $\mathcal{E}$ the theory of polynomial differential systems (1), by $\mathcal{R}$ the theory of commutative algebras and by $\mathcal{T}$ the theory of topological spaces. For any problem $s(1, m_1, m_2, ..., m_l)$ is determined the number $\{\beta\}$ as the set of solutions $\mathcal{S}(s(1, m_1, m_2, ..., m_l))$.

The correspondence $\Psi_A : \mathcal{D} \longrightarrow \mathcal{R}$, where $\Psi_A(s(1, m_1, m_2, ..., m_l)) = S(s(1, m_1, m_2, ..., m_l)$ and $\mathcal{D}$ is the set of all equations (1), is a representation of the class of problems $\mathcal{D}$ in the theory $\mathcal{R}$. We have $\Psi_A(\beta) = dk(S(1, m_1, m_2, ..., m_l))$ for any problem $s(1, m_1, m_2, ..., m_l)$ (Theorem 1).

The correspondence $\Psi_T : \mathcal{D} \longrightarrow \mathcal{T}$, where

$$\Psi_T(s(1, m_1, m_2, ..., m_l)) = Spec(S(1, m_1, m_2, ..., m_l))$$

is a representation of the class of problems $\mathcal{D}$ in the theory $\mathcal{T}$. We have

$$\Psi_a(\beta) = dk(Spec(S(1, m_1, m_2, ..., m_l)))$$

for any problem $s(1, m_1, m_2, ..., m_l)$ (Theorem 1).

Therefore the number $dk(S(1, m_1, m_2, ..., m_l)) = dk(Spec(S(1, m_1, m_2, ..., m_l)))$ is a generalized solution of the Center-Focus Problem of the system $s(1, m_1, m_2, ..., m_l)$.

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AVERAGING IN MULTIFREQUENCY SYSTEMS WITH LINEARLY TRANSFORMED ARGUMENTS AND WITH POINT AND INTEGRAL CONDITIONS

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Abstract. The review of the results of solvability research of multifrequency differential systems with linearly transformed arguments and multipoint and integral conditions is shown in this paper. The oscillation resonance condition, which depends on delay in fast variables, is introduced. The question of existence and uniqueness of the solution is considered, and the justification of averaging method on fast variables is investigated. The finest estimates of averaging method, which obviously depend on small parameter, are obtained.

Keywords: averaging method, multifrequency systems, linearly transformed argument, boundary conditions, Noether problem.
Universal Decimal Classification: 517.929.7

MEDIEREA ÎN SISTEMELE DE MULTIFRECVENȚĂ CU ARGUMENTE LINIAR TRANSFORMATE ȘI CU PUNCT ȘI CONDIȚIILE DE INTEGRARE

Rezumat. În această lucrare se prezintă o sinteză a rezultatelor ce țin de solvabilitatea sistemelor diferențiale de multifrecvență cu argumente liniar transformate și multipunct și condițiile de integrare. Este introdusă condiția de rezonanță a oscilației, care depinde de întârzierea în variabilele rapide. Se consideră problema de existență și unicitate a soluției și se justifică metoda de mediere pe variabile rapide. Sunt obținute cele mai bune estimări ale metodei de mediere, care, evident, depind de un parametru mic.

Cuvinte-cheie: metoda de mediere, sisteme de multifrecvență, argument liniar transformat, condiții de frontieră, problema Noether.

Introduction

Numerous oscillation processes in mechanics, physics, ecology, etc. are described with multifrequency nonlinear systems in the form [1]

$$\frac{da}{d\tau} = X(\tau, a, \varphi, \varepsilon), \quad \frac{d\varphi}{d\tau} = \frac{\omega(\tau, a)}{\varepsilon} + Y(\tau, a, \varphi, \varepsilon), \quad 0 \le \tau \le L, \tag{1}$$

where a and φ are $n-$ and $m-$dimensional vectors, respectively, $\tau = \varepsilon t$ is slow time, $0 < \varepsilon$ – small parameter, X, Y and vector of frequency ω belong to certain classes of smooth functions $2\pi-$periodic in φ.

As the system of equation (1) is complex both for research and for solution finding, then, in the times of Lagrange and Laplace, the procedure of averaging over fast variables φ is used. Much simpler system of equation is obtained

$$\frac{d\bar{a}}{d\tau} = X_0(\tau,\bar{a},\varepsilon), \qquad \frac{d\bar{\varphi}}{d\tau} = \frac{\omega(\tau,\bar{a})}{\varepsilon} + Y_0(\tau,\bar{a},\varepsilon), \tag{2}$$

where

$$[X_0,Y_0] = (2\pi)^{-m}\int_0^{2\pi}[X(\tau,\bar{a},\varphi,\varepsilon),Y(\tau,\bar{a},\varphi,\varepsilon)]d\varphi. \tag{3}$$

The main problem in investigation of system (1), where $m \ge 2$, is the problem of resonances. Here, the resonance case is understood as the case where the scalar product of the vector $\omega(a,\tau)$ and a nonzero vector with integer-valued coordinates turns into zero or becomes close to zero for certain values of a and τ.

The system (1) could remain in the neighborhood of the resonance quite long, and then deviation of the solutions could be

$$\|a(L,\varepsilon)-\bar{a}(L,\varepsilon)\| = O(1) \quad \text{for} \quad a(0,\varepsilon)=\bar{a}(0,\varepsilon).$$

For two-frequency system ($m=2$) when $\omega=\omega(a)$ averaging method was justified in the work of V. Arnold [2] and estimate $\|a(t,\varepsilon)-\bar{a}(t,\varepsilon)\| \le c\sqrt{\varepsilon}\ln^2\varepsilon$ was obtained for $0<\varepsilon\le\varepsilon_0<<1$ and $0\le t\le L\varepsilon^{-1}$.

Multifrequency systems (1) were investigated by E. Grebenikov [3], M. Khapaev [4], A. Neishtadt [5] and others.

Significant progress in investigation of multifrequency systems is achieved in the works of A. Samoilenko and R. Petryshyn. Such systems both with initial and with multipoint and integral conditions are investigated in [1].

The works of Ya. Bihun [6] and others are devoted to multifrequency systems with constant and variable delay. In particular, systems with integral conditions are investigated in [7]. Multifrequency systems with Noether boundary conditions are investigated by I. Krasnokutska [8]. Some new results for multifrequency systems with many linearly transformed arguments and with multipoint and/or integral conditions are also shown in [9].

Methods and materials used

Oscillation integrals, suggested in [1, 10], are used for averaging method justification. For system (1), when $\omega=\omega(\tau)$ the oscillation integral takes a form

$$I_k(t,\bar{t},\tau,\varepsilon) = \int_t^{t+\tau} f(y)\exp\left\{\frac{i}{\varepsilon}\int_{\bar{t}}^{y}(k,\omega(z))dz\right\}dy, \tag{4}$$

where $\tau\in[0,L]$, $t,\bar{t}\in \mathrm{R}$, $k\in Z^m$, $\|k\|\ne 0$, $(k,\omega)=k_1\omega_1+...+k_m\omega_m$.

The proving of existence and uniqueness of the solution is based on the Banach fixed-point theorem [11].

Obtained results and discussion

1. Multifrequency system of ODE

By $W_p(t)$ and $W_p^T(t)$ we denote the matrix $\left(\omega_\nu^{(j-1)}(t)\right)_{\nu,j=1}^{m,p}$ and its transpose, respectively.

Theorem 1 [1]. Let $\left\|(W_p^T(t)W_p(t))^{-1}W_p^T(t)\right\|$ be uniformly bounded and let the functions $\omega_\nu^{(j-1)}(t)$, $\nu=1,...,m$, $j=1,...,p$ be uniformly continuous for $t\in R$. Then one can indicate constants $\varepsilon_1>0$ and $c_1>0$ independent of $k,t,\bar t,\tau,\varepsilon$ and such that the following estimate holds for all $k\neq 0$, $t\in R$, $\bar t\in R$, $\tau\in[0,L]$, and $\varepsilon\in(0,\varepsilon_1]$:

$$\left\|I_k(t,\bar t,\tau,\varepsilon)\right\|\le c_1\varepsilon^{1/p}\left[\max_{[t,t+L]}\left\|f(y)\right\|+\frac{1}{\|k\|}\max_{[t,t+L]}\left\|f^{(1)}(y)\right\|\right].$$

Remark 1. If $p=m$ then $\det(W_m^T(t)W_m(t))=(\det W_m(t))^2$. Therefore, in this case, the condition that the Wronskian determinant of the functions $\omega_1(\tau),...\omega_m(\tau)$ is nonzero on $[0,L]$ is a sufficient condition for finding an efficient estimate for the oscillation integral $I_k(\tau,\varepsilon)$.

Let us consider the nonlinear multifrequency system (1), where $\omega=\omega(\tau)$, $\tau\in[0,L]$. Let $\omega\in C^l[0,L]$, $l\ge m+1$, $F:=[X,Y]$, $\frac{\partial F}{\partial\tau},\frac{\partial F}{\partial a}\in C^l(G)$, $\frac{\partial F}{\partial\varphi}\in C^{l+1}(G)$.

Theorem 2 [1]. Let us suppose that the following conditions are satisfied:

1) $\det\left(W_p^T(\tau)W_p(\tau)\right)\neq 0$ $\forall\tau\in[0,L]$ for certain minimal $m\le p\le l+1$;

2) X, Y and ω belong to certain classes of smooth functions;

3) for all $\tau\in[0,L]$, $y\in D_1\subset D$ and $\varepsilon\in(0,\varepsilon_0]$ the curve $\bar a=\bar a(\tau,y,\varepsilon)$, $\bar a(0,y,\varepsilon)=y$, lies in D together with its $\rho-$neighborhood.

Then one can find the constant $c_2>0$ independent on ε and such that, for sufficiently small $\varepsilon_2>0$ and for every $\tau\in[0,L]$, $y\in D_1$, $y\in D_1\subset D$ and $\psi\in R^m$, and $\varepsilon\in(0,\varepsilon_2]$ the following estimate holds:

$$\left\|a(\tau,y,\psi,\varepsilon)-\bar a(\tau,y,\varepsilon)\right\|+\left\|\varphi(\tau,y,\psi,\varepsilon)=\bar\varphi(\tau,y,\psi,\varepsilon)\right\|\le c_2\varepsilon^{1/p}, \qquad (5)$$

where $a(0,y,\psi,\varepsilon)=\bar a(0,y,\varepsilon)=y$, $\varphi(0,y,\psi,\varepsilon)=\bar\varphi(0,y,\psi,\varepsilon)=\psi$.

Theorem 2 is generalized for multifrequency systems with oscillation vector $\omega=\omega(\tau,a)$ and higher approximation systems. Result of theorem 2 is applied for the problem of existence of the solution and justification of averaging method for system (1) with boundary conditions of the form [1]

$$F\left(a\big|_{\tau=0},\varphi\big|_{\tau=0},a\big|_{\tau=L},\varphi\big|_{\tau=L},\varepsilon\right)=0 \tag{6}$$

and multipoint conditions, boundary-value problems with parameters.

2. Multifrequency Systems of Equations with Linearly Transformed Arguments

Let us suppose that λ_i and θ_j are numbers from semi-interval $(0,1]$,

$$0<\lambda_1<...<\lambda_r\le 1,\ 0<\theta_1<...<\theta_r\le 1,\ a_{\lambda_i}(\tau)=a(\lambda_i\tau),\ \varphi_{\theta_j}(\tau)=\varphi(\theta_j\tau),$$

$$a_\Lambda=(a_{\lambda_1},...,a_{\lambda_r}),\ \varphi_\Theta=(\varphi_{\theta_1},...,\varphi_{\theta_r}).$$

The system of equations is considered

$$\frac{da}{d\tau}=X(\tau,a_\Lambda,\varphi_\Theta),\qquad \frac{d\varphi}{d\tau}=\frac{\omega(\tau)}{\varepsilon}+Y(\tau,a_\Lambda,\varphi_\Theta), \tag{7}$$

where $a\in D\subset R^n$, $\varphi\in R^m$, $m\ge 1$, $\tau\in[0,L]$, $\varepsilon\in(0,\varepsilon_0]$.

In [12] the problem of existence of the solution of the system of equations (7), which satisfies integral conditions

$$\begin{gathered}\int_0^L f(\tau,a_\Lambda(\tau),\varphi_\Theta(\tau))d\tau=d_{1,}\\ \int_0^L\left[\sum_{j=1}^s b_j(\tau,a_\Lambda(\tau))\varphi_{\theta_j}(\tau)+g(\tau,a_\Lambda(\tau),\varphi_\Theta(\tau))\right]d\tau=d_2,\end{gathered} \tag{8}$$

is solved. Here vector-functions f,g,X and Y are $2\pi-$periodic in variables φ_{θ_j}, $d_1\in R^n$, $d_2\in R^m$.

In the problem (7), (8) both system (7) and vector-functions f and g in conditions (8) are averaged over fast variables. The averaged system takes the form

$$\begin{gathered}\frac{d\bar a}{d\tau}=X_0(\tau,\bar a_\Lambda),\qquad \frac{d\bar\varphi}{d\tau}=\frac{\omega(\tau)}{\varepsilon}+Y_0(\tau,\bar a_\Lambda),\\ \int_0^L f_0(\tau,\bar a_\Lambda(\tau))d\tau=d_{1,}\quad \int_0^L\left[\sum_{j=1}^s b_j(\tau,\bar a_\Lambda(\tau))\bar\varphi_{\theta_j}(\tau)+g_0(\tau,\bar a_\Lambda(\tau))\right]d\tau=d_2.\end{gathered} \tag{9}$$

The oscillation resonance condition in point τ, which depends on delay in fast variables in contradistinction to condition $(k,\omega)=0$ [13, 14], and takes the form

$$\sum_{j=1}^s\theta_j(k_j,\omega(\theta_j\tau))=0,\quad k_j\in Z^m,\quad \sum_{j=1}^s\|k_j\|\ne 0, \tag{10}$$

is found.

The existence of the solution of the problem (7), (8), is proved and the estimate of error of averaging method for slow variables is obtained

$$\|a(\tau,\bar y+\mu,\bar\psi+\xi,\varepsilon)-\bar a(\tau,\bar y,\varepsilon)\|\le c_3\varepsilon^\alpha,$$

where $0<\alpha\le(ms)^{-1}$, $\bar a(0,\bar y,\varepsilon)=\bar y$, $\|\mu\|\le c_4\varepsilon^\alpha$, $\|\eta\|\le c_5\varepsilon^{\alpha-1}$.

If $b_j = b_j(\tau)$, $j = 1,...,s$, then the solution of the problem (7), (8) exists and is unique.

In the work [9] there is investigated the system of equation (7), when for slow variables (amplitudes) the value

$$a(\tau_0) = a_0, \quad 0 \le \tau_0 \le L,$$

or linear combination of values, is set, and integral conditions have the form

$$\int_{\tau_1}^{\tau_2} \left[\sum_{j=1}^{s} b_j(\tau, a_\Lambda(\tau)) \varphi_{\theta_j}(\tau) + g(\tau, a_\Lambda(\tau), \varphi_\Theta(\tau)) \right] d\tau = d_1, \quad 0 \le \tau_1 < \tau_2 \le L.$$

Let us denote

$$S(\tau_1, \tau_2) := \sum_{j=1}^{s} \int_{\tau_1}^{\tau_2} b_j(\tau, \bar{a}_\Lambda(\tau, \bar{y})) d\tau,$$

$$S(\tau_0) := I - \sum_{j=1}^{r} \int_{0}^{\tau_0} \frac{\partial X_0((\tau, \bar{a}_\Lambda(\tau, \bar{y}))}{\partial \bar{a}_{\lambda_j}} \frac{\partial \bar{a}_{\lambda_j}(\tau, \bar{y})}{\partial \bar{y}} d\tau.$$

Theorem 3. Let us suppose that the following conditions are satisfied:

1) vector-fuctions X, Y, ω, f, g and matrix functions b_j belong to certain classes of smooth functions;

2) the Wronskian determinant of ms order of the functions $\{\omega(\theta_1\tau),..., \omega(\theta_s\tau)\}$ is not zero for $\tau \in [0, L]$;

3) the unique solution of averaged problem (9) for slow variables, which lies in D together with its $\rho-$neighborhood, exists;

4) the matrixes $S(\tau_1, \tau_2)$ and $S(\tau_0)$ are non-degenerate.

Then for sufficiently small $\varepsilon_3 > 0$ the unique solution of the problem (7), (8) exists and for every $\tau \in [0, L]$ and $\varepsilon \in (0, \varepsilon_3]$ the following estimate holds:

$$\|a(\tau, \bar{y} + \mu, \bar{\psi} + \xi, \varepsilon) - \bar{a}(\tau, \bar{y})\| + \|\varphi(\tau, \bar{y} + \mu, \bar{\psi} + \xi, \varepsilon) - \bar{\varphi}(\tau, \bar{y}, \bar{\psi}, \varepsilon) - \eta(\varepsilon)\| \le c_6 \varepsilon^\alpha,$$

where $\alpha = (ms)^{-1}$, $\|\eta(\varepsilon)\| \le c_7 \varepsilon^{\alpha - 1}$.

Remark 2. The asymptotic of estimates in theorems 1–3 under the imposed conditions is the finest.

Example 1. Let us consider the problem:

$$\frac{da}{d\tau} = 1 + \cos(\varphi - 2\varphi_\theta), \quad \theta = 0.5, \quad a(\tau_0) = a_0, \quad 0 < \tau_0 \le 1;$$

$$\frac{d\varphi}{d\tau} = \frac{1 + 2\tau}{\varepsilon}, \quad \tau \in [0,1], \quad \int_{\tau_1}^{\tau_2} \varphi(\tau) = d, \quad 0 \le \tau_1 < \tau_2 \le 1.$$

There is resonance $\omega(\tau) - 2\theta\omega(\theta\tau) = \tau$ in the point $\tau = 0$. The Wronskian determinant equals to -1. The estimate of error for slow variable is

$$\left|a(\tau,\varepsilon)-\bar{a}(\tau)\right| = \left|\int_{\tau_0}^{\tau} \cos\left(\frac{\tau^2}{\varepsilon}+\psi\right) d\tau\right| \le c_8\sqrt{\varepsilon}\ .$$

3. Averaging of Multifrequency System with Noether Boundary Conditions

Let us consider the system (7) with boundary conditions

$$A_0 a\big|_{\tau=0} + A_1 a\big|_{\tau=L} + \int_0^L f(s, a_\Lambda, \varphi_\Theta(s))ds = d, \qquad (11)$$

$$B_0\varphi\big|_{\tau=0} + B_1\varphi\big|_{\tau=L} + \int_0^L B(s)\varphi(s)ds = g_0 a\big|_{\tau=0} + g_1 a\big|_{\tau=L} + g_2\int_0^L a(s)ds, \qquad (12)$$

where f – preset $n-$measurable function $2\pi-$periodic in components φ_Θ, A_0, A_1 are constant $(n\times n)-$matrixes, B_0, B_1 – constant $(q\times m)-$matrixes, and B is vector-function of the same extension, d – preset n vector, g_0, g_1, g_2 – constant $(q\times n)-$matrixes.

Under the solution of problem (7), (11), (12) we will understand vector-function $\{a(\tau), \varphi(\tau)\}$, which satisfies the system of equations (7) and boundary condition (11) in classical understanding, and boundary condition (12) as pseudo solution [15], i.e. by substitution $\varphi = \varphi(\tau, y, \psi, \varepsilon)$, $\varphi(0, y, \psi, \varepsilon) = \psi$ in the condition (12), the initial value ψ is found as vector, which minimizes euclidean norm of discrepancy and the norm of which is the least under the conditions.

The oscillation resonance condition is condition (10).

Theorem 4. Let us suppose, that:

1) conditions 1), 2) of Theorem 3 are true;
2) the unique solution of averaged Noether problem for slow variables, which lies in D together with its $\rho-$neighborhood, exists;
3) matrix $M_1 = A_0 + A_1\dfrac{\partial\bar{a}(L,\bar{y})}{\partial\bar{y}} + \displaystyle\int_0^L f_0(s, \bar{a}(s,\bar{y}))\dfrac{\partial\bar{a}(s,\bar{y})}{\partial\bar{y}}ds$ is invertible, and $M_2 = B_0 + B_1 + \displaystyle\int_0^L B(s)ds$ is $(q\times m)$ full rank matrix, $q \ge m$.

Then there will be found constants $c_9 > 0, \varepsilon_4 > 0$ such that for every $\varepsilon \in (0, \varepsilon_2]$ the unique solution of the boundary problem (7), (11), (12) exists, moreover, for fast variables φ as pseudo solution, and for all $\tau \in [0, L]$ and $\varepsilon \in (0, \varepsilon_4]$, estimate performs

$$\left\|a(\tau, y, \psi, \varepsilon) - \bar{a}(\tau, y)\right\| + \left\|\phi(\tau, y, \psi, \varepsilon) - \bar{\varphi}(\tau, y, \psi, \varepsilon) - \eta(\varepsilon)\right\| \le c_9\varepsilon^\alpha.$$

Example 2. Let us consider the problem:

$$\frac{da}{d\tau} = 1 + \cos 4\varphi_\theta, \qquad \theta = 0.5, \qquad a(0) = a_0,$$
$$\frac{d\varphi}{d\tau} = \frac{2\tau}{\varepsilon}, \qquad \varphi(0) + \varphi(1) = 1, \quad 3\varphi(0) - 2\varphi(0) = -2.$$

There is resonance in the point $\tau = 0$ because $\omega_1(\tau) = 2\tau$. The pseudo solution $\varphi(0) = 0$ is found from boundary conditions. The estimate of error for slow variable for $\tau = 1$ is $\left|a(1,\varepsilon) - \bar{a}(1)\right| \le c_{10}\sqrt{\varepsilon}$.

The Case of the Classic Solution. Let us write boundary condition (12) in the form

$$\Omega\varphi = g_0 a\big|_{\tau=0} + g_1 a\big|_{\tau=L} + g_2 \int_0^L a(s)ds,$$

where $\Omega\varphi := B_0\varphi\big|_{\tau=0} + B_1\varphi\big|_{\tau=L} + \int_0^L B(s)\varphi(s)ds$ is linear bounded Noether operator.

Condition, which provides the existence of solving the system (7), which would satisfy the condition (12) in classic understanding was received in [15] and was written as

$$P_{\Omega^*}(g_0 a\big|_{\tau=0} + g_1 a\big|_{\tau=L} + g_2 \int_0^L a(s)ds) = 0,$$

where P_{Ω^*} is orthoprojector on the core $\ker\Omega^*$ of operator Ω^*, conjugated to Ω.

Conclusion

The results of research of multifrequency systems with linearly transformed arguments, with in the process of evolution pass through the resonances, are shown. The existence and uniqueness of solution of the boundary problems with multipoint and integral conditions are proved and the averaging method on fast variables is justified.

The obtained results are the basis for further investigation of new classes of systems, especially systems with frequencies depending on slow variables, and systems of higher approximation, and systems with transformed arguments.

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SELECTIONS OF SET-VALUED MAPPINGS AND LOCALLY FINITE PROPERTIES OF MAPPINGS

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Summary. Using some methods from the works of E. Michael [10, 11, 12], T. Dobrowol J. van Mill [5] and of one of the authors [1, 2, 3], two theorems of the existence of the selections with conditions of continuity are proved.

Keywords: set-valued mapping, selection, linear space.
2010 Mathematics Subject Classification: 54C65

SELECŢII ALE FUNCŢIILOR POLIFORME ŞI PROPRIETĂŢILE LOCAL FINITE ALE FUNCŢIILOR

Rezumat. Folosind unele metode din lucrările lui E. Michael [10, 11, 12], T. Dobrowolski şi J. van Mill [5] şi ale unui din autori [1, 2, 3], se demonstrează două teoreme de existenţă a selecţiilor cu condiţii de continuitate.

Cuvinte-cheie: funcţie poliformă, selecţie, spaţiu liniar.

1. Introduction

A single-valued mapping $f : X \longrightarrow Y$ of a space X into a space Y is said to be a selection of a given set-valued mapping $F : X \longrightarrow Y$ if $f(x) \in F(x)$ for each $x \in X$. Note that by the Axiom of Choice selections always exist. In the category of topological spaces and continuous single-valued mappings the situation is more complex.

The following problem is important: Under what conditions there exist continuous selections? There exist many theorems on continuous selections. One of them is the following classical Michael selection theorem for convex-valued mappings.

Theorem M. (E. Michael, [10]). *A multivalued mapping $F : X \longrightarrow B$ admits a continuous single-valued selection, provided that the following conditions are satisfied:*

(1) X is a paracompact space;

(2) B is a Banach space or a locally convex complete metrizable linear space;

(3) F is a lower semicontinuous mapping;

(4) for every point $x \in X$, $F(x)$ is a nonempty convex subset of B;

(5) for every point $x \in X$, $F(x)$ is a closed subset of B.

A natural question arises concerning the essentiality of each of conditions (1)-(5). There are lower semicontinuous convex-valued mappings $F : X \longrightarrow Y$ without any continuous single-valued selections, even for $X = [0; 1]$ (see Example 6.2 from [10]. An important example is published in [7]. It was proved that every convex-valued lower semicontinuous mapping mapping of a metrizable domain into a separable Banach space admits a selection, provided that all values are finite-dimensional ([10], special case of Theorem 3.1). Distinct results of this kind were proved in [4, 5, 6, 8, 13, 14, 15].

2. Main results

Any space is considered to be a Hausdorff space.

Let X and Y be topological spaces. We say that $F : X \longrightarrow Y$ is a set-valued mapping if $F(x)$ is a non-empty subset of Y for any point $x \in X$.

The set-valued mapping $F : X \longrightarrow Y$ is called:

- lower semicontinuous mapping if the set $F^{-1}(H) = \{x \in X : F(x) \cap H \neq \varnothing\}$ is an open subset of the space X for any open subset H of the space Y;

- upper semicontinuous mapping if the set $F^{-1}(H) = \{x \in X : F(x) \cap H \neq \varnothing\}$ is a closed subset of the space X for any closed subset H of the space Y;

- locally closed-valued if for any point $a \in X$ and any point $b \in F(x)$ there exist an open subset U of X and an open subset V of Y such that $F(x) \cap \cap cl_Y V$ is a closed subset of Y for each point $x \in U$;

- locally linear finite dimensional, where Y is a linear space, if for any point $a \in X$ and any point $b \in F(x)$ there exist an open subset U of X and an open subset V of Y such that $F(x) \cap V$ is a subset of some finite dimensional linear subspace of Y for each point $x \in U$.

Theorem 1. *Let $F : X \longrightarrow Y$ be a lower semicontinuous mapping of a normal metacompact or a hereditary metacompact space X into a complete metrizable space Y. If the mapping F is locally closed-valued, then there exists a lower semicontinuous compact-valued mapping $\phi : X \longrightarrow Y$ such that $\phi(x) \subset F(x)$ for each point $x \in X$.*

Proof. Let d be a complete metric on a space Y. For any point $a \in X$ we fix an open subset Ua of the space X and an open subset Va of Y such that $a \in Ua \subset F^{-1}(V_a)$ and $F(x) \cap cl_Y Va$ is a closed subset of Y for any point $x \in Ua$. Since X is a metacompact space, there exist a subset A of X and an open point-finite cover $\{W_a : a \in A\}$ of the space X such that $Wa \subset Ua$. If X is a normal space, then we can assume that Wa is an F_σ-subset of X for each $a \in A$. Hence Wa is a metacompact subspace of X for each $a \in A$. Since Va is an open subset of the complete space (Y, d), on Va there exists a complete metric d_a. For any $a \in A$ consider the lower semicontinuous closed-valued mapping $F_a : Wa \longrightarrow Va$, where $F - a(x)$ $= F9x) \cap Va$ for any $x \in Wa$, of a metacompact space Wa into a complete metrizable space (Va, d_a). Fix $a \in A$. As was proved in [1, 2], there exists a lower semicontinuous compact-valued mapping $\phi_a : Wa \longrightarrow Va$ such that $\phi_a(x) \subset F_a(x)$ for each point $x \in Wa$. Then $\phi(x) = \cup\{\phi_a(x) : a \in A, x \in Wa\}$ is the desired mapping. The proof is complete.

From the E.Michael result from [12] and Theorem 1 it follows

Corollary 1. *Let $F : X \longrightarrow Y$ be a lower semicontinuous mapping of a paracompact space X into a complete metrizable space Y. If the mapping F is locally closed-valued, then there exist a lower semicontinuous compact-valued mapping $\varphi : X \longrightarrow Y$ and a upper semicontinuous compact-valued mapping $\psi : X \longrightarrow Y$ such that $\varphi(x) \subset \psi(x) \subset F(x)$ for each point $x \in X$.*

Theorem 2. *Let $F : X \longrightarrow Y$ be a lower semicontinuous mapping of a normal metacompact or a hereditary metacompact space X into a linear metrizable locally convex space Y. If the mapping F is locally closed-valued and locally linear finite dimensional, then there exists a lower semicontinuous compact-valued mapping $\phi : X \longrightarrow Y$ such that $\phi(x) \subset F(x)$ for each point $x \in X$. Moreover, if the mapping F is convex-valued, then the mapping ϕ is convex-valued too.*

Proof. Let d be an invariant metric on a space Y. For any point $a \in X$ we fix an open subset Ua of the space X and an open subset Va of Y such that $a \in Ua \subset F^{-1}(V_a)$ and $F(x) \cap clYVa$ is a closed subset of some finite dimensional linear subspace $L(a, x)$ for any point $x \in Ua$. By virtue of the V.L. Klee theorem [9], the metric d is complete on any finite dimensional linear subspace L of Y. Hence the existence of the mapping ϕ follows from Theorem 1. Assume now that the sets $F(x)$ are convex. Then the $conv(\phi) : X \longrightarrow Y$ is lower semicontinuous too [10]. Fix $x \in X$. Then $\phi(x)$ is a compact subset of the finite dimensional subspace $L(a)$ which contains the linear subspaces $\{L(a, x) : x \in Wa\}$. Hence $conv(\phi)(x)$ is a compact convex subset of Y. The proof is complete.

From the E.Michael result [10] and Theorem 2 it follows

Corollary 2. *Let $F : X \longrightarrow Y$ be a lower semicontinuous mapping of a paracompact space X into a linear metrizable locally convex space Y. If the mapping F is locally closed-valued and locally linear finite dimensional, then there exists a single-valued continuous mapping $f : X \longrightarrow Y$ such that $f(x) \subset F(x)$ for each point $x \in X$.*

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INTEGRABILITY CONDITIONS FOR LOTKA-VOLTERRA DIFFERENTIAL SYSTEM WITH A BUNDLE OF TWO INVARIANT STRAIGHT LINES AND ONE INVARIANT CUBIC

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Abstract. For Lotka-Volterra differential system, we find conditions for the existence of a bundle of two invariant straight lines and one irreducible invariant cubic. We apply the Darboux theory to study the integrability of the obtained systems having three algebraic solutions.
Keywods: Lotka-Volterra differential system, invariant algebraic curves, integrability.
2010 Mathematics Subject Classification: 34C05

CONDIŢII DE INTEGRABILITATE PENTRU SISTEMUL DIFERENŢIAL LOTKA-VOLTERRA CU UN FASCICOL DIN DOUĂ DREPTE INVARIANTE ŞI O CUBICĂ INVARIANTĂ

Rezumat. Pentru sistemul diferenţial Lotka-Volterra sunt determinate condițiile de existență a unui fascicol format din două drepte invariante și o cubică invariantă ireductibilă. Aplicând teoria Darboux de integrabilitate se studiază integrabilitatea sistemelor obținute cu trei soluții algebrice.
Cuvinte-cheie: sistemul diferențial Lotka-Volterra, curbe invariante algebrice, integrabilitate.

1. Introduction

A planar polynomial differential system is a differential system of the form

$$\dot{x} = P(x,y), \ \ \dot{y} = Q(x,y), \tag{1}$$

where $P(x,y)$ and $Q(x,y)$ are real polynomials, $\dot{x} = \frac{dx}{dt}, \dot{y} = \frac{dy}{dt}$ denotes the derivatives with respect to independent variable t. We say that the polynomial differential system (1) has degree n, if $n = \max\{degP(x,y), degQ(x,y)\}$. In particular, when $n = 2$, a differential system (1) will be called *a quadratic system*.

In this paper we consider the quadratic system of differential equations

$$\dot{x} = x(a_1x + b_1y + c_1) \equiv P(x,y), \ \ \dot{y} = y(a_2x + b_2y + c_2) \equiv Q(x,y), \tag{2}$$

in which all coefficients $a_1, b_1, c_1, a_2, b_2, c_2$ and variables $x = x(t), \ y = y(t)$ are assumed to be real. The system (2) introduced by Lotka and Volterra appears in chemistry and ecology where it models two species in competition. It has been widely used in applied mathematics and in a large variety of physical topics such as laser physics, plasma physics, neural networks, hydrodynamics, etc [1]. Many authors have examined the integrability of system (2).

The Darboux integrability of (2) by using invariant straight lines and conics was investigated in [2]. The integrability of (2) via polynomial first integrals and polynomial inverse integrating factors was studied in [1]. The complete classification of systems (2) in the plane having a global analytic first integral was provided in [3]. The family of systems (2) according to their geometric properties encoded in the configurations of invariant straight lines which these systems possess was classified in [4].

The integrability conditions for some classes of quadratic systems (2) having an irreducible invariant cubic curve were obtained in [5] and [6].

In this paper we study the integrability of system (2) using invariant algebraic curves, two invariant straight lines and one irreducible invariant cubic curve, passing through one singular point, i.e. forming a bundle of invariant algebraic curves.

The integrability conditions will be found modulo the symmetry

$$(x, y, a_1, b_1, c_1, a_2, b_2, c_2) \to (y, x, b_2, a_2, c_2, b_1, a_1, c_1). \tag{3}$$

2. Invariant cubic curves

In this section we find the conditions under which the Lotka-Volterra system (2) has a bundle of two invariant straight lines and one irreducible invariant cubic.

Definition 2.1. An algebraic curve $\Phi(x, y) = 0$ in $\mathbb{C}^2$ with $\Phi(x, y) \in \mathbb{C}[x, y]$ is an *invariant algebraic curve* of a differential system (2) if the following identity holds

$$\frac{\partial \Phi(x, y)}{\partial x} P(x, y) + \frac{\partial \Phi(x, y)}{\partial y} Q(x, y) \equiv \Phi(x, y) K(x, y) \tag{4}$$

for some polynomial $K(x, y) \in \mathbb{C}[x, y]$ called the *cofactor* of the curve $\Phi(x, y) = 0$.

By Definition 2.1, a straight line $C + Ax + By = 0$, $A, B, C \in \mathbb{C}$, $(A, B) \neq 0$ is an *invariant straight line* of system (2) if and only if there exists a polynomial $K(x, y) = \gamma + \alpha x + \beta y$ such that the following identity holds

$$A \cdot P(x, y) + B \cdot Q(x, y) \equiv (C + Ax + By)(\gamma + \alpha x + \beta y). \tag{5}$$

If the quadratic system (2) has complex invariant straight lines then obviously they occur in complex conjugated pairs $C + Ax + By = 0$ and $\bar{C} + \bar{A}x + \bar{B}y = 0$.

By using the identity (5), it is easy to verify that the quadratic system (2) has always two invariant straight lines $x = 0$ and $y = 0$ with cofactors $K_1 = a_1 x + b_1 y + c_1$ and $K_2 = a_2 x + b_2 y + c_2$, respectively.

By Definition 2.1, a cubic curve

$$\Phi(x, y) \equiv a_{30}x^3 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3 + \\ + a_{20}x^2 + a_{11}xy + a_{02}y^2 + a_{10}x + a_{01}y = 0, \tag{6}$$

where $a_{ij} \in \mathbb{R}$, $i + j = 1,2,3$ and $(a_{30}, a_{21}, a_{12}, a_{03}) \neq 0$ is said to be an *invariant cubic curve* of system (2) if the identity (4) holds for some polynomial $K(x, y) = \gamma + \alpha x + \beta y$, called the cofactor of the invariant cubic curve $\Phi(x, y) = 0$.

Identifying the coefficients of the monomials $x^i y^j$ in (4) for cubic curve (6), we reduce this identity to an algebraic system of fourteen equations

$U_{40} \equiv a_{30}(3a_1 - \beta) = 0,$

$U_{31} \equiv a_{21}(2a_1 + a_2 - \beta) + a_{30}(3b_1 - \gamma) = 0,$

$U_{22} \equiv a_{12}(a_1 + 2a_2 - \beta) + a_{21}(2b_1 + b_2 - \gamma) = 0,$

$U_{13} \equiv\ a_{12}(b_1 + 2b_2 - \gamma) + a_{03}(3a_2 - \beta) = 0,$
$U_{04} \equiv a_{03}(3b_2 - \gamma) = 0,$
$U_{30} \equiv a_{20}(2a_1 - \beta) + a_{30}(3c_1 - \alpha) = 0,$
$U_{21} \equiv a_{11}(a_1 + a_2 - \beta) + a_{20}(2b_1 - \gamma) + a_{21}(c_2 + 2c_1 - \alpha) = 0,$ (7)
$U_{12} \equiv a_{11}(b_1 + b_2 - \gamma) + a_{02}(2a_2 - \beta) + a_{12}(2c_2 + c_1 - \alpha) = 0,$
$U_{03} \equiv a_{02}(2b_2 - \gamma) + a_{03}(3c_2 - \alpha) = 0,$
$U_{20} \equiv a_{10}(a_1 - \beta) + a_{20}(2c_1 - \alpha) = 0,$
$U_{11} \equiv a_{01}(a_2 - \beta) + a_{10}(b_1 - \gamma) + a_{11}(c_2 + c_1 - \alpha) = 0,$
$U_{02} \equiv a_{01}(b_2 - \gamma) + a_{02}(2c_2 - \alpha) = 0,$
$U_{10} \equiv a_{10}(c_1 - \alpha) = 0,$
$U_{01} \equiv a_{01}(c_2 - \alpha) = 0,$
for the unknowns $a_{30}, a_{21}, a_{12}, a_{03}, a_{20}, a_{11}, a_{02}, a_{10}, a_{01}$ and α, β, γ.

To simplify derivation of the invariant cubic curves from (7) we use the following assertion proved in [7] .

Lemma 2.1. Suppose that a polynomial system (1) of degree n has the invariant algebraic curve $\Phi(x,y) = 0$ of degree m. Let P_n, Q_n and Φ_m be the homogeneous components of P, Q and Φ of degree n and m, respectively. Then the irreducible factors of Φ_m must be factors of $yP_n - xQ_n$.

According to Lemma 2.1, the irreducible factors of Φ_3 must be the factors of

$$yP_2 - xQ_2 = xy[(a_1 - a_2)x + (b_1 - b_2)y].$$

The symmetry (3) implies $\Phi(x,y) = 0$ to have one of the following forms

$$\Phi(x,y) \equiv x^3 + a_{20}x^2 + a_{11}xy + a_{02}y^2 + a_{10}x + a_{01}y = 0, \quad (8)$$

$$\Phi(x,y) \equiv x^2y + a_{20}x^2 + a_{11}xy + a_{02}y^2 + a_{10}x + a_{01}y = 0, \quad (9)$$

$$\Phi(x,y) \equiv xy[(a_1 - a_2)x + (b_1 - b_2)y] + a_{20}x^2 + a_{11}xy + a_{02}y^2 + \\ + a_{10}x + a_{01}y = 0, \quad (10)$$

$$\Phi(x,y) \equiv x^2[(a_1 - a_2)x + (b_1 - b_2)y] + a_{20}x^2 + a_{11}xy + a_{02}y^2 + \\ + a_{10}x + a_{01}y = 0, \quad (11)$$

$$\Phi(x,y) \equiv x[(a_1 - a_2)x + (b_1 - b_2)y]^2 + a_{20}x^2 + a_{11}xy + a_{02}y^2 + \\ + a_{10}x + a_{01}y = 0, \quad (12)$$

$$\Phi(x,y) \equiv [(a_1 - a_2)x + (b_1 - b_2)y]^3 + a_{20}x^2 + a_{11}xy + a_{02}y^2 + \\ + a_{10}x + a_{01}y = 0, \quad (13)$$

where $a_{20}, a_{11}, a_{02}, a_{10}, a_{01}$ are unknown coefficients.

We study the consistency of system (7) for each cubic curve (8) - (13) and determine the conditions under which the Lotka-Volterra system (2) has an irreducible invariant cubic. We assume that

$$(a_1^2 + b_1^2 + a_2^2 + b_2^2)(a_1^2 + c_1^2)(b_2^2 + c_2^2) \neq 0 \quad (14)$$

and that

$$\frac{a_2}{a_1} = \frac{b_2}{b_1} = \frac{c_2}{c_1} \tag{15}$$

do not hold simultaneously. These conditions ensure the system (2) to be not linear and the vector field defined by (2) to be not constant.

There are proved the following theorems:

Theorem 2.1. The quadratic differential system (2) has an irreducible invariant cubic of the form (8) if and only if one of the following sets of conditions holds:

(i) $a_2 = \frac{3a_1}{2}, b_1 = b_2 = 0, c_2 = c_1;$ (ii) $b_2 = \frac{3b_1}{2}, c_2 = c_1;$

(iii) $a_2 = 3a_1, b_1 = b_2 = 0,\ c_2 = c_1;$ (iv) $a_2 = 3a_1, b_2 = \frac{3b_1}{2}, c_2 = c_1;$

(v) $a_2 = \frac{5a_1}{3}, b_2 = \frac{3b_1}{2}, c_2 = c_1;$ (vi) $a_2 = \frac{3a_1}{2},\ b_1 = b_2 = 0, c_1 = 2c_2;$

(vii) $a_2 = \frac{5a_1}{2}, b_2 = \frac{3b_1}{2}, c_1 = 2c_2;$ (viii) $a_2 = 2a_1,\ b_1 = b_2 = 0,\ c_2 = 3c_1;$

(ix) $a_2 = 3a_1,\ b_1 = b_2 = 0,\ c_2 = 2c_1;$ (x) $a_2 = 2a_1, b_2 = \frac{3b_1}{2},\ c_2 = 3c_1;$

(xi) $a_2 = \frac{15a_1}{8}, b_2 = \frac{3b_1}{2}, c_2 = 2c_1.$

Proof. Let $\Phi(x,y) = 0$ be of the form (8). We study the consistency of system (7) with $a_{30} = 1$, $a_{21} = a_{12} = a_{03} = 0$. In this case the equations $U_{40} = 0$, $U_{31} = 0$ of (7) yield $\beta = 3a_1, \gamma = 3b_1$ and $U_{10} \equiv a_{10}(\alpha - c_1) = 0, U_{01} \equiv a_{01}(\alpha - c_2) = 0$.

1) Assume that $a_{10} = a_{01} = 0$. In this case, the equations $U_{02} = 0$ and $U_{03} = 0$ imply $\alpha = 2c_2$ and $b_2 = (3b_1)/2$.

Let $c_2 = c_1$, then $a_{20} = c_1/a_1$. If $b_1 = 0$, then $a_2 = (3a_1)/2$, $a_{11} = 0$ and we get the invariant cubic

$$(a_1x + c_1)x^2 + a_{02}a_1y^2 = 0$$

with cofactor $K_3(x,y) = 3a_1x + 2c_1$, where $a_1c_1a_{02} \neq 0$. We obtain the set of conditions (i) of Theorem 2.1. If $b_1 \neq 0$ and $a_2 = 2a_1$, then $c_1 = 0$. In this case we obtain a set of conditions which is contained in (x).

Suppose that $b_1(2a_1 - a_2) \neq 0$. Then express a_{11} from $U_{12} = 0$ and a_{02} from $U_{21} = 0$. We get the invariant cubic

$$2(3a_1 - 2a_2)((a_1x + c_1)(2a_1 - a_2)x - b_1c_1y)x + b_1^2c_1y^2 = 0$$

with cofactor $K_3(x,y) = 3a_1x + 3b_1y + 2c_1$, where $a_1c_1b_1(2a_1 - a_2)(3a_1 - 2a_2) \neq 0$. We obtain the set of conditions (ii) of Theorem 2.1.

Let $c_2 \neq c_1$. Then $a_{20} = a_{11} = 0$. In this case the system (7) has no solutions.

2) Assume that $a_{10}a_{01} \neq 0$. Then $\alpha = c_1, c_2 = c_1$ and $c_1a_1 \neq 0$. We express a_{20} from $U_{30} = 0$, a_{10} from $U_{20} = 0$, a_{02} from $U_{02} = 0$ and a_{11} from $U_{11} = 0$. Then

$$U_{03} \equiv (3b_1 - b_2)(3b_1 - 2b_2) = 0.$$

Let $b_2 = 3b_1$ and $a_2 = 3a_1$. In this case $b_1 = 0$ and the system (2) has an invariant cubic curve

$$(a_1x + c_1)^2x + a_{01}a_1^2y = 0$$

with cofactor $K_3(x, y) = 3a_1x + c_1$, where $a_1c_1a_{01} \neq 0$. We get the conditions (iii).

When $b_2 = 3b_1$ and $a_2 \neq 3a_1$, the system (7) is not consistent.

Let $b_2 = (3b_1)/2$, $b_1 \neq 0$ and express a_{01} from $U_{12} = 0$. In this case we have $U_{21} \equiv (5a_1 - 3a_2)(3a_1 - a_2) = 0$. If $a_2 = 3a_1$, then we get the set of conditions (iv). The invariant cubic is

$$9a_1x(a_1x + c_1)^2 + 18a_1b_1c_1xy + 3b_1^2c_1y^2 + 2b_1c_1^2y = 0$$

with cofactor $K_3(x, y) = 3a_1x + 3b_1y + c_1$, where $a_1c_1b_1 \neq 0$.

If $a_2 = (5a_1)/3$, then we obtain the set of conditions (v). The invariant cubic is

$$a_1x(a_1x + c_1)^2 - 6a_1b_1c_1xy - 9b_1^2c_1y^2 - 6b_1c_1^2y = 0$$

with cofactor $K_3(x, y) = 3a_1x + 3b_1y + c_1$, where $a_1c_1b_1 \neq 0$.

3) Assume that $a_{10} \neq 0$ and let $a_{01} = 0$. Then $a_{02} \neq 0$ and $\alpha = c_1$. In this case we express c_1, b_2, a_{20}, a_{10}, a_{11} from the equations $U_{02} = 0$, $U_{03} = 0$, $U_{30} = 0$, $U_{20} = 0$, $U_{11} = 0$, respectively. If $b_1 = 0$ and $a_2 = (3a_1)/2$, then we obtain the invariant cubic

$$(a_1x + 2c_2)^2x + a_{02}a_1^2y^2 = 0$$

with cofactor $K_3(x, y) = 3a_1x + 2c_2$, where $a_1c_2a_{02} \neq 0$. We get the conditions (vi).

If $b_1 \neq 0$ and $a_2 = (5a_1)/2$, then we find the invariant cubic curve

$$a_1x(a_1x + 2c_2)^2 + 8a_1b_1c_2xy + 2b_1^2c_2y^2 = 0$$

with cofactor $K_3(x, y) = 3a_1x + 3b_1y + 2c_2$, where $a_1b_1c_2 \neq 0$. We determine the set of conditions (vii).

4) Assume that $a_{01} \neq 0$ and let $a_{10} = 0$. Suppose that $a_{02} = a_{20} = 0$. Then $c_2 = 3c_1$ and $b_2 = 3b_1$. If $a_{11} = 0$, then the system (7) is not consistent. If $a_{11} \neq 0$, then $b_1 = 0$ and $a_2 = 2a_1$. In this case we obtain the invariant cubic

$$a_1x^3 + a_{11}y(c_1 + a_1x) = 0$$

with cofactor $K_3(x, y) = 3(a_1x + c_1)$, where $a_1c_1a_{11} \neq 0$. We get the conditions (viii).

Let $a_{02} = 0$ and $a_{20} \neq 0$. In this case from the equations of (7) we find that $\alpha = c_2$, $c_2 = 2c_1$, $b_2 = 3b_1$, $a_{20} = c_1/a_1$, $a_{11} = a_{01}(3a_1 - a_2)/c_1$.

If $b_1 = 0$ and $a_2 = 3a_1$. Then (2) has an invariant cubic curve

$$a_1x^3 + c_1x^2 + a_{01}a_1y = 0$$

with cofactor $K_3(x, y) = 3a_1x + 2c_1$, where $a_1c_1a_{01} \neq 0$. We obtain the set of conditions (ix). If $b_1 = 0$ and $a_2 = 2a_1$, then the cubic curve (8) is reducible.

Let $a_{20} = 0$ and $a_{02} \neq 0$. In this case the equations of (7) yield $\alpha = c_2$, $c_2 = 3c_1$, $b_2 = (3b_1)/2$, $a_{01} = (2c_1a_{02})/b_1$, $a_{11} = 2a_{02}(3a_1 - a_2)/b_1$.

When $a_2 = 2a_1$, we get the set of conditions (x). The invariant cubic is

$$b_1x^3 + 2a_{02}a_1xy + a_{02}b_1y^2 + 2a_{02}c_1y = 0$$

with cofactor $K_3(x, y) = 3(a_1x + b_1y + c_1)$, where $a_1c_1b_1a_{02} \neq 0$.

Let $a_{20}a_{02} \neq 0$. In this case the equations of (7) yield $\alpha = c_2,\ c_2 = 2c_1,\ b_2 = (3b_1)/2, a_{20} = c_1/a_1, a_{02} = (3b_1a_{01})/(4c_1), a_{11} = a_{01}(3a_1 - a_2)/c_1$.

If $a_2 = (15a_1)/8$, then we obtain the set of conditions (xi). The invariant cubic is

$$9a_1^2x^2(a_1x + c_1) - 72a_1b_1c_1xy - 48b_1^2c_1y^2 - 64b_1c_1^2y = 0$$

with cofactor $K_3(x, y) = 3a_1x + 3b_1y + 2c_1$, where $a_1c_1b_1 \neq 0$. Theorem 2.1 is proved.

Theorem 2.2. The quadratic differential system (2) has an irreducible invariant cubic of the form (9) if and only if one of the following sets of conditions is realized:

(i) $a_2 = 0,\ b_2 = 2b_1,\ c_2 = c_1$; (ii) $a_2 = -a_1,\ b_1 = 0,\ c_2 = c_1$;

(iii) $a_2 = -a_1,\ b_2 = 2b_1,\ c_2 = c_1$; (iv) $a_2 = -a_1,\ b_2 = 2b_1,\ c_1 = 2c_2$;

(v) $a_2 = 0,\ b_2 = 2b_1,\ c_1 = 2c_2$; (vi) $a_2 = 0,\ b_1 = 0,\ c_2 = 2c_1$.

Proof. Let $\Phi(x, y) = 0$ be of the form (9). We study the consistency of system (7) with $a_{21} = 1,\ a_{30} = a_{12} = a_{03} = 0$. In this case the equations $U_{31} = 0,\ U_{22} = 0$ of (7) yield $\beta = 2a_1 + a_2, \gamma = 2b_1 + b_2$.

1) Assume that $a_{10} = a_{01} = 0$. Then $a_{20}a_{02} \neq 0$ and the equations of (7) yield $\alpha = 2c_1,\ c_2 = c_1, a_2 = 0, b_2 = 2b_1, a_{11} = (-2a_1a_{02})/b_1, a_{20} = (2a_{02}a_1^2 + b_1c_1)/(2b_1^2)$. In this case obtain the set of conditions (i) of Theorem 2.2. The invariant cubic is

$$x^2(2a_{02}a_1^2 + 2b_1^2y + b_1c_1) - 4a_1b_1a_{02}xy + 2b_1^2a_{02}y^2 = 0$$

with cofactor $K_3(x, y) = 2(a_1x + 2b_1y + c_1)$, where $a_{02}b_1 \neq 0$.

2) Assume that $a_{10}a_{01} \neq 0$. Then $\alpha = c_1, c_2 = c_1$. Let $a_{20} = 0$, then the equations $U_{20} = 0, U_{21} = 0, U_{12} = 0$ yield $a_2 = -a_1, a_{11} = (2c_1)/a_1,\ a_{02} = (-2b_1c_1)/(3a_1^2)$.

If $b_1 = 0$, then we get the set of conditions (ii). The invariant cubic is

$$(2a_1^2x + 4a_1c_1)xy + 2a_1^2a_{10}x + (2c_1^2 - a_1a_{10}b_2)y = 0$$

with cofactor $K_3(x, y) = a_1x + b_2y + c_1)$, where $a_{10}a_1(2c_1^2 - a_1a_{10}b_2) \neq 0$.

If $b_1 \neq 0$, then $b_2 = 2b_1$ and we obtain the set of conditions (iii) of Theorem 2.2. The invariant cubic (9) looks

$$9a_1b_1xy(a_1x + 2c_1) - 6b_1^2c_1y^2 + 8a_1c_1^2x - 3b_1c_1^2y = 0$$

with cofactor $K_3(x, y) = a_1x + 4b_1y + c_1$, where $a_1c_1b_1 \neq 0$.

Suppose that $a_{20} \neq 0$ and let $a_2 = 0$. Then the equations $U_{20} = 0, U_{02} = 0, U_{11} = 0$ yield $a_{20} = (a_1a_{10})/c_1,\ a_{02} = (2b_1a_{01})/c_1,\ a_{11} = (2a_1a_{01} + b_1a_{10} + b_2a_{10})/c_1$.

When $b_1 = 0$, the cubic is reducible. If $b_1 \neq 0$, then we express a_{01} from $U_{12} = 0$ and a_{10} from $U_{21} = 0$. In this case the cubic (9) is also reducible.

3) Assume $a_{01} = 0$ and let $a_{10} \neq 0$. Then $a_{02} \neq 0$ and $\alpha = c_1$. The equations $U_{02} = 0, U_{03} = 0$ yield $c_1 = 2c_2,\ b_2 = 2b_1$. If $a_{20} = 0$, then $a_2 = -a_1, a_{11} = (3c_2)/a_1$, $a_{02} = (-b_1c_2)/(a_1^2), a_{10} = c_2^2/(a_1b_1)$. In this case we get the set of conditions (iv) of Theorem 2.2. The invariant cubic (9) is

$$2a_1b_1xy(2a_1x + 3c_1) - 2b_1^2c_1y^2 + a_1c_1^2x = 0$$

with cofactor $K_3(x, y) = a_1x + 4b_1y + c_1$, where $a_1c_1b_1 \neq 0$.

If $a_{20} \neq 0$, then $a_2 = 0$. In this case $a_{10} = (2c_2a_{20})/a_1$, $a_{11} = (6b_1a_{20})/a_1$, $a_{20} = (3c_2)/(8b_1)$ and we obtain the set of conditions (v). The invariant cubic is

$$a_1b_1xy(8a_1x + 18c_2) + 3a_1^2c_2x^2 - 9b_1^2c_2y^2 + 6a_1c_2^2x = 0$$

with cofactor $K_3(x,y) = 2(a_1x + 2b_1y + c_2)$, where $a_1c_2b_1 \neq 0$.

4) Assume $a_{10} = 0$ and let $a_{01} \neq 0$. Then $a_{20} \neq 0$, $\alpha = c_2, a_2 = 0$ and $c_2 = 2c_1$.

Let $a_{02} = 0$. Then $b_1 = 0$ and $a_{11} = (2a_1a_{01})/c_1$. In this case we have the invariant cubic

$$2a_1^2x^2(y + a_{20}) + 2a_1(2c_1 - b_2a_{20})xy + c_1(2c_1 - b_2a_{20})y = 0$$

with cofactor $K_3(x,y) = 2a_1x + b_2y + 2c_1$, where $a_1c_1a_{20}(2c_1 - b_2a_{20}) \neq 0$. We get the set of conditions (vi) of Theorem 2.2.

Let $a_{02} \neq 0$. Then $b_2 = 2b_1$, $a_{02} = (b_1a_{01})/c_1$, $a_{11} = (2a_1a_{01})/c_1$, $a_1 = 0$ and $a_{20} = c_1/b_1$. In this case the cubic curve (9) is reducible. Theorem 2.2 is proved.

Theorem 2.3. The quadratic differential system (2) has an irreducible invariant cubic of the form (10) if and only if the following set of conditions is satisfied

(i) $a_2 = -a_1,\ b_1 = 0,\ \ c_2 = c_1$.

Proof. Let $\Phi(x,y) = 0$ be of the form (10). We study the consistency of system (7) with $a_{21} = a_1 - a_2$, $a_{12} = b_1 - b_2$, $a_{30} = a_{03} = 0$. In this case the equations $U_{31} = 0$, $U_{13} = 0$ of (7) yield $\beta = 2a_1 + a_2, \gamma = b_1 + 2b_2$.

1) Assume that $a_{10} = a_{01} = 0$. Then $a_{20}a_{02} \neq 0$ and the equations of (7) imply $\alpha = 2c_1$, $c_2 = c_1, a_2 = b_1 = 0$, $a_{11} = (-2a_1a_{02} - b_2c_1)/b_2$, $a_{20} = a_1(a_1a_{02} + b_2c_1)/b_2^2$. In this case the invariant cubic (10) is reducible.

2) Assume that $a_{10}a_{01} \neq 0$. Then $\alpha = c_1, c_2 = c_1$. When $a_{20} = a_{02} = 0$, the equations of (7) yield $a_2 = -a_1, b_2 = -b_1$, $a_{11} = c_1 = 0$, $a_{10} = (a_{01}a_1)/\,b_1$. In this case the cubic curve is reducible.

Suppose that $a_{02} \neq 0$ and let $a_{20} = 0$. Then from the equations of (7) we find that $a_2 = -a_1, b_1 = 0, a_{11} = 4c_1, a_{02} = (-2b_2c_1)/\,a_1$, $a_{01} = (-2c_1^2)/\,a_1$, $a_{10} = (4c_1^2)/\,b_2$. In this case the cubic curve (10) is reducible.

Suppose that $a_{20} \neq 0$ and let $a_{02} = 0$. Then from (7) we determine that $a_2 = 0$, $b_2 = -b_1, a_{11} = -4c_1, a_{10} = (-2c_1^2)/\,b_1, a_{20} = (-2a_1c_1)/\,b_1$, $a_{01} = (-4c_1^2)/\,a_1$. In this case the cubic curve (10) is also reducible.

When $a_{20}a_{02} \neq 0$, the equations of (7) yield $b_1 = a_2 = a_{11} = 0$, $a_{10} = c_1^2/\,b_2$, $a_{01} = (-c_1^2)/\,a_1$, $a_{20} = (a_1c_1)/\,b_2$ and $a_{02} = (-b_2c_1)/\,a_1$. The cubic (10) is reducible.

3) Assume $a_{10} \neq 0$ and let $a_{01} = 0$. Then $a_{02} \neq 0$, $b_1 = 0$ and $\alpha = c_1 = 2c_2$. We express a_{20}, a_{11}, a_{10} from the equations $U_{21} = 0, U_{12} = 0, U_{11} = 0$ of (7).

If $c_2 = 0$ or $a_2 = 0$, then the cubic curve (10) is reducible. Suppose that $a_2c_2 \neq 0$, then $a_2 = -a_1$ and $a_{02} = (-8b_2c_2)/(3a_1)$. In this case we get the set of conditions (i) of Theorem 2.3. The invariant cubic (10) looks

$$3a_1b_2xy(2a_1x - b_2y + 6c_2) - 8b_2^2c_2y^2 + 9a_1c_2^2x = 0$$

with cofactor $K_3(x,y) = a_1x + 2b_2y + 2c_2$, where $a_1c_2b_2 \neq 0$.

4) Assume $a_{01} \neq 0$ and let $a_{10} = 0$. This case is symmetric to the case 3) and we obtain the set of conditions symmetric to (i). Theorem 2.3 is proved.

Theorem 2.4. The quadratic differential system (2) has an irreducible invariant cubic of the form (11) if and only if one of the following sets of conditions holds:

(i) $a_2 = \frac{3a_1}{2}, b_2 = 2b_1,\ c_2 = c_1$; (ii) $a_2 = 3a_1,\ b_1 = 0,\ c_2 = c_1$;

(iii) $a_2 = \frac{5a_1}{3},\ b_2 = 2b_1,\ c_2 = c_1$; (iv) $b_1 = 0,\ c_2 = 3c_1$;

(v) $a_2 = \frac{15a_1}{7},\ b_2 = 2b_1,\ c_2 = 3c_1$; (vi) $a_2 = 3a_1,\ b_1 = 0,\ c_2 = 2c_1$;

(vii) $a_2 = \frac{5a_1}{2},\ b_2 = 2b_1,\ c_2 = \frac{c_1}{2}$; (viii) $a_2 = \frac{7a_1}{6},\ b_2 = 2b_1,\ c_2 = \frac{c_1}{2}$.

Proof. Let $\Phi(x,y) = 0$ be of the form (11). We study the consistency of system (7) with $a_{30} = a_1 - a_2, a_{21} = b_1 - b_2, a_{12} = a_{03} = 0$. In this case the equations $U_{40} = 0, U_{22} = 0$ of (7) yield $\beta = 3a_1, \gamma = 2b_1 + b_2$.

1) Assume that $a_{10} = a_{01} = 0$. Then $a_{02} \neq 0$ and $\alpha = 2c_2$. Suppose that $c_2 = c_1$, then $b_2 = 2b_1$. We express a_{11} and a_{20} from the equations $U_{12} = 0$ and $U_{21} = 0$ of (7).

If $a_2 = (3a_1)/2$, then we get the set of conditions (i). The invariant cubic is

$$a_1x^3 + 2b_1x^2y + c_1^2x^2 - 2a_{02}y^2 = 0$$

with cofactor $K_3(x,y) = 3a_1x + 4b_1y + 2c_1$, where $a_1b_1a_{02} \neq 0$. If $a_2 \neq (3a_1)/2$, then $a_{02} = (b_1^2c_1)/[a_1(2a_1 - a_2)]$. In this case the invariant cubic (11) is reducible.

Suppose that $c_2 \neq c_1$. Then $a_{20} = a_{11} = 0$, $b_2 = 2b_1$ and $a_2 = (3a_1)/2$. In this case the algebraic system (7) is not consistent.

2) Assume that $a_{10}a_{01} \neq 0$. Then $\alpha = c_1$, $c_2 = c_1$ and $c_1a_1 \neq 0$. We express a_{20}, a_{02}, a_{11}, a_{10} from the equations $U_{20} = 0, U_{02} = 0, U_{11} = 0, U_{30} = 0$ of (7).

Let $b_1 = 0$. If $a_2 = 2a_1$, then the invariant cubic (11) is reducible.

If $a_2 = 3a_1$, then we obtain the set of conditions (ii). The invariant cubic is

$$a_1x^2(2a_1x + b_2y + 4c_1) + 2b_2c_1xy + 2c_1^2x - a_{01}a_1y = 0$$

with cofactor $K_3(x,y) = 3a_1x + b_2y + c_1$, where $a_1c_1b_2a_{01} \neq 0$.

If $(a_2 - 2a_1)(a_2 - 3a_1) \neq 0$ and $a_{01} = (-b_2c_1^2)/a_1^2$, then (11) is reducible. Suppose that $b_1 \neq 0$. Then $U_{03} = 0$ yields $b_2 = 2b_1$. If $a_2 = 3a_1$, then $a_{01} = (-b_1c_1^2)/a_1^2$ and the invariant cubic (11) is reducible.

Let $a_2 \neq 3a_1$. Then express a_{01} from $U_{21} = 0$. In this case $U_{12} \equiv (3a_1 - 2a_2)(5a_1 - 3a_2) = 0$. If $a_2 = (3a_1)/2$, then the invariant cubic (11) is reducible.

If $a_2 = (5a_1)/3$, then we get the set of conditions (iii). The invariant cubic is

$$a_1^2x^2(2a_1x + 3b_1y + 4c_1) - 6a_1b_1c_1xy - 18b_1^2c_1y^2 + 2a_1c_1^2x - 9b_1c_1^2y = 0$$

with cofactor $K_3(x,y) = 3a_1x + 4b_1y + c_1$, where $a_1c_1b_1 \neq 0$.

3) Assume that $a_{01} \neq 0$ and let $a_{10} = 0$. Then $\alpha = c_2$. Suppose that $a_{20} = a_{02} = 0$, then $b_1 = 0$, $c_2 = 3c_1$ and $a_{11} = (2b_2c_1)/(a_2 - 2a_1)$. When $a_2 = 3a_1$, we get a set of conditions which is contained in (ii). When $a_2 \neq 3a_1$, we express a_{01} from $U_{11} = 0$. In this case we get the set of conditions (iv). The invariant cubic is

$$(3a_1 - a_2)(2a_1 - a_2)(2a_1x + b_2y)x^2 + 2b_2c_1^2y = 0$$

with cofactor $K_3(x, y) = 3a_1x + b_2y + 3c_1$, where $a_1c_1b_2(3a_1 - a_2)(2a_1 - a_2) \neq 0$.

Let $a_{20}a_{02} \neq 0$. Then $c_2 = 2c_1$ and $b_2 = 2b_1$. We express a_{11} from $U_{12} = 0$, a_{20} from $U_{21} = 0$, a_{01} from $U_{02} = 0$ and a_2 from $U_{11} = 0$. In this case the invariant cubic (11) is reducible.

Let $a_{20} = 0$ and $a_{02} \neq 0$. Then from the equations of (7) we find that $a_{02} = (2b_1a_{01})/(3c_1)$, $a_{11} = (6a_1a_{01})/(7c_1)$, $a_{01} = (49b_1c_1^2)/(3a_1^2)$ and $a_2 = (15a_1)/7$. In this case we get the set of conditions (v) of Theorem 2.4. The invariant cubic is

$$a_1^2(72a_1x + 63b_1y)x^2 - 882a_1b_1c_1xy - 686b_1^2c_1y^2 - 1029b_1c_1^2y = 0$$

with cofactor $K_3(x, y) = 3a_1x + 4b_1y + 3c_1$, where $a_1c_1b_1 \neq 0$.

Let $a_{20} \neq 0$ and $a_{02} = 0$. Then from the equations of (7) we find that

$$b_1 = 0,\ c_2 = 2c_1,\ a_{20} = (a_1a_{11})/b_2,\ a_{11} = a_{01}(3a_1 - a_2).$$

If $a_2 = 3a_1$, then we obtain the set of conditions (vi). The invariant cubic is

$$(2a_1x + b_2y + 2c_1)x^2 - a_{01}y = 0$$

with cofactor $K_3(x, y) = 3a_1x + b_2y + 2c_1$, where $a_1b_2c_1a_{01} \neq 0$.

If $a_2 \neq 3a_1$, then express a_{01} from $U_{30} = 0$. The invariant cubic is reducible.

4) Assume that $a_{10} \neq 0$ and let $a_{01} = 0$. Then $\alpha = c_1$, $b_2 = 2b_1$ and $c_2 = c_1/2$. We express a_{20}, a_{11}, a_{02} from the equations of (7) and obtain that

$$U_{21} \equiv (7a_1 - 6a_2)(5a_1 - 2a_2) = 0.$$

If $a_2 = (5a_1)/2$, then we get the set of conditions (vii). The invariant cubic is

$$a_1^2x^2(3a_1x + 2b_1y + 6c_1) + 18a_1b_1c_1xy + 3a_1c_1^2x + 9b_1^2c_1y^2 = 0$$

with cofactor $K_3(x, y) = 3a_1x + 4b_1y + c_1$, where $a_1c_1b_1 \neq 0$.

If $a_2 = (7a_1)/6$, then we get the set of conditions (viii). The invariant cubic is

$$a_1^2x^2(a_1x + 6b_1y + 2c_1) + 6a_1b_1c_1xy + a_1c_1^2x - 9b_1^2c_1y^2 = 0$$

with cofactor $K_3(x, y) = 3a_1x + 4b_1y + c_1$, where $a_1c_1b_1 \neq 0$. Theorem 2.4 is proved.

Theorem 2.5. The quadratic differential system (2) has an irreducible invariant cubic of the form (12) if and only if one of the following sets of conditions is realized:

(i) $a_2 = \frac{3a_1}{2},\ b_1 = 0,\ c_2 = c_1;$

(ii) $a_2 = 3a_1, b_2 = -b_1,\ c_2 = c_1;$

(iii) $a_2 = \frac{5a_1}{3},\ b_2 = -b_1,\ c_2 = c_1;$

(iv) $a_2 = \frac{5a_1}{3}, b_2 = -b_1, c_2 = 3c_1;$

(v) $a_2 = \frac{5a_1}{2},\ b_1 = 0,\ c_2 = 3c_1;$

(vi) $a_2 = \frac{3a_1}{2},\ b_1 = 0,\ c_2 = 3c_1;$

(vii) $a_2 = 7a_1,\ b_2 = -b_1, c_2 = 2c_1;$

(viii) $a_2 = \frac{5a_1}{3},\ b_2 = -b_1,\ c_2 = 2c_1;$

(ix) $a_2 = \frac{5a_1}{2},\ b_1 = 0,\ c_2 = 2c_1;$ (x) $a_2 = \frac{3a_1}{2},\ b_1 = 0,\ c_2 = \frac{c_1}{2};$

(xi) $a_2 = \frac{5a_1}{2},\ b_1 = 0,\ c_2 = \frac{c_1}{2}.$

Proof. Let $\Phi(x,y) = 0$ be of the form (12). We study the consistency of system (7) with $a_{30} = (a_1 - a_2)^2, a_{21} = 2(a_1 - a_2)(b_1 - b_2),\ a_{12} = (b_1 - b_2)^2\ , a_{03} = 0.$

In this case we have $U_{20} \equiv a_{20}(c_1 - c_2) = 0,\ U_{11} \equiv a_{11}(c_1 - c_2) = 0$ and the equations $U_{40} = 0,\ U_{31} = 0$ of (7) yield $\beta = 3a_1,\ \gamma = b_1 + 2b_2$. We divide the investigation into the following cases:

1) Assume that $a_{10} = a_{01} = 0$. Then $a_{02} \neq 0$ and $\alpha = 2c_2, b_1 = 0$.

Let $c_2 = c_1$. Then express a_{11} from $U_{12} = 0$ and a_{20} from $U_{30} = 0$. If $a_2 = 2a_1$, then the cubic curve (12) is reducible. If $a_2 = (3a_1)/2$, then we get the set of conditions (i) of Theorem 2.5. The invariant cubic is

$$x(a_1x + 2b_2y)^2 + a_1c_1x^2 + 4b_2c_1xy + 4a_{02}y^2 = 0$$

with cofactor $K_3(x,y) = 3a_1x + 2b_2y + 2c_1$, where $a_{02}a_1b_2 \neq 0$.

If $(a_2 - 2a_1)(2a_2 - 3a_1) \neq 0$, then $U_{21} = 0$ implies $a_{02} = (b_2^2c_1)/a_1$ and the invariant cubic (12) is reducible.

Let $c_2 \neq c_1$, then $a_{20} = a_{02} = 0$. In this case the system (7) is not consistent.

2) Assume that $a_{10}a_{01} \neq 0$. Then $\alpha = c_1, c_2 = c_1$ and $c_1b_2a_1 \neq 0$. We express a_{20} from $U_{20} = 0$, a_{02} from $U_{02} = 0$ and a_{11} from $U_{11} = 0$, then $U_{02} \equiv b_1(b_1 + b_2) = 0$.

Suppose that $b_1 = 0$ and express a_{10} from $U_{12} = 0$. If $a_2 = 2a_1$, then the cubic curve (12) is reducible. Let $a_2 \neq 2a_1$and express a_{01} from $U_{30} = 0$. If $a_2 = 3a_1$ or $a_2 = (3a_1)/2$, then the cubic (12) is reducible.

Suppose that $b_2 = -b_1,\ b_1 \neq 0$ and express a_{10} from $U_{12} = 0$. If $a_2 = 3a_1$, then we obtain the set of conditions (ii) of Theorem 2.5. The invariant cubic is

$$4x(a_1x - b_1y)^2 + 8a_1c_1x^2 - 8b_1c_1xy + 4c_1^2x + a_{01}y = 0$$

with cofactor $K_3(x,y) = 3a_1x - b_1y + c_1$, where $a_{01}a_1b_1c_1 \neq 0$.

If $a_2 \neq 3a_1$, then express a_{01} from $U_{21} = 0$. In this case $a_2 = (5a_1)/3$ and we get the set of conditions (iii) of Theorem 2.5. The invariant cubic is

$$a_1x(a_1x - 3b_1y)^2 + 2a_1^2c_1x^2 - 18a_1b_1c_1xy + a_1c_1^2x - 12b_1c_1^2y = 0$$

with cofactor $K_3(x,y) = 3a_1x - b_1y + c_1$, where $a_1b_1c_1 \neq 0$.

3) Assume that $a_{01} \neq 0$ and let $a_{10} = 0$. Then $\alpha = c_2$ and

$$U_{20} \equiv a_{20}(2c_1 - c_2) = 0.$$

Let $a_{20} = 0$. Then $U_{30} = 0$ yields $c_2 = 3c_1$. If $c_1 = 0$, then $b_2 = -b_1, a_2 = 3a_1$ and this case is contained in (ii). When $c_1 \neq 0$, we express a_{11} from $U_{11} = 0$, a_{02} from $U_{02} = 0$ and obtain that $U_{03} \equiv b_1(b_2 + b_1) = 0$.

If $b_2 = -b_1$, then express a_{01} from $U_{12} = 0$. In this case $a_2 = (5a_1)/3$ and we get the set of conditions (iv) of Theorem 2.5. The invariant cubic looks

$$a_1x(a_1x - 3b_1y)^2 - 36a_1b_1c_1xy - 27b_1c_1^2y = 0$$

with cofactor $K_3(x,y) = 3a_1x - b_1y + 3c_1$, where $a_1b_1c_1 \neq 0$.

If $b_2 \neq -b_1$, then $b_1 = 0$. We express a_{01} from $U_{12} = 0$ and obtain that

$$U_{21} \equiv (5a_1 - 2a_2)(3a_1 - 2a_2) = 0.$$

Suppose that $a_2 = (5a_1)/2$. In this case we obtain the set of conditions (v). The invariant cubic is

$$a_1x(3a_1x + 2b_2y)^2 - 48a_1b_2c_1xy - 32b_2^2y^2 - 96b_2c_1^2y = 0$$

with cofactor $K_3(x,y) = 3a_1x + 2b_2y + 3c_1$, where $a_1b_2c_1 \neq 0$.

Suppose that $a_2 = (3a_1)/2$, then we obtain the set of conditions (vi) of Theorem 2.5. The invariant cubic is

$$9a_1x(a_1x + 2b_2y)^2 + 144a_1b_2c_1xy + 32b_2^2c_1y^2 + 96b_2c_1^2y = 0$$

with cofactor $K_3(x,y) = 3a_1x + 2b_2y + 3c_1$, where $a_1b_2c_1 \neq 0$.

Let $a_{20} \neq 0$. Then $U_{30} = 0$ yields $c_2 = 2c_1$. We express a_{11} from $U_{11} = 0$, a_{02} from $U_{02} = 0$ and obtain that $U_{03} \equiv b_1(b_2 + b_1) = 0$. If $b_2 = -b_1$, then express a_{01} from $U_{12} = 0$, a_{20} from $U_{21} = 0$ and we get $U_{30} \equiv (a_2 - 7a_1)(3a_2 - 5a_1) = 0$.

When $a_2 = 7a_1$, we get the set of conditions (vii). The invariant cubic is

$$4a_1x(3a_1x - b_1y)^2 + 36a_1^2c_1x^2 - 12a_1b_1c_1xy + 3b_1c_1^2y = 0$$

with cofactor $K_3(x,y) = 3a_1x - b_1y + 2c_1$, where $a_1b_1c_1 \neq 0$.

When $a_2 = (5a_1)/3$, we obtain the set of conditions (viii). The invariant cubic is

$$4a_1x(a_1x - 3b_1y)^2 + 4a_1^2c_1x^2 - 108a_1b_1c_1xy - 81b_1c_1^2y = 0$$

with cofactor $K_3(x,y) = 3a_1x - b_1y + 2c_1$, where $a_1b_1c_1 \neq 0$.

If $b_2 \neq -b_1$, then $U_{03} = 0$ yields $b_1 = 0$. We express a_{01} from $U_{12} = 0$, a_{20} from $U_{21} = 0$ and we find that $U_{30} \equiv (5a_1 - 2a_2)(3a_1 - 2a_2)(3a_1 - a_2) = 0$.

When $a_2 = 3a_1$ or $a_2 = (3a_1)/2$, the cubic curve (12) is reducible. When $a_2 = (5a_1)/2$, we obtain the set of conditions (ix). The invariant cubic is

$$a_1x(3a_1x + 2b_2y)^2 + 9a_1^2c_1x^2 - 12a_1b_2c_1xy - 12b_2^2c_1y^2 - 24b_2c_1^2y = 0$$

with cofactor $K_3(x,y) = 3a_1x + 2b_2y + 2c_1$, where $a_1b_2c_1 \neq 0$.

4) Assume that $a_{10} \neq 0$ and let $a_{01} = 0$. Then $a_{02} \neq 0$, $\alpha = c_1$, $b_1 = 0$ and $c_2 = c_1/2$. We express a_{20} from $U_{20} = 0$, a_{11} from $U_{11} = 0$, a_{10} from $U_{30} = 0$ and obtain that $U_{21} \equiv (5a_1 - 2a_2)(3a_1 - 2a_2) = 0$.

If $a_2 = (3a_1)/2$, then we get the set of conditions (x). The invariant cubic is

$$x(a_1x + 2b_2y)^2 + 2a_1c_1x^2 + 4b_2c_1xy + 4a_{02}y^2 + c_1^2x = 0$$

with cofactor $K_3(x,y) = 3a_1x + 2b_2y + c_1$, where $a_1b_2c_1 \neq 0$.

If $a_2 = (5a_1)/2$, then $a_{02} = (4b_2^2c_1)/a_1$ and we obtain the set of conditions (xi). The invariant cubic is

$$a_1x(3a_1x + 2b_2y)^2 + 18a_1^2c_1x^2 + 36a_1b_2c_1xy + 16b_2^2c_1y^2 + 9a_1c_1^2x = 0$$

with cofactor $K_3(x,y) = 3a_1x + 2b_2y + c_1$, where $a_1b_2c_1 \neq 0$. Theorem 2.5 is proved.

Theorem 2.6. The quadratic differential system (2) has an irreducible invariant cubic of the form (13) if and only if one of the following sets of conditions holds:

(i) $a_2 = 2a_1,\ b_1 = 2b_2,\ c_2 = c_1 = 0$; (ii) $a_1(b_1 - 4b_2) + a_2(3b_2 - 2b_1) = 0,$ $c_2 = c_1$;

(iii) $a_2 = \frac{5a_1}{3},\ b_1 = 3b_2,\ c_2 = c_1$; (iv) $b_1 = \frac{7b_2}{3},\ a_2 = 5a_1,\ c_2 = 3c_1$;

(v) $b_1 = \frac{7b_2}{3},\ a_2 = \frac{9a_1}{5}, c_2 = 3c_1$; (vi) $b_1 = \frac{4b_2}{3},\ a_2 = 0,\ c_2 = 3c_1$;

(vii) $b_1 = \frac{5b_2}{2},\ a_2 = 4a_1,\ c_2 = 2c_1$; (viii) $b_1 = \frac{5b_2}{2},\ a_2 = -5a_1,\ c_2 = 2c_1$;

(ix) $a_2 = \frac{7a_1}{4},\ b_1 = \frac{5b_2}{2},\ c_2 = 2c_1$; (x) $a_2 = \frac{3a_1}{4},\ b_1 = \frac{7b_2}{6},\ c_2 = 2c_1$.

Proof. Let $\Phi(x, y) = 0$ be of the form (13). We study the consistency of system (7) with $a_{30} = (a_1 - a_2)^3$, $a_{12} = 3(a_1 - a_2)(b_1 - b_2)^2$, $a_{21} = 3(a_1 - a_2)^2(b_1 - b_2)$, $a_{03} = (b_1 - b_2)^3$. In this case the equations $U_{40} = 0$, $U_{31} = 0$ of (7) yield $\beta = 3a_1$, $\gamma = 3b_2$. We divide the investigation into the following cases:

1) Assume that $a_{10} = a_{01} = 0$. Let $a_{02} = 0$ and $a_{11} = 0$. Then $\alpha = 2c_2$, $c_2 = c_1 = 0$ and $a_1 = 0$. We obtain a contradiction with conditions (14).

If $a_{02} = 0$ and $a_{11} \neq 0$, then $c_1 = 2c_2$, $\alpha = c_1 + c_2$. In this case the system (7) is consistent only if $a_{20} = 0$. We get the set of conditions (i) of Theorem 2.6. The invariant cubic is

$$(a_1x - b_2y)^3 - a_{11}xy = 0$$

with cofactor $K_3(x, y) = 3(a_1x + b_2y)$, where $a_{11}a_1b_2 \neq 0$.

Let $a_{02} \neq 0$. Then $\alpha = 2c_2$ and $U_{20} \equiv (c_1 - c_2)a_{20} = 0$, $U_{11} \equiv (c_1 - c_2)a_{11} = 0$. Suppose that $c_2 = c_1$. Then $b_2a_1 \neq 0$. We express a_{20} from $U_{30} = 0$ and a_{02} from $U_{03} = 0$. If $a_2 = 2a_1$, then the cubic curve (13) is reducible. If $a_2 \neq 2a_1$, then express a_{11} from $U_{21} = 0$, and $U_{12} = 0$ becomes $U_{12} \equiv e_1e_2e_3 = 0$, where $e_1 = a_1b_1 - 2a_1b_2 + a_2b_2$, $e_2 = 2a_1b_1 - 3a_1b_2 - a_2b_1 + 2a_2b_2$, $e_3 = 3a_1b_1 - 4a_1b_2 - 2a_2b_1 + 3a_2b_2$.

If $e_1 = 0$ or $e_2 = 0$, then the cubic curve (13) is reducible. If $e_3 = 0$, then we obtain the set of conditions (ii) of Theorem 2.6. The invariant cubic looks

$$a_1b_2(2a_1 - a_2)[(a_1 - a_2)x - (b_1 - b_2)y]^3 + c_1b_2(a_1 - a_2)^3(2a_1 - a_2)x^2 +$$
$$+2a_1b_2c_1(b_1 - b_2)(a_1 - a_2)^2xy + a_1c_1(b_1 - b_2)^3(2a_1 - a_2)y^2 = 0,$$

where $K_3(x, y) = 3a_1x + 3b_2y + 2c_1$ and $c_1a_1b_2(2a_1 - a_2)(a_1 - a_2)(b_1 - b_2) \neq 0$.

Suppose $c_2 \neq c_1$, then $a_{20} = a_{11} = 0$ and the system (7) has no solutions.

2) Assume that $a_{10}a_{01} \neq 0$. Then $\alpha = c_1, c_2 = c_1$ and $c_1b_2a_1 \neq 0$. We express a_{20} from $U_{20} = 0$, a_{02} from $U_{02} = 0$, a_{11} from $U_{11} = 0$, a_{01} from $U_{03} = 0$ and a_{10} from $U_{30} = 0$. In this case the equations $U_{21} = 0$ and $U_{12} = 0$ have a common factor $h = a_1b_1 - 2a_1b_2 + a_2b_2$. If $h = 0$, then the cubic (13) is reducible.

Let $h \neq 0$ and suppose $b_1 = 3b_2$. Then we obtain an irreducible cubic curve of the form (13) only if $a_2 = (5a_1)/3$. We get the set of conditions (iii). The invariant cubic is

$$(a_1x - 3b_2y)^3 + 2a_1^2c_1x^2 - 36a_1b_2c_1xy - 54a_2^2c_1y^2 + a_1c_1^2x - 27b_2c_1^2y = 0$$

with cofactor $K_3(x,y) = 3a_1x + 3b_2y + c_1$, where $c_1a_1b_2 \neq 0$.

Suppose that $h(b_1 - 3b_2) \neq 0$ and let $a_2 = 3a_1$. Then (2) has an irreducible cubic curve only if $b_1 = (5b_2)/3$. In this case we obtain the set of conditions symmetric to (iii).

Let $h(b_1 - 3b_2)(a_2 - 3a_1) \neq 0$. Then the system of equations (7) ($U_{21} = 0$, $U_{12} = 0$) is not consistent.

3) Assume that $a_{01} \neq 0$ and let $a_{10} = 0$. Then $\alpha = c_2$ and

$$U_{20} \equiv a_{20}(2c_1 - c_2) = 0.$$

Suppose that $a_{20} = 0$. Then $U_{30} = 0$ yields $c_2 = 3c_1$ and $c_1b_2 \neq 0$. We express a_{02} from $U_{03} = 0$, a_{01} from $U_{02} = 0$ and a_{11} from $U_{11} = 0$. In this case

$$U_{21} \equiv (3b_1 - 7b_2)[(3a_1 - a_2)b_1 - 2(2a_1 - a_2)b_2] = 0.$$

If $b_1 = (7b_2)/3$ and $a_2 = 5a_1$, then we obtain the set of conditions (iv) of Theorem 2.6. The invariant cubic is

$$(3a_1x - b_2y)^3 + 18a_1b_2c_1xy - 6b_2^2c_1y^2 - 9b_2c_1^2y = 0$$

with cofactor $K_3(x,y) = 3(a_1x + b_2y + c_1)$, where $c_1a_1b_2 \neq 0$.

If $b_1 = (7b_2)/3$ and $a_2 = (9a_1)/5$, then we obtain the set of conditions (v) of Theorem 2.6. The invariant cubic is

$$(3a_1x - 5b_2y)^3 - 1350a_1b_2c_1xy - 750b_2^2c_1y^2 - 1125b_2c_1^2y = 0$$

with cofactor $K_3(x,y) = 3(a_1x + b_2y + c_1)$, where $c_1a_1b_2 \neq 0$.

Suppose that $3b_1 - 7b_2 \neq 0$ and let $(3a_1 - a_2)b_1 - 2(2a_1 - a_2)b_2 = 0$. Then $U_{12} = 0$ imply $a_2 = 0$. We get the set of conditions (vi). The invariant cubic looks

$$(3a_1x + b_2y)^3 + 27a_1b_2c_1xy + 6b_2^2c_1y^2 + 9b_2c_1^2y = 0$$

with cofactor $K_3(x,y) = 3(a_1x + b_2y + c_1)$, where $c_1a_1b_2 \neq 0$.

Suppose that $a_{20} \neq 0$, then $U_{20} = 0$ yields $c_2 = 2c_1$ and $a_1c_1b_2 \neq 0$. We express a_{02} from $U_{02} = 0$, a_{11} from $U_{11} = 0$, a_{01} from $U_{03} = 0$ and a_{20} from $U_{30} = 0$. In this case $U_{12} \equiv (2b_1 - 5b_2)[2b_1(3a_1 - a_2) - b_2(9a_1 - 5a_2)] = 0$.

If $b_1 = (5b_2)/2$ and $a_2 = 4a_1$, then we obtain the set of conditions (vii) of Theorem 2.6. The invariant cubic is

$$(2a_1x - b_2y)^3 + 8a_1^2c_1x^2 + 4a_1b_2c_1xy - 6b_2^2c_1y^2 - 4b_2c_1^2y = 0$$

with cofactor $K_3(x,y) = 3a_1x + 3b_2y + 2c_1$, where $c_1a_1b_2 \neq 0$.

If $b_1 = (5b_2)/2$ and $a_2 = -5a_1$, then we obtain the set of conditions (viii) of Theorem 2.6. The invariant cubic is

$$(4a_1x + b_2y)^3 + 64a_1^2c_1x^2 + 32a_1b_2c_1xy + 4b_2^2c_1y^2 + 4b_2c_1^2y = 0$$

with cofactor $K_3(x,y) = 3a_1x + 3b_2y + 2c_1$, where $c_1a_1b_2 \neq 0$.

If $b_1 = (5b_2)/2$ and $a_2 = (7a_1)/4$, then we obtain the set of conditions (ix) of Theorem 2.6. The invariant cubic is

$$(a_1x-2b_2y)^3+a_1^2c_1x^2-40a_1b_2c_1xy-32b_2^2c_1y^2-32b_2c_1^2y=0$$
with cofactor $K_3(x,y)=3a_1x+3b_2y+2c_1$, where $c_1a_1b_2\neq 0$.

Suppose that $2b_1-5b_2\neq 0$ and let $2b_1(3a_1-a_2)-b_2(9a_1-5a_2)=0$. Then $U_{21}\equiv a_2(3a_1-4a_2)=0$. If $a_2=0$, then the cubic curve (13) is reducible.

If $a_2=(3a_1)/4$, then we obtain the set of conditions (x). The invariant cubic is
$$(3a_1x+2b_2y)^3+27a_1^2c_1x^2+72a_1b_2c_1xy+32b_2^2c_1y^2+32b_2c_1^2y=0$$
with cofactor $K_3(x,y)=3a_1x+3b_2y+2c_1$, where $c_1a_1b_2\neq 0$.

4) Assume that $a_{10}\neq 0$ and let $a_{01}=0$. In this case we obtain the sets of conditions symmetric to (iv) - (x). Theorem 2.6 is proved.

3. Darboux theory of integrability

Let the polynomial differential system (1) have the invariant algebraic curves $\Phi_j(x,y)=0$, $j=1,\dots,q$ with cofactors $K_j(x,y)$. Then in most cases a first integral (an integrating factor) can be constructed in the Darboux form [8]
$$\Phi_1^{h_1}\Phi_2^{h_2}\cdots\Phi_q^{h_q}$$
and we say that the polynomial system (1) is *Darboux integrable*.

Theorem 3.1. The system (1) has a Draboux first integral
$$F(x,y)\equiv\Phi_1^{h_1}\Phi_2^{h_2}\cdots\Phi_q^{h_q}=\mathrm{C} \qquad (16)$$
if and only if there exists constants $\alpha_j\in\mathbb{C}$, not all identically zero, such that
$$h_1K_1(x,y)+h_2K_2(x,y)+\cdots+h_qK_q(x,y)\equiv 0, \qquad (17)$$
where $K_j(x,y)$ are the cofactors of $\Phi_j(x,y)=0$, $j=1,\dots,q$.

Following [8], the relation (16) is a first integral for system (1) if and only if
$$\frac{\partial F(x,y)}{\partial x}P(x,y)+\frac{\partial F(x,y)}{\partial y}Q(x,y)\equiv 0.$$

If a first integral cannot be found, Darboux proposed to search for an integrating factor μ of the same form.

Theorem 3.2. The system (1) has a Draboux integrating factor
$$\mu=\Phi_1^{h_1}\Phi_2^{h_2}\cdots\Phi_q^{h_q} \qquad (18)$$
if and only if there exists constants $\alpha_j\in\mathbb{C}$, not all identically zero, such that
$$h_1K_1(x,y)+h_2K_2(x,y)+\cdots+h_qK_q(x,y)+\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}\equiv 0, \qquad (19)$$
where $K_j(x,y)$ are the cofactors of $\Phi_j(x,y)=0$, $j=1,\dots,q$.

Following [8], the relation (18) is an integrating for system (1) if and only if
$$P(x,y)\frac{\partial\mu}{\partial x}+Q(x,y)\frac{\partial\mu}{\partial y}+\mu\left(\frac{\partial P}{\partial x}+\frac{\partial Q}{\partial y}\right)\equiv 0.$$

How many invariant algebraic curves $\Phi_j(x,y)=0$ must admit the system (1) to have a Daroux first integral or a Darboux integrating factor? Darboux proved

Theorem 3.3. Suppose system (1) has q distinct invariant algebraic curves $\Phi_j(x,y)=0$, $j=1,\dots,q$. If $q\geq n(n+1)/2$, then either we have a Darboux first integral or a Darboux integrating factor.

By Theorem 3.3, in the case of quadratic system (2), if $q\geq 3$, then either we have a Darboux first integral or a Darboux integrating factor.

The method of Darboux is very useful and elegant one to prove integrability for some classes of differential systems depending on parameters [8].

4. Darboux first integrals

In this section we determine the sets of conditions from Theorems 2.1 – 2.5, under which the quadratic system (2) has Darboux first integrals of the form

$$x^{h_1}y^{h_2}\Phi^{h_3}=C, \tag{20}$$

where $x=0, y=0$ are invariant straight lines, $\Phi=0$ is an irreducible invariant cubic of the form (6) and h_1, h_2, h_3 are real numbers.

To construct the first integrals (20) we take into account the cofactors $K_1(x,y), K_2(x,y)$ and $K_3(x,y)$ of these algebraic solutions, obtained in the proofs of Theorems 2.1 – 2.5. Then we apply the identity (17)

$$h_1K_1(x,y)+h_2K_2(x,y)+h_3K_3(x,y)\equiv 0 \tag{21}$$

to each set of conditions from Theorems 2.1 – 2.5. It was proved the following theorem.

Theorem 4.1. The Lotka-Volterra system (2) has a Darboux first integral of the form (20) if one of the following conditions is satisfied:

(i) $a_2=\frac{3a_1}{2}, b_1=b_2=0, c_2=c_1$;

(ii) $a_2=3a_1,\ b_1=b_2=0,\ c_2=c_1$;

(iii) $a_2=\frac{3a_1}{2},\ b_1=b_2=0,\ c_1=2c_2$;

(iv) $a_2=2a_1,\ b_1=b_2=0,\ c_2=3c_1$;

(v) $a_2=3a_1,\ b_1=b_2=0,\ c_2=2c_1$;

(vi) $a_2=2a_1, b_2=\frac{3b_1}{2},\ c_2=3c_1$;

(vii) $a_2=\frac{3a_1}{2}, b_2=2b_1,\ c_2=c_1$;

(viii) $a_2=3a_1,\ b_1=0,\ c_2=c_1$;

(ix) $a_2=\frac{5a_1}{3},\ b_2=2b_1,\ c_2=c_1$;

(x) $a_2=3a_1,\ b_1=0,\ c_2=2c_1$;

(xi) $a_2=\frac{3a_1}{2},\ b_1=0,\ c_2=c_1$;

(xii) $a_2=3a_1, b_2=-b_1,\ c_2=c_1$;

(xiii) $a_2=\frac{5a_1}{2},\ b_1=0,\ c_2=2c_1$;

(xiv) $a_2=\frac{3a_1}{2},\ b_1=0,\ c_2=\frac{c_1}{2}$;

(xv) $a_2=2a_1,\ b_1=2b_2,\ c_2=c_1=0$;

(xvi) $b_1=\frac{5b_2}{2},\ a_2=4a_1,\ c_2=2c_1$;

(xvii) $a_1(b_1-4b_2)+a_2(3b_2-2b_1)=0$,
$c_2=c_1$.

Proof. We use the identity (21) for each set of conditions from Theorems 2.1 – 2.5. Substituting in this identity the expressions of the cofactors and identifying the coefficients of x^0, x, y, we obtain systems of algebraic equations for the unknowns h_1, h_2 and h_3. Solving the obtained systems we determine the exponents h_1, h_2 and h_3.

Applying the identity (21) to the sets of conditions from Theorem 2.1, we obtain:

In case (i), $\Phi \equiv (a_1x + c_1)x^2 + a_{02}a_1y^2 = 0$ and $h_1 = 2,\ h_2 = 2,\ h_3 = -1$.

In case (ii), $\Phi \equiv (a_1x + c_1)^2x + a_{01}a_1^2y = 0$ and $h_1 = 2,\ h_2 = 1,\ h_3 = -1$.

In case (iii), $\Phi \equiv (a_1x + 2c_2)^2x + a_{02}a_1^2y^2 = 0$ and $h_1 = 2,\ h_2 = 2,\ h_3 = -1$.

In case (iv), $\Phi \equiv a_1x^3 + a_{11}y(c_1 + a_1x) = 0$ and $h_1 = 3,\ h_2 = 0,\ h_3 = -1$.

In case (v), $\Phi \equiv a_1x^3 + c_1x^2 + a_{01}a_1y = 0$ and $h_1 = 0,\ h_2 = 1,\ h_3 = -1$.

In case (vi), $\Phi \equiv b_1x^3 + 2a_{02}a_1xy + a_{02}b_1y^2 + 2a_{02}c_1y = 0$ and

$$h_1 = 3, \qquad h_2 = 0, \qquad h_3 = -1 .$$

Applying the identity (21) to the sets of conditions from Theorem 2.4, we have:

In case (vii), $\Phi \equiv a_1x^3 + 2b_1x^2y + c_1^2x^2 - 2a_{02}y^2 = 0$ and $h_1 = 0, h_2 = 2, h_3 = -1$.

In case (viii), $\Phi \equiv a_1x^2(2a_1x + b_2y + 4c_1) + 2b_2c_1xy + 2c_1^2x - a_{01}a_1y = 0$

and $h_1 = 0,\ h_2 = 1,\ h_3 = -1$.

In case (ix), $\Phi \equiv a_1^2x^2(2a_1x + 3b_1y + 4c_1) - 6a_1b_1c_1xy - 18b_1^2c_1y^2 + 2a_1c_1^2x -$

$-9b_1c_1^2y = 0$ and $h_1 = -2,\ h_2 = 3,\ h_3 = -1$.

In case (x), $\Phi \equiv (2a_1x + b_2y + 2c_1)x^2 - a_{01}y = 0$ and $h_1 = 0,\ h_2 = 1,\ h_3 = -1$.

Applying the identity (21) to the sets of conditions from Theorem 2.5, we get:

In case (xi), $\Phi \equiv x(a_1x + 2b_2y)^2 + a_1c_1x^2 + 4b_2c_1xy + 4a_{02}y^2 = 0$ and

$$h_1 = 0, \qquad h_2 = 2, \qquad h_3 = -1 .$$

In case (xii), $\Phi \equiv 4x(a_1x - b_1y)^2 + 8a_1c_1x^2 - 8b_1c_1xy + 4c_1^2x + a_{01}y = 0$ and

$$h_1 = 0, \qquad h_2 = 1, \qquad h_3 = -1 .$$

In case (xiii), $\Phi \equiv a_1x(3a_1x + 2b_2y)^2 + 9a_1^2c_1x^2 - 12a_1b_2c_1xy - 12b_2^2c_1y^2 -$

$-24b_2c_1^2y = 0$ and $h_1 = -2,\ h_2 = 2,\ h_3 = -1$.

In case (xiv), $\Phi \equiv x(a_1x + 2b_2y)^2 + 2a_1c_1x^2 + 4b_2c_1xy + 4a_{02}y^2 + c_1^2x = 0$ and

$$h_1 = 0, \qquad h_2 = 2, \qquad h_3 = -1 .$$

Applying the identity (21) to the set of conditions from Theorem 2.6, we obtain:

In case (xv), $\Phi \equiv (a_1x - b_2y)^3 - a_{11}xy = 0$ and $h_1 = 1,\ h_2 = 1,\ h_3 = -1$.

In case (xvi), $\Phi \equiv 4a_1x(3a_1x - b_1y)^2 + 36a_1^2c_1x^2 - 12a_1b_1c_1xy + 3b_1c_1^2y = 0$ and

$$h_1 = -2, \qquad h_2 = 2, \qquad h_3 = -1 .$$

In case (xvii), $\Phi \equiv a_1b_2(2a_1 - a_2)[(a_1 - a_2)x - (b_1 - b_2)y]^3 + c_1b_2(a_1 - a_2)^3 \cdot$

$\cdot (2a_1 - a_2)x^2 + 2a_1b_2c_1(b_1 - b_2)(a_1 - a_2)^2xy + a_1c_1(b_1 - b_2)^3(2a_1 - a_2)y^2 = 0$

and $h_1 = 2a_2 - 3a_1,\ h_2 = a_1,\ h_3 = a_1 - a_2$.

Theorem 4.1 is proved.

5. Darboux integrating factors

In this section we determine the sets of conditions from Theorems 2.1 – 2.5, under which the quadratic system (2) has Darboux integrating factors of the form

$$\mu = x^{h_1} y^{h_2} \Phi^{h_3}, \tag{22}$$

where $x = 0, y = 0$ are invariant straight lines, $\Phi = 0$ is an irreducible invariant cubic of the form (6) and h_1, h_2, h_3 are real numbers.

To construct the integrating factors (22) we take into account the cofactors $K_1(x,y), K_2(x,y)$ and $K_3(x,y)$ of these algebraic solutions, obtained in the proofs of Theorems 2.1 – 2.5. Then we apply the identity (19)

$$h_1 K_1(x,y) + h_2 K_2(x,y) + h_3 K_3(x,y) + \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \equiv 0 \tag{23}$$

for each set of conditions from Theorems 2.1 – 2.5. It was proved the following theorem.

Theorem 5.1. The Lotka-Volterra system (2) has a Darboux integrating factor of the form (22) if one of the following conditions is satisfied:

(i) $b_2 = \frac{3b_1}{2}, c_2 = c_1$;
(ii) $a_2 = 3a_1, b_2 = \frac{3b_1}{2}, c_2 = c_1$;
(iii) $a_2 = \frac{5a_1}{3}, b_2 = \frac{3b_1}{2}, c_2 = c_1$;
(iv) $a_2 = \frac{5a_1}{2}, b_2 = \frac{3b_1}{2}, c_1 = 2c_2$;
(v) $a_2 = \frac{15a_1}{8}, b_2 = \frac{3b_1}{2}, c_2 = 2c_1$;
(vi) $a_2 = 0,\ b_2 = 2b_1,\ c_2 = c_1$;
(vii) $a_2 = -a_1,\ b_1 = 0,\ c_2 = c_1$;
(viii) $a_2 = -a_1,\ b_2 = 2b_1,\ c_2 = c_1$;
(ix) $a_2 = -a_1,\ b_2 = 2b_1,\ c_1 = 2c_2$;
(x) $a_2 = 0,\ b_2 = 2b_1,\ c_1 = 2c_2$;
(xi) $a_2 = 0,\ b_1 = 0,\ c_2 = 2c_1$;
(xii) $a_2 = -a_1,\ b_1 = 0,\ c_2 = c_1$;
(xiii) $b_1 = 0,\ c_2 = 3c_1$;
(xiv) $a_2 = \frac{15a_1}{7},\ b_2 = 2b_1,\ c_2 = 3c_1$;
(xv) $a_2 = \frac{5a_1}{2},\ b_2 = 2b_1,\ c_2 = \frac{c_1}{2}$;
(xvi) $a_2 = \frac{7a_1}{6},\ b_2 = 2b_1,\ c_2 = \frac{c_1}{2}$;
(xvii) $a_2 = \frac{5a_1}{3},\ b_2 = -b_1,\ c_2 = c_1$;
(xviii) $a_2 = \frac{5a_1}{3}, b_2 = -b_1, c_2 = 3c_1$;
(xix) $a_2 = \frac{5a_1}{2},\ b_1 = 0,\ c_2 = 3c_1$;
(xx) $a_2 = \frac{3a_1}{2},\ b_1 = 0,\ c_2 = 3c_1$;
(xxi) $a_2 = 7a_1,\ b_2 = -b_1, c_2 = 2c_1$;
(xxii) $a_2 = \frac{5a_1}{3},\ b_2 = -b_1,\ c_2 = 2c_1$;
(xxiii) $a_2 = \frac{3a_1}{2},\ b_1 = 0,\ c_2 = \frac{c_1}{2}$;
(xiv) $a_2 = \frac{5a_1}{3},\ b_1 = 3b_2,\ c_2 = c_1$;
(xxv) $b_1 = \frac{7b_2}{3},\ a_2 = 5a_1,\ c_2 = 3c_1$;
(xxvi) $b_1 = \frac{7b_2}{3},\ a_2 = \frac{9a_1}{5}, c_2 = 3c_1$;
(xxvii) $b_1 = \frac{4b_2}{3},\ a_2 = 0,\ c_2 = 3c_1$;
(xxviii) $b_1 = \frac{5b_2}{2},\ a_2 = -5a_1, c_2 = 2c_1$;
(xxix) $a_2 = \frac{7a_1}{4},\ b_1 = \frac{5b_2}{2},\ c_2 = 2c_1$;
(xxx) $a_2 = \frac{3a_1}{4},\ b_1 = \frac{7b_2}{6},\ c_2 = 2c_1$.

Proof. We use the identity (23) for each set of conditions from Theorems 2.1 – 2.5. Substituting in this identity the expressions of the cofactors and identifying the coefficients of x^0, x, y, we obtain systems of algebraic equations for the unknowns h_1, h_2 and h_3. Solving the obtained systems we determine the exponents h_1, h_2 and h_3.

Applying the identity (23) to the sets of conditions from Theorem 2.1, we obtain:
In case (i), $\Phi \equiv 2(3a_1 - 2a_2)((a_1x + c_1)(2a_1 - a_2)x - b_1c_1y)x + b_1^2c_1y^2 = 0$ and

$$h_1 = 2, \qquad h_2 = \frac{2(a_2 - 2a_1)}{3a_1 - 2a_2}, \qquad h_3 = \frac{3a_2 - 4a_1}{3a_1 - 2a_2}.$$

In case (ii), $\Phi \equiv 9a_1x(a_1x + c_1)^2 + 18a_1b_1c_1xy + 3b_1^2c_1y^2 + 2b_1c_1^2y = 0$ and

$$h_1 = 2, \qquad h_1 = -1/2, \qquad h_3 = -5/6\,.$$

In case (iii), $\Phi \equiv a_1x(a_1x + c_1)^2 - 6a_1b_1c_1xy - 9b_1^2c_1y^2 - 6b_1c_1^2y = 0$ and

$$h_1 = -5/2, \qquad h_2 = 2, \qquad h_3 = -3/2\,.$$

In case (iv), $\Phi \equiv a_1x(a_1x + 2c_2)^2 + 8a_1b_1c_2xy + 2b_1^2c_2y^2 = 0$ and

$$h_1 = -1/4, \qquad h_2 = -1/2, \qquad h_3 = -1.$$

In case (v), $\Phi \equiv 9a_1^2x^2(a_1x + c_1) - 72a_1b_1c_1xy - 48b_1^2c_1y^2 - 64b_1c_1^2y = 0$ and

$$h_1 = -2, \qquad h_2 = 1/3, \qquad h_3 = -5/6\,.$$

Applying the identity (23) to each set of conditions from Theorem 2.2, we get:
In case (vi), $\Phi \equiv x^2(2a_{02}a_1^2 + 2b_1^2y{+}b_1c_1) - 4a_1b_1a_{02}xy + 2b_1^2a_{02}y^2 = 0$ and

$$h_1 = 1, \qquad h_2 = 0, \qquad h_3 = -3/2.$$

In case (vii), $\Phi \equiv (2a_1^2x + 4a_1c_1)xy + 2a_1^2a_{10}x + (2c_1^2 - a_1a_{10}b_2)y = 0$ and

$$h_1 = 0, \qquad h_2 = -1/2, \qquad h_3 = -3/2.$$

In case (viii), $\Phi \equiv 9a_1b_1xy(a_1x + 2c_1) - 6b_1^2c_1y^2 + 8a_1c_1^2x - 3b_1c_1^2y = 0$ and

$$h_1 = -2/3, \qquad h_2 = -1/2, \qquad h_3 = -5/6.$$

In case (ix), $\Phi \equiv 2a_1b_1xy(2a_1x + 3c_1) - 2b_1^2c_1y^2 + a_1c_1^2x = 0$ and

$$h_1 = -1/3, \qquad h_2 = -1/3, \qquad h_3 = -1.$$

In case (x), $\Phi \equiv a_1b_1xy(8a_1x + 18c_2) + 3a_1^2c_2x^2 - 9b_1^2c_2y^2 + 6a_1c_2^2x = 0$ and

$$h_1 = -1/3, \qquad h_2 = -2/3, \qquad h_3 = -5/6.$$

In case (xi), $\Phi \equiv 2a_1^2x^2(y + a_{20}) + 2a_1(2c_1 - b_2a_{20})xy + c_1(2c_1 - b_2a_{20})y = 0$ and

$$h_1 = 1, \qquad h_2 = -1/2, \qquad h_3 = -3/2.$$

Applying the identity (23) to the set of conditions from Theorem 2.3, we obtain the case (xii), with $\Phi \equiv 3a_1b_2xy(2a_1x - b_2y + 6c_2) - 8b_2^2c_2y^2 + 9a_1c_2^2x = 0$ and

$$h_1 = 0, \qquad h_2 = 0, \qquad h_3 = -1.$$

Applying the identity (23) to the sets of conditions from Theorem 2.4, we have:
In case (xiii), $\Phi \equiv (3a_1 - a_2)(2a_1 - a_2)(\,2a_1x + b_2y)x^2 + 2b_2c_1^2y = 0$ and

$$h_1 = 2, \qquad h_2 = \frac{a_2 - 2a_1}{3a_1 - a_2}, \qquad h_3 = \frac{a_2 - 4a_1}{3a_1 - a_2}.$$

In case (xiv), $\Phi \equiv a_1^2(72a_1x + 63b_1y)x^2 - 882a_1b_1c_1xy - 686b_1^2c_1y^2 -$
$-1029b_1c_1^2y = 0$ and $h_1 = -2,\ h_2 = 1/6,\ h_3 = -5/6.$

In case (xv), $\Phi \equiv a_1^2x^2(3a_1x + 2b_1y + 6c_1) + 18a_1b_1c_1xy + 3a_1c_1^2x + 9b_1^2c_1y^2 = 0$ and $h_1 = -1/3,\ h_2 = -2/3,\ h_3 = -5/6.$

In case (xvi), $\Phi \equiv a_1^2x^2(a_1x + 6b_1y + 2c_1) + 6a_1b_1c_1xy + a_1c_1^2x - 9b_1^2c_1y^2 = 0$ and $h_1 = -1/3,\ h_2 = -2,\ h_3 = -1/6.$

Applying the identity (23) to the sets of conditions from Theorem 2.5, we get:

In case (xvii), $\Phi \equiv a_1x(a_1x - 3b_1y)^2 + 2a_1^2c_1x^2 - 18a_1b_1c_1xy + a_1c_1^2x - -12b_1c_1^2y = 0$ and $h_1 = -1/2,\ h_2 = -1,\ h_3 = -1/2.$

In case (xviii), $\Phi \equiv a_1x(a_1x - 3b_1y)^2 - 36a_1b_1c_1xy - 27b_1c_1^2y = 0$ and $h_1 = -1/4, \quad h_2 = -1/4, \quad h_3 = -1.$

In case (xix), $\Phi \equiv a_1x(3a_1x + 2b_2y)^2 - 48a_1b_2c_1xy - 32b_2^2y^2 - 96b_2c_1^2y = 0$ and $h_1 = -5/2, \quad h_2 = 1, \quad h_3 = -3/2.$

In case (xx), $\Phi \equiv 9a_1x(a_1x + 2b_2y)^2 + 144a_1b_2c_1xy + 32b_2^2c_1y^2 + 96b_2c_1^2y = 0$ and $h_1 = -5/6,\ h_2 = -1/3,\ h_3 = -1/2.$

In case (xxi), $\Phi \equiv 4a_1x(3a_1x - b_1y)^2 + 36a_1^2c_1x^2 - 12a_1b_1c_1xy + 3b_1c_1^2y = 0$ and $h_1 = -1/3, \quad h_2 = -7/6, \quad h_3 = -1/6.$

In case (xxii), $\Phi \equiv 4a_1x(a_1x - 3b_1y)^2 + 4a_1^2c_1x^2 - 108a_1b_1c_1xy - 81b_1c_1^2y = 0$ and $h_1 = -1/3,\ h_2 = -1/2,\ h_3 = -5/6.$

In case (xxiii), $\Phi \equiv a_1x(3a_1x + 2b_2y)^2 + 18a_1^2c_1x^2 + 36a_1b_2c_1xy + 16b_2^2c_1y^2 + +9a_1c_1^2x = 0$ and $h_1 = -1/2,\ h_2 = -1,\ h_3 = -1/2.$

Applying the identity (23) to the sets of conditions from Theorem 2.6, we obtain:

In case (xxiv), $\Phi \equiv (a_1x - 3b_2y)^3 + 2a_1^2c_1x^2 - 36a_1b_2c_1xy - 54a_2^2c_1y^2 + a_1c_1^2x - -27b_2c_1^2y = 0$ and $h_1 = -5/6,\ h_2 = -1/2,\ h_3 = -2/3.$

In case (xxv), $\Phi \equiv (3a_1x - b_2y)^3 + 18a_1b_2c_1xy - 6b_2^2c_1y^2 - 9b_2c_1^2y = 0$ and $h_1 = -5/2, \quad h_2 = -3/2, \quad h_3 = 1.$

In case (xxvi), $\Phi \equiv (3a_1x - 5b_2y)^3 - 1350a_1b_2c_1xy - 750b_2^2c_1y^2 - 1125b_2c_1^2y = 0$ and $h_1 = -1/2,\ h_2 = -1/6,\ h_3 = -1.$

In case (xxvii), $\Phi \equiv (3a_1x + b_2y)^3 + 27a_1b_2c_1xy + 6b_2^2c_1y^2 + 9b_2c_1^2y = 0$ and $h_1 = -2,\ h_2 = -2/3,\ h_3 = 0.$

In case (xxviii), $\Phi \equiv (4a_1x + b_2y)^3 + 64a_1^2c_1x^2 + 32a_1b_2c_1xy + 4b_2^2c_1y^2 + +4b_2c_1^2y = 0$ and $h_1 = -5/3,\ h_2 = -5/6,\ h_3 = 1/6.$

In case (xxix), $\Phi \equiv (a_1x - 2b_2y)^3 + a_1^2c_1x^2 - 40a_1b_2c_1xy - 32b_2^2c_1y^2 - -32b_2c_1^2y = 0$ and $h_1 = -2/3,\ h_2 = -1/3,\ h_3 = -5/6.$

In case (xxx), $\Phi \equiv (3a_1x + 2b_2y)^3 + 27a_1^2c_1x^2 + 72a_1b_2c_1xy + 32b_2^2c_1y^2 + +32b_2c_1^2y = 0$ and $h_1 = -\frac{5}{3},\ h_2 = -5/9,\ h_3 = -2/9$.

Theorem 5.1 is proved.

Conclusion

For Lotka-Volterra system (2) with a bundle of two invariant straight lines and one irreducible invariant cubic, modulo the symmetry (3), there were obtained 47 sets of Darboux integrability conditions.

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CENTER-AFFINE INVARIANT CONDITIONS OF STABILITY OF UNPERTURBED MOTION FOR DIFFERENTIAL SYSTEM $s(1,2,3)$ WITH QUADRATIC PART OF DARBOUX TYPE

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Abstract. The Lie algebra, the Lyapunov series and the center-affine invariant conditions of stability of unperturbed motion have been determined by critical Lyapunov system with quadratic part of Darboux type.

Keywords: Differential system, stability of unperturbed motion, center-affine comitant and invariant, Lie algebra, Sibirsky graded algebra, group.
2010 Mathematics Subject Classification: 34C20, 34C45, 34D20

CONDIȚII CENTROAFIN-INVARIANTE DE STABILITATE A MIȘCĂRII NEPERTURBATE PENTRU SISTEMUL DIFERENȚIAL $s(1,2,3)$ CU PARTEA PĂTRATICĂ DE TIP DARBOUX

Rezumat. A fost determinată algebra Lie, seria Lyapunov și condițiile centroafin-invariante de stabilitate a mișcării neperturbate guvernate de sistemul critic de tip Lyapunov cu partea pătratică de tip Darboux.

Cuvinte-cheie: Sistem diferențial, stabilitatea mișcării neperturbate, comitanți și invatianți centro-afini, algebră Lie, algebră Sibirschi graduată, grup.

Introduction

A lot of papers were written in the field of stability of motion. The universal scientific literature, concerning the stability of motion contains thousands of papers, including hundreds of monographs and textbooks of many authors. This literature is rich in the development of this theory, as well as in its applications in practice.

Note that many problems on stability treated in these works are governed by two-dimensional (or multidimensional) autonomous polynomial differential systems. Methods of the theory of invariants for such systems were elaborated in the school of differential equations from Chișinău. Moreover, there was developed the theory of the Lie algebras and Sibirsky graded algebras [1-5] with applications in the qualitative theory of these equations.

With a special weight, in this domain, it is published the Lyapunov (1857-1918) PhD thesis concerning the stability of motion in 1882 [6]. This work contains many fruitful ideas and results of great importance. It is considered that all history related to the theory on stability of motion is divided into periods before and after Lyapunov.

First of all, A.M. Lyapunov gave a strict definition of the stability of motion, which was so successful that all scientists took it as fundamental one for their researches.

In this paper and [7], with these visions was studied the Lie algebra, was built the Lyapunov series and was determined the stability of the unperturbed motion for two-dimensional critical differential system $s(1,2,3)$ with quadratic part of Darboux type.

1. The Lie algebra allowed of Lyapunov canonical form of the differential system $s(1,2,3)$ with quadratic part of Darboux type

We will examine the differential system $s(1,2,3)$ with quadratic part of Darboux type of the form

$$\frac{dx^j}{dt} = a^j_\alpha x^\alpha + a^j_{\alpha\beta} x^\alpha x^\beta + a^j_{\alpha\beta\gamma} x^\alpha x^\beta x^\gamma \qquad (j, \alpha, \beta, \gamma = 1,2), \tag{1}$$

where $a^j_{\alpha\beta}$ and $a^j_{\alpha\beta\gamma}$ are a symmetric tensors in lower indices in which the total convolution is done. Coefficients and variables in (1) are given over the field of real numbers $\mathbb{R}$.

Remark 1.1. *The characteristic equation of system* (1) *has one zero root and the other ones real and negative if and only if the following invariant conditions* [7] *hold*

$$I_1^2 - I_2 = 0, \qquad I_1 < 0, \tag{2}$$

where

$$I_1 = a^\alpha_\alpha, \qquad I_2 = a^\alpha_\beta a^\beta_\alpha. \tag{3}$$

When the characteristic equation of (1) has one zero root and the other one is negative, i.e. the conditions (2) and $R_2 \equiv 0$ from (18) are satisfied, then this system by a center-affine transformation can be brought to its critical form

$$\begin{aligned} \frac{dx}{dt} &= x(gx + 2hy) + px^3 + 3qx^2y + 3rxy^2 + sy^3 \equiv P, \\ \frac{dy}{dt} &= ex + fy + y(gx + 2hy) + tx^3 + 3ux^2y + 3vxy^2 + wy^3 \equiv Q, \end{aligned} \tag{4}$$

where $a^1_1 = a^1_2 = a^1_{22} = a^2_{11} = 0$ and $a^2_1 = e, a^2_2 = f, a^1_{11} = 2a^2_{12} = g, a^1_{12} = \frac{1}{2} a^2_{22} = h$, $a^1_{111} = p,\ a^1_{112} = q, a^1_{122} = \ r,\ a^1_{222} = s,\ a^2_{111} = t,\ a^2_{112} = u,\ a^2_{122} = v,\ a^2_{222} = w$.

We examine the determined equations [8] for system (4)

$$\begin{aligned} \xi^1_x P + \xi^1_y Q &= \xi^1 P_x + \xi^2 P_y + D(P), \\ \xi^2_x P + \xi^2_y Q &= \xi^1 Q_x + \xi^2 Q_y + D(Q), \end{aligned} \tag{5}$$

where

$$\begin{aligned} D = {} & \eta^1 \frac{\partial}{\partial e} + \eta^2 \frac{\partial}{\partial f} + \eta^3 \frac{\partial}{\partial g} + \eta^4 \frac{\partial}{\partial h} + \eta^5 \frac{\partial}{\partial p} + \eta^6 \frac{\partial}{\partial q} + \eta^7 \frac{\partial}{\partial r} + \eta^8 \frac{\partial}{\partial s} + \\ & + \eta^9 \frac{\partial}{\partial t} + \eta^{10} \frac{\partial}{\partial u} + \eta^{11} \frac{\partial}{\partial v} + \eta^{12} \frac{\partial}{\partial w}. \end{aligned} \tag{6}$$

The polynomials P, Q are given in (4) and η^j $(j = \overline{1,12})$ are functions of the parameters *e*, *f*, *g*, *h*, *p*, *q*, *r*, *s*, *t*, *u*, *v*, *w*.

Let us consider

$$\xi^j = A^i x + B^i y \quad (i = \overline{1,2}), \tag{7}$$

where A^i, B^i are unknown parameters.

We write the operator

$$X = \xi^1 \frac{\partial}{\partial x} + \xi^2 \frac{\partial}{\partial y} + D, \tag{8}$$

where ξ^1, ξ^2 are given in (7) and *D* is defined in (6).

Solving the system of equations (5) with respect to the operators (6), (8) with coordinates (7) we obtain 3 independent linear operators

$$\begin{gathered} X_1 = x\frac{\partial}{\partial x} - e\frac{\partial}{\partial e} - g\frac{\partial}{\partial g} - 2p\frac{\partial}{\partial p} - q\frac{\partial}{\partial q} + s\frac{\partial}{\partial s} - 3t\frac{\partial}{\partial t} - 2u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v}, \\ X_2 = y\frac{\partial}{\partial y} + e\frac{\partial}{\partial e} - h\frac{\partial}{\partial h} - q\frac{\partial}{\partial q} - 2r\frac{\partial}{\partial r} - 3s\frac{\partial}{\partial s} + t\frac{\partial}{\partial t} - v\frac{\partial}{\partial v} - 2w\frac{\partial}{\partial w}, \\ X_3 = x\frac{\partial}{\partial y} - f\frac{\partial}{\partial e} - 2h\frac{\partial}{\partial g} - 3q\frac{\partial}{\partial p} - 2r\frac{\partial}{\partial q} - s\frac{\partial}{\partial r} + (p - 3u)\frac{\partial}{\partial t} + \\ +(q - 2v)\frac{\partial}{\partial u} + (r - w)\frac{\partial}{\partial v} + s\frac{\partial}{\partial w}. \end{gathered} \tag{9}$$

Remark 1.2. *The system* (4) *admits a solvable three-dimensional Lie algebra* L_3 *composed of operators* (9).

The following transformation of the phase plan

$$x = \bar{x}, \qquad y = -\alpha\bar{x} + \bar{y}$$

corresponds to the representation operator X_3 from (9) of the system (4) .

With this transformation, for $f \neq 0$, we can always get the equality *e = 0*.

Remark 1.3. *This property, for* $f \neq 0$*, is true for any Lyapunov canonical two-dimensional system.*

2. Invariant conditions of stability of unperturbed motion for critical system $s(1,2,3)$ of Lyapunov type (4) with quadratic part of Darboux type

According to Lyapunov's Theorem [6, §32], we examine the non-critical equation of the system (4)

$$ex + fy + gxy + 2hy^2 + tx^3 + 3ux^2y + 3vxy^2 + wy^3 = 0. \tag{10}$$

Then from this relation we express *y* and obtain

$$y = -\frac{e}{f}x - -\frac{g}{f}xy - 2\frac{h}{f}y^2 - \frac{t}{f}x^3 - 3\frac{u}{f}x^2y - 3\frac{v}{f}xy^2 - \frac{w}{f}y^3. \tag{11}$$

We seek *y* as a holomorphic function of *x*. Then we can write

$$y = -\frac{e}{f}x + B_2x^2 + B_3x^3 + B_4x^4 + B_5x^5 + B_6x^6 + B_7x^7 + B_8x^8 + B_9x^9 + \cdots \tag{12}$$

Substituting (12) into (11) and identifying the coefficients of the same powers of *x* in the obtained relation we have

$$B_2 = \frac{e}{f^2}\left(g - 2\frac{eh}{f}\right),$$

$$B_3 = -[\frac{1}{f}\left(g - 2\frac{eh}{f}\right)B_2 + \frac{1}{f}(t - 3\frac{eu}{f} + 3\frac{e^2v}{f^2} - \frac{e^3w}{f^3})],$$

$$B_4 = -[2\frac{h}{f}B_2^2 + \frac{1}{f}\left(g - 2\frac{eh}{f}\right)B_3 + \frac{3}{f}(u - 2\frac{ev}{f} + \frac{e^2w}{f^2})B_2],$$

$$B_5 = -[4\frac{h}{f}B_2B_3 + 3(\frac{v}{f} + \frac{ew}{f^2})B_2^2 + \frac{1}{f}\left(g - 2\frac{eh}{f}\right)B_4 + \frac{3}{f}(u - 2\frac{ev}{f} + \frac{e^2w}{f^2})B_3],$$

$$B_6 = -[2\frac{h}{f}(2B_2B_4 + B_3^2) + \frac{w}{f}B_2^3 + 6\left(\frac{v}{f} + \frac{ew}{f^2}\right)B_2B_3 + \frac{1}{f}\left(g - 2\frac{eh}{f}\right)B_5 +$$

$$+\frac{3}{f}(u - 2\frac{ev}{f} + \frac{e^2w}{f^2})B_4],$$

$$B_7 = -[4\frac{h}{f}(B_2B_5 + B_3B_4) + 3\frac{w}{f}B_2^2B_3 + 3\left(\frac{v}{f} + \frac{ew}{f^2}\right)(2B_2B_4 + B_3^2) +$$

$$+\frac{1}{f}\left(g - 2\frac{eh}{f}\right)B_6 + \frac{3}{f}(u - 2\frac{ev}{f} + \frac{e^2w}{f^2})B_5],$$

$$B_8 = -[2\frac{h}{f}(2B_2B_6 + 2B_3B_5 + B_4^2) + 3\frac{w}{f}(B_2^2B_4 + B_2B_3^2) +$$

$$+6\left(\frac{v}{f} + \frac{ew}{f^2}\right)(B_2B_5 + B_3B_4) + \frac{1}{f}\left(g - 2\frac{eh}{f}\right)B_7 + \frac{3}{f}(u - 2\frac{ev}{f} + \frac{e^2w}{f^2})B_6],$$

$$B_9 = -[4\frac{h}{f}(B_2B_7 + B_3B_6 + B_4B_5) + \frac{w}{f}(3B_2^2B_5 + 6B_2B_3B_4 + B_3^3) +$$

$$+3\left(\frac{v}{f} + \frac{ew}{f^2}\right)(2B_2B_6 + 2B_3B_5 + B_4^2) + \frac{1}{f}\left(g - 2\frac{eh}{f}\right)B_8 + \frac{3}{f}(u - 2\frac{ev}{f} + \frac{e^2w}{f^2})B_7],$$

$$B_{10} = -[2\frac{h}{f}(2B_2B_8 + 2B_3B_7 + 2B_4B_6 + B_5^2) + 3\frac{w}{f}(B_2^2B_6 + 2B_2B_3B_5 + B_2B_4^2 +$$

$$+B_3^2B_4) + 6\left(\frac{v}{f} + \frac{ew}{f^2}\right)(B_2B_7 + B_3B_6 + B_4B_5) + \frac{1}{f}\left(g - 2\frac{eh}{f}\right)B_9 +$$

$$+\frac{3}{f}(u - 2\frac{ev}{f} + \frac{e^2w}{f^2})B_8],$$

$$B_{11} = -[4\frac{h}{f}(B_2B_9 + B_3B_8 + B_4B_7 + B_5B_6) + 3\frac{w}{f}(B_2^2B_7 + 2B_2B_3B_6 +$$

$$+2B_2B_4B_5 + B_3^2B_5 + B_3B_4^2) + 3\left(\frac{v}{f} + \frac{ew}{f^2}\right)(2B_2B_8 + 2B_3B_7 + 2B_4B_6 + B_5^2) +$$

$$+\frac{1}{f}\left(g - 2\frac{eh}{f}\right)B_{10} + \frac{3}{f}(u - 2\frac{ev}{f} + \frac{e^2w}{f^2})B_9],$$

$$B_{12} = -[2\frac{h}{f}(2B_2B_{10} + B_3B_9 + B_4B_8 + B_5B_7 + B_6^2) + \frac{w}{f}(3B_2^2B_8 + 6B_2B_3B_7 +$$

$$+6B_2B_4B_6 + 3B_2B_5^2 + 3B_3^2B_6 + 6B_3B_4B_5 + B_4^3) + 6\left(\frac{v}{f} + \frac{ew}{f^2}\right)(B_2B_9 + B_3B_8 +$$

$$B_4B_7 + B_5B_6) + + \frac{1}{f}\left(g - 2\frac{eh}{f}\right)B_{11} + \frac{3}{f}(u - 2\frac{ev}{f} + \frac{e^2w}{f^2})B_{10}], \dots \qquad (13)$$

Substituting (12) into the right-hand side of the critical differential equation (4) we obtain

$$gx^2 + 2hxy + px^3 + 3qx^2y + 3rxy^2 + sy^3 =$$
$$= A_2x^2 + A_3x^3 + A_4x^4 + A_5x^5 + A_6x^6 + A_7x^7 + A_8x^8 + A_9x^9 + A_{10}x^{10} + \cdots$$

From this, taking into account (12) and (13), we get

$$A_2 = g - 2\frac{eh}{f},$$
$$A_3 = 2hB_2 + (t - 3\frac{eq}{f} + 3\frac{e^2r}{f^2} - \frac{e^3s}{f^3})],$$
$$A_4 = 2hB_3 + 3(q - 2\frac{er}{f} + \frac{e^2s}{f^2})B_2,$$
$$A_5 = 2hB_4 + 3(r - \frac{es}{f})B_2^2 + 3(q - 2\frac{er}{f} + \frac{e^2s}{f^2})B_3,$$
$$A_6 = sB_2^3 + 2hB_5 + 6(r - \frac{es}{f})B_2B_3 + 3(q - 2\frac{er}{f} + \frac{e^2s}{f^2})B_4,$$
$$A_7 = 3sB_2^2B_3 + 2hB_6 + 3\left(r - \frac{es}{f}\right)(2B_2B_4 + B_3^2) + 3(q - 2\frac{er}{f} + \frac{e^2s}{f^2})B_5,$$
$$A_8 = 3s(B_2^2B_4 + B_2B_3^2) + 2hB_7 + 6\left(r - \frac{es}{f}\right)(B_2B_5 + B_3B_4) + 3(q - 2\frac{er}{f} + \frac{e^2s}{f^2})B_6,$$
$$A_9 = s(3B_2^2B_5 + 6B_2B_3B_4 + B_3^3) + 2hB_8 + 3\left(r - \frac{es}{f}\right)(2B_2B_6 + 2B_3B_5 + B_4^2) +$$
$$+3\left(q - 2\frac{er}{f} + \frac{e^2s}{f^2}\right)B_7, \qquad (14)$$
$$A_{10} = 3s(B_2^2B_6 + 2B_2B_3B_5 + B_2B_4^2 + B_3^2B_4) + 2hB_9 + 6\left(r - \frac{es}{f}\right)(B_2B_7 +$$
$$+B_3B_6 + B_4B_5) + 3(q - 2\frac{er}{f} + \frac{e^2s}{f^2})B_8,$$
$$A_{11} = 3s(B_2^2B_7 + 2B_2B_3B_6 + 2B_2B_4B_5 + B_3^2B_5 + B_3B_4^2) + 2hB_{10} +$$
$$+3\left(r - \frac{es}{f}\right)(2B_2B_8 + 2B_3B_7 + 2B_4B_6 + B_5^2) + 3(q - 2\frac{er}{f} + \frac{e^2s}{f^2})B_9,$$
$$A_{12} = s3(B_2^2B_8 + 6B_2B_3B_7 + 6B_2B_4B_6 + 3B_2B_5^2 + 3B_3^2B_6 + 6B_3B_4B_5 + B_4^3) +$$
$$+2hB_{11} + 6\left(r - \frac{es}{f}\right)(B_2B_9 + B_3B_8 + B_4B_7 + B_5B_6) + 3\left(q - 2\frac{er}{f} + \frac{e^2s}{f^2}\right)B_{10}, \dots$$

We introduce the following notations:

$$N_1 = fg - 2eh; \quad N_2 = f^3p - 3ef^2q + 3\,e^2f\,r - e^3\,s;$$
$$N_3 = f^3t - 3ef^2u + 3\,e^2f\,v - e^3\,w; \quad N_4 = f^2q - 2\,e^2f\,r - e^3\,s; \qquad (15)$$
$$N_5 = fr - es.$$

Then, from (13) and (14) with this notations we obtain

$$B_2 = \frac{e}{f^3}N_1, \qquad B_3 = -(\frac{1}{f^2}B_2N_1 + \frac{1}{f^4}N_3),$$

$$A_2 = \frac{1}{f}N_1, \qquad A_3 = 2hB_2 + \frac{1}{f^3}N_2, \qquad A_4 = 2hB_3 + \frac{3}{f^2}B_2N_4,$$

$$A_5 = 2hB_4 + \frac{3}{f}B_2^2N_5 + \frac{3}{f^2}B_3N_4,$$

$$A_6 = sB_2^3 + 2hB_5 + \frac{6}{f}B_2B_3N_5 + \frac{3}{f^2}B_4N_4,$$

$$A_7 = 3sB_2^2B_3 + 2hB_6 + \frac{3}{f}(2B_2B_4 + B_3^2)N_5 + \frac{3}{f^2}B_5N_4,$$

$$A_8 = 3s(B_2^2B_4 + B_2B_3^2) + 2hB_7 + \frac{6}{f}(B_2B_5 + B_3B_4)N_5 + \frac{3}{f^2}B_6N_4,$$

$$A_9 = s(3B_2^2B_5 + 6B_2B_3B_4 + B_3^3) + 2hB_8 + \frac{3}{f}(2B_2B_6 + 2B_3B_5 + B_4^2)N_5 + \frac{3}{f^2}B_7N_4,$$

$$A_{10} = 3s(B_2^2B_6 + 2B_2B_3B_5 + B_2B_4^2 + B_3^2B_4) + 2hB_9 + \frac{6}{f}(B_2B_7 + B_3B_6 + B_4B_5)N_5 + \frac{3}{f^2}B_8N_4, \ldots \qquad (16)$$

Lemma 2.1. *The stability of unperturbed motion in the system of perturbed motion* (4) *is described by one of the following twelve possible cases, if for expressions* (15) $I_1 = f < 0$*) the following conditions are satisfied:*

I. $N_1 \neq 0$, *then the unperturbed motion is unstable;*

II. $N_1 = 0, N_2 > 0$, *then the unperturbed motion is stable;*

III. $N_1 = 0, N_2 < 0$, *then the unperturbed motion is unstable;*

IV. $N_1 = N_2 = 0,\ hN_3 \neq 0$, *then the unperturbed motion is unstable;*

V. $N_1 = N_2 = h = 0$; $N_3N_4 < 0$, *then the unperturbed motion is unstable;*

VI. $N_1 = N_2 = h = 0$; $N_3N_4 > 0$, *then the unperturbed motion is stable;*

VII. $N_1 = N_2 = N_4 = h = 0$, $N_3 \neq 0$; $N_5 > 0$, *then the unperturbed motion is stable;*

VIII. $N_1 = N_2 = N_4 = h = 0$, $N_3 \neq 0$; $N_5 < 0$, *then the unperturbed motion is unstable;*

IX. $N_1 = N_2 = N_4 = N_5 = h = 0$; $sN_3 < 0$, *then the unperturbed motion is unstable;*

X. $N_1 = N_2 = N_4 = N_5 = h = 0$; $sN_3 > 0$, *then the unperturbed motion is stable;*

XI. $N_1 = N_2 = N_3 = 0$, *then the unperturbed motion is stable;*

XII. $N_1 = N_2 = N_4 = N_5 = h = s = 0$, *then the unperturbed motion is stable.*

In the last two cases, the unperturbed motion belongs to some continuous series of stabilized motion. Moreover, this motion is also asymptotic stable in Cases II, VI, VII and X. The expressions N_i $(i = \overline{1,5})$ *are given in* (15).

Proof. According to Lyapunov Theorem [6, §32], the coefficients of the A_i series from (14) are analyzed.

If $A_2 \neq 0$, then from (16) we get $N_1 \neq 0$ (taking into account that $I_1 = f < 0$). According to Lyapunov Theorem [6, §32], we have proved the Case I.

If $A_2 = 0$, i.e. $N_1 = 0$ respectively $B_2 = 0$, then by (16) the stability or the instability of unperturbed motion is determined by the sign of the expression A_3 (the sign of the product N_2). Using the Lyapunov Theorem [6, §32] we obtain the Cases II and III.

If $N_1 = N_2 = 0$, then from (16) we get $A_4 = -2\frac{h}{f^4}N_3$. If $hN_3 \neq 0$. Then we obtain the Cases IV (see the Lyapunov Theorem [6, §32]).

Suppose $N_1 = N_2 = h = 0$. Then from (16) it results that $A_5 = -\frac{3}{f^6}N_3N_4$. So the stability or the instability of the unperturbed motion is determined by the sign of expression N_3N_4. Using the Lyapunov Theorem [6, §32] we get the Cases V and VI.

If $N_1 = N_2 = N_3 = 0$, then all $B_i = 0$ $(i \geq 3)$ and respectively $A_i = 0$ $(i \geq 5)$. By the Lyapunov Theorem [6, §32] we have the Case XI.

If $N_1 = N_2 = N_4 = h = 0$ and $N_3 \neq 0$, then $A_6 = 0$, but $A_7 = \frac{3}{f^9}N_3^2N_5$. So the stability or the instability of the unperturbed motion is determined by the sign of expression N_5. Using the Lyapunov Theorem [6, §32] we get the Cases VII and VIII.

If $N_1 = N_2 = N_4 = N_5 = h = 0$ and $N_3 \neq 0$, then $A_8 = 0$, but $A_9 = -\frac{s}{f^{12}}N_3^3$. So the stability or the instability of the unperturbed motion is determined by the sign of expression sN_3. Using the Lyapunov Theorem [6, §32] we get the Cases IX and X.

If $N_1 = N_2 = N_4 = N_5 = h = s = 0$ then all $A_i = 0$ $(\forall i)$ vanish. By the Lyapunov Theorem [6, §32] we get the Case XII. Lemma 2.1 is proved.

Let φ and ψ be homogeneous comitants of degree ρ_1 and ρ_2 respectively of the phase variables x and y of a two-dimensional polynomial differential system. Then the transvectant

$$(\varphi,\psi)^{(j)} = \frac{(\rho_1 - j)(\rho_2 - j)}{\rho_1!\,\rho_2!}\sum_{i=0}^{j}(-1)^j\binom{j}{i}\frac{\partial^j\varphi}{\partial x^{j-i}\partial y^i}\frac{\partial^j\psi}{\partial x^i\partial y^{j-i}} \tag{17}$$

is also a comitant for this system.

In the Iu. Calin's works, see for example [9], it is shown that by means of the transvectant (17) all generators of the Sibirsky algebras of comitants and invariant for any system of type (1) can be constructed.

According to [10] we write the following comitants of the system (1)

$$R_i = P_i(x,y)y - Q_i(x,y)x, \qquad S_i = \frac{1}{i}\left(\frac{\partial P_i(x,y)}{\partial x} + \frac{\partial Q_i(x,y)}{\partial y}\right), (i = \overline{1,3}). \tag{18}$$

Later on, we will need the following comitants and invariants from [10] of system (1) built by operations (17) and (18):

$$I_1 = S_1,\ \ I_2 = (R_1, R_1)^{(2)},\ \ I_3 = ((R_3, R_1)^{(2)}, R_1)^{(2)},\ \ I_4 = (S_3, R_1)^{(2)},$$
$$K_2 = R_1,\ \ K_5 = S_2,\ \ K_8 = R_3,\ \ K_9 = (R_3, R_1)^{(1)},\ \ K_{10} = (R_3, R_1)^{(2)}, \quad (19)$$
$$K_{11} = ((R_3, R_1)^{(2)}, R_1)^{(1)},\ \ K_{14} = (S_2, R_1)^{(1)},\ \ K_{15} = S_3,\ \ K_{16} = (S_3, R_1)^{(1)}.$$

We consider for system (1) the following expressions composed of comitants and invariants from (19) that can be written in the form:

$$\mathcal{N}_1 = 2K_{14} - I_1 K_5,$$
$$\mathcal{N}_2 = 2I_1^2\, K_{10} - 4\, I_1\, K_{11} - 3\, I_1\, I_2 K_{15} - 3\, I_1^2 K_{16} + \ 4I_3 K_2 + 3I_1\, I_4 K_2,$$
$$\mathcal{N}_3 = -12 I_1\, K_{10}\, K_2 + 8\, K_{11}\, K_2 + 3\, I_1^2\, K_{15}\, K_2 - 6\, I_1\, K_{16}\, K_2 + 6\, I_4\, K_2^2 -$$
$$-4\, I_1^3\, K_8 + \ 8\, I_1^2\, K_9,\ \ \mathcal{N}_4 = 2\, I_3\ +\ I_1\, I_4,\ \ \mathcal{N}_5 = 2\, K_{10}\ +\ I_1\, K_{15} - K_{16}, \qquad (20)$$
$$S = 3\, K_{15}\, K_2\ -\ 2\, I_1\, K_8\ -\ 4\, K_9.$$

Theorem [11]. *Let for system of perturbed motion* (1) *the invariant conditions* (2)-(3) and $R_2 \equiv 0$ from (18) *are satisfied. Then the stability of unperturbed motion is described by one of the following twelve possible cases:*

I. $\mathcal{N}_1 \not\equiv 0$, *then the unperturbed motion is unstable;*

II. $\mathcal{N}_1 \equiv 0, \mathcal{N}_2 > 0$, *then the unperturbed motion is stable;*

III. $\mathcal{N}_1 \equiv 0, \mathcal{N}_2 < 0$, *then the unperturbed motion is unstable;*

IV. $\mathcal{N}_1 \equiv \mathcal{N}_2 \equiv 0,\ K_5\mathcal{N}_3 \not\equiv 0$, *then the unperturbed motion is unstable;*

V. $\mathcal{N}_1 \equiv \mathcal{N}_2 \equiv\ K_5 \equiv 0;\ \mathcal{N}_3\mathcal{N}_4 < 0$, *then the unperturbed motion is unstable;*

VI. $\mathcal{N}_1 \equiv \mathcal{N}_2 \equiv\ K_5 \equiv 0;\ \mathcal{N}_3\mathcal{N}_4 > 0$, *then the unperturbed motion is stable;*

VII. $\mathcal{N}_1 \equiv \mathcal{N}_2 \equiv \mathcal{N}_4 \equiv\ K_5 \equiv 0, \mathcal{N}_3 \not\equiv 0;\ N_5 > 0$, *then the unperturbed motion is stable;*

VIII. $\mathcal{N}_1 \equiv \mathcal{N}_2 \equiv \mathcal{N}_4 \equiv\ K_5 \equiv 0$, $\mathcal{N}_3 \not\equiv 0$; $N_5 < 0$, *then the unperturbed motion is unstable;*

IX. $\mathcal{N}_1 \equiv \mathcal{N}_2 \equiv \mathcal{N}_4 \equiv\ \mathcal{N}_5 \equiv K_5 = 0;\ S\mathcal{N}_3 < 0$, *then the unperturbed motion is unstable;*

X. $\mathcal{N}_1 \equiv \mathcal{N}_2 \equiv \mathcal{N}_4 \equiv\ \mathcal{N}_5 \equiv K_5 \equiv 0;\ S\mathcal{N}_3 > 0$, *then the unperturbed motion is stable;*

XI. $\mathcal{N}_1 \equiv \mathcal{N}_2 \equiv \mathcal{N}_3 \equiv\ 0$, *then the unperturbed motion is stable;*

XII. $\mathcal{N}_1 \equiv \mathcal{N}_2 \equiv \mathcal{N}_4 \equiv\ \mathcal{N}_5 \equiv K_5 \equiv S \equiv 0$, *then the unperturbed motion is stable.*

In the last two cases, the unperturbed motion belongs to some continuous series of stabilized motion. Moreover, this motion is also asymptotic stable in Cases II, VI, VII and X. The expressions $S, K_5, \mathcal{N}_i\ (i = \overline{1,5})$ *are given in* (19)-(20).

Proof. Observe that the first three expressions from (20), for critical system (4), look as follows:

$$\mathcal{N}_1 = -3N_1 x,\ \ \mathcal{N}_2 = 4N_2 x^2,\ \ \mathcal{N}_3 = 8N_3 x^4 - 8N_2 x^3 y,\ \ \mathcal{N}_4 = 2N_4,$$
$$\mathcal{N}_5 = \frac{2}{f} N_5 (ex + fy)^2,\ \ K_5 = 3\frac{h}{f}(ex + fy),\ \ S = -4\frac{s}{f^3} N_5 (ex + fy)^4. \quad (21)$$

Using the expressions (21) and the last assertion together with Lemma 2.1, we obtain the Cases I-XII. We note that the comitants $\mathcal{N}_2, \mathcal{N}_3\mathcal{N}_4, \mathcal{N}_5, S\mathcal{N}_3$ from (20), used in the Cases II-X of Theorem, are even-degree comitants with respect to *x* and *y* and have the weights [1] equal to 0, 0, 0, -2, respectively. Moreover, each one of these comitants (in the case when it is applied) is a binary form with a well-defined sing. This ensures that any center-affine transformation cannot change their sign. Theorem is proved.

Conclusions

In this paper the Lie algebra allowed by differential system $s(1,2,3)$ of the Lyapunov canonical form with quadratic part of the Darboux type was determined, which is a solvable three-dimensional algebra. Based on the constructed Lyapunov series, all center-affine invariant conditions of stability of the unperturbed motion were obtained and they are included in twelve cases.

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THE COMMON HILBERT SERIES FOR SOME DIFFERENTIAL SYSTEMS WITH HOMOGENEOUS NONLINEARITIES OF ODD DEGREE

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Abstract. The Hilbert series for Sibirsky graded algebras of differential systems still now were examined using a generalized Sylvester method. These series have a special importance for some problems of qualitative theory of differential systems. For example, a problem related to the Hilbert series of differential systems is to determine relationships between them. Before finding some relations between Hilbert series, generalized or ordinary, it is necessary to build these Hilbert series. The article proposes the construction of Hilbert series of Sibirsky graded algebras using the residue method.
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SERII HILBERT OBIȘNUITE PENTRU UNELE SISTEME DIFERENȚIALE CU NELINIARITĂȚI IMPARE

Rezumat. Seriile Hilbert pentru algebrele graduate Sibirschi ale sistemelor diferențiale până în prezent au fost examinate utilizând metoda generalizată a lui Sylvester. Aceste serii au o importanță deosebită pentru unele probleme ale teoriei calitative ale sistemelor diferențiale. De exemplu, o problemă legată de seriile Hilbert corespunzătoare sistemelor diferențiale este determinarea unor relații între ele. Pentru a obține relații între serii Hilbert atât generalizate cât și obișnuite este nevoie de a construi aceste serii Hilbert. În articol se propune construirea seriilor Hilbert ale algebrelor graduate Sibirschi prin metoda reziduurilor.
Cuvinte-cheie: Serii Hilbert, algebre graduate Sibirschi, sisteme diferențiale.

1. Introduction

A problem related to the Hilbert series of differential systems is to determine the relationships between them. Some relations between generalized Hilbert series of differential systems with homogeneous nonlinearities of odd degree were found in [1].

Lemma 1 [1]. *The following relation*

$$H(SI_{1,3},b,d) = H(SI_1,b)H(S_3,u,d)\,|_{u^2=b} \tag{1}$$

exists between the generalized Hilbert series of algebras SI_1, S_3 *and* $SI_{1,3}$.

Lemma 2 [1]. *The following relation*

$$H(SI_{1,5},b,f) = H(SI_1,b)H(S_5,u,f)\,|_{u^2=b} \tag{2}$$

exists between the generalized Hilbert series of algebras SI_1, S_5 *and* $SI_{1,5}$.

According to (1) and (2) we can assume that between generalized Hilbert series of algebras SI_1, S_{2k+1} and $SI_{1,2k+1}$ there exists the next relation

$$H(SI_{1,2k+1},b,z) = H(SI_1,b)H(S_{2k+1},u,z)\,|_{u^2=b} \tag{3}$$

for any $k \geq 1$.

Before finding some relations between Hilbert series, generalized or ordinary, it is necessary to build these Hilbert series.

The construction of Hilbert series with generalized Sylvester method [2] is not always simple. The method of computing ordinary Hilbert series for invariants rings using the residues it is known from [3].

2. Hilbert series

Definition 1 [3]. *For a graded vector space* $V = \bigoplus_{d=k}^{\infty} V_d$ *with* V_d *finite dimensional for all d we define the Hilbert series of V as a formal Laurent series*

$$H(v,t) = \sum_{d=k}^{\infty} \dim(V_d) t^d .$$

Let G be a linearly reductive group over an algebraically closed field K and V be a n – dimensional rational representation. Through $H(K[V]^G, t)$ is denoted the Hilbert series of invariants ring $K[V]^G$ [3].

Theorem 1 (Molien's formula [3]). *Let G be a finite group acting on a finite dimensional vector space V over a field K of characteristic not dividing $|G|$. Then*

$$H(K[V]^G, t) = \frac{1}{|G|} \sum_{\sigma \in G} \frac{1}{\det_V^0 (1 - t\sigma)}.$$

If K has characteristic 0*, then* $\det_V^0 (1 - t\sigma)$ *can be taken as* $\det_V (1 - t\sigma)$.

Suppose that $char(K) = 0$. In Theorem 1 we have seen that for a finite group the Hilbert series of invariant ring can easily be computed. If G is a finite group and V is a finite dimensional representation, then according to [3] we have

$$H(K[V]^G, t) = \frac{1}{|G|} \sum_{\sigma \in G} \frac{1}{\det_V (1 - t\sigma)}. \tag{4}$$

This idea can be generalized to arbitrary reductive groups. Let us assume that K is the complex numbers $\mathbb{C}$. We can choose a Haar measure $d\mu$ on C and normalize it such that $\int_C d\mu = 1$. Let V be a finite dimensional rational representation of G. The proper generalization of (4) is given in [3]

$$H(\mathbb{C}[V]^G, t) = \int_C \frac{d\mu}{\det_V (1 - t\sigma)}. \tag{5}$$

We mention that the Hilbert series $H(\mathbb{C}[V]^G, t)$ converges for $|t| < 1$ because it is a rational function with poles only at $t = 1$. Since C is compact, there exist constants $A > 0$ such that for every $\sigma \in \mathbb{C}$ and every eigenvalue λ of σ we have $|\lambda| \leq A$. Since λ^{ℓ} is an eigenvalue of σ^{ℓ}, it follows that $|\lambda^{\ell}| \le A$ for all ℓ, so $|\lambda| \le 1$. It is clear that the integral on the right-hand side of (5) is also defined for $|t| < 1$ [3].

Assume that G is also connected. Let T be a maximal torus of G, and let D be a maximal compact subgroup of T. We may assume that C contains D. The torus can be identified with $(\mathbb{C}^*)^r$, where r is the rank of G, and D can be identified with the subgroup $(S^1)^r$ of $(\mathbb{C}^*)^r$, where $S^1 \subset \mathbb{C}^*$ is the unit circle. We can choose a Haar measure $d\mu$ on D such that $\int_D d\mu = 1$ [3].

Suppose that f is a continuous class function on C. An integral like $\int_C f(\sigma)d\mu$ can be viewed as an integral over D, since f is constant on conjugacy classes. More precisely, there exists a weight function $\varphi: D \to \mathbb{R}$, such that for every continuous class function f we have $\int_C f(\sigma)d\mu = \int_D \varphi(\sigma) f(\sigma)d\nu$.

So, from [3], we have

$$H(\mathbb{C}[V]^G, t) = \int_C \frac{d\mu}{\det_V(1 - t\sigma)} = \int_D \frac{\varphi(\sigma)d\nu}{\det_V(1 - t\sigma)}. \qquad (6)$$

3. The Residue Theorem

We recall the Residue Theorem in complex function theory. This theorem can be applied to compute the Hilbert series of invariant rings [3].

Suppose that $f(z)$ is a meromorphic function on $\mathbb{C}$. If $a \in \mathbb{C}$, then f can be written as a Laurent series around $z = a$

$$f(z) = \sum_{k=-d}^{\infty} c_k (z-a)^k.$$

If $d > 0$ and $c_{-d} \neq 0$, then f has a pole at $z = a$ and the pole order is d.

The residue of f at $z = a$ is denoted by $Res(f, a)$ and defined by

$$Res(f, a) = c_{-1}.$$

If the pole order of f at $z = a$ is $k \ge 1$, then the residue can be computed by

$$Res(f, a) = \frac{1}{(k-1)!} \lim_{z \to a} \frac{d^{k-1}}{dz^{k-1}} ((z-a)^k f(z)).$$

Suppose that $\gamma: [0,\ 1] \to \mathbb{C}$ is a smooth curve. The integral over the curve γ is defined by

$$\int_\gamma f(z)dz = \int_0^1 f(\gamma(t))\gamma'(t)dt.$$

Theorem 2 (The Residue Theorem [3]). *Suppose that D is a connected, simply connected compact region in $\mathbb{C}$ whose border is ∂D, and $\gamma:[0,1]\to\mathbb{C}$ is a smooth curve such that $\gamma([0,1])=\partial D$, $\gamma(0)=\gamma(1)$ and γ circles around D exactly once in counterclockwise direction. Assume that f is a meromorphic function on $\mathbb{C}$ with no poles in ∂D. Then we have*

$$\frac{1}{2\pi i}\int_{\gamma} f(z)dz=\sum_{a\in D} Res(f,a).$$

There are only finitely many points in the compact region D such that f has non-zero residue there. So we have

Theorem 3 [4].

$$H(K[V]^{G},t)=\frac{1}{2\pi i}\int_{S^1}\frac{1}{\det(I-t_{\rho V}(z))}\frac{dz}{z}, \qquad (7)$$

where $S^1\subset\mathbb{C}$ is the unit circle $\{z:|z|=1\}$.

4. Applications of the Residue Theorem to compute Hilbert series of Sibirsky graded algebras of differential systems

Using the Residue Theorem and corresponding generating function [2] the formula (7) can be adapted for computing ordinary Hilbert series for Sibirsky graded algebras of comitants and invariants of differential systems [5].

Theorem 4. *The ordinary Hilbert series for Sibirsky graded algebras of invariants of differential systems can be calculated using the formula*

$$H_{SI_\Gamma}(t)=\frac{1}{2\pi i}\int_{S^1}\frac{\varphi_\Gamma^{(0)}(z)}{z}dz, \qquad (8)$$

where $S^1\subset\mathbb{C}$ is the unit circle $\{z:|z|=1\}$, and $\varphi_\Gamma^{(0)}(z)$ is the corresponding generating function [2],

$$\varphi_\Gamma^{(0)}=(1-z^{-2})\psi_{m_0}^{(0)}(z)\psi_{m_1}^{(0)}(z)...\psi_{m_\ell}^{(0)}(z),$$

$$\psi_{m_i}^{(0)}(z)=\begin{cases}\dfrac{1}{(1-zt)(1-z^{-1}t)}, & \textit{for } m_i=0,\\[2ex] \dfrac{1}{(1-z^{m_i+1}t)(1-z^{-m_i-1}t)\prod_{k=1}^{m_i}(1-z^{m_i-2k+1}t)^2}, & \textit{for } m_i\neq 0,\end{cases}$$

$\Gamma=\{m_i\}_{i=0}^{\ell}$ *and consists of a finite number $(\ell<\infty)$ of distinct natural numbers.*

We mention that this method of computing ordinary Hilbert series for Sibirsky graded algebras of comitants and invariants for differential systems was verified for the following known Hilbert series H_{SI_1}, H_{S_2}, H_{SI_2}, $H_{SI_{0,2}}$, $H_{SI_{1,2}}$, $H_{SI_{1,3}}$, $H_{SI_{2,3}}$, H_{S_5} from [2] and $H_{S_{1,5}}$, $H_{SI_{1,5}}$ from [1].

Remark 1. *The ordinary Hilbert series of Sibirsky graded algebra of comitants are obtained from the ordinary Hilbert series of algebra of invariants in the following way*: $H_{S_\Gamma}(t) = H_{SI_{\Gamma\cup\{0\}}}(t),$ *where* $\Gamma = \{m_1, m_2, ..., m_\ell\} \not\ni \{0\}.$

From the paper [3] it is known the method of computing ordinary Hilbert series for invariants rings using the residues. This method was adapted for ordinary Hilbert series of Sibirsky graded algebras of comitants and invariants of differential systems. In contrast to the construction methods of these series, exposed in [2], with the help of residues [3], of the primary generating functions [2], we obtained the ordinary Hilbert series for Sibirsky graded algebras of the differential systems $s(7)$, $s(1,7)$, $s(1,2,3)$, $s(1,3,5)$, $s(1,3,7)$, $s(1,3,5,7)$.

Theorem 5. *For differential system* $s(7)$ *the following ordinary Hilbert series of the Sibirsky graded algebras of comitants* S_7 *and invariants* SI_7 *were obtained*

$$H_{S_7}(t) = \frac{1}{(1+t)^5(1-t)(1-t^3)^4(1-t^4)^2(1-t^5)^4(1-t^7)^3(1-t^9)}(1+4t+7t^2+6t^3+6t^4+$$
$$+ 28t^5 + 112t^6 + 325t^7 + 788t^8 + 1719t^9 + 3499t^{10} + 6716t^{11} + 12225t^{12} + 21205t^{13} +$$
$$+ 35194t^{14} + 56030t^{15} + 85698t^{16} + 126023t^{17} + 178425t^{18} + 243697t^{19} + 321789t^{20} +$$
$$+ 411501t^{21} + 510260t^{22} + 613944t^{23} + 717118t^{24} + 813553t^{25} + 896906t^{26} + 961309t^{27} +$$
$$+ 1002042t^{28} + 1015982t^{29} + 1002042t^{30} + 961309t^{31} + 896906t^{32} +$$
$$+ 813553t^{33} + 717118t^{34} + 613944t^{35} + 510260t^{36} + 411501t^{37} + 321789t^{38} +$$
$$+ 243697t^{39} + 178425t^{40} + 126023t^{41} +$$
$$+ 85698t^{42} + 56030t^{43} + 35194t^{44} + 21205t^{45} + 12225t^{46} + 6716t^{47} + 3499t^{48} + 1719t^{49} +$$
$$+ 788t^{50} + 325t^{51} + 112t^{52} + 28t^{53} + 6t^{54} + 6t^{55} + 7t^{56} + 4t^{57} + t^{58}),$$

$$H_{SI_7}(t) = \frac{1}{(1+t)^4(1-t)(1-t^3)^4(1-t^4)^3(1-t^5)^3(1-t^7)^2}(1+4t+4t^2+2t^3+2t^4+15t^5+$$
$$+ 59t^6 + 150t^7 + 312t^8 + 578t^9 + 1011t^{10} + 1673t^{11} + 2631t^{12} + 3917t^{13} + 5541t^{14} + 7450t^{15}$$
$$+ + 9551t^{16} + 11651t^{17} + 13543t^{18} + 15011t^{19} + 15933t^{20} + 16238t^{21} + 15933t^{22} +$$
$$+ 15011t^{23} + 13543t^{24} + 11651t^{25} + 9551t^{26} + 7450t^{27} + 5541t^{28} + 3917t^{29} + 2631t^{30} +$$
$$1673t^{31} + 1011t^{32} + 578t^{33} + 312t^{34} + 150t^{35} + 59t^{36} + 15t^{37} + 2t^{38} + 2t^{39} + 4t^{40} + 3t^{41} + t^{42}).$$

From this theorem it results that the Krull dimension [2] of the Sibirsky graded algebra S_7 (respectively SI_7) is equal to 15 (respectively 13).

Theorem 6. *For differential system* $s(1,7)$ *the following ordinary Hilbert series of the Sibirsky graded algebras of comitants* $S_{1,7}$ *and invariants* $SI_{1,7}$ *were obtained*

$$H_{S_{1,7}}(t) = \frac{1}{(1+t)^3(1-t^2)^3(1-t^3)^5(1-t^4)^3(1-t^5)^4(1-t^7)^3(1-t^9)}(1+4t+7t^2+7t^3+17t^4+$$
$$+ 85t^5 + 331t^6 + 1009t^7 + 2657t^8 + 6368t^9 + 14278t^{10} + 30208t^{11} + 60574t^{12} + 115441t^{13} +$$
$$+ 209688t^{14} + 363888t^{15} + 604838t^{16} + 965096t^{17} + 1481667t^{18} + 2193216\,t^{19} +$$
$$+ 3135942t^{20} + 4337738t^{21} + 5811835t^{22} + 7550176t^{23} + 9518852t^{24} + 11655892t^{25} +$$

$+ 13872730t^{26} + 16058633t^{27} + 18089130t^{28} + 19836497t^{29} + 21182751t^{30} +$
$+ 22032184t^{31} + 22322579t^{32} + 22032184t^{33} + 21182751t^{34} + 19836497t^{35} +$
$+ 18089130t^{36} + 16058633t^{37} + 13872730t^{38} + 11655892t^{39} + 9518852t^{40} + 7550176t^{41} +$
$+ 5811835t^{42} + 4337738t^{43} + 3135942t^{44} + 2193216t^{45} + 1481667t^{46} + 965096t^{47} +$
$+ 604838t^{48} + 363888t^{49} + 209688t^{50} + 115441t^{51} + 60574t^{52} + 30208t^{53} + 14278t^{54} +$
$+ 6368t^{55} + 2657t^{56} + 1009t^{57} + 331t^{58} + 85t^{59} + 17t^{60} + 7t^{61} + 7t^{62} + 4t^{63} + t^{64}),$

$$H_{SI_{1,7}}(t) = \frac{1}{(1+t)^5(1-t)^3(1-t^3)^5(1-t^4)^4(1-t^5)^3(1-t^7)^2}(1+3t+4t^2+2t^3+9t^4+53t^5+$$
$$+ 196t^6 + 525t^7 + 1214t^8 + 2558t^9 + 5097t^{10} + 9569t^{11} + 16975t^{12} + 28396t^{13} + 44981t^{14} +$$
$$+ 67577t^{15} + 96665t^{16} + 131839t^{17} + 171920t^{18} + 214631t^{19} + 257063t^{20} + 295599t^{21} +$$
$$+ 326684t^{22} + 346880t^{23} + 353937t^{24} + 346880t^{25} + 326684t^{26} + 295599t^{27} + 257063t^{28} +$$
$$+ 214631t^{29} + 171920t^{30} + 131839t^{31} + 96665t^{32} + 67577t^{33} + 44981t^{34} + 28396t^{35} +$$
$$+ 16975t^{36} + 9569t^{37} + 5097t^{38} + 2558t^{39} + 1214t^{40} + 525t^{41} + 196t^{42} + 53t^{43} +$$
$$+ 9t^{44} + 2t^{45} + 4t^{46} + 3t^{47} + t^{48}).$$

From this theorem it results that the Krull dimension [2] of the Sibirsky graded algebra $S_{1,7}$ (respectively $SI_{1,7}$) is equal to 19 (respectively 17).

Theorem 7. *For differential system* $s(1,2,3)$ *the following ordinary Hilbert series of the Sibirsky graded algebras of comitants* $S_{1,2,3}$ *and invariants* $SI_{1,2,3}$ *were obtained*

$$H_{S_{1,2,3}}(t) = \frac{1}{(1-t)^2(1-t^2)^2(1-t^3)^6(1-t^4)^3(1-t^5)^3(1-t^7)}(1-t+3t^2+9t^3+36t^4+90t^5+$$
$$+ 220t^6 + 459t^7 + 946t^8 + 1748t^9 + 3032t^{10} + 4845t^{11} + 7302t^{12} + 10268t^{13} + 13749t^{14} +$$
$$+ 17327t^{15} + 20781t^{16} + 23565t^{17} + 25460t^{18} + 26051t^{19} + 25460t^{20} + 23565t^{21} +$$
$$+20781t^{22} + 17327t^{23} + 13749t^{24} + 10268t^{25} + 7302t^{26} + 4845t^{27} + 3032t^{28} + 1748t^{29} +$$
$$+ 946t^{30} + 459t^{31} + 220t^{32} + 90t^{33} + 36t^{34} + 9t^{35} + 3t^{36} - t^{37} + t^{38}),$$

$$H_{SI_{1,2,3}}(t) = \frac{1}{(1-t)(1-t^2)^3(1-t^3)^5(1-t^4)^2(1-t^5)^3(1-t^7)}(1+t^2+6t^3+24t^4+57t^5+128t^6+$$
$$+ 244t^7 + 447t^8 + 756t^9 + 1203t^{10} + 1760t^{11} + 2433t^{12} + 3124t^{13} + 3800t^{14} + 4351t^{15} +$$
$$+ 4736t^{16} + 4854t^{17} + 4736t^{18} + 4351t^{19} +3800t^{20} + 3124t^{21} + 2433t^{22} + 1760t^{23} +$$
$$+1203t^{24} + 756t^{25} + 447t^{26} + 244t^{27} + 128t^{28} + 57t^{29} + 24t^{30} + 6t^{31} +t^{32} + t^{34}).$$

From this theorem it results that the Krull dimension [2] of the Sibirsky graded algebra $S_{1,2,3}$ (respectively $SI_{1,2,3}$) is equal to 17 (respectively 15).

Theorem 8. *For ifferential system* $s(1,3,5)$ *the following ordinary Hilbert series of the Sibirsky graded algebras of comitants* $S_{1,3,5}$ *and invariants* $SI_{1,3,5}$ *were obtained*

$$H_{S_{1,3,5}}(t) = \frac{1}{(1+t)^7(1-t)^6(1-t^3)^8(1-t^4)^4(1-t^5)^4(1-t^7)}(1+2t+3t^2+17t^3+102t^4+393t^5+$$
$$+ 1295t^6 + 3788t^7 + 10229t^8 + 25559t^9 + 59435t^{10} + 128624t^{11} + 260754t^{12} + 497142t^{13} +$$
$$+ 895543t^{14} + 1528784t^{15} + 2480535t^{16} + 3832821t^{17} + 5651535t^{18} + 7964888t^{19} +$$

$+ 10746190t^{20} + 13897132t^{21} + 17246232t^{22} + 20554573t^{23} + 23544429t^{24} +$
$+ 25932413t^{25} + 27476107t^{26} + 28009657t^{27} + 27476107t^{28} + 25932413t^{29} +$
$+ 23544429t^{30} + 20554573t^{31} + 17246232t^{32} + 13897132t^{33} + 10746190t^{34} +$
$+ 7964888t^{35} + 5651535t^{36} +497142t^{41} + 260754t^{42} + 3832821t^{37} + 2480535t^{38} +$
$+ 1528784t^{39} + 895543t^{40} + 128624t^{43} + 59435t^{44} + + 25559t^{45} + 10229t^{46} +$
$+ 3788t^{47} + 1295t^{48} + 393t^{49} + 102t^{50} + 17t^{51} + 3t^{52} + 2t^{53} +t^{54}),$

$$H_{SI_{1,3,5}}(t) = \frac{1}{(1+t)^6(1-t)^6(1-t^3)^7(1-t^4)^5(1-t^5)^3}(1+t+t^2+14t^3+77t^4+253t^5+781t^6+$$
$+ 2077t^7 + 5160t^8 + 11689t^9 + 24616t^{10} + 47739t^{11} + 86576t^{12} + 146479t^{13} + 233075t^{14} +$
$+ 348813t^{15} + 493340t^{16} + 659032t^{17} + 834212t^{18} + 1000116t^{19} + 1138132t^{20} +$
$+1228974t^{21} + 1261281t^{22} + 1228974t^{23} + 1138132t^{24} + 1000116t^{25} + 834212t^{26} +$
$+659032t^{27} + 493340t^{28} + 348813t^{29} + 233075t^{30} + 146479t^{31} + 86576t^{32} + 47739t^{33} +$
$+24616t^{34} +11689t^{35} + 5160t^{36} + 2077t^{37} + 781t^{38} + 253t^{39} + 77t^{40} +$
$+ 14t^{41} + 2t^{42} +t^{43} + t^{44}).$

From this theorem it results that the Krull dimension [2] of the Sibirsky graded algebra $S_{1,3,5}$ (respectively $SI_{1,3,5}$) is equal to 23 (respectively 21).

Theorem 9. *For differential system* $s(1,3,7)$ *the following ordinary Hilbert series of the Sibirsky graded algebras of comitants* $S_{1,3,7}$ *and invariants* $SI_{1,3,7}$ *were obtained*

$$H_{S_{1,3,7}}(t) = \frac{1}{(1+t)^3(1-t^2)^5(1-t^3)^8(1-t^4)^5(1-t^5)^5(1-t^7)^3(1-t^9)}(1+4t+8t^2+20t^3+119t^4+$$
$+ 630t^5 + 2704t^6 + 10022t^7 + 33698t^8 + 104818t^9 + 304181t^{10} + 826655t^{11} + 2112616t^{12} +$
$+ 5098405t^{13} + 11666106t^{14} + 25400587t^{15} +52790206t^{16} + 105011044t^{17} +$
$+ 200416900t^{18} + 367773321t^{19} + 650140950t^{20} + 1109089748t^{21} + 1828673257t^{22} +$
$+2918286116t^{23} + 4513317434t^{24} + 6772373326t^{25} + 9869976204t^{26} +$
$+ 13983988556t^{27} + 19277729149t^{28} + 25877612329t^{29} + 33848259389t^{30} +$
$+ 43167949995t^{31} + 53708076135t^{32} + 65220413010t^{33} + 77335714909t^{34} +$
$+ 89575940034t^{35} + 101380841773t^{36} + 112147463549t^{37} + 121279087722t^{38} +$
$+ 128238286339t^{39} + 132597788686t^{40} + 34082589969t^{41} + 132597788686t^{42} +$
$+ 128238286339t^{43} + 121279087722t^{44} + 112147463549t^{45} + 101380841773t^{46} +$
$+ 89575940034t^{47} + 77335714909t^{48} + 65220413010t^{49} + 53708076135t^{50} +$
$+ 43167949995t^{51} + 33848259389t^{52} + 25877612329t^{53} + 19277729149t^{54} +$
$+ 13983988556t^{55} + 9869976204t^{56} + 6772373326t^{57} + 4513317434t^{58} + 2918286116t^{59} +$
$+ 1828673257t^{60} + 1109089748t^{61} + 650140950t^{62} + 367773321t^{63} + 200416900t^{64} +$
$+ 105011044t^{65} + 52790206t^{66} + 25400587t^{67} + 11666106t^{68} + 5098405t^{69} +$
$+ 2112616t^{70} + 826655t^{71} + 304181t^{72} + 104818t^{73} + 33698t^{74} + 10022t^{75} + 2704t^{76} +$
$+ 630t^{77} + 119t^{78} + 20t^{79} + 8t^{80} + 4t^{81} +t^{82}),$

$$H_{SI_{1,3,7}}(t) = \frac{1}{(1+t)^3(1-t^2)^4(1-t^3)^8(1-t^4)^6(1-t^5)^5(1-t^7)^2}(1+4t+9t^2+22t^3+114t^4+$$

$+ 576t^{5} + 2433t^{6} + 8812t^{7} + 28787t^{8} + 86580t^{9} + 242349t^{10} + 633691t^{11} + 1554313t^{12} +$
$+ 3589873t^{13} + 7838767t^{14} + 16239174t^{15} + 32018338t^{16} + 60242752t^{17} + 108417618t^{18} +$
$+ 187010583t^{19} + 309738539t^{20} + 493386952t^{21} + 756961044t^{22} + 1119980967t^{23} +$
$+ 1599914185t^{24} + 2208870842t^{25} + 2949986298t^{26} + 3814040685t^{27} + 4777086279t^{28} +$
$+ 5799732655t^{29} + 6828681083t^{30} + 7800621224t^{31} + 8648294432t^{32} + 9307907390t^{33} +$
$+ 9726879111t^{34} + 9870564527t^{35} + 9726879111t^{36} + 9307907390t^{37} + 8648294432t^{38} +$
$+ 7800621224t^{39} + 6828681083t^{40} + 5799732655t^{41} + 4777086279t^{42} + 3814040685t^{43} +$
$+ 2949986298t^{44} + 2208870842t^{45} + 1599914185t^{46} + 1119980967t^{47} + 756961044t^{48} +$
$+ 493386952t^{49} + 309738539t^{50} + 187010583t^{51} + 108417618t^{52} + 60242752t^{53} +$
$+ 32018338t^{54} + 16239174t^{55} + 7838767t^{56} + 3589873t^{57} + 1554313t^{58} + 633691t^{59} +$
$+ 242349t^{60} + 86580t^{61} + 28787t^{62} + 8812t^{63} + 2433t^{64} + 576t^{65} + 114t^{66} + 22t^{67} + 9t^{68} +$
$+ 4t^{69} + t^{70}).$

From this theorem it results that the Krull dimension [2] of the Sibirsky graded algebra $S_{1,3,7}$ (respectively $SI_{1,3,7}$) is equal to 27 (respectively 25).

Theorem 10. *For differential system* $s(1,3,5,7)$ *the following ordinary Hilbert series of the Sibirsky graded algebras of comitants* $S_{1,3,5,7}$ *and invariants* $SI_{1,3,5,7}$ *were obtained*

$$H_{S_{1,3,5,7}}(t) = \frac{U(t) + 2298270315\ 143980746\ t^{60} + t^{120}U(t^{-1})}{(1+t)^{19}(1+t^{2})^{8}(1-t)^{14}(1-t^{3})^{12}(1-t^{5})^{8}(1-t^{7})^{4}(1-t^{9})},$$

where $U(t)=1 + 6t + 20t^{2} + 87t^{3} + 642t^{4} + 4481t^{5} + 26793t^{6} + 141973t^{7} + 684115t^{8} +$
$+ 3033350t^{9} + 12465139t^{10} + 47749507t^{11} + 171414077t^{12} + 579433144t^{13} +$
$+ 1852114710t^{14} + 5618767624t^{15} + 16230539293t^{16} + 44770726947t^{17} +$
$+ 118233818156t^{18} + 299625404135t^{19} + 730145608913t^{20} + 1714167261299t^{21} +$
$+ 3883773551652t^{22} + 8505306230645t^{23} + 18029418149708t^{24} + 37042309655531t^{25} +$
$+ 73851959357894t^{26} + 143039363140182t^{27} + 269416219454043t^{28} +$
$+ 493944596168225t^{29} + 882268074320900t^{30} + 1536543007952396t^{31} +$
$+ 2611196867637156t^{32} + 4333024660344442t^{33} + 7025611335473678t^{34} +$
$+ 11137398421309529t^{35} + 17271787147116907t^{36} + 26216525599773850t^{37} +$
$+ 38968364210329669t^{38} + 56747752371861786t^{39} + 80997424826732157t^{40} +$
$+ 113358368681589288t^{41} + 155617153462411693t^{42} + 209620178940739772t^{43} +$
$+ 277153165150321324t^{44} + 359788117447054402t^{45} + 458704770582751394t^{46} +$
$+ 574498645384155800t^{47} + 706992640391687667t^{48} + 855072713288320920t^{49} +$
$+ 1016569872742669961t^{50} + 1188209740459545784t^{51} + 1365646993055807450t^{52} +$
$+ 1543595104982837472t^{53} + 1716052512321252802t^{54} + 1876615582976246945t^{55} +$
$+ 2018857942986265569t^{56} + 2136746272693569424t^{57} + 2225056091622875140t^{58} +$
$+ 2279748435060291614t^{59},$

$$H_{SI_{1,3,5,7}}(t) = \frac{V(t) + 3293350250\ 5147932\ t^{52} + t^{104}V(t^{-1})}{(1+t)^{19}(1-t)^{15}(1+t^{2})^{9}(1-t^{3})^{12}(1-t^{5})^{7}(1-t^{7})^{3}},$$

where $V(t)=1 + 5t + 15t^{2} + 70t^{3} + 546t^{4} + 3691t^{5} + 21211t^{6} + 108097t^{7} + 501215t^{8} +$

$$+ 2135708t^9 + 8420376t^{10} + 30894213t^{11} + 106057925t^{12} + 342316946t^{13} +$$
$$+ 1043225615t^{14} + 3012988906t^{15} + 8273667765t^{16} + 21663519624t^{17} +$$
$$+ 54225659702t^{18} + 130054129145t^{19} + 299492368986t^{20} + 663439513913t^{21} +$$
$$+ 1416140486098t^{22} + 2917219852903t^{23} + 5807630254373t^{24} + 11187994444298t^{25} +$$
$$+ 20880385856690t^{26} + 37794195363608t^{27} + 66411190209119t^{28} +$$
$$+ 113391841520052t^{29} + 188282608991333t^{30} + 304271520124478t^{31} +$$
$$+ 478898737877115t^{32} + 734584562409596t^{33} + 1098797608776741t^{34} +$$
$$+ 1603661779481979t^{35} + 2284804664001899t^{36} + 3179293473234493t^{37} +$$
$$+ 4322594520474429t^{38} + 5744627532607767t^{39} + 7465155325802975t^{40} +$$
$$+ 9488929831214829t^{41} + 11801175204390804t^{42} + 14364091127469868t^{43} +$$
$$+ 17115070624832596t^{44} + 19967223601230372t^{45} + 22812575427180540t^{46} +$$
$$+ 25527987499683011t^{47} + 27983465544664079t^{48} + 30052140716959960t^{49} +$$
$$+ 31620895669339212t^{50} + 32600424240909358t^{51}.$$

From this theorem it results that the Krull dimension [2] of the Sibirsky graded algebra $S_{1,3,5,7}$ (respectively $SI_{1,3,5,7}$) is equal to 39 (respectively 37).

Remark 2. *The Theorem* 5 – 10 *are published for the first time into the papers* [6-11].

We note that the Krull dimension plays an important role in solving the center-focus problem for differential systems [12].

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MAXIMAL MULTIPLICITY OF THE LINE AT INFINITY FOR QUARTIC DIFFERENTIAL SYSTEMS

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Abstract. In this work we show that in the class of quartic differential systems the maximal algebraic multiplicity of the line at infinity is 10.

Keywords: quartic differential system, invariant straight line, algebraic multiplicity.

2010 Mathematics Subject Classification: 34G20, 34C45

MULTIPLICITATEA MAXIMALĂ A LINIEI DE LA INFINIT PENTRU SISTEMELE DIFERENŢIALE DE GRADUL PATRU

Rezumat. În această lucrare se arată că în clasa sistemelor diferenţiale de gradul patru multiplicitatea algebrică maximală a liniei de la infinit este egală cu 10.

Cuvinte-cheie: sistem diferenţial de gradul patru, dreaptă invariantă, multiplicitate algebrică.

1. Introduction and the statement of main result

We consider the real polynomial system of differential equations

$$\frac{dx}{dt} = P(x,y), \quad \frac{dy}{dt} = Q(x,y). \tag{1}$$

Denote $n = max\{deg(P), \deg(Q)\}$. If $n = 4$ then system (1) is called quartic.

At present, a great number of works are dedicated to the investigation of polynomial differential systems with invariant straight lines. The problem of the estimation of the number of invariant straight lines which can have a polynomial differential system was considered in [1].

In [2] it is given the estimation $3n - 2 \leq M_a(n) \leq 3n - 1$ of maximal algebraic multiplicity $M_a(n)$ of an invariant straight line for the class of two-dimensional polynomial differential systems of degree $n \geq 2$ and it was shown that in the class of cubic differential systems the maximal multiplicity of an affine real straight line (of the line at infinity) is seven.

In this paper we show that in the class of quartic differential systems the maximal algebraic multiplicity of the line at infinity is equal to 10.

Theorem. *For quartic differential systems the algebraic multiplicity of the line at infinity is at most ten. Any quartic system having the line at infinite of multiplicity 10 via affine transformations and time rescaling can be written in the form*

$$\dot{x} = -x, \ \dot{y} = x^4 + 3y. \tag{2}$$

2. The proof of the Theorem

We consider the real quartic system of differential equations

$$\dot{x} = \sum_{j=0}^{4} p_j(x,y) \equiv p(x,y), \quad \dot{y} = \sum_{j=0}^{4} q_j(x,y) \equiv q(x,y), \tag{3}$$

where $p_{0=}a_0$, $p_1(x,y) = a_1x + a_2y$, $p_2(x,y) = a_3x^2 + a_4xy + a_5y^2$, $p_3(x,y) = a_6x^3 + a_7x^2y + a_8xy^2 + a_9y^3$, $p_4(x,y) = a_{10}x^4 + a_{11}x^3y + a_{12}x^2y^2 + a_{13}xy^3 + a_{14}y^4$, $q_{0=}b_0$ $q_1(x,y) = b_1x + b_2y$, $q_2(x,y) = b_3x^2 + b_4xy + b_5y^2$, $q_3(x,y) = b_6x^3 + b_7x^2y + b_8xy^2 + b_9y^3$, $q_4(x,y) = b_{10}x^4 + b_{11}x^3y + b_{12}x^2y^2 + b_{13}xy^3 + b_{14}y^4$.

Suppose that the right-hand sides of (3) do not have the common divisors of degree greatest than 0, i.e.

$$gcd(p,q) = 1 \text{ and } yp_4(x,y) - xq_4(x,y) \not\equiv 0, \quad (4)$$

i.e. at infinity the system (3) has at most five distinct singular points.

The homogeneous system associated to the system (3) has the form

$$\dot{x} = \sum_{j=0}^{4} p_j(x,y)Z^{4-j} \equiv P(x,y,Z), \quad \dot{y} = \sum_{j=0}^{4} q_j(x,y)Z^{4-j} \equiv Q(x,y,Z). \quad (5)$$

Denote $\mathbb{X} = P(x,y,Z)\frac{\partial}{\partial x} + Q\,(x,y,Z)\frac{\partial}{\partial y}$.

We say that the line at infinity $Z = 0$ has *algebraic multiplicity* $m + 1$ if m is the greatest positive integer such that Z^m divides $\mathbb{E}_\infty = P \cdot \mathbb{X}(Q) - Q \cdot \mathbb{X}(P)$ (see [3]).

In this section, for quartic system (3) we determine the maximal algebraic multiplicity of the line at infinity $Z = 0$.

Because $p_4^2(x,y) + q_4^2(x,y)$ is not identically zero, by a centro-affine transformation and time rescaling we can make $b_{10} \neq 0$, and more that, $b_{10} = 1$.

For the homogenized system (4) we calculate the determinant $\mathbb{E}_\infty$ from the definition of the algebraic multiplicity. $\mathbb{E}_\infty$ is a polynomial of degree 11 in x, y, Z. We write it in the form:

$$\begin{aligned}\mathbb{E}_\infty = {} & A_0(x,y) + A_1(x,y)Z + A_2(x,y)Z^2 + A_3(x,y)Z^3 + \\ & + A_4(x,y)Z^4 + A_5(x,y)Z^5 + A_6(x,y)Z^6 + A_7(x,y)Z^7 + \\ & + A_8(x,y)Z^8 + A_9(x,y)Z^9 + A_{10}(x,y)Z^{10} + A_{11}(x,y)Z^{11}\end{aligned} \quad (6)$$

where $A_i(x,y)$, $i = 0,\dots,11$, are polynomials in x and y.

The algebraic multiplicity of the line at infinity is $m_\infty \in N^*$ if m_∞ is the maximal number such that $Z^{m_\infty - 1}$ divides $\mathbb{E}_\infty$.

The algebraic multiplicity m_∞ of the line at infinity is at least two if the identity $A_0(x,y) \equiv 0$ holds.

The polynomial $A_0(x,y)$ looks as: $A_0(x,y) = A_{01}(x,y)A_{02}(x,y)$ where

$$\begin{aligned}A_{01}(x,y) = {} & -x^5 + (a_{10} - b_{11})\,x^4y + (a_{11} - b_{12})\,x^3y^2 + (a_{12} - b_{13})\,x^2y^3 + \\ & + (a_{13} - b_{14})x\,y^4 + a_{14}y^5,\end{aligned}$$

$$\begin{aligned}A_{02}(x,y) = {} & (a_{11} - a_{10}b_{11})x^6 + 2(a_{12} - a_{10}b_{12})x^5y + (3a_{13} + a_{12}b_{11} - a_{11}b_{12} - \\ & -3\,a_{10}b_{13})\,x^4y^2 + 2(2\,a_{14} + a_{13}b_{11} - a_{11}b_{13} - 2\,a_{10}b_{14})\,x^3y^3 + \\ & + (3\,a_{14}b_{11} + a_{13} \cdot b_{12} - a_{12}b_{13} - 3\,a_{11}b_{14})x^2y^4 + \\ & + 2\,(a_{14}b_{12} - a_{12}b_{14})\,xy^5 + (a_{14}b_{13} - a_{13}b_{14})y^6.\end{aligned}$$

As $A_{01}(x,y) \not\equiv 0$ (see (4)), we require $A_{02}(x,y)$ to be identically equal to zero. The identity $A_{02}(x,y) \equiv 0$ holds if the following conditions

$a_{11} = a_{10}\, b_{11},\ a_{12} = a_{10}\, b_{12},\ a_{13} = a_{10}\, b_{13},\ a_{14} = a_{10}\, b_{14}$ are satisfied.

The algebraic multiplicity m_∞ of the line at infinity is at least three if $A_1(x,y) \equiv 0$.

Unde the above conditions we have $A_1(x,y) = -A_{11}(x,y)A_{12}(x,y)$, where

$$A_{11}(x,y) = x^4 + b_{11}\, x^3\, y + b_{12}\, x^2\, y^2 + b_{13}\, x\, y^3 + b_{14}\, y^4 \not\equiv 0,$$

$$A_{12}\,(x,y) = (a_7 - a_{10}a_6 - a_6 b_{11} + a_{10}^2\, b_6 + a_{10} b_{11} b_6 - a_{10} b_7)x^6 +$$
$$+(2a_8 - 2a_{10}\cdot a_7 - 2\, a_6\, b_{12} + 2\, a_{10}\, b_{12}\, b_6 + 2\, a_{10}^2 b_7 - 2\, a_{10} b_8)x^5 y +$$
$$+(3a_9 - 3a_{10}a_8 - a_{10}a_7 b_{11} + a_8\cdot b_{11} + a_{10}\, a_6 b_{12} - a_7\, b_{12} - 3\, a_6 b_{13} -$$
$$-a_{10}^2 b_{12} b_6 + 3\, a_{10} b_{13}\, b_6 + a_{10}^2\, b_{11} b_7 + a_{10} b_{12} b_7 + 3\, a_{10}^2 b_8 -$$
$$-a_{10}\, b_{11} b_8 - 3\, a_{10} b_9)x^4 y^2 +$$
$$+(-4\, a_{10}\, a_9 - 2\, a_{10} a_8 b_{11} + 2\, a_9\, b_{11} + 2\, a_{10}\, a_6\cdot b_{13} - 2\, a_7\, b_{13} -$$
$$-4\, a_6\, b_{14} - 2\, a_{10}^2\, b_{13} b_6 + 4\, a_{10} b_{14} b_6 + 2\, a_{10}\, b_{13}\, b_7 + 2\, a_{10}^2\, b_{11} b_8 +$$
$$+4\, a_{10}^2\, b_9 - 2\, a_{10}\, b_{11}\, b_9)\, x^3\, y^3 +$$
$$+(-\,3\, a_{10} a_9 b_{11} - a_{10} a_8 b_{12} + a_9 b_{12} + a_{10} a_7 b_{13} - a_8 b_{13} + 3\, a_{10} a_6 b_{14} -$$
$$-3\, a_7 b_{14} - 3\, a_{10}^2 b_{14} b_6 - a_{10}^2\, b_{13}\, b_7 + 3\, a_{10}\, b_{14}\, b_7 + a_{10}^2 b_{12}\, b_8 +$$
$$+a_{10} b_{13} b_8 + 3\, a_{10}^2 b_{11} b_9 - a_{10} b_{12} b_9)\, x^2\, y^4 +$$
$$+(-2\, a_{10}\, a_9\, b_{12} + 2\, a_{10}\, a_7\, b_{14} - 2\, a_8\, b_{14} - 2\, a_{10}^2\, b_{14}\, b_7 + 2a_{10} b_{14} b_8 +$$
$$+2\, a_{10}^2 b_{12} b_9)x\, y^5 +$$
$$+(-a_{10} a_9\, b_{13} + a_{10} a_8 b_{14} - a_9 b_{14} - a_{10}^2 b_{14} b_8 + a_{10}^2 b_{13} b_9 + a_{10}\, b_{14}\, b_9)\, y^6.$$

If $A_{12}\,(x,y) \equiv 0$ then we obtain the following two series of conditions:

1) $a_6 = a_{10} b_6,\ a_7 = a_{10}\, b_7,\ a_8 = a_{10}\, b_8,\ a_9 = a_{10}\, b_9$;

2) $a_7 = a_{10}\, b_7 - a_{10}\alpha - b_{11}\alpha,\ a_8 = a_{10} b_8 - a_{10}^2\, \alpha - a_{10}\, b_{11}\alpha - b_{12}\, \alpha,\ a_9 = a_{10}\, b_9 - a_{10}^3\alpha - a_{10}^2\, b_{11}\, \alpha - a_{10}\, b_{12}\, \alpha - b_{13}\, \alpha,\ b_{14} = -a_{10}\,(a_{10}^3 + a_{10}^2\, b_{11} + a_{10} b_{12} + b_{13}),\ \alpha = a_{10}\, b_6 - a_6,\ \alpha \neq 0.$

In the conditions 1) we have $A_2(x,y) = -A_{11}(x,y)A_{21}(x,y)$, where

$$A_{21}(x,y) = (a_4 - 2a_{10}a_3 - a_3\, b_{11} + 2a_{10}^2\, b_3 + a_{10} b_{11} b_3 - a_{10} b_4\,)x^5 +$$
$$+(2a_5 - 3a_{10}a_4 - a_{10}\, a_3\, b_{11} - 2\, a_3 b_{12} + a_{10}^2\, b_{11}\, b_3 + 2\, a_{10}\, b_{12}\, b_3 +$$
$$+3\, a_{10}^2 b_4 - 2\, a_{10} b_5)\, x^4\, y - (4\, a_{10} a_5 + 2a_{10}\, a_4\, b_{11} - a_5\, b_{11} + a_4\, b_{12} +$$
$$+3\, a_3\, b_{13} - 3\, a_{10}\, b_{13}\, b_3 - 2\, a_{10}^2\, b_{11}\, b_4 - a_{10}\, b_{12}\, b_4 - 4\, a_{10}^2 b_5 +$$
$$+a_{10} b_{11} b_5)\, x^3\, y^2 - (3a_{10} a_5 b_{11} + a_{10} a_4 b_{12} - a_{10} a_3 b_{13} + 2a_4 b_{13} + 4\, a_3 b_{14} +$$
$$+a_{10}^2\, b_{13} b_3 - 4\, a_{10}\, b_{14}\, b_3 - a_{10}^2\, b_{12}\, b_4 - 2\, a_{10}\, b_{13}\, b_4 - 3\, a_{10}^2\, b_{11} b_5)\, x^2 y^3 -$$
$$-(2\, a_{10} a_5 b_{12} + a_5 b_{13} - 2\, a_{10} a_3 b_{14} + 3\, a_4 b_{14} + 2\, a_{10}^2 b_{14} b_3 - 3a_{10} b_{14} b_4 -$$
$$-2a_{10}^2 b_{12} b_5 - a_{10} b_{13} b_5)\, x y^4 - (a_{10}\, a_5 b_{13} - a_{10}\, a_4\, b_{14} + 2\, a_5\, b_{14} + a_{10}^2\, b_{14}\, b_4 -$$
$$-a_{10}^2\, b_{13}\, b_5 - 2\, a_{10} b_{14}\, b_5)\, y^5\,.$$

If the identity $A_{21}(x,y) \equiv 0$ holds, then the multiplicity m_∞ is at least four. The identity $A_{21}(x,y) \equiv 0$ leads us to the following two series of conditions:

1.1) $a_3 = a_{10}\, b_3,\quad a_4 = a_{10}\, b_4\,,\quad a_5 = a_{10}\, b_5$;

1.2) $a_4 = a_{10}\, b_4 + 2\, a_{10}\beta + b_{11}\beta,\ a_5 = a_{10}\, b_5 + 3\, a_{10}^2\beta + 2\, a_{10} b_{11}\beta + b_{12}\beta,\quad b_{13} = -a_{10}\,(4a_{10}^2 + 3\, a_{10}\, b_{11} + 2b_{12}),\quad b_{14} = a_{10}^2\,(3\, a_{10}^2 + 2\, a_{10}\, b_{11} + b_{12}),\ \ \beta = a_3 - a_{10} b_3,\ \beta \neq 0;$

In the conditions 1.1) we have $A_3(x,y) = A_{11}(x,y)A_{31}(x,y)$, where

$$A_{31}(x,y) = (3\, a_1\, a_{10} - a_2 - 3a_{10}^2\, b_1 + a_1\, b_{11} - a_{10}\, b_1\, b_{11} + a_{10} b_2)\, x^4 + \\ +(4\, a_{10}\, a_2 + 2\, a_1\, a_{10}\, b_{11} - 2a_{10}^2 b_1\, b_{11} + 2\, a_1\, b_{12} - 2\, a_{10}\, b_1 b_{12} - 4\, a_{10}^2\, b_2) x^3 y + \\ +(3\, a_{10}\, a_2\, b_{11} + a_1 a_{10} b_{12} + a_2 b_{12} - a_{10}^2\, b_1\, b_{12} + 3\, a_1 b_{13} - 3\, a_{10}\, b_1\, b_{13} - \\ -3a_{10}^2\, b_{11}\, b_2 - a_{10}\, b_{12}\, b_2)\cdot x^2 y^2 + (2a_{10} a_2 b_{12} + 2a_2 b_{13} + 4a_1 b_{14} - 4a_{10} b_1 b_{14} - \\ -2a_{10}^2 b_{12} b_2 - 2a_{10} b_{13} b_2)\, x\, y^3 + (\, a_{10}\, a_2\, b_{13} - a_1\, a_{10}\, b_{14} + 3\, a_2\, b_{14} + a_{10}^2\, b_1\, b_{14} - \\ -a_{10}^2\, b_{13}\, b_2 - 3\, a_{10}\, b_{14}\, b_2)\, y^4.$$

The identity $A_{31}(x,y) \equiv 0$ holds if one of the following two sets of conditions is satisfied:

1.1.1) $a_1 = a_{10}\, b_1,\ a_2 = a_{10}\, b_2;$

1.1.2) $a_2 = a_{10}\, b_2 + 3\, a_{10}\gamma + b_{11}\gamma,\ \ b_{12} = -3\, a_{10}\,(2\, a_{10} + b_{11}),\ \ b_{13} = a_{10}^2\,(8\, a_{10} + 3\, b_{11}),\ \ b_{14} = -a_{10}^3\,(3\, a_{10} + b_{11}),\ \gamma = a_1 - a_{10}\, b_1,\ \gamma \neq 0.$

If one of the conditions 1.1.1) or 1.1.2) is satisfied, then the multiplicity $m_\infty \geq 5$.

In the conditions 1.1.1) we have $A_4(x,y) = \delta A_{11}(x,y)\cdot A_{41}(x,y)$, where $\delta = a_0 - a_{10}\, b_0$ and $A_{41}(x,y) = 4\, a_{10}\, x^3 + b_{11}\, x^3 + 3\, a_{10}\, b_{11}\, x^2\, y + 2b_{12}\, x^2\, y + 2\, a_{10}\, b_{12}\, x\, y^2 + 3\, b_{13}\, x\, y^2 + a_{10} b_{13}\, y^3 + 4\, b_{14}\, y^3.$

If $\delta = 0$, then $deg(gcd(P,Q)) > 0$ (see (4)). Let $\delta \neq 0$ and $A_{41}(x,y) \equiv 0 \Rightarrow b_{11} = -4\, a_{10},\ \ b_{12} = 6a_{10}^2,\ b_{13} = -4\, a_{10}^3,\ \ b_{14} = a_{10}^4,$ then $A_5(x,y) = \delta A_{11}(x,y)\cdot A_{51}(x,y)$, where $A_{51}(x,y) = 3\, a_{10} b_6\, x^2 + b_7 x^2 + 2a_{10} b_7\, xy + 2b_8\, x\, y + a_{10} b_8\, y^2 + 3b_9 y^2.$

The identity $A_{51}(x,y) \equiv 0$ holds if $b_7 = -3\, a_{10}\, b_6,\ b_8 = 3\, a_{10}^2\, b_6,\ b_9 = -a_{10}^3\, b_6.$ In these conditions $A_6(x,y) = \delta\, A_{11}(x,y)(2\, a_{10} b_3\, x + b_4\, x + a_{10} b_4\, y + 2\, b_5\, y) \equiv 0 \ \Rightarrow\ b_4 = -2\, a_{10} b_3,\ b_5 = a_{10}^2 b_3 \ \Rightarrow A_7(x,y) = \delta(a_{10} b_1 + b_2)\cdot A_{11}(x,y) \equiv 0 \Rightarrow b_2 = -a_{10} b_1 \ \Rightarrow A_8(x,y) = 4\,\delta^2 (x - a_{10} y)^3 \not\equiv 0.$

Thus, we have obtain $\mathbb{E}_\infty = Z^8(4\, x^3 - 12a_{10}\, x^2 y + 12\, a_{10}^2\, xy^2 - 4\, a_{10}^3\, y^3 + 3\, b_6\, x^2 Z - 6\, a_{10} b_6\, xyZ + 3\, a_{10}^2 b_6\, y^2 Z + 2\, b_3\, xZ^2 + 2\, a_{10} b_3\, yZ^2 + b_1 Z^3)\delta^2$ and the algebraic multiplicity $m_\infty = 9$.

The quartic system {(3), (4)} takes the form:

$$\dot{x} = a_{10} x^4 - 4\, a_{10}^2\, x^3 y + 6\, a_{10}^3\, x^2\, y^2 - 4a_{10}^4\, xy^3 + a_{10}^5\, y^4 + a_{10} b_6\, x^3 - 3\, a_{10}^2 b_6\cdot x^2 y \\ + 3\, a_{10}^3 b_6\, xy^2 - a_{10}^4 b_6 y^3 + a_{10} b_3\, x^2 - 2\, a_{10}^2 b_3\, xy + a_{10}^3 b_3\, y^2 + \\ + a_{10} b_1 x - a_{10}^2 b_1 y + a_{10} b_0 + \delta, \qquad (7)$$

$$\dot{y} = x^4 - 4a_{10} x^3 y + 6\, a_{10}^2\, x^2 y^2 - 4\, a_{10}^3\, x\, y^3 + a_{10}^4 y^4 + b_6 x^3 - 3\, a_{10} b_6 x^2\, y + \\ +3\, a_{10}^2\, b_6\, x\, y^2 - a_{10}^3\, b_6\, y^3 + b_3\, x^2 - 2\, a_{10}\, b_3\, x\, y + a_{10}^2\, b_3\, y^2 + b_1\, x - a_{10} b_1\, y + b_0.$$

In the conditions 1.1.2) we have $A_4(x,y) = A_{11}(x,y)\cdot A_{41}(x,y)$, where

$A_{41}(x,y) = -4a_0\, a_{10}\, x^3 + 4\, a_{10}^2\, b_0\, x^3 - a_0\, b_{11}\, x^3 + a_{10}\, b_0\, b_{11}\, x^3 + 12\, a_0\, a_{10}^2\, x^2\, y -$

$$-12\,a_{10}^3\,b_0\,x^2\,y+3\,a_0a_{10}\,b_{11}\,x^2\,y-3\,a_{10}^2b_0b_{11}x^2\,y-12\,a_0\,a_{10}^3\,x\,y^2+$$
$$+12\,a_{10}^4\,b_0\,xy^2-3\,a_0\,a_{10}^2\,b_{11}\,x\,y^2+3\,a_{10}^3\,b_0b_{11}\,x\;\,y^2+4\,a_0a_{10}^4\,y^3-\;4\,a_{10}^5b_0\,y^3+$$
$$+a_0a_{10}^3b_{11}\,y^3-a_{10}^4b_0\,b_{11}\,y^3+a_{10}b_6\,x^3\,\gamma+\;b_{11}b_6\,x^3\,\gamma-b_7\,x^3\,\gamma-9\,a_{10}^2\,b_6\,x^2y\,\gamma-$$
$$-3\,a_{10}b_{11}b_6\,x^2y\,\gamma-a_{10}\,b_7\,x^2\,y\,\gamma-2\,b_8\,x^2\,y\,\gamma-6\,a_{10}^2\,b_7\,x\,y^2\,\gamma-$$
$$-2a_{10}b_{11}b_7\,x\,y^2\gamma-3\,a_{10}\,b_8\,x\,y^2\gamma-b_{11}b_8\,xy^2\gamma-3\,b_9\,xy^2\,\gamma-3\,a_{10}^2b_8\cdot\,y^3\gamma-$$
$$-a_{10}b_{11}b_8\,y^3\gamma-5\,a_{10}b_9\,y^3\gamma-2b_{11}b_9\,y^3\gamma\;.$$

The identity $A_{41}(x,y)\equiv 0$ yields

$$b_7=(-4a_0a_{10}+4a_{10}^2b_0-a_0b_{11}+a_{10}b_0b_{11}+a_{10}b_6\,\gamma+b_{11}b_6\,\gamma)/\gamma,$$
$$b_8=a_{10}(8\,a_0a_{10}-8a_{10}^2b_0+2a_0b_{11}-2a_{10}b_0b_{11}-5a_{10}b_6\gamma-2b_{11}\cdot b_6\,\gamma)/\gamma,$$
$$b_9=a_{10}^2\,(-4\,a_0a_{10}+4\,a_{10}^2\,b_0-a_0\,b_{11}+a_{10}\,b_0\,b_{11}+3\,a_{10}\,b_6\,\gamma+b_{11}\,b_6\,\gamma)/\gamma.$$

In these conditions $A_5(x,y)=-A_{11}(x,y)\cdot A_{51}(x,y)/\gamma$, where

$$A_{51}(x,y)=4\,a_0^2\,a_{10}\,x^2-8\,a_0a_{10}^2b_0\,x^2+4\,a_{10}^3\,b_0^2\,x^2+a_0^2\,b_{11}\,x^2-$$
$$-2\,a_0\,a_{10}\,b_0\,b_{11}\,x^2+a_{10}^2\,b_0^2\,b_{11}\,x^2-8\,a_0^2a_{10}^2\,x\,y+\;16\,a_0\,a_{10}^3\,b_0\,x\,y\;-$$
$$-8\,a_{10}^4b_0^2\,x\,y\;-2a_0^2\,a_{10}b_{11}\,x\,y\;+\;4\,a_0a_{10}^2b_0b_{11}\,x\,y\;-2\,a_{10}^3b_0^2b_{11}\,x\,y\;+$$
$$+4a_0^2a_{10}^3\,y^2-8\,a_0a_{10}^4b_0\,y^2+4\,a_{10}^5b_0^2\,y^2+a_0^2a_{10}^2b_{11}\,y^2-2\,a_0a_{10}^3b_0b_{11}y^2+$$
$$+a_{10}^4b_0^2b_{11}\,y^2\;-\;4\,a_0a_{10}b_6\,x^2\gamma+4\,a_{10}^2\,b_0b_6\,x^2\gamma-a_0b_{11}b_6\,x^2\,\gamma+a_{10}b_0b_{11}b_6\,x^2\,\gamma+$$
$$+8\,a_0\,a_{10}^2\,b_6x\,y\,\gamma-8\,a_{10}^3b_0b_6\,x\,y\,\gamma+2\,a_0a_{10}b_{11}b_6\,x\,y\,\gamma\;-2\,a_{10}^2b_0b_{11}b_6\,x\,y\,\gamma-$$
$$-\,4\,a_0a_{10}^3b_6\,y^2\,\gamma+4\,a_{10}^4b_0b_6\cdot\,y^2\,\gamma-a_0a_{10}^2\,b_{11}\,b_6\,y^2\,\gamma+\;a_{10}^3\,b_0\,b_{11}\,b_6\,y^2\,\gamma+$$
$$+\,2a_{10}\,b_3\,x^2\,\gamma^2+b_{11}\,b_3\,x^2\,\gamma^2-b_4x^2\,\gamma^2-6\,a_{10}^2\,b_3\,x\,y\,\gamma^2-2\,a_{10}\,b_{11}\,b_3\,x\,y\,\gamma^2-$$
$$-2\,b_5\,x\,y\,\gamma^2-\;3\,a_{10}^2\,b_4\,y^2\,\gamma^2\;-\;a_{10}\,b_{11}\,b_4\,y^2\,\gamma^2\;-\;2\,a_{10}\,b_5\,y^2\,\gamma^2\;-b_{11}\,b_5\,y^2\,\gamma^2.$$

The identity $A_{51}\equiv\;0\Longrightarrow b_4=(4\,a_0^2\,a_{10}-8\,a_0\,a_{10}^2\,b_0+4\,a_{10}^3\,b_0^2+\;a_0^2\,b_{11}-$

$$-\,2a_0\cdot a_{10}b_0b_{11}+a_{10}^2b_0^2\,b_{11}-4a_0a_{10}b_6\,\gamma\;+4a_{10}^2b_0\,b_6\,\gamma-a_0b_{11}b_6\,\gamma$$
$$+\;\;a_{10}b_0b_{11}b_6\,\gamma+2\,a_{10}\cdot\;b_3\,\gamma^2+b_{11}\,b_3\,\gamma^2)/\gamma^2,$$
$$b_5=-a_{10}\,(4\,a_0^2\,a_{10}-8\,a_0\,a_{10}^2\,b_0+4\,a_{10}^3\,b_0^2+a_0^2\,b_{11}-\;2a_0a_{10}\cdot b_0b_{11}+a_{10}^2b_0^2b_{11}-$$
$$-4a_0a_{10}b_6\,\gamma+4\,a_{10}^2b_0b_6\,\gamma-a_0b_{11}b_6\,\gamma+a_{10}\,b_0\,b_{11}\,b_6\,\gamma+$$
$$3\,a_{10}\,b_3\cdot\;\gamma^2+b_{11}\,b_3\;\;\gamma^2)/\gamma^2\Rightarrow$$
$$A_6(x,y)=-(x-\;a_{10}\,y)^2\,(x\;+\;3\,a_{10}\,y\;+\;b_{11}\,y)\cdot A_{61}(x,y)/\gamma^2,\text{ where}$$
$$A_{61}(x,y)=-4\,a_0^3\,a_{10}\,x^2+12\,a_0^2\,a_{10}^2\,b_0\,x^2-12\,a_0\,a_{10}^3\,b_0^2\,x^2+\;\;4\,a_{10}^4\,b_0^3\,x^2-$$
$$-a_0^3\,b_{11}\,x^2+3\,a_0^2\,a_{10}\,b_0\,b_{11}\,x^2-3\,a_0\,a_{10}^2\,b_0^2\,b_{11}\,x^2+a_{10}^3\,b_0^3\,b_{11}\,x^2+$$
$$+8\,a_0^3\,a_{10}^2\,xy-24a_0^2\;\cdot a_{10}^3\,b_0\,x\,y+24a_0\,a_{10}^4\,b_0^2\,x\,y-8\,a_{10}^5\,b_0^3\,x\,y\;+$$
$$+2\,a_0^3\,a_{10}\,b_{11}\,x\,y-6\,a_0^2\,a_{10}^2\,b_0\,b_{11}\,xy+6a_0\cdot a_{10}^3\,b_0^2\,b_{11}\,x\,y-2a_{10}^4\,b_0^3\,b_{11}x\,y-$$
$$-4a_0^3\,a_{10}^3\,y^2+12\,a_0^2\,a_{10}^4\,b_0\,y^2-12\,a_0a_{10}^5\,b_0^2\,y^2+4\,a_{10}^6\cdot\;b_0^3y^2-a_0^3a_{10}^2\,b_{11}\,y^2+$$
$$+\,3\,a_0^2\,a_{10}^3\,b_0\,b_{11}\,y^2-3a_0\,a_{10}^4\,b_0^2\,b_{11}\,y^2+a_{10}^5\,b_0^3\,b_{11}\,y^2+4\,a_0^2\,a_{10}\cdot\;b_6\,x^2\,\gamma-$$
$$-8\,a_0\,a_{10}^2\,b_0\,b_6\,x^2\,\gamma+\;4\,a_{10}^3\,b_0^2\,b_6\,x^2\gamma+a_0^2\,b_{11}b_6\,x^2\gamma-2\,a_0\,a_{10}\,b_0\,b_{11}\,b_6\,x^2\,\gamma+$$
$$+a_{10}^2\,b_0^2\,b_{11}b_6\,x^2\;\;\gamma-8\,a_0^2\,a_{10}^2\,b_6\,x\,y\;\;\gamma+16\,a_0\,a_{10}^3\,b_0\,b_6\,x\,y\;\;\gamma-$$
$$-8\,a_{10}^4\,b_0^2\,b_6\,x\,y\;\;\gamma-2\,a_0^2\;\;\cdot a_{10}\,b_{11}\,b_6\,x\,y\;\;\gamma+4\,a_0\,a_{10}^2\,b_0\,b_{11}\,b_6\,x\,y\;\;\gamma-$$
$$-2\,a_{10}^3\,b_0^2\,b_{11}\,b_6\,x\,y\;\;\gamma+4\,a_0^2\,a_{10}^3\,b_6\,y^2\;\;\gamma-\;8\,a_0\,a_{10}^4\,b_0\,b_6\,y^2\;\;\gamma+$$
$$+4\,a_{10}^5\,b_0^2\,b_6\,y^2\;\;\gamma+\;a_0^2\,a_{10}^2\,b_{11}\,b_6\,y^2\;\;\gamma-2\,a_0\,a_{10}^3\,b_0\,b_{11}\,b_6\,y^2\;\;\gamma+$$

$$+a_{10}^4 b_0^2 b_{11} b_6 y^2 \gamma - 4 a_0 a_{10} b_3 x^2 \gamma^2 + 4 a_{10}^2 b_0 b_3 x^2 \gamma^2 - a_0 b_{11} b_3 x^2 \gamma^2 +$$
$$+a_{10} b_0 b_{11} \cdot b_3 x^2\gamma^2 + 8 a_0 a_{10}^2 b_3 x y \gamma^2 - 8 a_{10}^3 b_0 b_3 xy \gamma^2 +$$
$$+ 2 a_0 a_{10} b_{11} b_3 xy \gamma^2 - a_{10}^2 b_0 b_{11} \cdot b_3 xy \gamma^2 - 4 a_0 a_{10}^3 b_3 y^2\gamma^2 +$$
$$+4 a_{10}^4 b_0 b_3 y^2 \gamma^2 - a_0 a_{10}^2 b_{11} b_3 y^2 \gamma^2 + a_{10}^3 b_0 b_{11} b_3 \cdot y^2 \gamma^2 + 3 a_{10} b_1 x^2 \gamma^3 +$$
$$+b_1 b_{11} x^2\gamma^3 - b_2 x^2\gamma^3 - 6 a_{10}^2 b_1 x y \gamma^3 - 2 a_{10} b_1 b_{11} x y \gamma^3 + 2 a_{10} b_2 xy \gamma^3 +$$
$$+3 a_{10}^3 b_1 y^2\gamma^3 + a_{10}^2 b_1 b_{11} y^2\gamma^3 - a_{10}^2 b_2 y^2\gamma^3 - 3 x^2\gamma^4 - 18 a_{10} xy \gamma^4 -$$
$$-6 b_{11} y\gamma^4 - 27 a_{10}^2 y^2 \gamma^4 - 18 a_{10} b_{11} y^2 \gamma^4 - 3 b_{11}^2 y^2\gamma^4).$$

The identity $A_{61}(x,y) \equiv 0$ holds if $b_2 = -(a_{10} b_1 + 3\gamma)$, $b_{11} = -4a_{10}$. In these conditions we have $A_7(x,y) = \gamma(x - a_{10} y)^4 (4 a_0 - 4 a_{10} b_0 - b_6 \gamma) \equiv 0 \Rightarrow a_0 = (4 a_{10} b_0 + b_6 \gamma)/4 \Rightarrow A_8(x,y) = \gamma^2(8 b_3 - 3 b_6^2)(-x + a_{10} y)^3/4 \Rightarrow b_3 = 3b_6^2/8 \Rightarrow A_9(x,y) = 3\gamma^2(-x + a_{10} y)(16 b_1 x - b_6^3 x - 16 a_{10} b_1 y + a_{10} b_6^3 y - 64 y \gamma)/16 \not\equiv 0$.

So, $\mathbb{E}_\infty = -Z^9 \gamma^2(-4x + 4 a_{10} y - b_6 Z)(-48 b_1 x + 3 b_6^3 x + 48 a_{10} b_1 y - 3 a_{10} b_6^3 y - 64 b_0 Z + 4 b_1 b_6 Z + 192 y \gamma)/64$ and $m_\infty = 10$.

In this case the quartic system {(3), (4)} looks as:

$$\dot{x} = 8a_{10} x^4 - 32a_{10}^2 x^3 y + 48a_{10}^3 x^2y^2 - 32 a_{10}^4 xy^3 + 8 a_{10}^5 y^4 +$$
$$+8 a_{10} b_6 x^3 - 24 a_{10}^2 b_6 x^2 y + 24 a_{10}^3 b_6 xy^2 - 8 a_{10}^4 b_6 y^3 + 3 a_{10} \cdot b_6^2 x^2 -$$
$$-6 a_{10}^2 b_6^2 xy + 3a_{10}^3 b_6^2 y^2 + 8 a_{10} b_1 x - 8 a_{10}^2 b_1 y + 8 a_{10} b_0 +$$
$$+ 8 x \gamma - 32 a_{10} y\gamma + 2 b_6 \gamma)/8, \qquad (8)$$
$$\dot{y} = (8 x^4 - 32 a_{10} x^3 y + 48 a_{10}^2 x^2y^2 - 32 a_{10}^3 xy^3 + 8 a_{10}^4 y^4 +$$
$$+8 b_6 x^3 - 24 a_{10} b_6 x^2 y + 24 a_{10}^2 b_6 xy^2 - 8 a_{10}^3 b_6 y^3 + 3 b_6^2 x^2 -$$
$$- 6 a_{10} b_6^2 xy + 3 a_{10}^2 b_6^2 y^2 + 8 b_1 x - 8 a_{10} b_1 y + 8 b_0 - 24 y \gamma)/8.$$

The transformation of coordinates $X = b_6 + 4x - 4 a_{10} y$, $Y = 4(64 b_0 - 4 b_1 b_6 + (48 b_1 - 3 b_6^3)x - (48 a_{10} b_1 - 3 a_{10} b_6^3 + 192\gamma)y)/3$ and time rescaling $t = -\tau/\gamma$ reduce the system (8) to the system

$$\dot{X} = -X, \quad \dot{Y} = X^4 + 3Y. \qquad (9)$$

In the conditions 1.2) we have $A_3(x,y) = -A_{11}(x,y) \cdot A_{31}(x,y)$, where

$$A_{31}(x,y) = (-3 a_1 a_{10} + a_2 + 3 a_{10}^2 b_1 - a_1 b_{11} + a_{10} b_1 b_{11} - a_{10} b_2)x^4 -$$
$$-(4 a_{10} a_2 + 2 a_1 a_{10} b_{11} - 2a_{10}^2 b_1 b_{11} + 2a_1 b_{12} - 2a_{10} b_1 b_{12} - 4 a_{10}^2 b_2)x^3 y +$$
$$+(12a_1 a_{10}^3 - 12a_{10}^4 b_1 + 9a_1 a_{10}^2 b_{11} - 3 a_{10} a_2 b_{11} - 9a_{10}^3 b_1 b_{11} + 5a_1 a_{10} b_{12} -$$
$$a_2 b_{12} - 5 a_{10}^2 b_1 b_{12} + 3a_{10}^2 b_{11} b_2 + a_{10} b_{12} b_2) x^2y^2 -$$
$$-(12 a_1 a_{10}^4 - 8a_{10}^3 a_2 - 12a_{10}^5 b_1 + 8a_1 a_{10}^3 b_{11} - 6a_{10}^2 a_2 b_{11} - 8 a_{10}^4 b_1 b_{11} +$$
$$+4 a_1 a_{10}^2 b_{12} - 2 a_{10} a_2 b_{12} - 4 a_{10}^3 b_1 b_{12} + 8a_{10}^4 b_2 + 6a_{10}^3 b_{11} b_2 + 2a_{10}^2 b_{12} b_2) \cdot xy^3 +$$
$$+(3a_1 a_{10}^5 - 5a_{10}^4 a_2 - 3 a_{10}^6 b_1 + 2a_1 a_{10}^4 b_{11} - 3a_{10}^3 a_2 b_{11} - 2a_{10}^5 b_1 b_{11} +$$
$$+a_1 a_{10}^3 b_{12} - a_{10}^2 a_2 b_{12} - a_{10}^4 b_1 b_{12} + 5 a_{10}^5 b_2 + 3 a_{10}^4 b_{11} b_2 + a_{10}^3 b_{12} b_2)y^4 +$$
$$+ \beta(a_{10} b_6 + b_{11} b_6 - b_7)x^4 +$$
$$+ \beta(2 a_{10}^2 b_6 + 2a_{10} b_{11} b_6 + 2 b_{12} b_6 - 2 b_8) x^3 y - \beta(9 a_{10}^3 b_6 + 6 a_{10}^2 b_{11} b_6 +$$
$$+3 a_{10} b_{12} b_6 - a_{10}^2 b_7 - a_{10} b_{11} b_7 - b_{12} b_7 + a_{10} b_8 + b_{11} b_8 + 3 b_9)x^2y^2 -$$

$$-\beta\,(6\,a_{10}^3 b_7 + 4\,a_{10}^2 b_{11} b_7 + 2\,a_{10} b_{12} b_7 + 2a_{10} b_9 + 2b_{11} b_9) x y^3 -$$
$$-\beta(3\,a_{10}^3 b_8 + 2\,a_{10}^2\,b_{11}\,b_8 + a_{10}\,b_{12}\,b_8 + a_{10}^2\,b_9 + a_{10}\,b_{11} b_9 + b_{12}\,b_9) y^4 .$$

As $A_{11}(x,y) \not\equiv 0$, we require that $A_{31}(x,y) \equiv 0$. The identity $A_{31}(x,y) \equiv 0$ holds if

$$a_2 = 3a_1 a_{10} - 3a_{10}^2 b_1 + a_1 b_{11} - a_{10} b_1 b_{11} + a_{10} b_2 - a_{10} b_6 \beta - b_{11} b_6\,\beta + b_7\,\beta,$$
$$b_8 = (-6a_1 a_{10}^2 + 6a_{10}^3 b_1 - 3\,a_1 a_{10} b_{11} + 3\,a_{10}^2 b_1 b_{11} - a_1 b_{12} + a_{10} b_1 b_{12} +$$
$$+3a_{10}^2 b_6\,\beta + 3a_{10} b_{11} b_6\,\beta + b_{12} b_6 \beta - 2a_{10} b_7 \beta)/\beta,$$
$$b_9 = -a_{10}\,(-6\,a_1 a_{10}^2 + 6a_{10}^3 b_1 - 3a_1 a_{10} b_{11} + 3\,a_{10}^2 b_1 b_{11} - a_1 b_{12} + a_{10} b_1 b_{12} +$$
$$+4a_{10}^2 b_6 \beta + 3a_{10} b_{11} b_6 \beta + b_{12} b_6 \beta - a_{10} b_7 \beta)/\beta.$$

In these conditions $A_4(x,y) \equiv 0 \;\Rightarrow$

$$b_4 = -(4\,a_0 a_{10} - 4a_{10}^2 b_0 + a_0 b_{11} - a_{10} b_0 b_{11} - a_1 a_{10} b_6 + a_{10}^2 b_1 b_6 - a_1 b_{11} b_6 +$$
$$+a_{10} b_1 b_{11} b_6 + a_1 b_7 - a_{10} b_1 b_7 - 2a_{10} b_3\,\beta - b_{11} b_3 \beta + a_{10} b_6^2 \beta +$$
$$+b_{11} b_6^2 \beta - b_6 b_7 \beta + 2\beta^2)/\beta,$$
$$b_5 = (4a_0 a_{10}^2 - 4a_{10}^3 b_0 + a_0 a_{10} b_{11} - a_{10}^2 b_0 b_{11} - a_1 a_{10}^2 b_6 + a_{10}^3 b_1 b_6 - a_1 a_{10} b_{11} b_6 +$$
$$+a_{10}^2 b_1 b_{11} b_6 + a_1 a_{10} b_7 - a_{10}^2 b_1 b_7 - 3a_{10}^2 \cdot b_3 \beta - a_{10} b_{11} b_3 \beta + a_{10}^2 b_6^2 \beta +$$
$$+a_{10} b_{11} b_6^2 \beta - a_{10} b_6 b_7 \beta - 6a_{10} \beta^2 - 2b_{11} \beta^2)\beta,$$
$$b_{12} = -3(2a_{10}^2 + a_{10} b_{11}) \Rightarrow A_5(x,y) = -A_{11}(x,y) \cdot A_{51}(x,y)/\beta, \text{ where}$$
$$A_{51} = (-4a_0 a_1 a_{10} + 4a_1 a_{10}^2 b_0 + 4a_0 a_{10}^2 b_1 - 4a_{10}^3 b_0 b_1 - a_0 a_1 b_{11} + a_1 a_{10} b_0 b_{11} +$$
$$+a_0 a_{10} b_1 b_{11} - a_{10}^2 b_0 b_1 b_{11} + a_1^2 a_{10} b_6 - 2a_1 a_{10}^2 b_1 b_6 + a_{10}^3 b_1^2 b_6 + a_1^2 b_{11} b_6 -$$
$$-2a_1 a_{10} b_1 b_{11} b_6 + a_{10}^2 b_1^2 b_{11} b_6 - a_1^2 b_7 + 2a_1 \cdot a_{10} b_1 b_7 - a_{10}^2 b_1^2 b_7) x^2 + (8\,a_0 a_1 a_{10}^2 -$$
$$-8\,a_1 a_{10}^3 b_0 - 8\,a_0 a_{10}^3 b_1 + 8\,a_{10}^4 b_0 b_1 + 2\,a_0 a_1\,a_{10} \cdot b_{11} - 2a_1 a_{10}^2 b_0 b_{11} -$$
$$-2a_0 a_{10}^2 b_1 b_{11} + 2a_{10}^3 b_0 b_1 b_{11} - 2a_1^2 a_{10}^2 b_6 + 4a_1 a_{10}^3 b_1 b_6 - 2a_{10}^4 b_1^2 b_6 - 2a_1^2 a_{10} b_{11} b_6 +$$
$$+4a_1 a_{10}^2 b_1 b_{11} b_6 - 2a_{10}^3 b_1^2 b_{11} b_6 + 2a_1^2 a_{10} b_7 - 4a_1 a_{10}^2 b_1 b_7 + 2a_{10}^3 b_1^2 b_7) x\,y +$$
$$+(4a_1 a_{10}^4 b_0 - 4a_0 a_1 a_{10}^3 + 4a_0 a_{10}^4 b_1 - 4a_{10}^5 b_0 b_1 - a_0 a_1 a_{10}^2 b_{11} + a_1 a_{10}^3 b_0 b_{11} +$$
$$+a_0 a_{10}^3 b_1 b_{11} - a_{10}^4 b_0 b_1 b_{11} + a_1^2\,a_{10}^3 b_6 - 2a_1 a_{10}^4 b_1 b_6 + a_{10}^5 b_1^2 b_6 + a_1^2 a_{10}^2 b_{11} b_6 -$$
$$-2a_1 a_{10}^3 b_1 b_{11} b_6 + a_{10}^4 b_1^2 b_{11} \cdot b_6 - a_1^2 a_{10}^2 b_7 + 2a_1 a_{10}^3 b_1 b_7 - a_{10}^4 b_1^2 b_7)\,y^2 +$$
$$+\beta\,(3\,a_0 a_{10} b_6 - 3\,a_{10}^2 b_0 b_6 - a_1 a_{10} b_6^2 + a_{10}^2 b_1 \cdot b_6^2 - a_1 b_{11} b_6^2 + a_{10} b_1 b_{11} b_6^2 +$$
$$+a_0 b_7 - a_{10} b_0 b_7 + a_1 b_6 b_7 - a_{10} b_1 b_6 b_7) x^2 + \beta(6\,a_{10}^3 b_0 b_6 - 6a_0 a_{10}^2 b_6 + 2a_1 a_{10}^2 b_6^2 -$$
$$-2a_{10}^3 b_1 b_6^2 + 2a_1 a_{10} b_{11} b_6^2 - 2a_{10}^2\,b_1 b_{11} b_6^2 - 2a_0 a_{10} b_7 + 2a_{10}^2 b_0 b_7 - 2a_1 a_{10} b_6 b_7 +$$
$$+2a_{10}^2 b_1 b_6 b_7)\,x\,y + \beta(3\,a_0 a_{10}^3\,b_6 - 3\,a_{10}^4\,b_0\,b_6 - a_1\,a_{10}^3\,b_6^2 + a_{10}^4\,b_1\,b_6^2 -$$
$$-a_1 a_{10}^2\,b_{11}\,b_6^2 + a_{10}^3\,b_1\,b_{11}\,b_6^2 + a_0\,a_{10}^2\,b_7 - a_{10}^3\,b_0\,b_7 + a_1\,a_{10}^2\,b_6\,b_7 -$$
$$-a_{10}^3\,b_1\,b_6\,b_7) y^2 + \beta^2\,(3a_1 - 6\,a_{10}\,b_1 - b_1\,b_{11} + b_2 + a_{10}\,b_3\,b_6 + b_{11}\,b_3\,b_6 -$$
$$-b_3\,b_7) x^2 + \beta^2\,(18\,a_1 a_{10} - 12\,a_{10}^2\,b_1 + 6\,a_1 b_{11} - 4\,a_{10} b_1 b_{11} - 2a_{10} b_2 -$$
$$-2a_{10}^2\,b_3\,b_6 - 2a_{10}\,b_{11} b_3\,b_6 + 2a_{10} b_3 b_7) \cdot x\,y + \beta^2\,(27\,a_1\,a_{10}^2 - 30\,a_{10}^3\,b_1 +$$
$$+18\,a_1\,a_{10}\,b_{11} - 19\,a_{10}^2\,b_1\,b_{11} + 3\,a_1\,b_{11}^2 - 3\,a_{10}\,b_1 b_{11}^2 + a_{10}^2\,b_2 + a_{10}^3\,b_3\,b_6 +$$
$$+a_{10}^2\,b_{11}\,b_3\,b_6 - a_{10}^2\,b_3\,b_7) y^2 - \beta^3 b_6\,x^2 - \beta^3(10\,a_{10}\,b_6 + 6\,b_{11}\,b_6 - 4\,b_7)\,x\,y -$$
$$-\beta^3(13\,a_{10}^2 b_6 + 12\,a_{10} b_{11} b_6 + 3\,b_{11}^2 b_6 - 4\,a_{10} b_7 - 2\,b_{11}\,b_7) y^2.$$

The identity $A_5(x,y) \equiv 0$ holds if

$$b_2 = (-12a_1^3 a_{10} + 36a_1^2 a_{10}^2 b_1 - 36a_1 a_{10}^3 b_1^2 + 12a_{10}^4 b_1^3 - 3a_1^3 b_{11} + 9a_1^2 a_{10} b_1 b_{11} - 9a_1 a_{10}^2 b_1^2 b_{11} + 3a_{10}^3 b_1^3 b_{11} + 20a_0 a_1 a_{10}\beta - 20a_1 a_{10}^2 b_0 \beta - 20a_0 a_{10}^2 b_1 \beta + +20a_{10}^3 b_0 b_1 \beta + 5a_0 a_1 b_{11}\beta - 5a_1 a_{10} b_0 b_{11}\beta - 5a_0 a_{10} b_1 b_{11}\beta + 5a_{10}^2 b_0 b_1 b_{11}\beta + +16a_1^2 a_{10} b_6 \beta - 32a_1 a_{10}^2 b_1 b_6 \beta + 16a_{10}^3 b_1^2 b_6 \beta + 4\, a_1^2 b_{11} b_6 \beta - 8a_1 a_{10} b_1 \cdot b_{11} b_6 \beta + +4a_{10}^2 b_1^2 b_{11} b_6 \beta - 12a_1 a_{10} b_3 \beta^2 + 12a_{10}^2 b_1 b_3 \beta^2 - 3a_1 b_{11} b_3 \beta^2 + 3a_{10} b_1 b_{11} b_3 \beta^2 - +12a_0 a_{10} b_6 \beta^2 + 12a_{10}^2 b_0 b_6 \beta^2 - 3a_0 b_{11} b_6 \beta^2 + 3a_{10} b_0 b_{11} b_6\, \beta^2 - 4a_1 a_{10} b_6^2 \beta^2 + +4a_{10}^2 b_1 b_6^2 \cdot \beta^2 - a_1 b_{11} b_6^2 \beta^2 + a_{10} b_1 b_{11} b_6^2 \beta^2 - 6a_1 \beta^3 + 12a_{10} b_1 \beta^3 + 2b_1 b_{11}\beta^3 + +4a_{10} b_3 b_6 \beta^3 + b_{11} b_3 \cdot b_6 \beta^3 + 2b_6 \beta^4)/(2\beta^3) \text{ and}$$

$$b_7 = -3(4a_1 a_{10} - 4a_{10}^2 b_1 + a_1 b_{11} - a_{10} b_1 b_{11} - 2a_{10} b_6 \beta - b_{11} b_6 \beta)/(2\beta).$$

In these conditions $A_6(x,y) \not\equiv 0$, therefore $m_\infty = 7$.

In the conditions 2) the identity $A_2(x,y) \equiv 0$ leads us to the following conditions

$$a_4 = 2a_{10} a_3 + a_3 b_{11} - 2a_{10}^2 b_3 - a_{10} b_{11} b_3 + a_{10} b_4 + a_{10} b_6 \alpha + b_{11} b_6 \alpha - b_7 \alpha + \alpha^2,$$

$$a_5 = 3a_{10}^2 a_3 + 2a_{10} a_3 b_{11} + a_3 b_{12} - 3a_{10}^3 b_3 - 2a_{10}^2 b_{11} b_3 - a_{10} b_{12} b_3 + a_{10} b_5 + +2a_{10}^2 b_6 \alpha + 2a_{10} \cdot b_{11} b_6 \alpha + b_{12} b_6 \alpha - a_{10} b_7 \alpha - b_8 \alpha + 3a_{10}\alpha^2 + b_{11}\alpha^2,$$

$$b_9 = -a_{10}^3 b_6 - a_{10}^2 b_7 - a_{10} b_8 + 6a_{10}^2 \alpha + 3a_{10} b_{11}\alpha + b_{12}\alpha,$$

$$b_{13} = -a_{10}(4a_{10}^2 + 3a_{10} b_{11} + 2b_{12}).$$

In the above conditions we have: $A_3(x,y) \equiv 0 \;\Rightarrow$

$$a_2 = 3a_1 a_{10} - 3a_{10}^2 b_1 + a_1 b_{11} - a_{10} b_1 b_{11} + a_{10} b_2 - a_{10} a_3 b_6 - a_3 b_{11} b_6 + a_{10}^2 b_3 b_6 + +a_{10} b_{11} b_3 b_6 + a_3 b_7 - a_{10} b_3 b_7 - 3a_3 \alpha + 5a_{10} b_3 \alpha + b_{11} b_3 \alpha - b_4 \alpha - a_{10} b_6^2 \alpha - -b_{11} b_6^2 \alpha + b_6 b_7 \alpha - 2b_6 \alpha^2,$$

$$b_5 = -4a_{10} a_3 - a_3 b_{11} + 3a_{10}^2 b_3 + a_{10} b_{11} b_3 - a_{10} b_4 - a_{10} b_6 \alpha - b_{11} b_6 \alpha + b_7 \alpha - 2\alpha^2,$$

$$b_8 = -3a_{10}^2 b_6 - 2a_{10} b_7 + 8a_{10}\alpha + 2b_{11}\alpha, \quad b_{12} = -3a_{10}(2a_{10} + b_{11}) \Rightarrow$$

$$\Rightarrow A_4(x,y) \equiv 0 \Rightarrow b_2 = -a_1,\; b_4 = -2a_3, \quad b_7 = 3(-a_{10} b_6 + \alpha),\; b_{11} = -4a_{10} \Rightarrow$$

$$\Rightarrow A_5(x,y) \not\equiv 0,\; m_\infty = 6.$$

Thus, the maximal algebraic multiplicity of the line at infinity is not greater than ten (see the case 1.1.2). In this way we have proved the Theorem.

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INVARIANT CONDITIONS OF STABILITY OF MOTION FOR SOME FOUR-DIMENSIONAL DIFFERENTIAL SYSTEMS

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Abstract. Center-affine invariant conditions of the stability of unperturbed motion were determined for four-dimensional quadratic differential system of Darboux type in non-degenerate invariant condition.
Keywords: differential system, unperturbed motion, invariant, comitant, Lie algebra, stability.
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CONDIȚIILE INVARIANTE DE STABILITATE ALE MIȘCĂRII PENTRU UNELE SISTEME DIFERENȚIALE PATRUDIMENSIONALE

Rezumat. Au fost obținute condițiile centroafin invariante de stabilitate a mișcării neperturbate pentru sistemul diferențial patru dimensional pătratic de tip Darboux în condiția invariantă nedegenerată.
Cuvinte-cheie: sistemul diferențial, mișcarea neperturbată, invariant, comitant, algebra Lie, stabilitatea.

1. Introduction

In mathematics, stability theory addresses the stability of solutions of differential equations and of trajectories of dynamical systems under small perturbations of initial conditions.

The differential systems with polynomial nonlinearities are important in various applied problems. For example: the Van der Pol oscillator; the Fitzhugh–Nagumo model for action potentials of neurons; in seismology to model the two plates in a geological fault; in studies of phonation to model the right and left vocal fold oscillators as well as many other applications.

The stability of unperturbed motions using the theory of algebras, of invariants and of Lie algebras was studied for the first time in [1].

In [2] the center-affine invariant conditions of stability of unperturbed motion, described by critical two-dimensional differential systems with quadratic nonlinearities s(1; 2), cubic nonlinearities s(1; 3) and fourth-order nonlinearities s(1; 4), were obtained.

In this paper, the similar investigations are done for some four-dimensional differential systems with quadratic nonlinearities.

2. Center-affine invariants and mixt comitants for four-dimensional differential system with quadratic nonlinearities

We consider the system of differential equations

$$\frac{dx^j}{dt} = a^j_\alpha x^\alpha + a^j_{\alpha\beta} x^\alpha x^\beta \equiv P^j(x,a) \quad \left(j,\alpha,\beta=\overline{1,4}\right), \tag{1}$$

where $a^j_{\alpha\beta}$ is a symmetric tensor in lower indices in which the total convolution is done, and the group of center-affine transformations $GL(4,\mathbb{R})$ given by formulas

$$\overline{x}^r = q^r_j x^j, \ \ \det\left(q^r_j\right) \neq 0; \ \ \left(r, j = \overline{1,4}\right). \tag{2}$$

Coefficients and variables in (1) are given over the field of real numbers $\mathbb{R}$. The phase variables vector $x = \left(x^1, x^2, x^3, x^4\right)$ of system (1), which changes by formulas (2), is usually called *contravariant* [3]. Any other vector $y = \left(y^1, y^2, y^3, y^4\right)$ which changes by formulas (2), is called *cogradient* with vector x. The vector $u = \left(u_1, u_2, u_3, u_4\right)$, which changes by formulas

$$\overline{u}_r = p^j_r u_j, \quad \left(r, j = \overline{1,4}\right), \tag{3}$$

where $p^r_j q^j_s = \delta^r_s$ is the Kronecker's symbol, is called *covariant*. The vector u is also called *contragradient* with vector x.

Applying the transformation (2), the system (1) will be brought to the system

$$\frac{d\overline{x}^j}{dt} = \overline{a}^j_\alpha \overline{x}^\alpha + \overline{a}^j_{\alpha\beta} \overline{x}^\alpha \overline{x}^\beta \quad \left(j, \alpha, \beta = \overline{1,4}\right), \tag{4}$$

in which the coefficients are linear functions of the coefficients of system (1) and are rational functions of parameters of transformation (2). We will denote the set of coefficients of system (1) by a, the set of coefficients of transformed system (4) by $\overline{a}$, and the set of parameters of transformation (2) by q.

According to [3], we say that the polynomial $k(x,u,a)$ of the coefficients of system (1) and of the coordinates of vectors x and u is call *mixt comitant* of the system (1) with respect to $GL(4,\mathbb{R})$ group, if the following identity holds

$$k(\overline{x}, \overline{u}, \overline{a}) = \Delta^{-g} \cdot k(x,u,a), \tag{5}$$

for all q from $GL(4,\mathbb{R})$ and every coordinates of vectors x and u, as well as all the coefficients a of system (1), where g is an integer number called the *weight of comitant*. If the mixt comitant k does not depend on the coordinates of the vector u, then we call it simply *comitant*, but if k does not depend on the coordinates of the vector x we call it *contravariant*. If k does not depend on x and u, then we will call it *invariant* of system (1) with respect to $GL(4,\mathbb{R})$ group.

The following center-affine invariant polynomials of the system (1) are known from [4]:

$$I_{1,4} = a^\alpha_\alpha, \ \ I_{2,4} = a^\alpha_\beta a^\beta_\alpha, \ \ I_{3,4} = a^\alpha_\gamma a^\beta_\alpha a^\gamma_\beta, \ \ I_{4,4} = a^\alpha_\delta a^\beta_\alpha a^\gamma_\beta a^\delta_\gamma,$$

$$P_{1,4} = a^\alpha_{\alpha\beta}\, x^\beta, \ \ P_{2,4} = a^\alpha_\beta a^\beta_{\alpha\gamma}\, x^\gamma, \ \ P_{3,4} = a^\alpha_\gamma a^\beta_\alpha\, a^\gamma_{\beta\delta}\, x^\delta, \ \ P_{4,4} = a^\alpha_\delta\, a^\beta_\alpha\, a^\gamma_\beta\, a^\delta_{\gamma\mu} x^\mu,$$

$$K_{6,4} = a^\alpha_\theta\, a^\beta_\gamma\, a^\gamma_\varphi\, a^\delta_\mu a^\mu_\nu\, a^\nu_\psi x^\theta x^\varphi x^\psi x^\tau \varepsilon_{\alpha\beta\delta\tau}, \ \ S_{0,4} = u_\alpha\, x^\alpha, \ \ S_{1,4} = a^\alpha_\beta x^\beta u_\alpha\,,$$

$$S_{2,4}=a_{\gamma}^{\alpha}a_{\alpha}^{\beta}x^{\gamma}u_{\beta}\,,\;\; S_{3,4}=a_{\delta}^{\alpha}a_{\alpha}^{\beta}a_{\beta}^{\gamma}x^{\delta}u_{\gamma}\,,\;\; \bar{R}_{6,4}=a_{p}^{\alpha}\,a_{q}^{\beta}\,a_{\beta}^{\gamma}\,a_{r}^{\delta}a_{\delta}^{\mu}\,a_{\mu}^{\nu}u_{s}u_{\alpha}u_{\gamma}u_{\nu}\varepsilon^{pqrs}\,,$$

$$\bar{R}_{6,4}=\det\left(\frac{\partial S_{i-1,4}}{\partial x^{j}}\right)_{i,j=\overline{1,4}}\,,\quad \tilde{K}_{1,4}=a_{\beta\gamma}^{\alpha}\,x^{\beta}x^{\gamma}y^{\delta}z^{\mu}\varepsilon_{\alpha\gamma\delta\mu}, \tag{6}$$

$I_{i,4}$ $\left(i=\overline{1,4}\right)$ are invariants, $P_{i,4}$ $\left(i=\overline{1,4}\right)$ and $K_{6,4}$ are comitants, $S_{j,4}$ $\left(j=\overline{0,3}\right)$ are mixed comitants, $\bar{R}_{6,4}$ is contravariant, and $\tilde{K}_{1,4}$ is comitant of cogradient vectors x, y, z [3]. The vectors $\varepsilon_{\alpha\beta\delta\tau}$ and ε^{pqrs} are four-dimensional unit vector with coordinates 1 when an even permutation of the indices holds, -1 when an odd permutation of the indices holds and 0 in other cases.

Remark 1. The characteristic equation of the system (1) has the form

$$\rho^{4}+L_{1,4}\,\rho^{3}+L_{2,4}\,\rho^{2}+L_{3,4}\,\rho+L_{4,4}=0, \tag{7}$$

where the coefficients of equation (7) are invariants of system (1) and have the following form:

$$L_{1,4}=-I_{1,4},\quad L_{2,4}=\frac{1}{2}\left(I_{1,4}^{2}-I_{2,4}\right),\quad L_{3,4}=\frac{1}{6}\left(3I_{1,4}I_{2,4}-2I_{3,4}-I_{1,4}^{3}\right),$$

$$L_{4,4}=\frac{1}{24}\left(8I_{1,4}I_{3,4}-6I_{4,4}-6I_{1,4}^{2}I_{2,4}+3I_{2,4}^{2}+I_{1,4}^{4}\right), \tag{8}$$

where $I_{1,4}$ $\left(i=\overline{1,4}\right)$ from (6).

3. Invariant conditions of stability of unperturbed motion for system (1) in case when the roots of the characteristic equation have nonzero real parts

Definition 1. If for any small positive value ε, however small, one can find a positive number δ such that for all perturbations $x^{j}(t_0)$ satisfying the condition

$$\sum_{j=1}^{2}\left(x^{j}(t_0)\right)^{2}\le\delta, \tag{9}$$

the inequality $\sum_{j=1}^{2}\left(x^{j}(t)\right)^{2}<\varepsilon,$ is valid for any $t\ge t_0$, then the unperturbed motion $x^{j}=0$ $\left(j=\overline{1,4}\right)$ is called *stable*, otherwise it is called unstable. If the unperturbed motion is stable and the number δ can be found however small such that for any perturbed motions satisfying (9) the condition $\lim_{t\to\infty}\sum_{j=1}^{2}\left(x^{j}(t)\right)^{2}=0,$ is valid, then the unperturbed motion is called *asymptotically stable.*

By means of the Lyapunov theorems on stability of unperturbed motion by the signs of the roots of the characteristic equation (7) of system (1) and the Hurwitz theorem on the signs of the roots of an algebraic equation (see, for example, [5]) we have

Theorem 1. Assume that the center-affine invariants (8) of system (1) satisfy inequalities

$$L_{i,4} > 0 \ \left(i = \overline{1,4}\right), \quad L_{1,4}L_{2,4}L_{3,4} - L_{3,4}^2 - L_{1,4}^2 L_{4,4} > 0 .$$

Then the unperturbed motion $x^j = 0 \ \left(j = \overline{1,4}\right)$ of this system is asymptotically stable.

Theorem 2. If at least one of the center-affine invariant expressions (8) of system (1) is negative, then the unperturbed motion $x^j = 0 \ \left(j = \overline{1,4}\right)$ of this system is unstable.

4. Invariant conditions of stability of unperturbed motion for system (1) in case when the characteristic equation has one zero root in conditions $\overline{R}_{6,4} \neq 0,\ \tilde{K}_{1,4} \equiv 0$

Lemma 1. [4] If in (6) we have $\tilde{K}_{1,4} \equiv 0$ then the system (1) takes the form

$$\frac{dx^j}{dt} = a_\alpha^j x^\alpha + 2x^j \left(a_{1\alpha}^1 x^\alpha\right) \ \left(j, \alpha = \overline{1,4}\right). \tag{10}$$

The system (10) is called *four-dimensional differential system of Darboux type*.

Remark 2. The expression $K_{6,4} = 0$ from (6) is the invariant partucular $GL(4, \mathbb{R})-$ integral of system (10).

Remark 3. For any center-affine transformation of the system (6), its quadratic part retains its form changing only the variables and coefficients. This follows from the fact that the identity $\tilde{K}_{1,4} \equiv 0$ is preserved under any center-affine transformation.

From [4] with considering Remark 3 it follows

Lemma 2. If in system (10) we have $\overline{R}_{6,4} \neq 0$, then by the center-affine transformation

$$\overline{x}^1 = S_{0,4}, \ \overline{x}^2 = S_{1,4}, \ \overline{x}^3 = S_{2,4}, \ \overline{x}^4 = S_{3,4},$$

the system (10) can be brought to the following form :

$$\dot{x}^1 = x^2 + 2x^1\left(a_{1\alpha}^1 x^\alpha\right), \ \dot{x}^2 = x^3 + 2x^2\left(a_{1\alpha}^1 x^\alpha\right), \ \dot{x}^3 = x^4 + 2x^3\left(a_{1\alpha}^1 x^\alpha\right),$$
$$\dot{x}^4 = -L_{4,4}x^1 - L_{3,4}x^2 - L_{2,4}x^3 - L_{1,4}x^4 + 2x^4\left(a_{1\alpha}^1 x^\alpha\right), \tag{11}$$

where $S_{i,4} \ \left(i = \overline{0,3}\right)$ are from (6) and $L_{j,4} \ \left(j = \overline{1,4}\right)$ are from (8).

Definition 2. The differential system (1) will be called *a critical system of Lyapunov type* if the characteristic equation of the system has one zero root and all other roots have negative real parts.

Notice that for system (11) the characteristic equation coincides with equation (7).

Lemma 3. The system (1) or (11) is critical of Lyapunov type if and only if the following invariant conditions hold:

$$L_{4,4}=0,\quad L_{i,4}>0\quad (i=1,2,3),\quad L_{1,4}L_{2,4}-L_{3,4}>0, \tag{12}$$

where $L_{j,4}\ \left(j=\overline{1,4}\right)$ are from (8).

The proof of Lemma 3 follows from the Hurwitz theorem on the signs of the roots of an algebraic equation and from equation (7) (see, for example [5]).

Notice that the system (11) in invariant conditions (12) by the center-affine transformation

$$\bar{x}^1=L_{3,4}x^1+L_{2,4}x^2+L_{1,4}x^3+x^4,\quad \bar{x}^2=x^2,\quad \bar{x}^3=x^3,\quad \bar{x}^4=x^1,$$

can be brought to the canonical form

$$\dot{x}=2x\left(ax+by+cz+du\right),\ \dot{y}=z+2y\left(ax+by+cz+du\right),$$

$$\dot{z}=x-L_{2,4}y-L_{1,4}z-L_{3,4}u+2z\left(ax+by+cz+du\right),\ \dot{u}=y+2u\left(ax+by+cz+du\right). \tag{13}$$

According to Lyapunov’s theorem [6], we will build the power series by which we can determine the stability of unperturbed motion of system (13). The first equation in system (13) is called the *critical equation*, and the other three are called *non-critical equations.* Using the algorithm from Lyapunov’s theorem [6] we examine the equations generated by right-hand sides of latest three equations of system (13). We have non-critical equations $z+2y\left(ax+by+cz+du\right)=0$, $x-L_{2,4}y-L_{1,4}z-L_{3,4}u+2z\left(ax+by+cz+du\right)=0$, $y+2u\left(ax+by+cz+du\right)=0$.

We express x, y and z from non-critical equations in the following way:

$$y=-2u\left(ax+by+cz+du\right),\quad z=-2y\left(ax+by+cz+du\right),$$

$$u=\frac{x}{L_{3,4}}-\frac{L_{2,4}}{L_{3,4}}y-\frac{L_{1,4}}{L_{3,4}}z+\frac{2z}{L_{3,4}}\left(ax+by+cz+du\right) \tag{14}$$

We will seek x, y and z as a holomorphic function on x. Then we can write

$$y(x)=A_1x+A_2x^2+A_3x^3+\ldots,\ z(x)=B_1x+B_2x^2+B_3x^3+\ldots,\ u(x)=C_1x+C_2x^2+C_3x^3+\ldots \tag{15}$$

Substituting (15) into (14) we get

$$A_1x+A_2x^2+A_3x^3+\ldots=-2(C_1x+C_2x^2+C_3x^3+\ldots)[ax+b(A_1x+A_2x^2+A_3x^3+\ldots)+$$
$$+c(B_1x+B_2x^2+B_3x^3+\ldots)+d(C_1x+C_2x^2+C_3x^3+\ldots)],$$
$$B_1x+B_2x^2+B_3x^3+\ldots=-2(A_1x+A_2x^2+A_3x^3+\ldots)[ax+b(A_1x+A_2x^2+A_3x^3+\ldots)+$$
$$+c(B_1x+B_2x^2+B_3x^3+\ldots)+d(C_1x+C_2x^2+C_3x^3+\ldots)],$$
$$C_1x+C_2x^2+C_3x^3+\ldots=\frac{x}{L_{3,4}}-\frac{L_{2,4}}{L_{3,4}}(A_1x+A_2x^2+A_3x^3+\ldots)-\frac{L_{1,4}}{L_{3,4}}(B_1x+B_2x^2+B_3x^3+\ldots)+$$
$$+2(B_1x+B_2x^2+B_3x^3+\ldots)[ax+b(A_1x+A_2x^2+A_3x^3+\ldots)+c(B_1x+B_2x^2+B_3x^3+\ldots)+$$
$$+d(C_1x+C_2x^2+C_3x^3+\ldots)].$$

This implies that $A_1=0,\ B_1=0,\quad C_1=\dfrac{1}{L_{3,4}},\ A_2=-2C_1(a+dC_1),\ B_2=0,$

$$C_2=\frac{2C_1L_{2,4}(a+dC_1)}{L_{3,4}},\ \ A_3=-2[bA_2C_1+C_2(a+2dC_1)],\ \ B_3=-2A_2(a+dC_1),$$

$$C_3=\frac{2}{L_{3,4}}[A_2L_{1,4}(a+dC_1)+bA_2C_1L_{2,4}+C_2L_{2,4}(a+2dC_1)],$$

$$A_4=-2[C_1(bA_3+cB_3)+C_2(bA_2+dC_2)+C_3(a+2dC_1)],\quad B_4=-2[A_3(a+dC_1)+A_2(bA_2+dC_2)],$$

$$C_4=\frac{2}{L_{3,4}}[(B_3+A_3L_{1,4})(a+dC_1)+(A_2L_{1,4}+C_2L_{2,4})(bA_2+dC_2)+C_1L_{2,4}(bA_3+cB_3)+$$

$$+C_3L_{2,4}(a+2dC_1)],... \qquad (16)$$

Substituting (15) into right-hand side of the critical equation (13) we get

$$2x\left(ax+by+cz+du\right)=D_1x+D_2x^2+D_3x^3+...,$$

or in expanded form we get

$$2x[ax+b(A_1x+A_2x^2+A_3x^3+...)+c(B_1x+B_2x^2+B_3x^3+...)+d(C_1x+C_2x^2+C_3x^3+...)]=$$
$$=D_1x+D_2x^2+D_3x^3+...,$$

This implies that

$$D_1=0,\ D_2=2(a+dC_1),\ D_3=2(bA_2+dC_2),\ D_4=2(bA_3+cB_3+dC_3),$$
$$D_5=2(bA_4+cB_4+dC_4),\ D_6=2(bA_5+cB_5+dC_5),\ D_7=2(bA_6+cB_6+dC_6),... \qquad (17)$$

Using the Lyapunov's theorem, in [7] was obtained

Lemma 4. The stability of the unperturbed motion corresponding to system (13) is described by one of the following two possible cases:

1) $L_{3,4}a+d\neq 0,$ then the unperturbed motion is unstable ;

2) $L_{3,4}a+d=0,$ then the unperturbed motion is stable.

In the latter case the unperturbed motion belongs to some continuous series of stabilized motions, and moreover, if perturbations are small enough then perturbed motion will tend Asymptotically to one of stabilized motions.

Proof. According to Lyapunov's theorem on stability of unperturbed motion in critical case [6], we examine the coefficients D_i from (17) taking into account (16). If $D_2\neq 0$, then we have first case from Lemma 4. If $D_2=0$, then we obtain $A_i=B_i=C_i=0$ $(i\geq 2)$ from (16), therefore $D_i=0,\ i=1,2,3,...$. According to Lyapunov's theorem we have the second case of this lemma. Lemma 4 is proved.

Theorem 3. Let for differential system of the perturbed motion (1) the invariant conditions $\bar{R}_{6,4}\neq 0,\ \tilde{K}_{1,4}\equiv 0$ be satisfied. Then in conditions (12) the stability of unperturbed motion corresponding to this system is described by one of the following two possible cases:

1) $4(I_{1,4}^3P_{1,4}-3I_{1,4}I_{2,4}P_{1,4}+2I_{3,4}P_{1,4})-15(I_{1,4}^2P_{2,4}-I_{2,4}P_{2,4}-2I_{1,4}P_{3,4}+2P_{4,4})\neq 0,$ then the unperturbed motion is unstable;

1) $4(I_{1,4}^3 P_{1,4} - 3I_{1,4} I_{2,4} P_{1,4} + 2I_{3,4} P_{1,4}) - 15(I_{1,4}^2 P_{2,4} - I_{2,4} P_{2,4} - 2I_{1,4} P_{3,4} + 2P_{4,4}) = 0,$ then the unperturbed motion is stable.

In the latter case the unperturbed motion belongs to some continuous series of stabilized motions, and moreover, if perturbations are small enough then perturbed motion will tend Asymptotically to one of stabilized motions. The invariant polynomials $I_{i,4}$ $(i = \overline{1,4})$ and $P_{j,4}$ $(j = \overline{1,4})$ are given in (6).

Proof. Using the system (13), obtained as a result of center-affine transformation in conditions $\bar{R}_{6,4} \neq 0, \tilde{K}_{1,4} \equiv 0$ and (12) with the help of the invariant polynomials $I_{i,4}$ $(i = \overline{1,4})$ and $P_{j,4}$ $(j = \overline{1,4})$ from (6), we obtain

$$4(I_{1,4}^3 P_{1,4} - 3I_{1,4} I_{2,4} P_{1,4} + 2I_{3,4} P_{1,4}) - 15(I_{1,4}^2 P_{2,4} - I_{2,4} P_{2,4} - 2I_{1,4} P_{3,4} + 2P_{4,4}) = 30(L_{3,4} a + d)x.$$

Consequently taking into account Lemma 4 we obtain truth of this theorem. Theorem 3 is proved.

5. Invariant conditions of stability of unperturbed motion for system (1) in case when the characteristic equation (7) has two pure imaginary roots in conditions $\bar{R}_{6,4} \neq 0, \tilde{K}_{1,4} \equiv 0$

Lemma 5. The characteristic equation (7) has two pure imaginary roots $\lambda\sqrt{-1}$ and $-\lambda\sqrt{-1}$ and the other two real and negative if and only if the following invariant conditions

$$L_{1,4} > 0, \quad L_{3,4} > 0, \quad L_{1,4} L_{2,4} - L_{3,4} > 0, \quad L_{1,4}^2 L_{4,4} + L_{3,4}^2 - L_{1,4} L_{2,4} L_{3,4} = 0 \qquad (18)$$

hold, where $L_{i,4}$ $(i = \overline{1,4})$ are from (8).

Proof. Denote by ρ_i $(i = \overline{1,4})$ the roots of characteristic equation (7). According to Vieta's theorem we have

$$\rho_1 + \rho_2 + \rho_3 + \rho_4 = -L_{1,4}, \quad \rho_1\rho_2 + \rho_1\rho_3 + \rho_1\rho_4 + \rho_2\rho_3 + \rho_2\rho_4 + \rho_3\rho_4 = L_{2,4},$$
$$\rho_1\rho_2\rho_3 + \rho_1\rho_2\rho_4 + \rho_1\rho_3\rho_4 + \rho_2\rho_3\rho_4 = -L_{3,4}, \quad \rho_1\rho_2\rho_3\rho_4 = L_{4,4}. \qquad (19)$$

Let us suppose that $\rho_1 = \lambda i$ and $\rho_2 = -\lambda i$ $(i^2 = -1)$, where $\lambda \neq 0$ is real number. From (19) we obtain

$$\rho_3 + \rho_4 = -L_{1,4}, \quad \lambda^2 + \rho_3\rho_4 = L_{2,4}, \; \lambda^2(\rho_3 + \rho_4) = -L_{3,4}, \; \lambda^2 \rho_3\rho_4 = L_{4,4}. \qquad (20)$$

From the first and third equalities (20) we get

$$\lambda = \pm\sqrt{\frac{L_{3,4}}{L_{1,4}}} \quad (L_{1,4} L_{3,4} > 0). \qquad (21)$$

Taking into account the first and second equalities from (20) we obtain

$$\rho_j^2 + L_{1,4}\rho_j + L_{2,4} - \frac{L_{3,4}}{L_{1,4}} = 0 \quad (j = 3, 4). \tag{22}$$

Using the Hurwitz theorem on the signs of the roots of an algebraic equation [5] and the inequality (21) we get first three conditions from (18). Substituting $\rho_3\rho_4$ from second equality (20) into last equality (20) we obtain equality from (18). Lemma 5 is proved.

Lemma 6. The characteristic equation (7) has two pure imaginary roots $\lambda\sqrt{-1}$ and $-\lambda\sqrt{-1}$ of multiplicity 2 if and only if the following invariant conditions

$$L_{2,4} > 0, \quad L_{1,4} = L_{3,4} = L_{2,4}^2 - 4L_{4,4} = 0, \tag{23}$$

hold, where $L_{i,4}$ $(i = \overline{1,4})$ are from (8).

Proof. Let us suppose that

$$\rho_1 = \rho_2 = \lambda i, \quad \rho_3 = \rho_4 = -\lambda i, \tag{24}$$

where $\lambda \neq 0$ is real number. From (19) we obtain

$$L_{1,4} = L_{3,4} = 0, \quad 2\lambda^2 = L_{2,4}, \quad \lambda^4 = L_{4,4}. \tag{25}$$

Because $\lambda \neq 0$ is real number, from (25) we get

$$\lambda = \pm\sqrt{\frac{1}{2}L_{2,4}} \quad (L_{2,4} > 0), \tag{26}$$

and

$$L_{2,4}^2 - 4L_{4,4} = 0. \tag{27}$$

The conditions (25)-(27) coincide with (23). Lemma 6 is proved.

Theorem 4. Let for differential system of the perturbed motion (1) the invariant conditions $\overline{R}_{6,4} \neq 0, \tilde{K}_{1,4} \equiv 0$ be satisfied. Then this system by center-affine transformation can be reduced to the form $(x = x^1,\ y = x^2,\ z = x^3,\ u = x^4)$

a) in conditions (18):

$$\dot{x} = -\lambda y + 2x\cdot\psi \equiv P, \quad \dot{y} = \lambda x + 2y\cdot\psi \equiv Q, \quad \dot{z} = u + 2z\cdot\psi \equiv R, \tag{28}$$
$$\dot{u} = y + (\lambda^2 - L_{2,4})z - L_{1,4}u + 2u\cdot\psi \equiv S,$$

where λ is from (21), $L_{i,4}$ is from (8) and $\psi = Ax + By + Cz + Du$ with A, B, C, D real constants.

b) in conditions (23):

$$\dot{x} = -\lambda y + 2x\cdot\psi, \quad \dot{y} = \lambda x + 2y\cdot\psi, \quad \dot{z} = u + 2z\cdot\psi, \quad \dot{u} = y - \lambda^2 z + 2u\cdot\psi, \tag{29}$$

where λ is from (26), $L_{i,4}$ is from (8) and $\psi = Ax + By + Cz + Du$ with A, B, C, D real constants.

Proof. a) As shown in the Lemmas 1 and 2 in conditions $\bar{R}_{6,4} \neq 0,\ \tilde{K}_{1,4} \equiv 0$ the system (1) by the center affine transformation is reduced to the form (11). In the case (18) the system (11) has the form $(x = x^1,\ y = x^2,\ z = x^3,\ u = x^4,\ a^1_{11} = \alpha,\ a^1_{12} = \beta,\ \ a^1_{13} = \gamma,\ a^1_{14} = \delta)$

$$\dot{x} = y + 2x\cdot\Phi,\quad \dot{y} = z + 2y\cdot\Phi,\quad \dot{z} = u + 2z\cdot\Phi,\ \ \dot{u} = \frac{b^2 + bcd}{d^2}x + by + cz + du + 2u\cdot\Phi, \tag{30}$$

where

$$b = -L_{3,4},\ \ c = -L_{2,4},\ \ d = -L_{1,4},\quad \Phi = \alpha x + \beta y + \gamma z + \delta u\quad (\alpha,\beta,\gamma,\delta \in \mathbb{R}). \tag{31}$$

Let’s consider the transformation

$$X = -(c + \lambda^2)y - dz + u,\ \ Y = -\lambda(c + \lambda^2)x - d\lambda y + \lambda z,\ \ Z = \lambda x,\ \ U = \lambda y, \tag{32}$$

where according to (21) and (31) we have $\lambda^2 = \dfrac{b}{d}$ and determinant $\Delta = -\lambda^3 \neq 0.$

Making the transformation (32) in the system (30)-(31) we obtain for it the form (28).

b) In the case (23) the system (11) has the form

$$\dot{x} = y + 2x\cdot\Phi,\quad \dot{y} = z + 2y\cdot\Phi,\quad \dot{z} = u + 2z\cdot\Phi,\ \ \dot{u} = -\lambda^4 x - 2\lambda^2 z + 2u\cdot\Phi, \tag{33}$$

where

$$\Phi = ax + by + cz + du,\ \ \lambda = \pm\sqrt{\frac{L_{2,4}}{2}},\ \ L_{1,4} = L_{3,4} = 0,\ \ L^2_{2,4} = 4L_{4,4}. \tag{34}$$

Let’s consider the transformation

$$X = \lambda^2 y + u,\ \ Y = \lambda^3 x + \lambda z,\ \ Z = \lambda x,\ \ U = \lambda y. \tag{35}$$

According to (16) the determinant of transformation (35) is $\Delta = -\lambda^3 \neq 0.$

Making the transformation (35) in the system (33)-(34) we obtain for it the form (29). Theorem 4 is proved.

6. The theorem on the integrating factor for a four-dimensional differential system

Let's suppose that the system (1) admits the $(n-1)$ - dimensional commutative Lie algebra with operators

$$X_\alpha = \xi^j_\alpha(x)\frac{\partial}{\partial x^j}\ \ \left(j = \overline{1,4};\ \alpha = \overline{1,3}\right), \tag{36}$$

and

$$\Lambda = P^j(x,a)\frac{\partial}{\partial x^j}\ \ \left(j = \overline{1,4}\right). \tag{37}$$

Let’s consider the determinant constructed on coordinates of operators (36)-(37)

$$\Delta = \begin{vmatrix} \xi^1_1 & \xi^2_1 & \xi^3_1 & \xi^4_1 \\ \xi^1_2 & \xi^2_2 & \xi^3_2 & \xi^4_2 \\ \xi^1_3 & \xi^2_3 & \xi^3_3 & \xi^4_3 \\ P^1 & P^2 & P^3 & P^4 \end{vmatrix} \tag{38}$$

Theorem 5. [4] If the four-dimensional differential system (1) admits three-dimensional commutative Lie algebra of operators (36), then the function $\mu = \frac{1}{\Delta}$ where $\Delta \neq 0$ from (38) is the integrating factor for Pfaff equations

$$\begin{vmatrix} \xi_2^2 & \xi_2^3 & \xi_2^4 \\ \xi_3^2 & \xi_3^3 & \xi_3^4 \\ P^2 & P^3 & P^4 \end{vmatrix} dx^1 - \begin{vmatrix} \xi_2^1 & \xi_2^3 & \xi_2^4 \\ \xi_3^1 & \xi_3^3 & \xi_3^4 \\ P^1 & P^3 & P^4 \end{vmatrix} dx^2 + \begin{vmatrix} \xi_2^1 & \xi_2^2 & \xi_2^4 \\ \xi_3^1 & \xi_3^2 & \xi_3^4 \\ P^1 & P^2 & P^4 \end{vmatrix} dx^3 - \begin{vmatrix} \xi_2^1 & \xi_2^2 & \xi_2^3 \\ \xi_3^1 & \xi_3^2 & \xi_3^3 \\ P^1 & P^2 & P^3 \end{vmatrix} dx^4 = 0,$$

$$\begin{vmatrix} \xi_1^2 & \xi_1^3 & \xi_1^4 \\ \xi_3^2 & \xi_3^3 & \xi_3^4 \\ P^2 & P^3 & P^4 \end{vmatrix} dx^1 - \begin{vmatrix} \xi_1^1 & \xi_1^3 & \xi_1^4 \\ \xi_3^1 & \xi_3^3 & \xi_3^4 \\ P^1 & P^3 & P^4 \end{vmatrix} dx^2 + \begin{vmatrix} \xi_1^1 & \xi_1^2 & \xi_1^4 \\ \xi_3^1 & \xi_3^2 & \xi_3^4 \\ P^1 & P^2 & P^4 \end{vmatrix} dx^3 - \begin{vmatrix} \xi_1^1 & \xi_1^2 & \xi_1^3 \\ \xi_3^1 & \xi_3^2 & \xi_3^3 \\ P^1 & P^2 & P^3 \end{vmatrix} dx^4 = 0,$$

$$\begin{vmatrix} \xi_1^2 & \xi_1^3 & \xi_1^4 \\ \xi_2^2 & \xi_2^3 & \xi_2^4 \\ P^2 & P^3 & P^4 \end{vmatrix} dx^1 - \begin{vmatrix} \xi_1^1 & \xi_1^3 & \xi_1^4 \\ \xi_2^1 & \xi_2^3 & \xi_2^4 \\ P^1 & P^3 & P^4 \end{vmatrix} dx^2 + \begin{vmatrix} \xi_1^1 & \xi_1^2 & \xi_1^4 \\ \xi_2^1 & \xi_2^2 & \xi_2^4 \\ P^1 & P^2 & P^4 \end{vmatrix} dx^3 - \begin{vmatrix} \xi_1^1 & \xi_1^2 & \xi_1^3 \\ \xi_2^1 & \xi_2^2 & \xi_2^3 \\ P^1 & P^2 & P^3 \end{vmatrix} dx^4 = 0, \quad (39)$$

that determine the general integral of system (1).

7. The Lie algebra of operators admitted by the system (28). Some particular integrals and one first integral of Darboux type

Lemma 7. The Lie algebra of operators admitted by the system (28) has the form

$$X_1 = [(Bd-D)\lambda(c+\lambda^2)x + Ad\lambda(c+\lambda^2)y - 2\varphi_1 x^2 + 2\varphi_2\varphi_3 xz + 2C\varphi_2 xu]\frac{\partial}{\partial x} +$$

$$+[-Ad\lambda(c+\lambda^2)x + (Bd-D)\lambda(c+\lambda^2)y - 2\varphi_1 xy + 2\varphi_2\varphi_3 yz + 2C\varphi_2 yu]\frac{\partial}{\partial y} +$$

$$+[A(c+\lambda^2)y + (c+\lambda^2)\varphi_2 z - 2\varphi_1 xz + 2\varphi_2\varphi_3 z^2 + 2C\varphi_2 zu]\frac{\partial}{\partial z} +$$

$$+[A\lambda(c+\lambda^2)x + (c+\lambda^2)\varphi_2 u - 2\varphi_1 xu + 2\varphi_2\varphi_3 zu + 2C\varphi_2 u^2]\frac{\partial}{\partial u},$$

$$X_2 = [\lambda(c+\lambda^2)\varphi_5 x + A\lambda(c+\lambda^2)(c+2\lambda^2)y - 2\varphi_4 x^2 + 2\lambda\varphi_3\varphi_6 xz + 2C\lambda\varphi_6 xu]\frac{\partial}{\partial x} +$$

$$+[-A\lambda(c+\lambda^2)(c+2\lambda^2)x + \lambda(c+\lambda^2)\varphi_5 y - 2\varphi_4 xy + 2\lambda\varphi_3\varphi_6 yz + 2C\lambda\varphi_6 yu]\frac{\partial}{\partial y} +$$

$$+[A\lambda(c+\lambda^2)x + \lambda(c+\lambda^2)\varphi_6 z - 2\varphi_4 xz + 2\lambda\varphi_3\varphi_6 z^2 + 2C\lambda\varphi_6 zu]\frac{\partial}{\partial z} +$$

$$+[-A\lambda^2(c+\lambda^2)y + \lambda(c+\lambda^2)\varphi_6 u - 2\varphi_4 xu + 2\lambda\varphi_3\varphi_6 zu + 2C\lambda\varphi_6 u^2]\frac{\partial}{\partial u},$$

$$X_3 = [-B\lambda(c+\lambda^2)x - A\lambda(c+\lambda^2)y + 2\varphi_7 x^2 + 2\varphi_8 xz + 2\varphi_9 xu]\frac{\partial}{\partial x} +$$

$$+[A\lambda(c+\lambda^2)x - B\lambda(c+\lambda^2)y + 2\varphi_7 xy + 2\varphi_8 yz + 2\varphi_9 yu]\frac{\partial}{\partial y} +$$

$$+[-B\lambda(c+\lambda^2)z+A(c+\lambda^2)u+2\varphi_7 xz+2\varphi_8 z^2+2\varphi_9 zu]\frac{\partial}{\partial z}+$$

$$+[A(c+\lambda^2)y+A(c+\lambda^2)^2 z+(Ad-B\lambda)(c+\lambda^2)u+2\varphi_7 xu+2\varphi_8 zu+2\varphi_9 u^2]\frac{\partial}{\partial u},$$

$$X_4=[\lambda(c+\lambda^2)x-2(\varphi_5-B\lambda^2)x^2+2A(c+\lambda^2)xy-2\lambda\varphi_{10}xz+2C\lambda xu]\frac{\partial}{\partial x}+$$

$$+[\lambda(c+\lambda^2)y-2(\varphi_5-B\lambda^2)xy+2A(c+\lambda^2)y^2-2\lambda\varphi_{10}yz+2C\lambda yu]\frac{\partial}{\partial y}+$$

$$+[\lambda(c+\lambda^2)z-2(\varphi_5-B\lambda^2)xz+2A(c+\lambda^2)yz-2\lambda\varphi_{10}z^2+2C\lambda zu]\frac{\partial}{\partial z}+$$

$$+[\lambda(c+\lambda^2)u-2(\varphi_5-B\lambda^2)xu+2A(c+\lambda^2)yu-2\lambda\varphi_{10}zu+2C\lambda u^2]\frac{\partial}{\partial u}, \qquad (40)$$

where

$$\varphi_1=(A^2+B^2)cd-BCd-BDc+CD-AC\lambda+(A^2d+B^2d-BD)\lambda^2,$$

$$\varphi_2=Ac+(Bd-D)\lambda+2A\lambda^2,\ \varphi_3=-Cd+(c+\lambda^2)D,$$

$$\varphi_4=-2BCc+C^2+A(Cd-Dc)\lambda+3(A^2c+B^2c-BC)\lambda^2-AD\lambda^3+(A^2+B^2)(c^2+2\lambda^4),$$

$$\varphi_5=B(c+2\lambda^2)-C,\ \varphi_6=B(c+2\lambda^2)-C-Ad\lambda,\ \varphi_7=(A^2+B^2)(c+\lambda^2)-BC,$$

$$\varphi_8=AC(c+\lambda^2)+B(Cd-Dc)\lambda-BD\lambda^3,\ \varphi_9=AD(c+\lambda^2)-BC\lambda,\ \varphi_{10}=Cd-cD-D\lambda^2. \qquad (41)$$

Proof. Writing the operators (36) in a general form $X=\xi^j(x)\frac{\partial}{\partial x^j}$ and solving the determining equations

$$\xi^j_{x^1}P^1+\xi^j_{x^2}P^2+\xi^j_{x^3}P^3+\xi^j_{x^4}P^4=\xi^1P^j_{x^1}+\xi^2P^j_{x^2}+\xi^3P^j_{x^3}+\xi^4P^j_{x^4},\ (j=\overline{1,4})$$

we obtain that the system (28) admits the operators (40)-(41).

The operators $X_i\ (i=1,2,3,4)$ are linearly independent, since the determinant of fourth order constructed on coordinates of these operators is different from zero. Notice that commutators $[X_i,X_j]=0,\ (i,j=\overline{1,4})$. Therefore operators $X_i\ (i=\overline{1,4})$ form a four-dimensional Lie algebra. Further, using the theorem 5 on integrating factor we calculate determinant μ which is constructed on the coordinates of three operators $X_i\ (i=1,2,3,4)$ and on the right-hand sides of the system (28), we obtain

$$\mu_{134}=\mu_{234}=0,\ \ \mu_{123}=A^2B\lambda(c+\lambda^2)^2\varsigma_1\varsigma_2\varsigma_3,\ \mu_{124}=-A^2\lambda(c+\lambda^2)^2\varsigma_1\varsigma_2\varsigma_3,$$

where

$$\varsigma_1=x^2+y^2,\ \ \varsigma_2=\lambda^3+c\lambda-2(Bc-C+B\lambda^2)x+2A(c+\lambda^2)y+2\lambda(-Cd+cD+D\lambda^2)z+2C\lambda u,$$

$$\varsigma_3=\lambda^2x^2+d\lambda xy+cd\lambda xz+\lambda(2c+d^2+4\lambda^2)xu-(c+\lambda^2)y^2-[2c^2+(6c+d^2)\lambda^2+4\lambda^2]yz-$$

$$-cdyu-[c^3+c(5c+d^2)\lambda^2+(8c+d^2)\lambda^4+4\lambda^6]z^2-[c^2+(4c+d^2)\lambda^2+4\lambda^4](dzu-u^2). \qquad (42)$$

We denote the operator of system (28) by $\Lambda=P\frac{\partial}{\partial x}+Q\frac{\partial}{\partial y}+R\frac{\partial}{\partial z}+S\frac{\partial}{\partial u}$. Then we obtain

$$\Lambda(\varsigma_1)=4\varsigma_1\psi,\quad \Lambda(\varsigma_2)=2\varsigma_2\psi,\quad \Lambda(\varsigma_3)=\varsigma_3(d+4\psi),\quad \Lambda(\varsigma_1^{\alpha}\varsigma_2^{\beta})=2(2\alpha+\beta)\varsigma_1^{\alpha}\varsigma_2^{\beta}\psi,$$

where $\psi = Ax+By+Cz+Du.$

From the last equalities we get

Theorem 6. The functions $\varsigma_1, \varsigma_2, \varsigma_3$ from (42) are particular integrals of the system (28) and the function $F=\varsigma_1\varsigma_2^{-2}$ is a first integral of Darboux type for this system.

Remark 4. The comitant $K_{6,4}$ from (6) for the system (28) has the form $K_{6,4}=\lambda\varsigma_1\varsigma_3$, where λ from (21) and ς_1, ς_3 are from (42).

8. The Lie algebra of operators admitted by the system (29). Some particular integrals and one first integral of Darboux type

Lemma 8. The Lie algebra of operators admitted by the system (28) has the form

$$Y_1=[\lambda^3x-2(C+B\lambda^2)x^2+2A\lambda^2xy+2D\lambda^3xz-2C\lambda xu]\frac{\partial}{\partial x}+$$

$$+[\lambda^3y-2(C+B\lambda^2)xy+2A\lambda^2y^2+2D\lambda^3yz-2C\lambda yu]\frac{\partial}{\partial y}+$$

$$+[\lambda^3z-2(C+B\lambda^2)xz+2A\lambda^2yz+2D\lambda^3z^2-2C\lambda zu]\frac{\partial}{\partial z}+$$

$$+[\lambda^3u-2(C+B\lambda^2)xu+2A\lambda^2yu+2D\lambda^3zu-2C\lambda u^2]\frac{\partial}{\partial u},$$

$$Y_2=[-D\lambda^3x+2(CD-AC\lambda+BD\lambda^2)x^2-2D^2\lambda^3xz+2CD\lambda xu]\frac{\partial}{\partial x}+$$

$$+[-D\lambda^3y+2(CD-AC\lambda+BD\lambda^2)xy-2D^2\lambda^3yz+2CD\lambda yu]\frac{\partial}{\partial y}+$$

$$+[A\lambda^2y-D\lambda^3z+2(CD-AC\lambda+BD\lambda^2)xz-2D^2\lambda^3z^2+2CD\lambda zu]\frac{\partial}{\partial z}+$$

$$+[A\lambda^3x-D\lambda^3u+2(CD-AC\lambda+BD\lambda^2)xu-2D^2\lambda^3zu+2CD\lambda u^2]\frac{\partial}{\partial u},$$

$$Y_3=[-B\lambda^3x-A\lambda^3y+2Ex^2+2\lambda^2(AC-BD\lambda)xz+2\lambda(BC+AD\lambda)xu]\frac{\partial}{\partial x}+$$

$$+[A\lambda^3x-B\lambda^3y+2Exy+2\lambda^2(AC-BD\lambda)yz+2\lambda(BC+AD\lambda)yu]\frac{\partial}{\partial y}+$$

$$+[-B\lambda^3z+A\lambda^2u+2Exz+2\lambda^2(AC-BD\lambda)z^2+2\lambda(BC+AD\lambda)zu]\frac{\partial}{\partial z}+$$

$$+A\lambda^2y-A\lambda^4z-B\lambda^3u+2Exu+2\lambda^2(AC-BD\lambda)zu+2\lambda(BC+AD\lambda)u^2]\frac{\partial}{\partial u},$$

$$Y_4=[-\lambda^3(C+B\lambda^2)x-A\lambda^5y+2Hx^2-2\lambda^3Fxz+2\lambda Gxu]\frac{\partial}{\partial x}+$$

$$+[A\lambda^5x-\lambda^3(C+B\lambda^2)y+2Hxy-2\lambda^3Fyz+2\lambda Gyu]\frac{\partial}{\partial y}+$$

$$+[A\lambda^3x-\lambda^3(C+B\lambda^2)z+A\lambda^4u+2Hxz-2\lambda^3Fz^2+2\lambda Gzu]\frac{\partial}{\partial z}+$$

$$+-A\lambda^6z-\lambda^3(C+B\lambda^2)u+2Hxu-2\lambda^3Fzu+2\lambda Gu^2]\frac{\partial}{\partial u}, \qquad (43)$$

where $E=BC+(A^2+B^2)\lambda^2$, $F=CD-AC\lambda+BD\lambda^2$, $G=C^2+BC\lambda^2+AD\lambda^3$, $H=C^2+BC\lambda^2+AD\lambda^3+(A^2+B^2)\lambda^4.$

The proof of Lemma 8 is similarly with the proof of Lemma 7.

The operators Y_i $(i=1,2,3,4)$ are linearly independent, since the determinant of fourth order constructed on coordinates of these operators is different from zero. Notice that commutators $[Y_i,Y_j]=0,$ $(i,j=\overline{1,4})$. Therefore operators Y_i $(i=\overline{1,4})$ form a four-dimensional Lie algebra. Further, using the theorem 5 on integrating factor we calculate determinant μ which is constructed on the coordinates of three operators Y_i $(i=1,2,3,4)$ and on the right-hand sides of the system (29), we obtain

$$\mu_{123}=\mu_{134}=0,\ \ \mu_{124}=-A^2\lambda^7\varphi^2\phi,\ \ \mu_{234}=-A^2B\lambda^7\varphi^2\phi,$$

where

$$\varphi=x^2+y^2,\quad \phi=\lambda^3-2(C+B\lambda^2)x+2A\lambda^2y+2D\lambda^3z-2C\lambda u, \qquad (44)$$

Direct calculation of the operator Λ for the system (29) gives

$$\Lambda(\varphi)=4\varphi\psi,\quad \Lambda(\phi)=2\phi\psi,\quad \Lambda(\varphi^\alpha\phi^\beta)=2(2\alpha+\beta)\varphi^\alpha\phi^\beta\psi,$$

where $\psi=Ax+By+Cz+Du.$

From the last equalities we get

Theorem 7. The functions φ and ϕ from (44) are particular integrals of the system (29) and the function $F=\varphi\phi^{-2}$ is a first integral of Darboux type for this system.

Remark 5. The comitant $K_{6,4}$ from (6) for the system (29) has the form $K_{6,4}=\lambda^3\varphi^2$, where λ from (26) and φ are from (44).

Remark 6. The first integral $F=\varsigma_1\varsigma_2^{-2}$ of the system (28) is the holomorphic integral of Lyapunov type, i.e. this integral can be written in the form $F=x^2+y^2+\tilde{F}(x,y,z,u)$, where $\tilde{F}(x,y,z,u)$ is the polynomial of the order more than two.

From [4] it is known the comitant of system (1) in the form

$$\Phi_{4,4}=L_{4,4}-2\left(\frac{4}{5}L_{3,4}P_{1,4}+L_{2,4}P_{2,4}+L_{1,4}P_{3,4}+P_{4,4}\right), \qquad (45)$$

where $P_{j,4}$ $(j=\overline{1,4})$ are from (6) and $L_{i,4}$ $(i=\overline{1,4})$ are from (8).

Remark 7. The comitant $\Phi_{4,4}$ for the system (28) has the form $\Phi_{4,4}=-\lambda\varsigma_2$, where ς_2 is from (42).

Using the Lyapunov's theorem [6], the theorems 6-7 and remarks 6-7, we obtain

Theorem 7. [8] Assume for the system (1) with $\tilde{K}_{1,4} \equiv 0$ and $\bar{R}_{6,4} \neq 0$ under center-affine invariant conditions (18), the comitant (45) is not identically zero. Then the system has a periodic solution containing an arbitrary constant, and varying this constant one can obtain a continuous sequence of periodic motions, which comprises the studied unperturbed motion. This motion is stable and any perturbed motion, sufficiently close to the unperturbed motion, will tend asymptotically to one of the periodic motions.

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THE PROJECTIVE SERIES OF PENCILS OF CONICS

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Abstract. In this paper there are discussed some results which will be of help in the future, to classify and prove certain theorems of the cubic curves in the projective plane**.**
Keywords: Projective plan, conics, projective series, pencils of conics.
2010 Mathematics Subject Classification: 34C20, 34C45

PROIECTIVITATEA FASCICOLELOR DE SECȚIUNI CONICE

Abstract. În această lucrare sunt discutate câteva rezultate care vor fi de ajutor în viitor, de a clasifica și de a demonstra anumite teoreme ale curbelor cubice în planul proiectiv.
Cuvinte cheie: plan proiectiv, secțiuni conice, serii proiective, fascicole de secțiuni conice.

We are working in the projective plane.

Definition 1. *A series* (*or range*) *is a bijective function which has as an image a line from the plane.*

Definition 2. *Let* f, g *be two series and* r *a bijective function, such that* $Dom(r) = Dom(g)$ *and* $Im(r) = Dom(r)$. *Then we define the series* f *and* g (*in this order*) *to be* r *- projective, written simply as* $f \wedge_r g$, *if and only if for any distinct points* $\{A, B, C, D\} \subset Im(g)$;

$$(A, B; C, D) = (frg^{-1}(A), frg^{-1}(B); frg^{-1}(C), frg^{-1}(D))$$ *- as cross-ratios* [1, p. 33].

Because of bijectivity if the above equality is true, then also

$$(A, B; C, D) = (grf^{-1}(A), grf^{-1}(B); grf^{-1}(C), grf^{-1}(D))$$

is true. Hence $f \wedge_r g \to g \wedge_r f$. *Similarly* $g \wedge_r f \to f \wedge_r g$. *Therefore the order does not matter, and we will simply denote* $f \wedge_r g \to g \wedge_r f$ *to mean that* f *and* g *are* r *- projective.*

Definition 3. *Let* A, B, C, D *be four distinct any three non-collinear points in the projective plane.* P_{ABCD} *is the set that contains all the conics that pass through* A, B, C *and* D *also named a pencil of conics. Let* x *be a line that passes through only one of the points* A, B, C *or* D. *Suppose it passes through* A (*the same procedure is undertaken for the other points*). *Then any conic from the pencil* P_{ABCD} *intersects the line* x *in another second point, let it be* X. X *is different from* A *in all cases except the case when the conic is tangent* x, *and* $A = X$ *will be a double point. Now, for any* $X \in x$ *there is, respectively, the conic* $XABCD \in P_{ABCD}$, *the conic that passes through the points* X, A, B, C *and* D *when* $X = A$ *it will be the conic from the pencil tangent to* x.

This establishes a bijective correspondence between points $X \in x$ and conics from P_{ABCD}, in particular a function $f : P_{ABCD} \to x$. This series will be denoted by $s_{x,A,B,C,D}$ or simply s_x, when there is no confusion.

Before going forward with the main theorem, we need a lemma, which is a well-known result in projective plane geometry.

Lemma 1. *Let A and B be two points, a_i and b_i will represent lines passing through A and respectively B, $i \in \mathbb{N}$.*

1. *If $(a_0,a_1;a_2,a_3) = (b_0,b_1;b_2,b_3)$ (this is the cross-ratio of lines), $(a_0,a_1;a_2,a_4) = (b_0,b_1;b_2,b_4)$, $(a_0,a_1;a_2,a_5) = (b_0,b_1;b_2,b_5)$ and finally $(a_0,a_1;a_2,a_6) = (b_0,b_1;b_2,b_6)$, then $(a_3,a_4;a_5,a_6) = (b_3,b_4;b_5,b_6)$.*
2. *In this part every line passes through A. If $(a,a';n,m) = (b,b';n,m) = (c,c';n,m) = (d,d';n,m)$ then $(a,b;c,d) = (a',b';c',d')$.*

Theorem 1. *Let A,B,C,D be four distinct non-collinear points in the projective plane, see Figure 1. Let x,y be lines that pass through only one of the points A,B,C or D. Then $s_x \wedge_{id} s_y$ where id is the identity function on P_{ABCD}.*

Proof.

Let $X \in x$ and $Y = XABCD \cap y$, where Y is the second point of intersection on line y. There are two cases, either the lines pass through the same point or through two different points.

First case. Suppose, without loss of generality, that $x \cap y = A$. Then

$$A(X,Y;D,C) = B(X,Y;D,C)$$

by the conic's general properties. As X varies on x, the cross-ratio of $A(X,Y;D,C)$ is constant, as the lines x,y are fixed, results that the cross-ratio of $B(X,Y;D,C)$ also must be constant. So as X varies on x, Y moves accordingly on y. Because $B(X,Y;D,C)$ is constant for any $X \in x$, by the lemma (here $n = BD$, $m = BC$) from above, we have for X_1,X_2,X_3,X_4 (distinct points on x) and their corresponding Y_1,Y_2,Y_3,Y_4 on y, that

$$B(X_1,X_2;X_3,X_4) = B(Y_1,Y_2;Y_3,Y_4)$$

which means exactly

$$(X_1,X_2;X_3,X_4) = (Y_1,Y_2;Y_3,Y_4)$$

therefore $s_x \wedge_{id} s_y$.

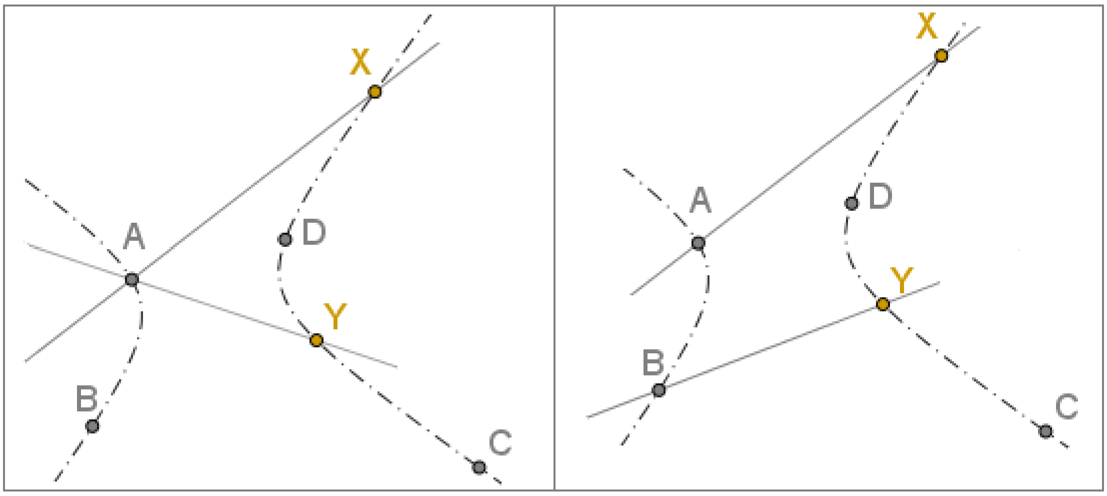

Figure 1. The two cases

Second case. Suppose, without loss of generality, that $A \in x,\ B \in y$. Then

$$A(X,Y;D,C) = B(X,Y;D,C).$$

Furthermore, the cross-ratio of $A(X,Y;D,C)$, depends only on Y, as the lines $AX,\ AD,\ AC$ are fixed. Same way, the cross-ratio of $B(X,Y;D,C)$ depends only on X. So as X is varies on x, Y moves accordingly on y. By the cross-ratio properties, we have also that

$$A(D,C;X,Y) = B(X,Y;D,C).$$

By the lemma (here $a_0 = AD,\ a_1 = AC,\ a_2 = AX$ and $b_0 = BC,\ b_1 = BD,\ b_2 = BY$), we have for X_1, X_2, X_3, X_4 (distinct points on x) and their corresponding Y_1, Y_2, Y_3, Y_4 on y, that

$$A(X_1, X_2; X_3, X_4) = B(Y_1, Y_2; Y_3, Y_4)$$

which means exactly

$$(X_1, X_2; X_3, X_4) = (Y_1, Y_2; Y_3, Y_4)$$

therefore $s_x \wedge_{id} s_y$.

This theorem shows that it does not matter which line x (as in the theorem) is chosen, the series is projectively "invariant". In conclusion, any pencil of conics gives a unique projective series.

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CUBIC DIFFERENTIAL SYSTEMS WITH AFFINE REAL INVARIANT STRAIGHT LINES OF TOTAL PARALLEL MULTIPLICITY SIX AND CONFIGURATIONS $\big(3(m),1,1,1\big)$

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Abstract. We classify all cubic differential systems with exactly six affine real invariant straight lines (taking into account their parallel multiplicity) of four slopes. One invariant strait line of the first slope has parallel multiplicity m, $m = 1, 2, 3$. We proove that there are five distinct classes of such systems. For every class we carried out the qualitative investigation on the Poincaré disk.

Keywords: Cubic differential system, invariant straight line, phase portrait.
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SISTEMELE DIFERENŢIALE CUBICE CU DREPTE INVARIANTE AFINE REALE DE MULTIPLICITATE PARALELĂ TOTALĂ ŞASE ŞI DE CONFIGURAŢIA $\big(3(m),1,1,1\big)$

Rezumat. Sunt clasificate sistemele diferenţiale cubice cu exact şase drepte afine reale invariante (ţinându-se cont de multiplicitatea paralelă) de patru pante. O dreaptă de prima pantă are multiplicitatea paralelă m, $m = 1, 2, 3$. Se arată că există cinci clase distincte de astfel de sisteme. Fiecare clasă este studiată din punct de vedere calitativ şi pe discul Poincaré sunt construite portretele de fază.

Cuvinte-cheie: Sistem diferenţial cubic, dreaptă invariantă, portret de fază.

1. Introduction and statement of main results

We consider the real polynomial system of differential equations

$$\frac{dx}{dt} = P(x,y),\ \frac{dy}{dt} = Q(x,y),\ \ \gcd(P,Q) = 1 \tag{1}$$

and the vector field

$$\mathbb{X} = P(x,y)\frac{\partial}{\partial x} + Q(x,y)\frac{\partial}{\partial y} \tag{2}$$

associated to system (1).

Denote $n = \max\big\{\deg(P), \deg(Q)\big\}$. If $n = 2$ ($n = 3$) then system (1) is called quadratic (cubic).

An algebraic curve $f(x,y) = 0,\ f \in \mathbb{C}[x,y]$ $\big($a function $f = \exp(\frac{g}{h}),\ g,h \in \mathbb{C}[x,y]\big)$ is called invariant algebraic curve (exponential factor) of the system (1) if there exists a polynomial
$K_f \in \mathbb{C}[x,y]$, $\deg(K_f) \le n-1$ such that the identity $\mathbb{X}(f) \equiv f(x,y)K_f(x,y),\ \ (x,y) \in \mathbb{R}^2$ holds. In particular, a straight line $l \equiv \alpha x + \beta y + \gamma = 0,\ \alpha,\beta,\gamma \in \mathbb{C}$ is invariant for (1) if there exists a polynomial $K_l \in \mathbb{C}[x,y]$ such that the identity

$$\alpha P(x,y) + \beta Q(x,y) \equiv (\alpha x + \beta y + \gamma)K_l(x,y),\ (x,y) \in \mathbb{R}^2 \tag{3}$$

holds. The polynomial $K_f(x,y)$ is called cofactor of the invariant algebraic curve (exponential factor) f. If m is the greatest natural number such that l^m divides $\mathbb{X}(l)$ then we say that l has parallel multiplicity m. In the case of cubic systems we have $m \in \{1,2,3\}$. If l has the parallel multiplicity m, then $f_1 = \exp(\frac{1}{l}), ..., f_{m-1} = \exp(\frac{1}{l^{m-1}})$ are exponential factors.

Let $f_1,\ldots,f_r$ $\big(f_{r+1}=\exp(g_{r+1}/h_{r+1}),\ldots,f_s=\exp(g_s/h_s)\big)$ are invariant algebraic curves (exponential factors) of (1) with cofactors $K_{f_1}(x,y),...,K_{f_s}(x,y)$, respectively. The system (1) is called *Darboux integrable* if there exists a non-constant function of the form $F=f_1^{\alpha_1}\cdots f_s^{\alpha_s}$,
$\alpha_j\in\mathbb{C}$, $j=\overline{1,s}$, such that either F is a first integral or F is an integrating factor for (1) (about the theory of Darboux, presented in the context of planar polynomial differential systems on the affine plane, see [23]). The function of the form

$$f_1^{\alpha_1}\cdots f_s^{\alpha_s}, \tag{4}$$

where $\alpha_j\in\mathbb{C},\ |\alpha_1|+\cdots+|\alpha_s|\neq 0$, is a first integral (an integrating factor) for (1) if and only if in x and y the identity

$$\alpha_1K_{f_1}(x,y)+\alpha_2K_{f_2}(x,y)+...+\alpha_sK_{f_s}(x,y)\equiv 0 \tag{5}$$

$$\left(\sum_{j=1}^{s}\alpha_jK_{f_j}(x,y)\equiv-\frac{\partial P(x,y)}{\partial x}-\frac{\partial Q(x,y)}{\partial y}\right) \tag{6}$$

holds.

By present a great number of works have been dedicated to the investigation of polynomial differential systems with invariant straight lines.

The problem of estimating the number of invariant straight lines which a polynomial differential system can have was considered in [2]; the problem of coexistence of the invariant straight lines and limit cycles was examined in $\{[22]:n=2\}$, $\{[11], n=3\}$, [10].

The classification of all cubic systems with the maximum number of invariant straight lines, including the line at infinity, and taking into account their geometric multiplicities, is given in [13].

In [2] it was proved that the non-degenerate cubic system (1) can have at most eight affine invariant straight lines. The cubic systems with exactly eight and exactly seven distinct affine invariant straight lines have been studied in [13], [15]; with invariant straight lines of total geometric (parallel) multiplicity eight (seven) - in [3], [4], [5] $\big($[19], [30]$\big)$, and with six real invariant straight lines along two (three) directions - in [17], [18]. The cubic systems with degenerate infinity and invariant straight lines of total parallel multiplicity six and total parallel multiplicity five were investigated in [20], [27], [28]. In [31] it was shown that in the class of cubic differential systems the maximal $\big($algebraic, geometric, integrable or infinitesimal, see [6]$\big)$ multiplicity of an affine real straight line (of the line at infinity) is seven. In [32] the cubic systems with two affine real non-parallel invariant straight lines of maximal multiplicity are classified.

In this paper a qualitative investigation of real cubic systems of the form

$$\begin{cases}\dot{x}=P_0+P_1(x,y)+P_2(x,y)+P_3(x,y)\equiv P(x,y),\\ \dot{y}=Q_0+Q_1(x,y)+Q_2(x,y)+Q_3(x,y)\equiv Q(x,y),\ \gcd(P,Q)=1,\end{cases} \tag{7}$$

where $P_k=\sum\limits_{j+l=k}a_{jl}x^jx^l,\ Q_k=\sum\limits_{j+l=k}b_{jl}x^jx^l\ (k=\overline{0,3})$ and $|P_3(x,y)|+|Q_3(x,y)|\not\equiv 0$, with affine real invariant straight lines of total parallel multiplicity six and of four distinct slopes, is given. Only one invariant straight line from these lines can have the parallel multiplicity greater or equal two. Our main result is the following one:

Theorem 1.1. *Assuming that a cubic system* (7) *possesses affine real invariant straight lines of total parallel multiplicity six with four distinct directions and at least three of these lines have multiplicity one. Then via an affine transformation and time rescaling this system can be brought to one of the five systems* (8)$-$(12) *given in Table 1.1. Also, in this table for each system* (8)$-$(12) *the invariant straight lines, Darboux first integral* $F(x,y)$ $\big($*or integrating factor* $\mu(x,y)\big)$ *and phase portrait in the Poincaré disk are given.*

Table 1.1. Canonical forms and qualitative investigation of the cubic systems with invariant straight lines of configurations $(3,1,1,1)$, $(3(2),1,1,1)$ and $(3(3),1,1,1)$

Systems, invariant straight lines l_j, first integral (F) or integrating factor (μ)	Fig./ Tab.
Configuration (3,1,1,1). (8) $\begin{cases} \dot{x} = x(x+1)(x-a), \\ \dot{y} = (y-1)\big(ay+(1-b)x^2+(a-1)bxy+aby^2\big), \\ (b-1)(a+b+ab)(1+b+ab) \neq 0,\ a>0,\ b\in\mathbb{R}, \end{cases}$ $l_1 = x,\ l_2 = x+1,\ l_3 = x-a,\ l_4 = y-1,\ l_5 = x-ay,\ l_6 = x+y;$ $\mu(x,y) = x^{\alpha_1}(x+1)^{\alpha_2}(x-a)^{\alpha_3}(y-1)^{\alpha_4}(x-ay)^{\alpha_5}(x+y)^{\alpha_6}$ where $\alpha_4 = (1-b)\alpha_1 = \dfrac{1-b}{b},\ \alpha_2 = a\alpha_3 = -\dfrac{a}{b(a+1)},\ \alpha_5 = (a+b+ab)\alpha_3,$ $\alpha_6 = (1+b+ab)\alpha_3$ if $b\neq 0$; $F_1(x,y) = \dfrac{(x+1)(x-ay)}{x(y-1)}$ if $b=0$;	1.1/ 4.1
Configuration (3,1,1,1). (9) $\begin{cases} \dot{x} = x(x+1)(x-a),\ -1<a\leq 1, a\neq 0,\ b>0,\ c\in\mathbb{R}^*, \\ \dot{y} = y\big(-a+(1-a)x+(1-bc)x^2+(b-1)cxy+cy^2\big), \\ \big(\lvert a+b+ab\rvert+\lvert ac-(a+1)^2\rvert\big)\big(\lvert 1+a+ab\rvert+\lvert c-a\rvert\big)\neq 0, \text{ if } -1<a<0, \\ \text{and } \big(\lvert b-a\rvert+\lvert ac-1\rvert\big)\big(\lvert c-a\rvert+\lvert ab-1\rvert\big)\neq 0, \text{ if } 0<a\leq 1, \end{cases}$ $l_1 = x,\ l_2 = x+1,\ l_3 = x-a,\ l_4 = y,\ l_5 = y-x,\ l_6 = y+bx;$ $F_2(x,y) = (x+1)^{-\frac{(b+1)bc}{a+1}}(x-a)^{-\frac{(b+1)abc}{a+1}}y^{-(b+1)}(y-x)^b(y+bx);$	1.2/ 4.2
Configuration $\big(3(2),1,1,1\big)$. (10) $\begin{cases} \dot{x} = x^2(x+1), \\ \dot{y} = y\big(x+(1-bc)x^2+(b-1)cxy+cy^2\big), \\ b\in\mathbb{R}^*_+,\ c\in\mathbb{R}^*, \end{cases}$ $l_{1,2} = x,\ l_3 = x+1,\ l_4 = y,\ l_5 = y-x,\ l_6 = y+bx;$ $F_3(x,y) = (x+1)^{-(b+1)bc}y^{-(b+1)}(y-x)^b(y+bx);$	1.3

Table 1.1 (continued)

Systems, invariant straight lines l_j, first integral (F) or integrating factor (μ)	Fig./ Tab.
Configuration $\big(3(2),1,1,1\big)$. (11) $\begin{cases} \dot{x} = x^2(x+1), \\ \dot{y} = y\big(-bc - 2bcx + (b-1)cy + (1-bc)x^2 + (b-1)cxy + cy^2\big), \\ b \in \mathbb{R}_+^*,\ c \in \mathbb{R}^*, \end{cases}$ $l_{1,2} = x,\ l_3 = x+1,\ l_4 = y,\ l_5 = y-x-1,\ l_6 = y + b(x+1);$ $F_4(x,y) = x^{-(b+1)bc} e^{(b+1)bc/x} y^{-(b+1)} (y-x-1)^b \big(y + b(x+1)\big);$	1.4
Configuration $\big(3(3),1,1,1\big)$. (12) $\begin{cases} \dot{x} = x^3, \\ \dot{y} = y\big((1-bc)x^2 + (b-1)cxy + cy^2\big), \\ c(bc-1)(bc+c+1)(b^2+bc+1) \neq 0,\ b > 0,\ c \in \mathbb{R}, \end{cases}$ $l_{1,2,3} = x,\ l_4 = y,\ l_5 = y - x,\ l_6 = y + bx;$ $F_5(x,y) = x^{-(b+1)bc} y^{-(b+1)} (y-x)^b (y+bx).$	1.5

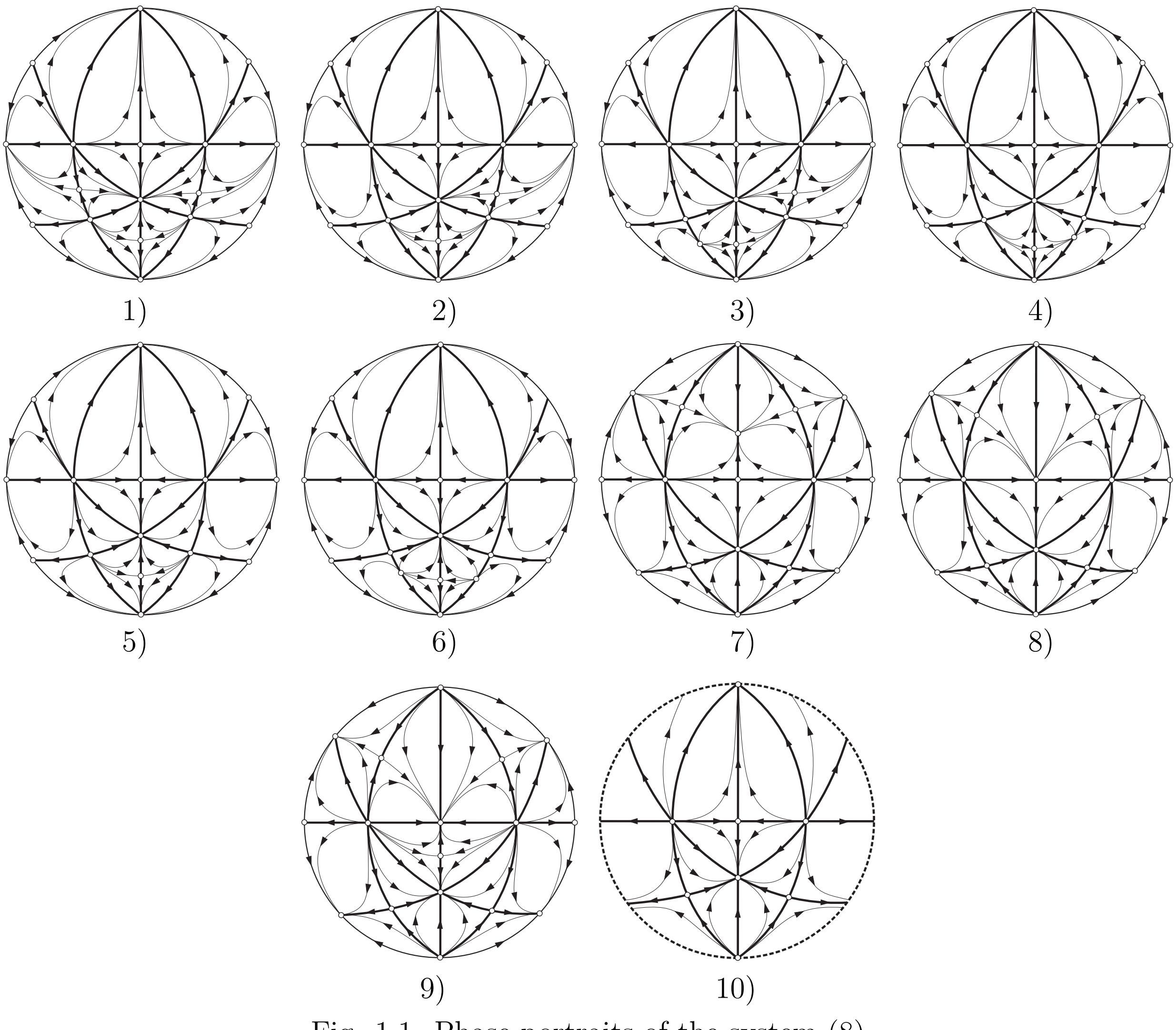
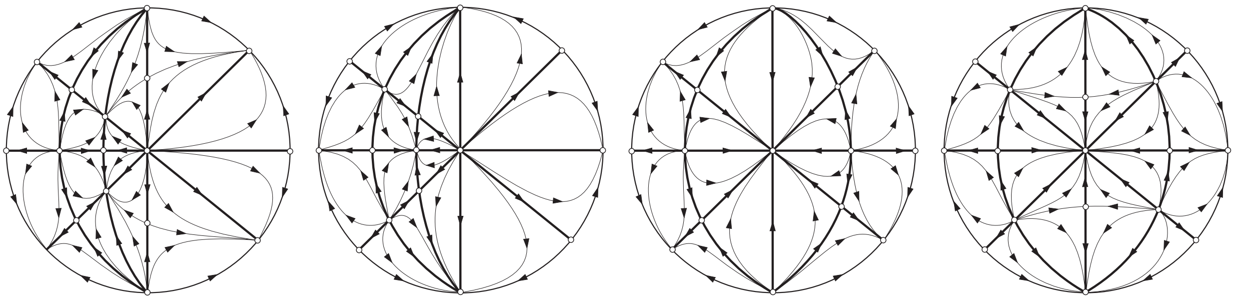
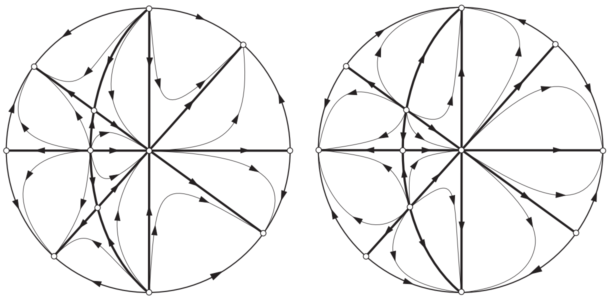


Fig. 1.1. Phase portraits of the system (8)



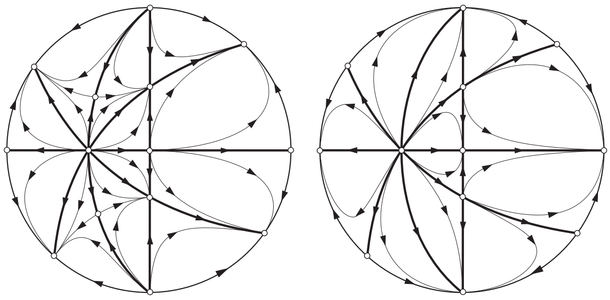
1) 2) 3) 4)

Fig. 1.2. Phase portraits of the system (9)



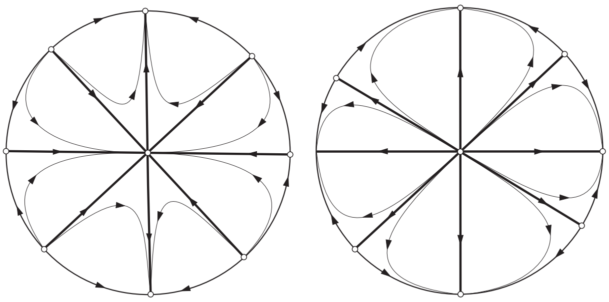
1) 2)

Fig. 1.3. Phase portraits of the system (10)



1) 2)

Fig. 1.4. Phase portraits of the system (11)



1) 2)

Fig. 1.5. Phase portraits of the system (12)

2. Some properties of the cubic systems with invariant straight lines

By a *straight lines configuration of invariant straight lines* of a cubic system we understand the set of all its invariant affine straight lines, each endowed with its own parallel multiplicity.

The goal of this section is to enumerate such properties for invariant straight lines that will allow the construction of configurations of straight lines realizable for (7). Some of these properties are obvious or easy to prove and others were proved in [29].

Properties:

2.1) *In the finite part of the phase plane each system* (7) *has at most nine singular points.*

2.2) *In the finite part of the phase plane, on any straight line there are at most three singular points of the system* (7)*.*

2.3) *The system* (7) *has no more than eight affine invariant straight lines* $\big([2]\big)$.

2.4) *At infinity the system* (7) *has at most four distinct singular points if* $yP_3(x,y) - xQ_3(x,y) \not\equiv 0$. *In the case* $yP_3(x,y) - xQ_3(x,y) \equiv 0$ *the infinity is degenerate, i.e. consists only of singular points.*

2.5) *If $yP_3(x,y) - xQ_3(x,y) \not\equiv 0$, then the infinity represents for* (7) *a non-singular invariant straight line, i.e. a line that is not filled up with singularities.*

2.6) *Through one point cannot pass more than four distinct invariant straight lines of the system* (7).

We say that the straight lines $l_j \equiv \alpha_j x + \beta_j y + \gamma_j \in \mathbb{C}[x,y]$, $(\alpha_j, \beta_j) \neq (0,0), j = 1,2,$ are *parallel* if $\alpha_1\beta_2 - \alpha_2\beta_1 = 0$. Otherwise the straight lines are called *concurrent.* If an affine invariant straight line l has the parallel multiplicity equal to m, then we will consider that we have m parallel invariant straight lines identical with l.

2.7) *The intersection point (x_0, y_0) of two concurrent invariant straight lines l_1 and l_2 of the system* (7) *is a singular point for this system.*

By a triplet of parallel affine invariant straight lines we shall mean a set of parallel affine invariant straight lines of total parallel multiplicity 3.

2.8) *If the cubic system* (7) *has a triplet of parallel affine invariant straight lines, then all its finite singular points lie on these straight lines.*

2.9) *The parallel multiplicity of an affine invariant straight line of the cubic system* (7) *is at most three.*

2.10) *If the cubic system* (7) *has two concurrent affine invariant straight lines l_1, l_2 and l_1 has the parallel multiplicity equal to m, $1 \leq m \leq 3$, then this system cannot have more than $3 - m$ singular points on $l_2 \setminus l_1$.*

We say that three affine straight lines are in generic position if no pair of these lines are parallel and no more that two lines are passing through the same point.

2.11) *For the cubic system* (7) *the total parallel multiplicity of three affine invariant straight lines in generic position is at most four.*

Proposition 2.1. *If $l \equiv \alpha x + \beta y + \gamma = 0$, $\alpha \neq 0$ $(\beta \neq 0)$ is a real invariant straight line of the system* (7) *then the transformation $X = \alpha x + \beta y + \gamma$, $Y = y$ $\big(X = \alpha x + \beta y + \gamma$, $Y = x\big)$ reduce* (7) *to a system of the form*

$$\begin{cases} \dot{X} = X(a_0 + a_1 X + a_2 Y + a_3 X^2 + a_4 XY + a_5 Y^2), \\ \dot{Y} = b_0 + b_1 X + b_2 Y + b_3 X^2 + b_4 XY + b_5 Y^2 + \\ \qquad +b_6 X^3 + b_7 X^2 Y + b_8 XY^2 + b_9 Y^3. \end{cases} \tag{13}$$

Indeed, in the case $\alpha \neq 0$ $(\beta \neq 0)$, from (7) and (3), we have:

$$\begin{aligned} &\dot{X} = \alpha\dot{x} + \beta\dot{y} = (\alpha x + \beta y + \gamma)K_l(x,y) = X \cdot K_l\big((X - \beta Y - \gamma)/\alpha,\ Y\big), \\ &\dot{Y} = \dot{y} = Q(x,y) = Q\big((X - \beta Y - \gamma)/\alpha,\ Y\big) \end{aligned}$$

$$\left(\begin{aligned} &\dot{X} = \alpha\dot{x} + \beta\dot{y} = (\alpha x + \beta y + \gamma)K_l(x,y) = X \cdot K_l\big(Y, (X - \alpha Y - \gamma)/\beta\big), \\ &\dot{Y} = \dot{y} = Q(x,y) = Q\big(Y, (X - \alpha Y - \gamma)/\beta\big) \end{aligned}\right).$$

Denote that the polynomial $K_l(x,y)$ has degree less or equal to two and, consequently, $K_l\big((X - \beta Y - \gamma)/\alpha,\ Y\big)$ has the same degree. □

Proposition 2.2. *If $l_j \equiv \alpha_j x + \beta_j y + \gamma_j = 0$, $j = 1,2$, $\Delta \equiv \alpha_1\beta_2 - \alpha_2\beta_1 \neq 0$ are two real invariant straight lines of the system* (7) *then the transformation $X = \alpha_1 x + \beta_1 y + \gamma_1$,*

$Y = \alpha_2 x + \beta_2 y + \gamma_2$ *reduce* (7) *to a system of the form*

$$\begin{cases} \dot{X} = X(a_0 + a_1 X + a_2 Y + a_3 X^2 + a_4 XY + a_5 Y^2), \\ \dot{Y} = Y(b_0 + b_1 X + b_2 Y + b_3 X^2 + b_4 XY + b_5 Y^2). \end{cases} \tag{14}$$

Indeed,

$$\begin{aligned} \dot{X} &= \alpha_1 \dot{x} + \beta_1 \dot{y} = (\alpha_1 x + \beta_1 y + \gamma_1) K_{l_1}(x, y) = \\ &= X \cdot K_{l_1}\big((\beta_2 X - \beta_1 Y + \beta_1 \gamma_2 - \beta_2 \gamma_1)/\Delta,\ (-\alpha_2 X + \alpha_1 Y + \alpha_2 \gamma_1 - \alpha_1 \gamma_2)/\Delta\big), \\ \dot{Y} &= \alpha_2 \dot{x} + \beta_2 \dot{y} = (\alpha_2 x + \beta_2 y + \gamma_2) K_{l_2}(x, y) = \\ &= Y \cdot K_{l_2}\big((\beta_2 X - \beta_1 Y + \beta_1 \gamma_2 - \beta_2 \gamma_1)/\Delta,\ (-\alpha_2 X + \alpha_1 Y + \alpha_2 \gamma_1 - \alpha_1 \gamma_2)/\Delta\big). \ \square \end{aligned}$$

3. Canonical forms

Let the system (7) hase a triplet $\{l_1, l_2, l_3\}$ of parallel invariant straight lines. Then:

3.1) l_j, $j = 1, 2, 3$ are distinct and $l_1 \parallel l_2 \parallel l_3$, or

3.2) l_1 has parallel multiplicity two, $l_2 \equiv l_1 \not\equiv l_3$ and $l_1 \parallel l_3$, or

3.3) $l_1 \equiv l_2 \equiv l_3$ and l_1 has parallel multiplicity three.

Along four directions there are only three possible configurations of six invariant straight lines, three of which form a triplet of parallel invariant straight lines:

1) $(3, 1, 1, 1)$, **2)** $\big(3(2), 1, 1, 1\big)$, **3)** $\big(3(3), 1, 1, 1\big)$.

Notation $(3, 1, 1, 1)$ means that there are six distinct real invariant straight lines of four directions and three of these lines form a triplet of parallel straight lines (the case 3.1)). Configurations $\big(3(2), 1, 1, 1\big)$ and $\big(3(3), 1, 1, 1\big)$ correspond to the cases 3.2) and 3.3), respectively.

3.1. Configuration $(3, 1, 1, 1)$.

Without loss of generality we can consider that one straight line of these six is parallel with to Ox axis and the straight lines from triplet are parallel with to Oy axis of coordinates. Taking into account the properties **2.2), 2.7)** and **2.8)** from Section 2, the straight lines can have (up to some affine transformations) one of the following three positions given in Fig. 3.1.

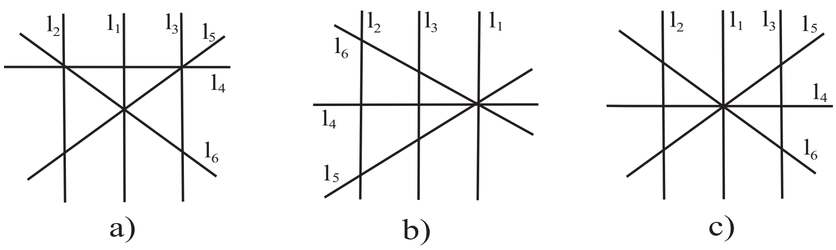


Fig. 3.1. Configurations of the type (3,1,1,1)

It is clear that the set of cubic systems which have the invariant straight lines of configuration $(3, 1, 1, 1)$ is a subset of the set of all cubic systems which have invariant straight lines of configuration $(3, 1)$.

In the case a) of Fig. 3.1 we can consider $l_1 = x$, $l_1 \cap l_5 \cap l_6 = (0, 0)$, $l_2 = x + 1$, $l_3 = x - a$, $a > 0$, $l_4 = y - 1$. Then, using an affine transformation and time rescaling, the cubic system for which $(0, 0)$ is a singular point and l_j, $j = 1, 2, 3, 4$ are invariant straight lines can be written in the form

$$\begin{cases} \dot{x} = x(x+1)(x-a) \equiv P(x,y),\ a>0, \\ \dot{y} = (y-1)(b_1x+b_2y+b_3x^2+b_4xy+b_5y^2) \equiv Q(x,y),\ \gcd(P,Q)=1. \end{cases} \tag{15}$$

Note that the straight line $l \equiv y - Ax - B = 0$, $A, B \in \mathbb{R}$ is invariant for polynomial differential system (1) if and only if the polynomial in x:

$$\Psi_l(x) = Q(x, Ax+B) - A \cdot P(x, Ax+B)$$

is identically zero. According to [16] if the straight line $l \equiv y - Ax - B = 0$, $A, B \in \mathbb{R}$ is invariant for (1) then l divides

$$E(\mathbb{X}) = P \cdot \mathbb{X}(Q) - Q \cdot \mathbb{X}(P), i.e.$$

$$E(\mathbb{X}) = P(x,y)\Big(P(x,y)\tfrac{\partial Q(x,y)}{\partial x} + Q(x,y)\tfrac{\partial Q(x,y)}{\partial y}\Big) - Q(x,y)\Big(P(x,y)\tfrac{\partial P(x,y)}{\partial x} + Q(x,y)\tfrac{\partial P(x,y)}{\partial y}\Big).$$

The polynomial $E(\mathbb{X})$ has in x and y the degree $3(n-1)+2$. In particular, in the case of cubic systems we have $\deg\big(E(\mathbb{X})\big) = 8$. Let $l_1, ..., l_6$ be the invariant straight lines of (1) and $l = y - Ax - B$. Suppose that the lines l, l_j, $j = 1, ..., 6$ are distinct. Denote $E_l(x) = \big(E(\mathbb{X})/(l_1 \cdots l_6)\big)|_{y=Ax+B}$. The straight line $l = y - Ax - B$ is invariant for (1) if and only if in the same time the identities $\Psi_l(x) \equiv 0$ and $E_l(x) \equiv 0$ take place.

The straight line l_5 (l_6) passes through the singular points $(0,0)$ and $(a,1)$ $\big((-1,1)\big)$, therefore it is described by equation $x - ay = 0$ $(x+y=0)$. The lines l_5 and l_6 are invariant if

$$\begin{cases} \Psi_{l_5}(x) = x(a-x)\big(a(b_2 - ab_1 - a) + (b_5 + ab_4 + a^2b_3 - a^2)x\big) \equiv 0, \\ \Psi_{l_6}(x) = x(x+1)\big(b_2 - b_1 - a + (1 - b_3 + b_4 - b_5)x\big) \equiv 0, \end{cases}$$

i.e. if the following series of conditions is satisfied: $b_1 = 0$, $b_2 = a$, $b_4 = b(a-1)$, $b_5 = ab$, where $b = 1 - b_3$. In these conditions the system (15) looks as

$$\begin{cases} \dot{x} = x(x+1)(x-a) \equiv P(x,y),\ a>0, \\ \dot{y} = (y-1)\big(ay + (1-b)x^2 + (a-1)bxy + aby^2\big) \equiv Q(x,y),\ \gcd(P,Q)=1, \end{cases} \tag{16}$$

i.e. we obtain the system (8) from Table 1.1.

Let $l = y - Ax - B$. For (16) we have

$$\begin{aligned} E_l(x) = &-\big(a(1+bB)(2-2b+3bB) + b(a-1+5aA+b-ab-2aAb-2bB+2abB+ \\ &+6aAbB)x + b(1-b-2Ab+2aAb+3aA^2b)x^2\big), \\ \Psi_l(x) = &\ aB(B-1)(1+bB) + B(2aA+b-ab-2aAb-bB+abB+3aAbB)x+ \\ &+\big((1-b)(1+A)(aA-1) + B(1-b-2Ab+2aAb+3aA^2b)\big)x^2+ \\ &+bA(1+A)(aA-1)x^3. \end{aligned}$$

In conditions $a > 0$ and $\deg\big(\gcd(P,Q)\big) = 0$ the identities $\{\Psi_l(x) \equiv 0,\ E_l(x) \equiv 0\}$ hold if $(b-1)\big(a+b(a+1)\big)\big(1+b(a+1)\big) = 0$. In this case (15) has more than six invariant straight lines. Indeed, in the case $b = 1$ (respectively, $a + b(a+1) = 0$; $1 + b(a+1) = 0$) the system (15) has the invariant straight line $l_7 = y$ (respectively, $l_7 = x - ay + a + 1$; $l_7 = 1 + b(x+y)$).

In the case b) and c) of Fig. 3.1 we can consider $l_1 = x$, $l_2 = x+1$, $l_3 = x-a$ and $l_4 = y$. It is clear that in the case b) (c)) of Fig. 3.1 we have $-1 < a < 0$ $(a > 0)$. Moreover, in the case c) we can consider $0 < a \le 1$. The cubic system for which l_j, $j = 1,2,3,4$ are invariant straight lines looks as

$$\begin{cases} \dot{x} = x(x+1)(x-a), \ (-1 < a < 0 \text{ or } 0 < a \le 1), \\ \dot{y} = y(b_0 + b_1 x + b_2 y + b_3 x^2 + b_4 xy + b_5 y^2). \end{cases} \tag{17}$$

The straight lines $l_{5,6}$ pass through the singular point $(0,0)$. Therefore, they are described by an equation of the form $y - bx = 0$, $b \in \mathbb{R} \setminus \{0\}$. Using the transformation $x \to x$, $y \to \alpha y$, $\alpha > 0$ we can choose $l_5 = y - x$. Then, $l_6 = y + bx$, $b > 0$. Solving the system of identities

$$\begin{cases} \Psi_{l_5}(x) = x(a + b_0 + (b_1 + b_2 + a - 1)x + (b_3 + b_4 + b_5 - 1)x^2) \equiv 0, \\ \Psi_{l_6}(x) = -bx(a + b_0 - (b \cdot b_2 - b_1 - a + 1)x + (b^2 \cdot b_5 - b \cdot b_4 + b_3 - 1)x^2) \equiv 0 \end{cases}$$

we obtain that the straight lines $l_{5,6}$ are invariant for (17) if $b_0 = -a$, $b_1 = 1 - a$, $b_2 = 0$, $b_3 = 1 - bc$, $b_4 = c(b-1)$, where $c = b_5$, i.e. if the system (17) has the form

$$\begin{cases} \dot{x} = x(x+1)(x-a) \equiv P(x,y), \ -1 < a \le 1, \ a \ne 0, \ b > 0, \\ \dot{y} = y\big[-a + (1-a)x + (1-bc)x^2 + c(b-1)xy + cy^2\big] \equiv Q(x,y). \end{cases} \tag{18}$$

Let $l = y - Ax - B$. For (18) we have

$$E_l(x) = c(3(cB^2 - a) + 2(1 - a - cB + 3cAB + bcB)x + (1 - 2cA + 3cA^2 - bc + 2cAb)x^2),$$
$$\Psi_l(x) = B(B^2c - a) + B(1 - a - cB + 3cAB + bcB)x + B(1 - 2cA + 3cA^2 - bc + 2bcA)x^2 + $$
$$+ cA(A-1)(A+b)x^3.$$

If $c = 0$, then (18) is degenerate, i.e. $\deg\big(\gcd(P,Q)\big) > 0$. Let $c \ne 0$. Then, the system of identities $\big\{E_l(x) \equiv 0,\ \Psi_l(x) \equiv 0\big\}$ is equivalent to the system of equalities $\big\{A(A-1)(A+b) = 0,\ cB^2 - a = 0,\ 1 - a - cB + 3cAB + bcB = 0,\ 1 - 2cA + 3cA^2 - bc + 2cAb = 0\big\}$.

In the case $A = 0$ we obtain $b - a = ac - 1 = 0$, $B = 1/a$ or $c - a = ab - 1 = 0$, $B = -1$. Therefore, if $0 < a \le 1$ then the system (18) has the seventh invariant straight line $l_7 \equiv y - a = 0$ if $b - a = ac - 1 = 0$, $B = 1/a$ and $l_7 \equiv y + 1 = 0$ if $c - a = ab - 1 = 0$, $B = -1$. Let $\big(|b - a| + |ac - 1|\big)\big(|c - a| + |ab - 1|\big) \ne 0$ and $A = 1$. Then $\{c - a = ab + a + 1 = 0,\ B = 1\}$ $\Rightarrow -1 < a < 0$ and we have the invariant straight line $l_7 \equiv y - x - 1 = 0$. At last, if $A = -b$ then $a + b(a+1) = ac - (a+1)^2 = 0, A = B = a/(a+1)$. Taking into account that $b > 0$ these equalities imply $-1 < a < 0$. Thus, if $-1 < a < 0$ then the system (18) has exactly six distinct invariant straight lines if and only if the following inequality $\big(|c - a| + |ab + a + 1|\big)\big(|a + b(a+1)| + |ac - (a+1)^2|\big) \ne 0$ holds.

The above description leads us to the system (9) from Table 1.1 and to the inequalities associated with it.

3.2. Configuration $\big(3(2),1,1,1\big)$. Let the system (7) have six invariant straight lines of the considered configuration of which l_1 has parallel multiplicity two, $l_2 \equiv l - 1$, and $l_3 \parallel l_{1,2}$. Taking into account Properties **2.8)** and **2.10)** the invariant straight lines l_j, $j = 1, ..., 6$ have (up to some affine transformations) one of the following two positions given in Fig. 3.2.

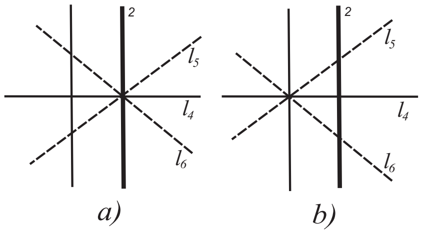


Fig. 3.2. Configuration $\big(3(2),1,1,1\big)$

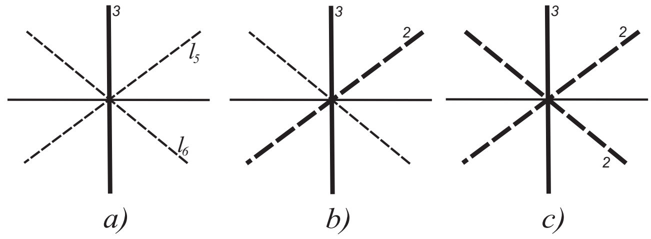


Fig. 3.3. Configuration $\big(3(3),1,1,1\big)$

Without loss of generality we can consider that $l_{1,2} = x$, $l_3 = x+1$, $l_4 = y$. The cubic system for which these lines are invariant looks as

$$\begin{cases} \dot{x} = x^2(x+1) \equiv P(x,y), \\ \dot{y} = y(b_0 + b_1x + b_2y + b_3x^2 + b_4xy + b_5y^2) \equiv Q(x,y), \ \gcd(P,Q) = 1. \end{cases} \tag{19}$$

In the case $a)$ $\big(b)\big)$ of Fig. 3.2 via the transformation $x \to x$, $y \to \gamma y$, $\gamma \neq 0$ we make the line l_5 to be described by the equation $y - x = 0$ $\ (y - x - 1 = 0)$. The equation of l_6 has the form $y = -bx$ $(y = -bx - b)$, $b > 0$. In this case, i.e. $a)$ $\big(b)\big)$ of Fig. 3.2, the straight lines $l_{5,6}$ are invariant for (19) if the identities hold:

$$\begin{cases} \Psi_{l_5} = x\big[b_0 + (b_1 + b_2 - 1)x + (b_3 + b_4 + b_5 - 1)x^2\big] \equiv 0, \\ \Psi_{l_6} = bx\big[-b_0 + (bb_2 - b_1 + 1)x + (bb_4 - b^2b_5 - b_3 + 1)x^2\big] \equiv 0 \end{cases}$$

$$\left(\begin{cases} \Psi_{l_5} = b_0 + b_2 + b_5 + (b_0 + b_1 + 2b_2 + b_4 + 3b_5)x + (b_1 + b_2 + b_3 + 2b_4 + 3b_5 - 1)x^2 + \\ \qquad +(b_3 + b_4 + b_5 - 1)x^3 \equiv 0, \\ \Psi_{l_6} = b\big[-b_0 - b^2b_5 + bb_2 + (-3b^2b_5 + 2bb_2 + bb_4 - b_0 - b_1)x + (-3b^2b_5 + bb_2 + \\ \qquad +2bb_4 - b_1 - b_3 + 1)x^2 + (-b^2b_5 + bb_4 - b_3 + 1)x^3\big] \equiv 0 \end{cases}\right).$$

These identities give us

$$b_0 = b_2 = 0,\ b_1 = 1,\ b_3 = 1 - bc,\ b_4 = c(b-1)$$

$$\big(b_0 = -bc,\ b_1 = -2bc,\ b_2 = b_4 = c(b-1),\ b_3 = 1 - bc\big),$$

where $c = b_5$. We obtained the system (10) $\big((11)\big)$ from Table 1.1. For both systems the equality $c = 0$ is in contradiction with the condition $\gcd(P,Q) = 1$.

3.3. Configuration $\big(3(3),1,1,1\big)$. For the first step, without loss of generality, we consider the system

$$\begin{cases} \dot{x} = x^3, \\ \dot{y} = y(b_0 + b_1x + b_2y + b_3x^2 + b_4xy + b_5y^2). \end{cases} \tag{20}$$

The system (20) has the invariant straight lines: $l_{1,2,3} = x$ and $l_4 = y$. The other invariant straight lines l_5 and l_6 of (20) (if exist) must pass through singular point $(0,0)$. Moreover, we can consider that l_5 $\ (l_6)$ is described by the equation $y - x = 0$ $\ (y + bx = 0,\ b > 0)$. The identities

$$\Psi_{l_5} = x\big[b_0 + (b_1 + b_2)x + (b_3 + b_4 + b_5 - 1)x^2\big] \equiv 0,$$
$$\Psi_{l_6} = -bx\big[b_0 + (b_1 - bb_2)x + (b^2b_5 - bb_4 + b_3 - 1)x^2\big] \equiv 0$$

have the solution

$$b_0 = b_1 = b_2 = 0,\ b_3 = 1 - bc,\ b_4 = c(b-1), \tag{21}$$

where $c = b_5$.

In the conditions $\{(21), b_5 = c\}$ the straight lines $l_{5,6}$ are invariant for the system (20). The cofactors of lines l_4, l_5, l_6 are respectively: $K_4(x,y) = (1-bc)x^2 + c(b-1)xy + cy^2$, $K_5(x,y) = x^2 + bcxy + cy^2$ and $K_6(x,y) = x^2 - cxy + cy^2$. From these, $K_4(x,0) = (1-bc)x^2$, $K_5(x,x) = (bc+c+1)x^2$ and $K_6(x,-bx) = (b^2c+bc+1)x^2$. Therefore, if $(bc-1)(bc+c+1)(b^2c+bc+1) = 0$ then at least one of the invariant straight lines has parallel multiplicity greater than one but this is not allowed in the examined configuration. If in the system $\{(21),(21), b_5 = c\}$ the parameter c vanishes then the condition $\gcd(P,Q) = 1$ is not met. Thus, the system (12) from Table 1.1 of Theorem 1.1 and its associated conditions are obtained.

4. Darboux integrability

In this section we construct the first integrals (F) or the integrating factors (μ) for systems (8)$-$(12).

4.1. Integrability of the system (8):

$$\begin{cases} \dot{x} = x(x+1)(x-a) \equiv P(x,y), \\ \dot{y} = (y-1)\big(ay + (1-b)x^2 + (a-1)bxy + aby^2\big) \equiv Q(x,y), \\ (b-1)(a+b+ab)(1+b+ab) \neq 0,\ a > 0,\ b \in \mathbb{R}. \end{cases}$$

The cofactors of the invariant straight lines: $l_1 = x$, $l_2 = x+1$, $l_3 = x-a$, $l_4 = y-1$, $l_5 = x - ay$, $l_6 = x + y$ of this system are, respectively:

$$K_{l_1}(x,y) = (x+1)(x-a),\ K_{l_2}(x,y) = x(x-a),\ K_{l_3}(x,y) = x(x+1),$$
$$K_{l_4}(x,y) = ay + (1-b)x^2 + (a-1)bxy + aby^2,$$
$$K_{l_5}(x,y) = -a + (1-ab)x + a(1-b)y + x^2 + abxy + aby^2,$$
$$K_{l_6}(x,y) = -a + (b-a)x + a(1-b)y + x^2 - bxy + aby^2.$$

Putting $s = 6$, $f \equiv l$ and $K_{l_j}(x,y)$, $j = \overline{1,6}$ in (6) and identifying the coefficients near the same powers of x and y, we get the system

$$\begin{cases} \alpha_1 + \alpha_5 + \alpha_6 = -2, \\ (1-a)\alpha_1 - a\alpha_2 + \alpha_3 + (1-ab)\alpha_5 + (b-a)\alpha_6 = (a-1)(b+2), \\ \alpha_4 + (1-b)(\alpha_5 + \alpha_6) = 2(b-1), \\ \alpha_1 + \alpha_2 + \alpha_3 + (1-b)\alpha_4 + \alpha_5 + \alpha_6 = b - 4, \\ b\big(2a - 2 + (a-1)\alpha_4 + a\alpha_5 - \alpha_6\big) = 0, \\ b(3 + \alpha_4 + \alpha_5 + \alpha_6) = 0. \end{cases}$$

If $b \neq 0$ then this system has the following solution in $\alpha_1, ..., \alpha_6$:

$$\alpha_1 = \frac{1}{b},\ \alpha_2 = -\frac{a}{(a+1)b},\ \alpha_3 = -\frac{1}{(a+1)b},\ \alpha_4 = \frac{1-b}{b},\ \alpha_5 = -\frac{a+(a+1)b}{(a+1)b},\ \alpha_6 = -\frac{1+(a+1)b}{(a+1)b}.$$

Therefore,

$$\mu(x,y) = x^{\frac{1}{b}}(x+1)^{-\frac{a}{(a+1)b}}(x-a)^{-\frac{1}{(a+1)b}}(y-1)^{\frac{1-b}{b}}(x-ay)^{-\frac{a+(a+1)b}{(a+1)b}}(x+y)^{-\frac{1+(a+1)b}{(a+1)b}}$$

is a Darboux integrating factor of the system (8) (see, (4)).

For these cofactors in the case $b \neq 0$, the identity (5) takes place if and only if $\alpha_1 = 0, ..., \alpha_6 = 0$. If $b = 0$, then the identity (5) is equivalent to the system

$$\begin{cases} \alpha_1+\alpha_5+\alpha_6=0, \\ (1-a)\alpha_1-a\alpha_2+\alpha_3+\alpha_5+a\alpha_6=0, \\ \alpha_4+\alpha_5+\alpha_6=0, \\ \alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5+\alpha_6=0; \end{cases} \Leftrightarrow \begin{cases} \alpha_1=-(\alpha_5+\alpha_6), \\ \alpha_2=\alpha_5, \\ \alpha_3=\alpha_6, \\ \alpha_4=-(\alpha_5+\alpha_6). \end{cases}$$

This system has the solution $\alpha_1=\alpha_4=-\alpha_2=-\alpha_5=-1$. Thus,

$$F(x,y)=\frac{(x+1)(x-ay)}{x(y-1)}$$

is a first integral of the system $\{(8),\ b=0\}$.

4.2. Integrability of the system (9):

$$\begin{cases} \dot{x}=x(x+1)(x-a),\ a>-1,\ b>0,\ c\in\mathbb{R}^*, \\ \dot{y}=y\big(-a+(1-a)x+(1-bc)x^2+(b-1)cxy+cy^2\big), \\ \big(|a+b+ab|+|ac-(a+1)^2|\big)\big(|1+a+ab|+|c-a|\big)\neq 0,\ \text{if } -1<a<0, \\ \text{and } \big(|b-a|+|ac-1|\big)\big(|c-a|+|ab-1|\big)\neq 0,\ \text{if } 0<a\leq 1. \end{cases}$$

The invariant straight lines: $l_1=x$, $l_2=x+1$, $l_3=x-a$, $l_4=y$, $l_5=y-x$, $l_6=y+bx$ of (9) have the cofactors, respectively:

$$K_{l_1}(x,y)=(x+1)(x-a),\ K_{l_2}(x,y)=x(x-a),\ K_{l_3}(x,y)=x(x+1),$$
$$K_{l_4}(x,y)=-a+(1-a)x+(1-bc)x^2+(b-1)cxy+cy^2,$$
$$K_{l_5}(x,y)=-a+(1-a)x+x^2+bcxy+cy^2,\ K_{l_6}(x,y)=-a+(1-a)x+x^2-cxy+cy^2.$$

Putting $K_{l_i}(x,y)$, $i=\overline{1,6}$ in the identity (5) we obtain in α_i, $i=\overline{1,6}$ the system:

$$\begin{cases} \alpha_1+\alpha_4+\alpha_5+\alpha_6=0, \\ (1-a)(\alpha_1+\alpha_4+\alpha_5+\alpha_6)-a\alpha_2+\alpha_3=0, \\ \alpha_1+\alpha_2+\alpha_3+(1-bc)\alpha_4+\alpha_5+\alpha_6=0, \\ (b-1)\alpha_4+b\alpha_5-\alpha_6=0, \\ \alpha_4+\alpha_5+\alpha_6=0; \end{cases} \Leftrightarrow \begin{cases} \alpha_1=0, \\ \alpha_2=-(b+1)bc\alpha_6/(a+1), \\ \alpha_3=-(b+1)abc\alpha_6/(a+1), \\ \alpha_4=-(b+1)\alpha_6, \\ \alpha_5=b\alpha_6. \end{cases}$$

Considering $\alpha_6=1$, the solution of this system gives us the following first integral of (9):

$$F(x,y)=(x+1)^{-\frac{(b+1)bc}{a+1}}(x-a)^{-\frac{(b+1)abc}{a+1}}y^{-(b+1)}(y-x)^b(y+bx).$$

4.3. Integrability of the systems (10)−(12).

Similarly to subsections **4.1** and **4.2** for each system (10)−(12) we calculate the cofactors $K_{l_j}(x,y)$, $j=\overline{1,6}$ $\big($see, (3) $\big)$ of invariant straight lines and the exponents α_j, $j=\overline{1,6}$ $\big($see, (5)$\big)$ of the first integrals $F(x,y)$ of the form (4). The obtained results are given in Table 4.1.

Table 4.1. First integrals of systems (10) − (12)

Syst.	$l_i,\ i=\overline{1,6}$	$K_i(x,y),\ i=\overline{1,6}$	$\alpha_i,\ i=\overline{1,6}$	F
(10)	$l_1=x$, $l_2=e^{1/x}$, $l_3=x+1$, $l_4=y$, $l_5=y-x$, $l_6=y+bx$,	$K_{l_1}=x(x+1)$, $K_{l_2}=-x-1$, $K_{l_3}=x^2$, $K_{l_4}=x+(1-bc)x^2+(b-1)cxy+cy^2$, $K_{l_5}=x+x^2+bcxy+cy^2$, $K_{l_6}=x+x^2-cxy+cy^2$,	$\alpha_1=0$, $\alpha_2=0$, $\alpha_3=-(b+1)bc\alpha_6$, $\alpha_4=-(b+1)\alpha_6$, $\alpha_5=b\alpha_6$;	F_3

Table 4.1 (continued)

Syst.	$l_i,\ i=\overline{1,6}$	$K_i(x,y),\ i=\overline{1,6}$	$\alpha_i,\ i=\overline{1,6}$	F
(11)	$l_1=x,$ $l_2=e^{1/x},$ $l_3=x+1,$ $l_4=y,$ $l_5=y-x-1,$ $l_6=y+bx+b,$	$K_{l_1}=x(x+1),$ $K_{l_2}=-x-1,$ $K_{l_3}=x^2,$ $K_{l_4}=bc(y-2x-1)-cy+(1-bc)x^2+$ $+(b-1)cxy+cy^2,$ $K_{l_5}=bcy+x^2+bcxy+cy^2,$ $K_{l_6}=-cy+x^2-cxy+cy^2,$	$\alpha_1=-(b+1)bc\alpha_6,$ $\alpha_2=(b+1)bc\alpha_6,$ $\alpha_3=0,$ $\alpha_4=-(b+1)\alpha_6,$ $\alpha_5=b\alpha_6;$	F_4
(12)	$l_1=x,$ $l_2=e^{1/x},$ $l_3=e^{1/x^2},$ $l_4=y,$ $l_5=y-x,$ $l_6=y+bx,$	$K_{l_1}=x^2,$ $K_{l_2}=-x,$ $K_{l_3}=-2,$ $K_{l_4}=(1-bc)x^2+(b-1)cxy+cy^2,$ $K_{l_5}=x^2+bcxy+cy^2,$ $K_{l_6}=x^2-cxy+cy^2,$	$\alpha_1=-(b+1)bc\alpha_6,$ $\alpha_2=0,$ $\alpha_3=0,$ $\alpha_4=-(b+1)\alpha_6,$ $\alpha_5=b\alpha_6.$	F_5

5. Qualitative investigation of the systems (8)−(12)

In this section, the qualitative study of the systems (8)−(12) from Theorem 1.1 will be done. For this purpose, in order to determine the topological behavior of trajectories, the singular points in the finite and infinite part of the phase plane will be examined. This information and the information provided by the existence of invariant straight lines will be taken into account when the phase portraits of the system (8)−(12) on the Poincaré disk will be constructed.

We set the abbrevation SP for a singular point and use here the following notations: λ_1 and λ_2 for eigenvalues of SP; S for a saddle ($\lambda_1\lambda_2<0$); N^s for a stable node ($\lambda_1,\lambda_2<0$); N^u for a unstable node ($\lambda_1,\lambda_2>0$); $S-N^{s(u)}$ for a saddle-node with a stable (unstable) parabolic sector, $P^{s(u)}$ for a stable (unstable) parabolic sector, H for a hyperbolic sector.

5.1. System (8) (configuration (3,1,1,1)), i.e. the system

$$\begin{cases} \dot{x}=x(x+1)(x-a)\equiv P(x,y),\\ \dot{y}=(y-1)\big(ay+(1-b)x^2+(a-1)bxy+aby^2\big)\equiv Q(x,y),\\ (b-1)(a+b+ab)(1+b+ab)\neq 0,\ a>0,\ b\in\mathbb{R}, \end{cases}$$

which has the invariant straight lines: $l_1=x$, $l_2=x+1$, $l_3=x-a$, $l_4=y-1$, $l_5=x-ay$ and $l_6=x+y$. This system has in the finite part of the phase plane nine singular points if $b\neq 0$ and six if $b=0$. The semi-plane of parameters $a,b;\ a>0$ is divided in thirteen sectors I_j by straight lines $a=0$, $b=0$, $b=\pm1$ and the hyperbolas $(a+1)b=\pm1$, $(a+1)b=\pm a$ (see, Fig. 5.1).

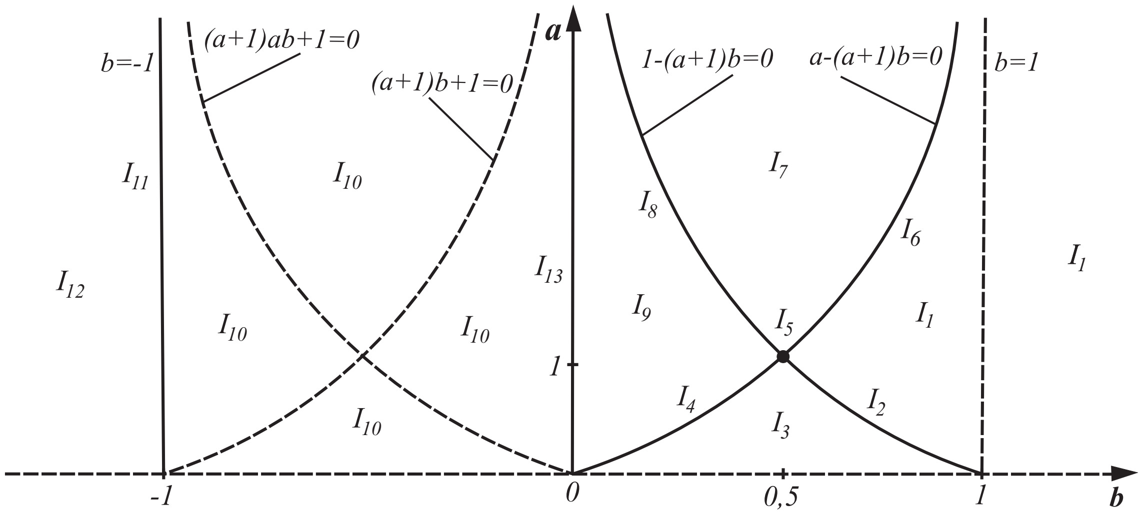


Fig. 5.1. Bifurcation diagram of the system (8)

In each of these sectors we calculate the eigenvalues of singular points and bring them in Table 5.1. In Table 5.1 we used the notations: $\alpha = a + 1$ and $\beta = a(b + 1)$.

Taking into account that $a > 0$, from Table 5.1 it is easy to see that the singular points O_1 and O_3 (respectively, point O_5) of the system (8) are unstable nodes (is a stable nod). If $b < -1$ or $b > 0$ then the O_9 is a saddle, and if $-1 < b < 0$ then O_9 is a stable node and so on.

Further we will study non-hyperbolic singular points of the system (8): O_2 in the sector I_{11}; O_4 in the sectors I_4–I_6 and O_6 in I_2, I_5 and I_8. In the other cases the singular points are hyperbolic.

Table 5.1. System (8): singular points, eigenvalues and types of SP

SP	λ_1; λ_2	$I_1/I_2/I_3$	$I_4/I_5/I_6$	$I_7/I_8/I_9$	$I_{10}/I_{11}/I_{12}$	I_{13}
$O_1(-1,1)$	α; α	N^i				
$O_2(0,1)$	$-a$; β	S	S	S	$S/S{-}N^s/N^s$	S
$O_3(a,1)$	$a\alpha$; $a\alpha$	N^u				
$O_4(-1,-\frac{1}{a})$	α; $\frac{\alpha(\alpha b-a)}{a}$	N^u	$S{-}N^u$	S	S	S
$O_5(0,0)$	$-a$; $-a$	N^s				
$O_6(a,-a)$	$a\alpha$; $a\alpha(\alpha b-1)$	$N^u/S{-}N^u/S$	$S/S{-}N^u/N^u$	$N^u/S{-}N^i/S$	S	S
$O_7(-1,\frac{b-1}{b})$	α; $\frac{a-\alpha b}{b}$	S	$-$	N^u	S	$-$
$O_8(a,\frac{b-1}{b})$	$a\alpha$; $\frac{a-\alpha ab}{b}$	$S/-/N^u$	$N^u/-/S$	$S/-/N^u$	S	$-$
$O_9(0,-\frac{1}{b})$	$-a$; $\frac{\beta}{b}$	S	S	S	$N^s/-/S$	$-$
Fig. 1.1:		1)/2)/3)	4)/5)/2)	3)/4)/6)	7)/8)/9)	10)

1) *Singular point* $O_2(0,1)$. Sector I_{11} is the semi-straight line of the semi-plane bOa, $a > 0$ given by equation $b = -1$. On I_{11} the eigenvalues of O_2 are $\lambda_1 = -a$ and $\lambda_2 = 0$, therefore it is a semi-hyperbolic singular point. The transformation $(x, y) \to (x, y-1)$ translate O_2 in the origin of the system of coordinates xOy. Then, changing x by y and y by x, i.e. $x = Y$,

$y = X$ and rescaling the time $\tau = -at$, the system $\{(8),\ b = -1\}$ takes the form

$$\begin{cases} \dot{X} = \frac{1}{a}X\big(aX + (a-1)Y + aX^2 + (a-1)XY - 2Y\big) = P(X,Y), \\ \dot{Y} = Y - \frac{1-a}{a}Y^2 - \frac{1}{a}Y^3 = Y + Q(X,Y). \end{cases}$$

The function $Y = \varphi(X) = \sum_{i\geq 1} c_i X^i$ is an analitic solution of equation $Y + Q(X,Y) = 0$ if and only if it vanishes $Y = \varphi(X) \equiv 0$. Putting $Y = 0$ in $P(X,Y)$ we obtain $\psi(X) = X^2 + X^3$. According to [1] the singular point $O_2(0,1)$ is a stable saddle-node.

2) *Singular point* $O_4\big(-1, -\frac{1}{a}\big)$. In this case the sectors I_4, I_5 and I_6 are placed on the hyperbola $a - (1+a)b = 0$, i.e. $b = \frac{a}{a+1}$, where $a > 0$. The eigenvalues of O_4 are $\lambda_1 = 1 + a$ and $\lambda_2 = 0$, therefore O_4 is a semi-hyperbolic singular point. Translating O_4 in the origin $\big((x,y) \to (x+1, y+1/a)\big)$ and putting $b = \frac{a}{a+1}$ in (8) we obtain

$$\dot{x} = x(x-1)(x-a-1),\ \dot{y} = (ay - a - 1)\big(-(a+1)x + x^2 + a(a-1)xy + a^2y^2\big)/(a^2+a).$$

The nondegenerate transformation $(x,y) \to \big(Y, X + Y/a\big)$ and the time rescaling $\tau = (a+1)t$ reduce the last system to the form

$$\begin{cases} \dot{X} = -\frac{1}{a+1}X\big(aX + (a+2)Y - \frac{a^2}{a+1}X^2 - \frac{(a+2)a}{a+1}XY - 2Y^2\big) = P(X,Y), \\ \dot{Y} = Y - \frac{a+2}{a+1}Y^2 + \frac{1}{a+1}Y^3 = Y + Q(X,Y). \end{cases}$$

From the equation $Y + Q(X,Y) = 0$ we find $Y = \varphi(X) = 0$. Putting $Y = 0$ in $P(X,Y)$ we obtain $\psi(X) = -\frac{a}{a+1}X^2 + \frac{a^2}{(a+1)^2}X^3$. According to [1], the singular point $O_4\big(-1, -\frac{1}{a}\big)$ is an unstable saddle-node.

3) *Singular point* $O_6(a, -a)$, $a > 0$. The sectors I_2, I_5 and I_6 are placed on the hyperbola $(a+1)b = 1$, i.e. $b = \frac{1}{a+1}$. The eigenvalues of O_6 are $\lambda_1 = (1+a)a$ and $\lambda_2 = 0$, thus the singular point O_6 is semi-hyperbolic. Proceeding in the same way as in the case 2) for O_6 we obtain $\psi(X) = -\frac{1}{a+1}X^2 + \frac{1}{(a+1)^2}X^3$. According to [1] the point O_6 is of saddle-node type.

Proposition 5.1. At infinity the system (8) has the following singular points:

a) $X_{1\infty}(1,0,0)$ – saddle; $X_{2\infty}(1,-1,0)$, $X_{3\infty}\big(1, \frac{1}{a}, 0\big)$ – stable nodes and $Y_\infty(0,1,0)$ – unstable node, if $b < 0$;

b) $X_{1\infty}(1,0,0)$ – stable node; $X_{2\infty}(1,-1,0)$, $X_{3\infty}\big(1, \frac{1}{a}, 0\big)$ – saddles and $Y_\infty(0,1,0)$ – stable node, if $b > 0$;

c) if $b = 0$ then the infinity is degenerate for (8), i.e. consists only of singular points. The singular points situated at the ends of the Oy axis are nodes. Through each of every other singular point at the infinity passes only one trajectory.

Proof. In the case $b \neq 0$ ($b = 0$) the first Poincaré transformation $x = 1/z$, $y = u/z$ and the time rescaling $\tau = t/z^2$ ($\tau = t/z$) reduce (8) to the system

$$\dot{z} = z(z+1)(az-1),\ \dot{u} = (u+1)(au-1)\big(bu + (1-b)z\big)$$

$$\big(\dot{z} = (z+1)(az-1),\ \dot{u} = (u+1)(au-1)\big),$$

and the second transformation: $x = v/z$, $y = 1/z$ and $\tau = t/z^2$ ($\tau = zt$) give us

$$\dot{v} = v(v+1)(v-a)\big(b + (1-b)z\big),\ \dot{z} = z(z-1)\big(ab + (a-1)bv + az + (1-b)v^2\big)$$

$$\big(\dot v = v(v+1)(v-a),\ \dot z = (z-1)(az+v^2)\big).$$

Putting $z = 0$ in the right-hand sides of these systems and equaling them with zero we obtain the following singular points, respectively: $X_{1_\infty}(1,0,0) : \{\lambda_1 = -1, \lambda_2 = -b\}$, $X_{2_\infty}(1,-1,0) : \{\lambda_1 = -1, \lambda_2 = b(a+1)\}$, $X_{3_\infty}\big(1,\frac{1}{a},0\big) : \big\{\lambda_1 = -1, \lambda_2 = \frac{b(a+1)}{a}\big\}$ and $Y_\infty(0,1,0) : \{\lambda_1 = \lambda_2 = -ab\}$ $\big(Y_\infty(0,1,0) : \{\lambda_1 = \lambda_2 = -a\}\big)$. The types of these singular points are completely determined by their eigenvalues: λ_1 and λ_2. □

In Fig. 5.2 are illustrated the singular points from Proposition 5.1.

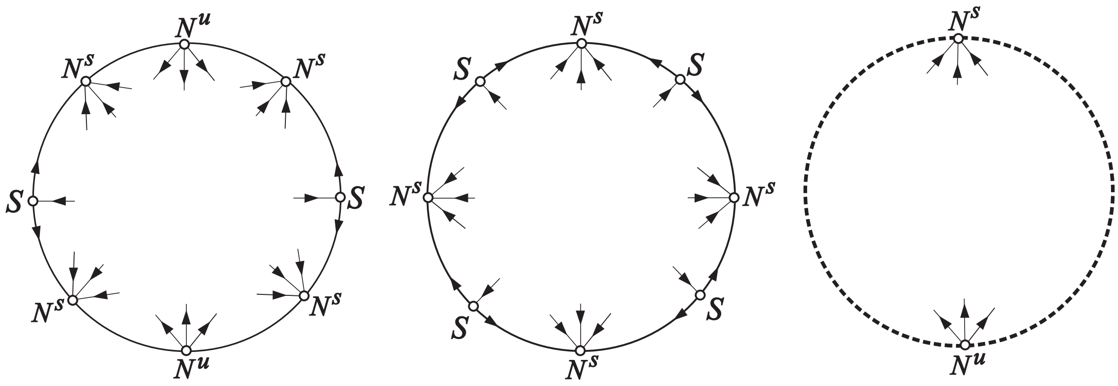


Fig. 5.2. Singular points at the infinity of the system (8)

The qualitative study in the finite part of phase plane and at the infinity leads us to the portraits given in Fig. 1.1.

5.2. System (9) (configuration (3,1,1,1)):

$$\begin{cases} \dot x = x(x+1)(x-a),\ a>-1,\ b>0,\ c\in\mathbb{R}^*, \\ \dot y = y\big(-a+(1-a)x+(1-bc)x^2+(b-1)cxy+cy^2\big), \\ \big(|a+b+ab|+|ac-(a+1)^2|\big)\big(|1+a+ab|+|c-a|\big)\neq 0,\ \text{if } -1<a<0, \\ \text{and } \big(|b-a|+|ac-1|\big)\big(|c-a|+|ab-1|\big)\neq 0,\ \text{if } 0<a\le 1. \end{cases}$$

For this system the straight lines: $l_1 = x$, $l_2 = x+1$, $l_3 = x-a$, $l_4 = y$, $l_5 = y-x$, $l_6 = y+bx$ are invariant. At the infinity it has four singular points and in the finite part of the phase plan it has nine (seven). All singular point are hyperbolic. Their eigenvalues and their types are given in Table 5.2. The information from the Table 5.2 are sufficiently to sketch phase portraits (see Fig. 1.2).

In the Table 5.2 we used the notations: $\alpha = a+1$ and $\beta = b+1$.

Table 5.2. System (9): singular points, eigenvalues and types of SP

SP	$\lambda_1; \lambda_2$	$-1 < a < 0$		$0 < a \leq 1$	
		$c < 0$	$c > 0$	$c < 0$	$c > 0$
$O_1(-1, 0)$	$\alpha;\ -bc$	N^i	S	N^i	S
$O_2(0, 0)$	$-a;\ -a$	N^i	N^i	N^s	N^s
$O_3(a, 0)$	$a\alpha;\ -a^2bc$	S	N^s	N^i	S
$O_4(-1, -1)$	$\alpha;\ c\beta$	S	N^i	S	N^i
$O_5(a, a)$	$a\alpha;\ a^2c\beta$	N^s	S	S	N^i
$O_6(-1, b)$	$\alpha;\ bc\beta$	S	N^i	S	N^i
$O_7(a, -ab)$	$a\alpha;\ a^2bc\beta$	N^s	S	S	N^i
$O_{8,9}\left(0, \pm\sqrt{\frac{a}{c}}\right)$	$-a;\ 2a$	S	$-$	$-$	S
$X_{1_\infty}(1, 0, 0)$	$-1;\ -bc$	S	N^s	S	N^s
$X_{2_\infty}(1, 1, 0)$	$-1;\ c\beta$	N^s	S	N^s	S
$X_{3_\infty}(1, -b, 0)$	$-1;\ bc\beta$	N^s	S	N^s	S
$Y_\infty(0, 1, 0)$	$-c;\ -c$	N^i	N^s	N^i	N^s
see Fig. 1.2:		1)	2)	3)	4)

5.3. System (10) (configuration $\big(3(2),1,1,1\big)$):

$$\begin{cases} \dot{x} = x^2(x+1), \\ \dot{y} = y\big(x + (1-bc)x^2 + (b-1)cxy + cy^2\big), \\ b \in \mathbb{R}_+^*,\ \ c \in \mathbb{R}^*. \end{cases}$$

The straight lines: $l_{1,2} = x$, $l_3 = x+1$, $l_4 = y$, $l_5 = y - x$ şi $l_6 = y + bx$ are invariant for (10).

The lines l_1, l_5, l_4 and l_6 divide the neighborhood of $O(0,0)$ in eight sectors. We enumerate these sectors from positive Ox semi-axis in counterclockwise direction. The notation $P^uH4P^sHP^u$ means that the first sector is unstable parabolic, the second sector is of hyperbolic type, the 3,4,5,6 sectors are stable parabolic, the 7 sector is hyperbolic and the 8 sector is unstable parabolic.

Proposition 5.2. In the finite part of the phase plane the system (10) has the following singular points:

1) $O_1(0,0)$ $-$ $P^uH4P^sHP^u$ if $c < 0$, and $2P^uH2P^sH2P^u$, if $c > 0$;

2) $O_2(-1,0)$ $-$ unstable node if $c < 0$, and saddle if $c > 0$;

3) $O_{3,4}(-1,-1)$ $-$ saddle if $c < 0$, and unstable node if $c > 0$.

Proof. We will examine separately every singular point O_1–O_4.

a) *Singular point* $O_1(0,0)$. Both eigenvalues of the point O_1 are null. We will study the behavior of the trajectories in a neighborhood of this point using blow-up method. First we apply in (10) the transformation $x = X$, $y = XY$:

$$\begin{cases} \dot{X} = \dot{x} = x^2(x+1) = X^2(X+1), \\ \dot{Y} = \dot{y}/x - y\dot{x}/x^2 = bX^2Y(Y-1)(Y+a). \end{cases}$$

Then, rescaling the time $\tau = X^2 t$ and using the substitution $(X, Y) \to (X+1, Y)$, the last system takes the form:

$$\dot{X} = X, \ \dot{Y} = bY(Y-1)(Y+a). \tag{22}$$

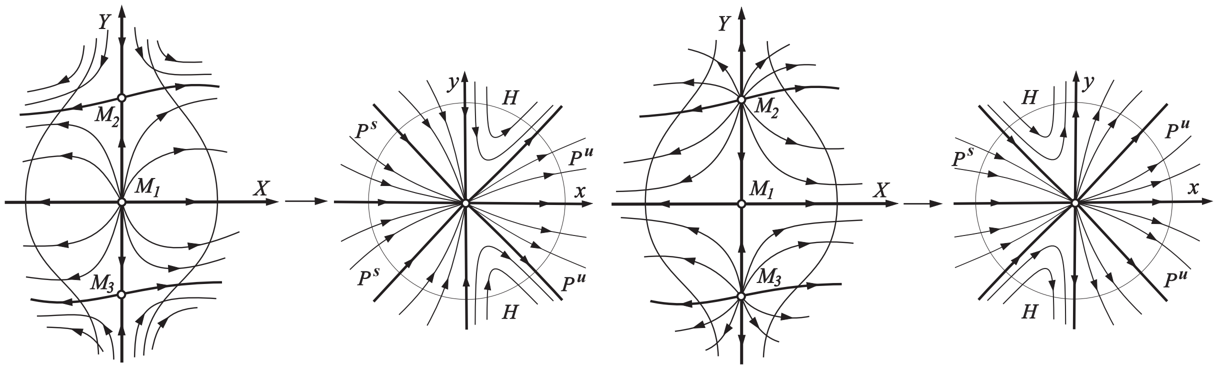
The singular points of the system (22) and their eigenvalues are:

$\{M_1(0,0) : \lambda_1 = 1, \lambda_2 = -bc\}$ – unstable node if $c < 0$, and saddle if $c > 0$;

$\{M_2(0,1) : \lambda_1 = 1, \lambda_2 = (b+1)c\}$ – saddle if $c < 0$, and unstable node if $c > 0$;

$\{M_3(0,-b) : \lambda_1 = 1, \lambda_2 = (b+1)bc\}$ – saddle if $c < 0$, and unstable node if $c > 0$.

The behavior of the trajectories near the points: M_1, M_2, and $(0,0)$ is illustrated in Fig. 5.3a (Fig. 5.3b) if $c < 0$ $(c > 0)$.



a) $c < 0$ b) $c > 0$

Fig. 5.3. System (10): the type of the singular point $(0,0)$

b) *Singular points* $O_2(-1,0)$, $O_3(-1,-1)$ *and* $O_4(-1,0)$. These points have the eigenvalues respectively:

O_2: $\lambda^2 + (bc-1)\lambda - bc = 0$; $\lambda_1 = 1$; $\lambda_2 = -bc$;

O_3: $\lambda^2 - (1+(b+1)c)\lambda + (b+1)c = 0$; $\lambda_1 = 1$; $\lambda_2 = (b+1)c$;

O_4: $\lambda^2 - (1+(b+1)bc)\lambda + (b+1)bc = 0$; $\lambda_1 = 1$; $\lambda_2 = (b+1)bc$.

Each of the point O_2, O_3 and O_4 are hyperbolic and is not difficult to determine their types. □

Because the cubic nonlinearities of (9) and (10) coincide, these systems have the same singular points at the infinity: $X_{1_\infty}(1,0,0)$, $X_{2_\infty}(1,1,0)$, $X_{3_\infty}(1,-b,0)$, $Y_\infty(0,1,0)$. Moreover, for both systems the eigenvalues λ_1, λ_2 are the same, respectively, and their types are completely determined by the value of parameter c (see, Tab. 5.2).

The arguments outlined above are enough to be able to draw the phases portraits of the system (10)
(see, Fig. 1.3,1) if $c < 0$ and Fig. 1.3,2) if $c > 0$.)

5.4. System (11) (configuration (3(2),1,1,1)):

$$\begin{cases} \dot{x} = x^2(x+1), \\ \dot{y} = y\big(-bc - 2bcx + (b-1)cy + (1-bc)x^2 + (b-1)cxy + cy^2\big), \\ b \in \mathbb{R}_+^*, \ c \in \mathbb{R}^*. \end{cases}$$

For the system (11) the straight lines: $l_{1,2} = x$, $l_3 = x+1$, $l_4 = y$, $l_5 = y - x - 1$ and $l_6 = y + b(x+1)$ are invariant.

Proposition 5.3. If $c < 0$ $(c > 0)$, then the system (11) has in the finite parte of the phase plane the following six (four) singular points:

1) $O_1(0,0)$, $O_2(0,1)$, $O_3(0,-b)$ – saddle-nodes;
2) $O_4(-1,0)$ – unstable node;
3) $O_{5,6}\left(-1, \pm\frac{1}{\sqrt{-c}}\right)$ – saddles.

Proof. a) *Singular point* $O_1(0,0)$. This point has the eigenvalues: $\lambda_1 = 0$ and $\lambda_2 = -bc$. Therefore, O_1 is a semi-hyperbolic. Rescaling in (11) the time $\tau = -bct$ we obtain the system

$$\begin{cases} \dot{x} = -\frac{1}{bc}x^2(x+1) = P(x,y), \\ \dot{y} = y - \frac{b-1}{b}y^2 + 2xy + \frac{bc-1}{bc}x^2y + \frac{1-b}{b}xy^2 - \frac{1}{b}y^3 = y + Q(x,y). \end{cases}$$

The equation $\{y + Q(x,y) = 0,\ y(0) = 0\}$ has the solution $y = 0$. Putting $y = 0$ in $P(x,y)$ we have $\psi(x) = P(x,0) = -\frac{1}{bc}(x^2 + x^3)$. According to [1], the singular point $O_1(0,0)$ is of saddle-node type.

b) *Singular point* $O_2(0,1)$ has the eigenvalues: $\lambda_1 = 0$ and $\lambda_2 = (b+1)c$, i.e. O_2 is semi-hyperbolic. At the beginning, via substitution $(x,y) \to (x, y-1)$ we translate O_2 in origin, then rescaling in (11) the time $\tau = (b+1)ct$, we obtain the system

$$\begin{cases} \dot{x} = \frac{1}{(b+1)c}x^2(x+1) = P(x,y), \\ \dot{y} = y + \frac{2}{(b+1)c}x^2(1-x) + 2xy + \frac{b+2}{b+1}y^2(x+1) + \frac{1+(b+1)c}{(b+1)c}x^2y + \frac{1}{b+1}y^3 = y + Q(x,y). \end{cases}$$

The solution $y = \varphi(x) = \sum_{i\geq 1} c_i x^i$ of the equation $y + Q(x,y) = 0$ has the form $\varphi(x) = -\frac{2}{(b+1)c}x^2 + \frac{2}{(b+1)c}x^3 + \cdots$. Putting $\varphi(x)$ in $P(x, \varphi(x))$ we come to the function $\psi(x) = \frac{1}{(b+1)c}(x^2 + x^3)$. Therefore, the singular point $O_2(0,1)$ is of saddle-node type (see, [1]).

c) *Singular point* $O_3(0,-b)$. Similarly as in b), for $O_3(0,-b)$ we get $\varphi = \frac{2}{(b+1)c}x^2 + \cdots$ and $\psi(x) = \frac{1}{(b+1)bc}(x^2 + x^3)$. Thus, O_3 is of saddle-node type $\big([1]\big)$.

d) *Singular points* $O_4(-1,0)$ *and* $O_{5,6}(0, \pm 1/\sqrt{-c})$. The eigenvalues of O_4 $(O_{5,6})$ are $\lambda_1 = \lambda_2 = 1$ ($\lambda_1 = -2$ and $\lambda_2 = 1$). Therefore, O_4 $(O_{5,6})$ is (are) unstable node (saddles). □

Because the systems (9) and (11) have the same cubic non-linearities, their singular points at the infinity coincide. The qualitative characteristics of these points are given in Tab. 5.2.

The investigations allowed us to draw the phase portraits of the system (11) (see, Fig. 1.4).

5.5. System (12) (configuration (3(3),1,1,1)):

$$\begin{cases} \dot{x} = x^3, \\ \dot{y} = y\big((1-bc)x^2 + (b-1)cxy + cy^2\big), \\ c(bc-1)\big((b+1)c+1\big)\big((b+1)bc+1\big) \neq 0,\ b > 0,\ bc \in \mathbb{R}. \end{cases}$$

This system has the following invariant straight lines: $l_{1,2,3} = x$, $l_4 = y$, $l_5 = y - x$ and $l_6 = y + bx$.

Proposition 5.4. *If* $c < 0$ $(c > 0)$ *then in the finite parte of the phase plane the system* (12) *has only one singular point* $(0,0)$ *which is of the type* $P^u 2H2P^u 2HP^u$ *(unstable topological node) if* $c < 0$ $(c > 0)$*.*

Proof. Both eigenvalues of the singular point $O(0,0)$ are null. Therefore, $O(0,0)$ is nilpotent. We will study the behavior of the trajectories in a neighborhood of this point using blow-up method. In the polar coordinates $x = \rho cos\theta$, $y = \rho sin\theta$ the system (12) takes the form:

$$\begin{cases} \frac{d\rho}{d\tau} = \rho\big(\cos^4\theta + b\sin^4\theta + (1-bc)\sin^2\theta\cos^2\theta + c(b-1)\sin^3\theta\cos\theta\big), \\ \frac{d\theta}{d\tau} = c\sin\theta\cos\theta(\sin\theta - \cos\theta)(\sin\theta + b\cos\theta), \end{cases} \quad (23)$$

where $\tau = \rho^2 t$. Taking into account that the system (12) is symmetric with respect to the origin, it is sufficient to consider $\theta \in [0,\pi)$. The singular points of the system (23) with first coordinate $\rho = 0$ and the second $\theta \in [0,\pi)$, their eigenvalues and types respectively are:

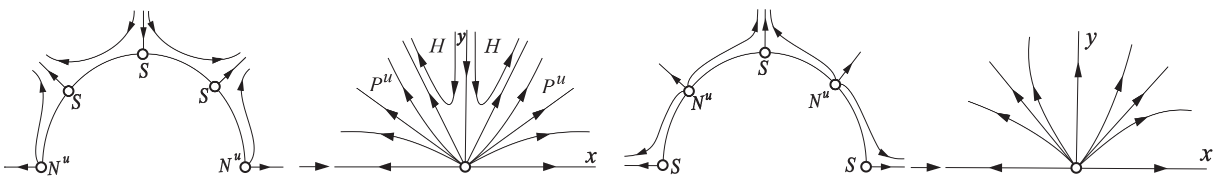
$M_1(0,0)$: $\{\lambda_1 = 1, \lambda_2 = -bc\}-$ unstable node, if $c < 0$, and saddle, if $c > 0$;

$M_2(0,\pi/2)$: $\{\lambda_{1,2} = \pm c\}-$ saddle;

$M_3(0,\pi/4)$: $\Big\{\lambda_1 = \dfrac{1}{2}, \lambda_2 = \dfrac{(b+1)c}{2}\Big\}-$ saddle, if $c < 0$, and unstable node, if $c > 0$;

$M_4(0,-\mathrm{arctg}\, b)$: $\Big\{\lambda_1 = \dfrac{1}{b^2+1}, \lambda_2 = \dfrac{(b+1)bc}{b^2+1}\Big\}-$ saddle, if $c < 0$, and unstable node, if $c > 0$.

We obtain Fig. 5.4a), if $c < 0$, and Fig. 5.4b), if $c > 0$. In the case $c < 0$ we have the following partition in sectors of the neighborhood of the origin: $P^u 2H2P^u 2HP^u$ and in the case $c > 0$ the neighborhood of the origin is an unstable topological node. □



a) $c < 0$ b) $c > 0$

Fig. 5.4. System (12): the type of the singular point $(0,0)$

The systems (9) and (12) have the same qualitative characteristic at the infinity.

The phase portraits of the system (12) are given in Fig. 1.5.

The results obtained in the Sections **3** $-$ **5** prove the Theorem 1.1.

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ON COMPACTNESS OF SOME INTEGRAL OPERATORS WITH CAUCHY KERNELS

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Abstract. In this paper, it is proved that the integral operator $S^* - S$ is compact if the contour of integration is of the Lyapunov type. An example is brought to show that this property of the operator $S^* - S$ becomes false if the contour of integration has angular points.
Keywords: singular integral operator, compact operator, piecewise Lyapunov contour.
2010 Mathematics Subject Classification: 34G10

ASUPRA COMPACTICITAȚII UNOR OPERATORI INTEGRALI CU NUCLEE DE TIP CAUCHY

Rezumat. În lucrare se demonstrează că operatorul integral singular $S^* - S$ este compact în cazul în care conturul de integrare este de tip Lyapunov. Se construiește un exemplu care arată că această proprietate a operatorului $S^* - S$ devine falsă dacă conturul are puncte unghiulare.
Cuvinte-cheie: operator integral singular, operator compact, contur Lyapunov pe porțiuni.

1. Introducere

Fie Γ un contur compus pe planul complex $\mathbb{C}$ și S operatorul integral singular cu nucleul Cauchy

$$(S\varphi)(t) = \frac{1}{\pi i}\int_{\Gamma} \frac{\varphi(\tau)}{\tau - t} d\tau,\ t \in \Gamma. \tag{1.1}$$

În lucrare se demonstrează că în cazul în care conturul Γ este de tip Lyapunov atunci operatorul $S - S^*$ este compact în spațiul $L_p(\Gamma, \rho)$, unde

$$\rho(t) = \prod_{k=1}^{n} |t - t_k|^{\beta_k},\ -1 < \beta_k < p - 1,\ k = 1,2,..., n. \tag{1.2}$$

Se construiește și se analizează un exemplu care demonstrează că dacă conturul Γ are puncte unghiulare, atunci operatorul $S - S^*$ încetează a mai fi compact. Din faptul că proprietatea operatorului $S - S^*$ de a fi compact depinde de netezimea conturului de integrare rezultă că normele esențiale ale operatorilor $S, P = \frac{1}{2}(I + S)$ și $P = \frac{1}{2}(I - S)$ depind de mărimile unghiurilor formate de contur în punctele sale unghiulare. Astfel, în consecință, metodele de cercetare elaborate de către matematicianul I. Simonenko în cazul ecuațiilor integrale singulare pe contururi cu puncte unghiulare necesită unele precizări și modificări.

2. Operatorul S^*

Vom stabili câteva proprietăți ale operatorului S^* începând cu determinarea formei explicite al acestui operator (a se vedea [1] și [2]). Fie $\varphi \in L_p(\Gamma,\rho)$ și $\psi \in L_q(\Gamma,\rho^{1-q})$, unde $p^{-1}+q^{-1}=1$, atunci

$$\left|\int_\Gamma \varphi(t)\ \bar{\psi}(t)|dt|\right| = \left|\int_\Gamma \varphi(t)\rho^{1/p}(t)\ \bar{\psi}(t)\rho^{-1/p}(t)|dt|\right| \le$$

$$\Big(\int_\Gamma |\varphi(t)|^p\ \rho\ (t)\ |dt|\Big)^{1/p}\Big(\int_\Gamma |\bar{\psi}(t)|\,\rho^{-q/p}(t)|dt|\Big)^{1/q} = \|\varphi\|_{L_p(\Gamma,\rho)}\|\psi\|_{L_q(\Gamma,\rho^{1-q})}.$$

Din teorema lui Riesz despre forma generală a operatorului liniar și mărginit în spațiul $L_p(\Gamma)$ rezultă următoarea afirmație.

Spațiul conjugat al spațiului $L_p(\Gamma,\rho)$ *este spațiul* $L_q(\Gamma,\rho^{1-q}),\ p^{-1}+q^{-1}=1.$

În mod obișnuit aceasta înseamnă că toate funcționalele liniare și continue din $L^*_p(\Gamma,\rho)$ au următoarea formă

$$\Psi(\varphi)=\int_\Gamma \varphi(t)\bar{\psi}(t)|dt| \quad (\varphi\in L_p(\Gamma,\rho)),$$

unde $\psi \in L_q(\Gamma,\rho^{1-q})$ și, în plus,

$$\|\Psi\|_{L^*_p(\Gamma,\rho)} = \|\psi\|_{L_q(\Gamma,\rho^{1-q})}.$$

Menționăm că dacă ponderea $\rho(t)=\prod_{k=1}^{n}|t-t_k|^{\beta_k}$ verifică condițiile

$$-1<\beta_k<p-1,\ k=1,2,...,n, \tag{2.1}$$

atunci ponderea $\rho^{1-q}(t)=\prod_{k=1}^{n}|t-t_k|^{(1-q)\beta_k}$ verifică condițiile

$$-1<(1-q)\beta_k<q-1. \tag{2.2}$$

Așadar, dacă condițiile (2.1) și (2.2) sunt verificate, atunci din teorema lui B. Hvedelidze [3] rezultă că operatorul S este mărginit și în spațiul $L_q(\Gamma,\rho^{1-q})$.

Fie $t\in\Gamma$. Se vede ușor că are loc egalitatea

$$dt = h(t)|dt|,$$

unde $h(t)=exp(i\theta(t))$, iar $\theta(t)$ este unghiul format de tangenta la curba Γ cu semiaxa pozitivă reală. Funcția $h(t)$ este definită în orice punct nesingular și este mărginită și continuă pe porțiuni.

Teorema 2.1. *Fie* Γ *un contur compus și pentru funcția* $\rho(t)$ *sunt verificate condițiile (2.1). În spațiul* $L_q(\Gamma,\rho^{1-q})$ *operatorul* S^* *este legat de operatorul* S *prin relația*

$$S^* = -HSH, \tag{2.3}$$

unde operatorul H *este definit de egalitatea*

$$(H\varphi)(t)=\overline{h(t)\varphi(t)}.$$

Demonstrație. Fie φ și ψ funcții raționale pe conturul Γ. În integrala iterată

$$(S\varphi,\psi)=\frac{1}{\pi i}\int_\Gamma \overline{\psi(t)}|dt|\int_\Gamma \frac{\varphi(\tau)}{\tau-t}d\tau,$$

în care o integrală este obișnuită iar alta singulară, avem dreptul [3] să schimbăm ordinea de integrare. Obținem

$$(S\varphi,\psi)=\frac{1}{\pi i}\int \varphi(t)h(t)dt\int_\Gamma \frac{\overline{\psi(\tau)}h^{-1}(\tau)}{\tau-t}d\tau=-(\varphi,\overline{H\,S\,H\,\psi}).$$

Prin urmare $S^*=-HSH$. Teorema este demonstrată.

Considerăm câteva exemple. Fie Γ un arc de cerc, atunci $\tau=t_0+Re^{i\theta}, d\tau=Re^{i\theta}id\theta=i(\tau-t_0)|d\tau|$. Prin urmare, $h(\tau)=i(\tau-t_0)$. În acest caz are loc egalitatea

$$(S^*\varphi)(t)=\frac{-1}{\pi(t-t_0)}\int_\Gamma \frac{i(\tau-t_0)\varphi(\tau)}{R^2(\overline{\tau}-t)}\overline{d\tau}=\frac{1}{\pi i}\int_\Gamma \frac{\varphi(\tau)d\tau}{\tau-t}=(S\varphi)(t).$$

În mod similar, dacă $\Gamma=[a,b]$, atunci $|d\tau|=d\tau, h(t)\equiv 1$ și $S^*=S$. Vom arăta că, într-un anumit sens, cu aceste exemple au fost epuizate toate curbele în care operatorul S este autoadjunct în spațiul $L_2(\Gamma)$. Are loc următoarea teoremă.

Teorema 2.2. *Dacă operatorul* S *este autoadjunct în spațiul* $L_2(\Gamma)$ *, atunci* Γ *este un cerc, un arc de cerc, sau o parte a unei drepte.*

Demonstrație. Fie $S^*=S$, atunci pentru orice funcție φ din $L_2(\Gamma)$ are loc egalitatea

$$\frac{1}{\pi i}\int_\Gamma\left(\frac{1}{\tau-t}-\frac{h^{-1}(t)h^{-1}(\tau)}{\overline{\tau}-\overline{t}}\right)\varphi(\tau)d\tau=((S-S^*)\varphi)(t)\equiv 0.$$

Din această relație rezultă că

$$(\overline{\tau}-\overline{t})h(t)h(\tau)\equiv\tau-t. \qquad (2.4)$$

Fie s abscisa de arc și $t=t(s)-$ ecuația (naturală) a curbei Γ. Așa cum $dt=h(t)ds$, atunci $h(t(s))=t'(s)$ și egalitatea (2.4) poate fi transcrisă sub forma

$$t'(s)t'(s_0)=\frac{t(s)-t(s_0)}{\overline{t(s)}-\overline{t(s_0)}}. \qquad (2.5)$$

Din această egalitate rezultă existența derivatei de orice ordin a funcției $t(s)$. Derivând ambele părți ale egalității

$$(\overline{t(s)}-\overline{t(s_0)})t'(s)t'(s_0)=t(s)-t(s_0)$$

odată în raport cu s, apoi în raport cu s_0, obținem

$$\overline{(t'(s)}t'(s)t'(s_0)-(\overline{t(s_0)}-\overline{t(s)})t''(s)t'(s_0)=t'(s)$$

și

$$\overline{(t'(s_0)}t'(s)t'(s_0)+(\overline{t(s_0)}-\overline{t(s)})t'(s)t''(s_0)=t'(s_0).$$

Ținând cont de egalitatea $t'(s)\overline{t'(s)}\equiv 1,$atunci din ultimele două relații obținem

$$(\overline{t(s_0)}-\overline{t(s)})(t'(s)t''(s_0)-t''(s)t'(s_0))=0.$$

Din egalitatea

$$t''(s)/(t'(s)=t''(s_0)/t'(s_0) \qquad (2.6)$$

și din faptul că funcția $t=t(s)$ nu poate fi constantă obținem că raportul $t''(s)/(t'(s)$ este constant $=k$). De aici, pentru $k\neq 0$ rezultă că $t(s)=ce^{ks}+c_1$. Deoarece $|t'(s)|\equiv 1,$atunci Re$k\;=\;0$ ceea ce înseamnă că funcția $t=t(s)$ reprezintă ecuația unui cerc, sau a unui arc de cerc. Pentru $k=0$ soluția ecuației (2.6) este funcția $t=cs+c_1$, în care $|c|=1$.

Teorema este demonstrată.

Menționăm că din cele demonstrate mai sus operatorul S este autoadjunct în spațiul $L_2(\Gamma)$ și în cazul în care Γ este orice dreaptă sau o parte a unei drepte.

3. Compacticitatea operatorului S^*-S

În caz general operatorii S și S^* nu coincid, însă pentru o clasă vastă de curbe acești operatori diferă printr-un termen compact. Această afirmație se conține în următoarea teoremă.

Teorema 3.1. *Fie* Γ *un contur compus de tip Lyapunov și* S^* *conjugatul operatorului* S *care acționează în spațiul* $L_p(\Gamma,\rho)$. *Atunci operatorul* S^*-S *este compact în spațiul* $L_q(\Gamma,\rho^{1-q})$.

Demonstrație. Pentru început considerăm Γ un contur simplu închis de tip Lyapunov. Notăm cu Γ_0 cercul unitate, iar prin $t=\beta(z)$ funcția lui Riemann care transformă conform discul unitate în domeniul G^+, mărginit de Γ. Operatorul S poate fi (a se vedea [2]) exprimat sub forma

$$S=B^{-1}S_0B+T_1, \qquad (3.1)$$

unde

$$(S_0\varphi)(z)=\frac{1}{\pi i}\int_{\Gamma_0}\frac{\varphi(\xi)}{\xi-z}d\xi,\ z\in\Gamma_0, \qquad (3.2)$$

$$(T_1\varphi)(\xi)=\frac{1}{\pi i}\int_{\Gamma_0}\left(\frac{\beta'(\xi)}{\beta(\xi)-\beta(z)}-\frac{1}{\xi-z}\right)\varphi(\beta(\xi))d\xi, \qquad (3.3)$$

$$(B\varphi)(z)=\varphi(\beta(z)),\ \ (B^{-1}\psi)(t)=\varphi(\omega(t)),$$

iar $z=\omega(t)$ este funcția inversă funcției $t=\beta(z)$. Nucleul operatorului integral (3.3) are singularități slabe pe conturul Γ_0 și, prin urmare [4], este compact în spațil $L_q(\Gamma,\rho^{1-q})$. Determinăm operatorii B^* și $(B^{-1})^*$. Fie φ și ψ funcții raționale pe conturul Γ_0, atunci

$$(B\varphi,\psi)=\int_{\Gamma_0}\varphi(\beta(z))\psi(z)|dz|=\int_{\Gamma}\varphi(t)\psi(\omega(t))\|\omega'(t)|dt|=(\varphi,\left|\frac{d\omega}{dt}\right|B^{-1}\psi).$$

Din această egalitate rezultă că

$$B^*=\left|\frac{d\omega}{dt}\right|B^{-1} \text{ și } (B^{-1})^*=\left|\frac{d\beta}{dz}\right|B.$$

Operatorul $S_0|\beta'(t)|-|\beta'(t)|S_0$ este compact, aceasta rezultă din teorema 4.3 din lucrarea [2]. Deoarece $S_0^*-S_0$ este compact, atunci

$$S^*-S=|\omega'(t)|B^{-1}S_0|\beta'(t)|B+T_1^*-B^{-1}S_0B-T_1=$$
$$|\omega'(t)||\beta'(\omega(t))|B^{-1}S_0B-B^{-1}S_0B+T_2=T_2,$$

unde T_2 este un operator compact. Vom considera acum cazul în care conturul Γ este un arc simplu deschis. Fie $\tilde{\Gamma}$ un contur simplu închis care conține arcul Γ. Notăm cu $\chi(t)$ funcția caracteristică al arcului Γ:

$$\chi(t)=\begin{cases}1, & t\in\Gamma\\0, & t\in\tilde{\Gamma}\setminus\Gamma\end{cases}.$$

Spațiul $L_p(\Gamma,\rho)$ în mod obișnuit poate fi identificat cu subspațiul funcțiilor **N** de forma $\chi\tilde{\varphi}$ $(\tilde{\varphi}\in L_p(\tilde{\Gamma},\rho)$. Subspațiul **N** este invariant în raport cu operatorul $A=\chi\tilde{S}\chi I$,unde

$$(\tilde{S}h)(z)=\frac{1}{\pi i}\int_{\tilde{\Gamma}}\frac{h(\xi)}{\xi-z}d\xi,\ \ z\in\tilde{\Gamma},$$

iar restricția acestui operator pe spațiul $L_p(\Gamma,\rho)$ coincide cu operatorul S. În baza celor demonstrate, avem $A^*=\chi\tilde{S}^*\chi I=\chi(\tilde{S}+T)\chi I$, unde T este un operator compact. De aici rezultă că operatorul S^*-S este compact.

Considerăm acum cazul general în care Γ este alcătuit dintr-un număr finit de arce și curbe închise $\Gamma_1,\Gamma_2,...,\Gamma_2$. Fie $\chi_j(t)$ funcția caracteristică a curbei Γ_j și R_j operatorul definit în spațiul $L_p(\Gamma,\rho)$ prin relația $R_j=\chi_j I$. Atunci $S=\sum_{j,k=1}^{n}R_jSR_k$. Operatorii $R_jSR_k(j\neq k)$ sunt operatori integrali cu nuclee continue (amintim că curbele Γ_j și Γ_k $(j\neq k)$ nu au puncte comune), prin urmare sunt compacți. Restricția operatorilor R_jSR_k pe spațiul $L_p(\Gamma_j,\rho)(=R_jL_p(\Gamma,\rho))$ coincide cu operatorul S. În baza celor deja demonstrate avem $(R_jSR_j)^*=R_jSR_j+T_j$, unde T_j sunt operatori compacți. De aici rezultă că S^*-S este compact. Teorema este demonstrată.

Teorema 3.1 devine falsă dacă cel puțin într-un punct al conturului Γ nu este îndeplinită condiția lui Lyapunov. Presupunem, de exemplu, că $\Gamma=\Gamma_1\cup\Gamma_2$, unde Γ_1 și Γ_2 sunt segmente de dreaptă care unesc punctul $z=0$ cu $z=1$ și, respectiv $z=0$ cu $z=i$. În punctul $z=0\in\Gamma$ conturul formează un unghi de măsură $\pi/2$. Vom arăta că în acest caz operatorul S^*-S nu este compact în spațiul $L_2(\Gamma)$.

Fie $t \in \Gamma$. Așa cum $|dt| = dt$ pentru $t \in \Gamma_1$ și $dt = i|dt|$ pentru $t \in \Gamma_2$, atunci $h(t) = 1$ pentru $t \in \Gamma_1$ și $h(t) = i$ pentru $t \in \Gamma_2$. În baza teoremei 2.1 avem $S^* = -HSH$.

Admitem că operatorul $S^* - S$ este compact, atunci operatorul $T = \mathrm{X}(HSH + S)$, unde X este funcția caracteristică a lui Γ_2, de asemenea este compact. În spaţiul $L_2(\Gamma)$ considerăm şirul $\{\varphi_n(t)\}$ normat de funcţii definit prin relaţiile

$$\varphi_n(t) = \begin{cases} \sqrt{n}, & \text{pentru} \quad t \in \left[0, \frac{1}{n}\right]; \\ 0, & \text{pentru} \quad t \in \Gamma \setminus \left[0; \frac{1}{n}\right]. \end{cases} \qquad n \in N,$$

și vom arăta că din șirul $\psi_n = T\varphi_n$ nu se poate extrage nici un subşir convergent. În baza definiţiei operatorului T avem

$$(T\varphi_n)(t) = \mathrm{X}(t)(S + HSH)\varphi_n = \frac{\mathrm{X}(t)\sqrt{n}}{\pi i} \int_0^{1/n} \left(\frac{1}{\tau - t} - \frac{1}{\tau - \bar{t}} \right) d\tau =$$

$$= \frac{\mathrm{X}(t)(1+i)}{\pi i} \sqrt{n} \int_0^{1/n} \frac{t + |t|}{\tau^2 + |t|^2} d\tau =$$

$$\frac{\mathrm{X}(t)(1+i)}{\pi i} (ln(1 + \frac{1}{n^2 |t|^2}) + 2arctg \frac{1}{|t| n}).$$

Fie $u_n(t) = \mathrm{X}(t) ln(1 + \frac{1}{n^2 |t|^2}$, $v_n(t) = \mathrm{X}(t) \sqrt{n} arctg \frac{1}{|t| n}$ și $\psi_n = \frac{1}{2} u_n + v_n$.

Din relațiile

$$\|u_n\|_{L_p(\Gamma)}^p = n^{p/2} \int_0^1 ln^p (1 + \frac{1}{n^2 x^2}) dx =$$

$$n^{p/2} \int_0^n ln^p (1 + \frac{1}{y^2}) \frac{dy}{n} \le n^{\frac{p-2}{p}} \int_0^n ln^p (1 + \frac{1}{y^2}) dy = c_1 n^{\frac{p-2}{p}},$$

$$|v_n|_{L_p(\Gamma)}^p = n^{p/2} \int_0^1 (\mathrm{arctg} \frac{1}{nx})^p dx = n^{p/2} \int_0^n \left(\mathrm{arctg} \frac{1}{y} \right)^p \frac{dy}{n} \le$$

$$n^{\frac{p-2}{p}} \int_0^\infty \left(\mathrm{arctg} \frac{1}{y} \right)^p dy = c_2 n^{\frac{p-2}{p}}$$

rezultă că $\lim_{n \to \infty} \|T\varphi_n\|_{L_p(\Gamma)} = 0$, pentru $1 < p < 2$.

Astfel, dacă şirul $\psi_n = T\varphi_n (\in L_2(\Gamma))$ ar conţine un subşir convergent, atunci acest subşir în mod necesar ar converge la zero. Deoarece $|\psi_n(t)| \ge v_n(t)$, atunci

$$\|\psi_n\|_{L_2(\Gamma)} \ge \|v_n\|_{L_2(\Gamma)} = \int_0^n arctg^2 \frac{1}{y} dy \ge \int_0^1 arctg^2 \frac{1}{y} dy > 0.$$

De aici rezultă că $\{\psi_n\}$ nu conţine nici un subşir convergent în spaţiul $L_2(\Gamma)$. Aşadar, operatorul T nu este compact în spaţiul $L_2(\Gamma)$.

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CANONICAL FORMS OF CUBIC DIFFERENTIAL SYSTEMS WITH REAL INVARIANT STRAIGHT LINES OF TOTAL MULTIPLICITY SEVEN ALONG ONE DIRECTION

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Abstract. Consider the general cubic differential system $\dot{x} = P(x, y)$, $\dot{y} = Q(x, y)$, where $P, Q \in \mathbf{R}[x, y]$, $\max\{\deg P, \deg Q\} = 3$, $GCD(P, Q) = 1$. If this system has enough invariant straight lines considered with their multiplicities, then, according to [1], we can construct a Darboux first integral. In this paper we obtain 26 canonical forms for cubic differential systems which possess real invariant straight lines along one direction of total multiplicity seven including the straight line at the infinity.
Keywords: cubic differential system, invariant straight line, Darboux integrability.
2010 Mathematics Subject Classification: 34C05

FORMELE CANONICE ALE SISTEMELOR DIFERENȚIALE CUBICE CE POSEDĂ DREPTE INVARIANTE REALE DE-A LUNGUL UNEI DIRECȚII A CĂROR MULTIPLICITATE TOTALĂ ESTE EGALĂ CU ȘAPTE

Rezumat. Fie sistemul diferențial cubic general $\dot{x} = P(x, y)$, $\dot{y} = Q(x, y)$, unde $P, Q \in \mathbf{R}[x, y]$, $\max\{\deg P, \deg Q\} = 3$, $GCD(P, Q) = 1$. Conform [1], pentru un sistem diferențial cubic se poate de construit o integrală primă de tip Darboux, dacă sistemul dat posedă un număr suficient de drepte invariante considerate cu multiplicitățile lor. În această lucrare se obțin 26 sisteme ce reprezintă formele canonice ale sistemelor diferențiale cubice ce posedă drepte invariante reale de-a lungul unei direcții și a căror multiplicitate totală este egală cu șapte împreună cu dreapta de la infinit.
Cuvinte cheie: sistem diferențial cubic, dreaptă invariantă, integrabilitate Darboux.

1. Introduction

We consider the real polynomial system of differential equations

$$\begin{cases} \dfrac{dx}{dt} = P(x, y) \\ \dfrac{dy}{dt} = Q(x, y) \end{cases}, \qquad GCD(P, Q) = 1 \tag{1}$$

and the vector field

$$\mathbb{X} = P(x, y)\frac{\partial}{\partial x} + Q(x, y)\frac{\partial}{\partial y} \tag{2}$$

associated to system (1). Denote $n = \max\{\deg(P), \deg(Q)\}$. If $n = 2$ $(n = 3)$, then the system (1) is called a quadratic (cubic) system.

Definition 1. An algebraic curve $f(x, y) = 0$, $f \in \mathbf{C}[x, y]$, is called *invariant algebraic curve* for the system (1), if there exists a polynomial $K_f \in \mathbf{C}[x, y]$, such that the identity

$$\mathbb{X}(f) = f(x, y)K_f(x, y) \tag{3}$$

holds.

The invariant algebraic curves $f(x,y)=\alpha x+\beta y+\gamma$ of the order one of system (1) are called *invariant straight lines* of the system (1).

System (1) is called *Darboux integrable* if there exists a non-constant function of the form $F=f_1^{\lambda_1}\cdot f_2^{\lambda_2}\cdot\ldots\cdot f_s^{\lambda_s}$, where f_j is an invariant algebraic curve and $\lambda_j\in\mathbf{C}$, $j=\overline{1,s}$, such that F is a first integral or F is an integrating factor for (1). The function $F=f_1^{\lambda_1}\cdot f_2^{\lambda_2}\cdot\ldots\cdot f_s^{\lambda_s}$ is called a *Darboux first integral*. If a polynomial differential system has enough invariant straight lines (including their multiplicity), then, according to [1], a Darboux first integral can be constructed for this system.

In the theory of dynamic systems, the investigation of polynomial differential systems with invariant straight lines is done using different types of multiplicities of these invariant straight lines, for example: parallel multiplicity, geometric multiplicity; algebraic multiplicity; etc [2]. In this paper we will use the notion of algebraic multiplicity of an invariant straight line.

Definition 2. Let $\mathbf{C}_m[x]$ be the $\mathbf{C}$-vector space of polynomials in $\mathbf{C}[x]$ of degree at most m. Then it has dimension $R=C_{n+m}^n$. Let $v_1,v_2,\ldots,v_R$ be a base of $\mathbf{C}_m[x]$. If k is the greatest positive integer such that the k-th power of $f(x,y)$ divides $\det M_R$, where

$$M_R=\begin{pmatrix} v_1 & v_2 & \ldots & v_R \\ \mathbb{X}(v_1) & \mathbb{X}(v_2) & \ldots & \mathbb{X}(v_R) \\ \ldots & \ldots & \ldots & \ldots \\ \mathbb{X}^{R-1}(v_1) & \mathbb{X}^{R-1}(v_2) & \ldots & \mathbb{X}^{R-1}(v_R) \end{pmatrix},$$

then the invariant algebraic curve f of degree m of the vector field $\mathbb{X}$ has algebraic multiplicity k.

In the above definition, the expression $\mathbb{X}^{R-1}(v_1)$ means that the operator $\mathbb{X}$ is applied $R-1$ times on vector v_1, i.e. $\mathbb{X}^{k+1}(v_i)=\mathbb{X}(\mathbb{X}^k(v_i))$.

There are a great number of articles dedicated to the investigation of polynomial differential systems with invariant straight lines. In [3] the authors estimate the number of invariant straight lines that a polynomial differential system can have. The problem of coexistence of invariant straight lines and limit cycles has been studied in [4,5], and the problem of coexistence of invariant straight lines and singular points of the center type for cubic system has been studied in [6,7]. The classification of all cubic systems which have the maximum number of invariant straight lines including their multiplicities was performed in [8,9]. In [10] were studied the cubic systems with exactly eight invariant straight lines. The cubic systems with six real invariant straight lines along two and three directions were studied in [11,12].

In this paper we obtain all canonical forms of cubic differential systems with real invariants straight lines along one direction with their total multiplicity equal to seven including the multiplicity of the invariant straight line at the infinity.

2. The algebraic multiplicity of invariant straight lines

We will study the following cubic differential system

$$\begin{cases} \dot{x} = a_0 + P_1(x,y) + P_2(x,y) + P_3(x,y), \\ \dot{y} = b_0 + Q_1(x,y) + Q_2(x,y) + Q_3(x,y), \end{cases} \tag{4}$$

where $P_i(x,y), Q_i(x,y), i=\overline{1,3}$ are homogenous polynomials of degree i, and the coefficients are arbitrary parameters $P_i(x,y) = \sum_{j=0}^{i} a_{i-j,j} x^{i-j} y^j$, $Q_i(x,y) = \sum_{j=0}^{i} b_{i-j,j} x^{i-j} y^j$, $i=\overline{1,3}$.

When the system (4) has an invariant straight line of the form $\alpha x + \beta y + \gamma = 0$, we can bring this straight line to the form $\overline{x} = 0$ using the affine transformation $\overline{x} = \alpha x + \beta y + \gamma, \overline{y} = y$. It is obvious that the conditions for the existence of invariant straight line $x = 0$ for system (4) are simpler than the conditions for the existence of invariant straight line $\alpha x + \beta y + \gamma = 0$ for the same system.

Besides the existence of invariant straight lines, we are interested the invariant straight lines to have a certain algebraic multiplicity. According to Definition 2, for the cubic differential system with invariant straight lines we have $R = C_3^2 = 3$. As the basis of the vector space of polynomials $\mathbf{C}_1[x]$ we can choose $v_1 = 1, v_2 = x, v_3 = y$. Then the matrix M_R has the form

$$M_R = \begin{pmatrix} 1 & x & y \\ 0 & P(x,y) & Q(x,y) \\ 0 & \mathbb{X}(P) & \mathbb{X}(Q) \end{pmatrix}.$$

In this case, the polynomial $\det M_R$ looks $\det M_R = P\mathbb{X}(Q) - Q\mathbb{X}(P)$ and is a polynomial of degree 8 with respect to x and y. According to Definition 2, the straight line $x = 0$ is invariant if and only if the polynomial $\det M_R$ can be written as $\det M_R = x\sum_{i=0}^{7} A_i(y) x^i$, where $\deg\{A_i(y)\} = 7 - i$. Moreover, if the polynomials $A_0(y), A_1(y), ..., A_k(y)$, $k \in \overline{0,6}$, are identically zero, then the straight line $x = 0$ has the algebraic multiplicity $k + 2$.

To study the multiplicity of an invariant straight line at infinity we carry out the Poincaré transformation $x = \frac{1}{\overline{x}}$, $y = \frac{\overline{y}}{\overline{x}}$. The multiplicity of an invariant straight line at infinity is equal to the multiplicity of the invariant straight line $\overline{x} = 0$ of the following system

$$\begin{cases} \dot{\bar{x}} = \overline{y}\overline{x}^3 P\left(\dfrac{1}{\overline{x}}, \dfrac{\overline{y}}{\overline{x}}\right) - \overline{x}^3 Q\left(\dfrac{1}{\overline{x}}, \dfrac{\overline{y}}{\overline{x}}\right), \\ \dot{\bar{y}} = \overline{x}^4 P\left(\dfrac{1}{\overline{x}}, \dfrac{\overline{y}}{\overline{x}}\right). \end{cases}$$

3. Obtaining the canonical forms of cubic differential systems

We emphasize that the calculations used in determination of canonical forms are quite large, so we will show in detail only formulas used for obtaining a single canonical form, and the rest will be omitted. Let us note by $\left(d_1(m_1)+d_2(m_2)+d_3\right)$ a configuration of invariant straight lines, where d_i is the number of straight lines, and m_i is their corresponding multiplicity. If $m_i = 1$, then m_i is not written. For example, the notation (1(4)+2) indicates that there are three parallel invariant straight lines, where one of them has the multiplicity equal to four and the other two have the multiplicity equal to one. Depending on the multiplicity of the invariant straight lines at infinity, we divide the investigation into seven cases.

Case 1: The straight line at infinity has multiplicity equal to 1. The real invariant straight lines from the finite plane can have the following configurations: **a)** (1(6)); **b)** (1(4)+2); **c)** (1(4)+1(2)); **d)** (1(3)+1(3)); **e)** (1(3)+1(2)+1); **f)** (1(2)+1(2)+1(2)).

1.a) (1(6)) Conditioning the system (4) to have the invariant straight line $x = 0$ and applying Definition 1, we obtain the following conditions on the parameters of the system (4):

$$a_{00} = 0; a_{01} = 0; a_{02} = 0; a_{03} = 0.$$

According to Definition 2, the condition for the invariant straight line $x = 0$ to have algebraic multiplicity equal to two is equivalent with the condition $A_0(y) = 0$, i.e. the following system of equations hold

$$\begin{cases} b_{00}\left(a_{10} + a_{11}b_{00} - a_{10}b_{01}\right) = 0; \\ -2a_{10}a_{11}b_{00} - 2a_{12}b_{00}^2 - a_{10}^2 b_{01} - a_{11}b_{00}b_{01} + a_{10}b_{01}^2 + 2a_{10}b_{00}b_{02} = 0; \\ -a_{11}^2 b_{00} - 2a_{10}a_{12}b_{00} - 2a_{10}a_{11}b_{01} - 3a_{12}b_{00}b_{01} - a_{10}^2 b_{02} + 3a_{10}b_{01}b_{02} + 3a_{10}b_{00}b_{03} = 0; \\ -2a_{11}a_{12}b_{00} - a_{11}^2 b_{01} - 2a_{10}a_{12}b_{01} - a_{12}b_{01}^2 - 2a_{10}a_{11}b_{02} - 2a_{12}b_{00}b_{02} + a_{11}b_{01}b_{02} + 2a_{10}b_{02}^2 - a_{10}^2 b_{03} + a_{11}b_{00}b_{03} + 4a_{10}b_{01}b_{03} = 0; \\ -a_{12}^2 b_{00} - 2a_{11}a_{12}b_{01} - a_{11}^2 b_{02} - 2a_{10}a_{12}b_{02} - a_{12}b_{01}b_{02} + a_{11}b_{02}^2 - 2a_{10}a_{11}b_{03} - a_{12}b_{00}b_{03} + 2a_{11}b_{01}b_{03} + 5a_{10}b_{02}b_{03} = 0; \\ -a_{12}^2 b_{01} - 2a_{11}a_{12}b_{02} - a_{11}^2 b_{03} - 2a_{10}a_{12}b_{03} + 3a_{11}b_{02}b_{03} + 3a_{10}b_{03}^2 = 0; \\ \left(a_{12} - b_{03}\right)\left(a_{12}b_{02} + 2a_{11}b_{03}\right) = 0; \\ a_{12}b_{03}\left(a_{12} - b_{03}\right) = 0. \end{cases}$$

By solving this system, we get four sets of conditions, i.e. the system (4) with the invariant straight line $x = 0$ implies four cubic differential systems which have this invariant straight line of algebraic multiplicity equal to two.

By asking the invariant straight line $x = 0$ of the system (4) to have algebraic multiplicity equal to three, i.e. the condition $A_1(y) = 0$ must hold for each one of those four systems, we obtain eight differential systems.

By asking the invariant straight line $x=0$ of the system (4) to have algebraic multiplicity equal to four, i.e. the condition $A_2(y)=0$ to be realized for each one of those eight systems, we obtain 11 cubic differential systems.

By asking the invariant straight line $x=0$ of the system (4) to have algebraic multiplicity equal to five, i.e. the condition $A_3(y)=0$ must hold for each one of those 11 systems, we obtain two differential systems.

Finally, by conditioning the invariant straight line $x=0$ to have multiplicity equal to six for the system (4), we obtain one set of conditions, i.e. the system satisfying these conditions has the form

$$\begin{cases} \dot{x} = a_{30}x^3, b_{00} \neq 0, \\ \dot{y} = b_{00} + b_{10}x + b_{20}x^2 + b_{30}x^3 + 3a_{30}x^2y. \end{cases}$$

Carrying out the transformations $x \to x, y \to \dfrac{-b_{30}x + 2b_{20}y}{2a_{30}b_{00}b_{20}} - \dfrac{b_{20}}{3a_{30}}$, $t = \dfrac{\tau}{a_{30}}$ and using the notation $b_{10} = ab_{00}$, we obtain the canonical form of the cubic differential system with invariant straight lines of total algebraic multiplicity equal to seven including the straight line at infinity:

$$\begin{cases} \dot{x} = x^3, \mathrm{a} \geq 0, \\ \dot{y} = 1 + ax + 3x^2y, \end{cases} \qquad \textbf{(s1)}$$

where $a \geq 0$, as the transformation $x \to -x, a \to -a$ doesn't change the system **(s1)**.

1.b) (1(4)+2) There are 11 systems with the invariant straight line $x=0$ of total multiplicity equal to 4, but only one of them can have the invariant straight lines $x+1=0$ and $x-a=0$. This system can be brought to the form:

$$\begin{cases} \dot{x} = x(x+1)(x-a), a \geq 1, \\ \dot{y} = -ay + (1-a)xy + x^3 + x^2y. \end{cases} \qquad \textbf{(s2)}$$

1.c) (1(4)+1(2)) We have established that 11 systems have the invariant straight line $x=0$ with total multiplicity equal to 4. Asking that the straight line $x+1=0$ to be invariant with multiplicity equal to 2, we obtain only one system, which can be brought to the form:

$$\begin{cases} \dot{x} = x^2(x+1), a \in \mathbf{R}, \\ \dot{y} = 1 + ax^2 + 2xy + 3x^2y. \end{cases} \qquad \textbf{(s3)}$$

1.d) (1(3)+1(3)) There are 8 systems that have the invariant straight line $x=0$ with algebraic multiplicity equal to 3. Asking that the straight line $x+1=0$ to be invariant with multiplicity equal to 3, we obtain only one system, which can be brought to the form:

$$\begin{cases} \dot{x} = x(x+1)^2, a \in \mathbf{R} \setminus \{1\}, \\ \dot{y} = y + ax^2 + 2xy + x^3 + x^2y. \end{cases} \qquad \textbf{(s4)}$$

1.e) (1(3)+1(2)+1) There are 8 systems that have the invariant straight line $x=0$ with multiplicity equal to 3. Asking that the straight line $x+1=0$ to be invariant with multiplicity equal to 2 and the straight line $x-a=0$ to be invariant, we obtain only one system, which can be brought to the form:

$$\begin{cases} \dot{x}=x(x+1)(x-a), a\in\mathbf{R}\setminus\{-1;0\}, \\ \dot{y}=-ay+(a+1)x^2+(1-a)xy+(a+1)x^2y. \end{cases} \quad \textbf{(s5)}$$

1.f) (1(2)+1(2)+1(2)) There are 4 systems that have the invariant straight line $x=0$ with multiplicity equal to 2. Asking that the straight lines $x+1=0$ and $x-a=0$ to be invariant with multiplicity equal to 2, we obtain only one differential system that can be brought to the form:

$$\begin{cases} \dot{x}=x(x+1)(x-a), a>1, b\in\mathbf{R}, \\ \dot{y}=bx-ay+x^2+2(1-a)xy+3x^2y. \end{cases} \quad \textbf{(s6)}$$

Case 2: The straight line at infinity has multiplicity equal to 2. Asking that the invariant straight line at infinity of the system (4) to have multiplicity equal to two, we obtain 5 sets of conditions, i.e. there are 5 cubic differential systems that satisfy this condition. For the total algebraic multiplicity to be equal to 7, we must search for real planar invariant straight lines with total multiplicity equal to 5. The planar invariant straight lines can have the following configurations: **a)** (1(5)); **b)** (1(4)+1); **c)** (1(3)+1(2)); **d)** (1(3)+1+1); **e)** (1(2)+1(2)+1).

2.a) (1(5)) For the five differential systems which have the invariant straight line at infinity with multiplicity equal to two we require that the invariant straight line $x=0$ to be invariant with multiplicity five. As a result, we obtain the following system:

$$\begin{cases} \dot{x}=x^3, \mathrm{a}\neq 0, \\ \dot{y}=a+x-x^2. \end{cases} \quad \textbf{(s7)}$$

2.b) (1(4)+1) In this case we obtain two differential systems:

$$\begin{cases} \dot{x}=x(x+1), \\ \dot{y}=y+xy+x^3. \end{cases} \quad \textbf{(s8)} \qquad \begin{cases} \dot{x}=x^2(x+1), \mathrm{a}\neq 0, \\ \dot{y}=a+x^2+2xy. \end{cases} \quad \textbf{(s9)}$$

2.c) (1(3)+1(2)) In this case we obtain three differential systems:

$$\begin{cases} \dot{x}=x(x+1), \\ \dot{y}=y+ax^2+xy-x^2y. \end{cases} \textbf{(s10)} \quad \begin{cases} \dot{x}=x^2(x+1), \mathrm{a}\in\mathbf{R}, \\ \dot{y}=1+ax^2-xy. \end{cases} \textbf{(s11)} \quad \begin{cases} \dot{x}=x(x+1)^2, \mathrm{a}\in\mathbf{R}, \\ \dot{y}=y+ax^2+2xy. \end{cases} \textbf{(s12)}$$

2.d) (1(3)+1+1) This configuration corresponds to the system

$$\begin{cases} \dot{x}=x(x+1)(a-x), |a|>1, \\ \dot{y}=ay+x^2+(a-1)xy. \end{cases} \quad \textbf{(s13)}$$

2.e) (1(2)+1(2)+1) In this case we obtain the system

$$\begin{cases} \dot{x}=x(x+1)(a-x), \mathrm{a}\in(-1;+\infty)\setminus\{0\}, b\in\mathbf{R}, \\ \dot{y}=bx+ay+x^2+(1+2a)xy. \end{cases} \quad \textbf{(s14)}$$

Case 3: The straight line at the infinity has multiplicity equal to 3. There are 10 cubic differential systems that have the invariant straight line at infinity with the multiplicity equal to 3. It follows that the real invariant straight lines must have total algebraic multiplicity equal with four, therefore we can have the following configurations:

a) (1(4)); **b)** (1(3)+1); **c)** (1(2)+1(2)); **d)** (1(2)+1+1).

3.a) (1(4)) Asking the straight line $x=0$ to be invariant with algebraic multiplicity equal to four, we establish that only one of those 10 systems satisfies this condition and it can be brought to the form

$$\begin{cases} \dot{x}=x^2, a\neq 0, \\ \dot{y}=a+x^2+2xy+x^3. \end{cases} \quad \textbf{(s15)}$$

3.b) (1(3)+1) In this case we obtain the system

$$\begin{cases} \dot{x}=x(x+1), a\in \mathbf{R}, \\ \dot{y}=y+ax^2+xy+x^3. \end{cases} \quad \textbf{(s16)}$$

3.c) (1(2)+1(2)) By solving the remaining system of algebraic equations, we will obtain several sets of condition. By performing affine transformations and time rescaling, we can bring the obtained systems to one of the following two canonical forms:

$$\begin{cases} \dot{x}=x(x+1), a\in \mathbf{R}, \\ \dot{y}=ax+y+2xy+x^3. \end{cases} \quad \textbf{(s17)} \qquad \begin{cases} \dot{x}=x(x+1)^2, \\ \dot{y}=x+y. \end{cases} \quad \textbf{(s18)}$$

3.d) (1(2)+1+1) In this case we obtain the following canonical form:

$$\begin{cases} \dot{x}=x(x+1)(a-x), \\ \dot{y}=x+ay, a>1. \end{cases} \quad \textbf{(s19)}$$

Case 4: The straight line at infinity has multiplicity equal to 4. By asking the invariant straight line at infinity of the system (4) to have multiplicity equal to four, we obtain 13 cubic differential systems. Therefore, the real planar invariant straight lines must have total multiplicity equal to three, so they can have the following configurations:

a) (1(3)); **b)** (1(2)+1); **c)** (1+1+1).

4.a) (1(3)) By asking the straight line $x=0$ to be invariant with multiplicity equal to three, we obtain the following two systems:

$$\begin{cases} \dot{x}=x^2, a>0, \\ \dot{y}=a-xy+x^3. \end{cases} \quad \textbf{(s20)} \qquad \begin{cases} \dot{x}=x, \\ \dot{y}=y+x^2+x^3. \end{cases} \quad \textbf{(s21)}$$

4.b) (1(2)+1) If the straight line $x=0$ is invariant with multiplicity equal to two and the straight line $x+1=0$ is invariant, then only one system from those 13 systems satisfies these conditions, and he can be brought it to the following form:

$$\begin{cases} \dot{x}=x(x+1), a\in \mathbf{R}, \\ \dot{y}=ax+y-xy+x^3. \end{cases} \quad \textbf{(s22)}$$

4.c) (1+1+1) In this case, for each one of these 13 systems we determine three invariant straight lines of the form $x=0$, $x+1=0$ and $x-a=0, a>1$. Only one system satisfy these conditions and it can be brought to the form:

$$\begin{cases} \dot{x} = x(x+1)(a-x), \\ \dot{y} = 1, a>1. \end{cases} \quad \textbf{(s23)}$$

Case 5: The straight line at infinity has multiplicity equal to 5. In this case we have seven cubic differential systems. It follows that the real planar invariant straight lines must have total algebraic multiplicity equal to two, therefore they can have one of the following two configurations: **a)** (1(2)); **b)** (1+1).

5.a) (1(2)) For these 7 systems, we obtain that only one system can be brought to the canonical form:

$$\begin{cases} \dot{x} = x, \mathrm{a} \in \mathbf{R}, \\ \dot{y} = ax + y + x^2 + x^3. \end{cases} \quad \textbf{(s24)}$$

5.b) (1+1) By asking the straight lines $x=0$ and $x+1=0$ to be invariant for the cubic differential systems with an invariant straight line at infinity which have algebraic multiplicity equal to five, we obtain that there are no parameter values satisfying these conditions. Therefore, there are no cubic differential systems of such configuration.

Case 6: The straight line at infinity has multiplicity equal to 6. By asking the invariant straight line at infinity of the system (4) to have multiplicity equal to six, we obtain three cubic differential systems. We can have only one real planar invariant straight line. Therefore, for each of these systems we condition the straight line $x=0$ to be invariant. Thus, we obtain a single system, which can be brought to the following form:

$$\begin{cases} \dot{x} = x, \\ \dot{y} = -2y + x^2 + x^3. \end{cases} \quad \textbf{(s25)}$$

Case 7: The straight line at infinity has multiplicity equal to 7. By asking the invariant straight line at infinity of the system (4) to have multiplicity equal to seven, we get the system:

$$\begin{cases} \dot{x} = 1, \mathrm{a} \in \mathbf{R}, \\ \dot{y} = x(a + x^2). \end{cases} \quad \textbf{(s26)}$$

According to the above obtained results, we have proved the following theorem:

Theorem. Any cubic differential system with real invariant straight lines along one direction with total algebraic multiplicity equal to seven, including the invariant straight line at the infinity, by an affine transformation and time rescaling can be brought to one of the systems **(s1)** – (**s26)**.

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THE CARTESIAN PRODUCT OF TWO SUBCATEGORIES

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Abstract. We examine a categorial construction which permits to obtained a new reflective subcategorie with a special properties.

Key words: Reflective subcategories, pairs of conjugated subcategories, right product of the two subcategories.
2010 Mathematics Subject Classification: 18B30, 18B40

PRODUSUL CARTEZIAN A DOUĂ SUBCATEGORII

Rezumat. Se examinează o construcţie categorială care permite de a obţine noi subcategorii reflective cu anumite proprietăţi.

Cuvinte-cheie: Subcategorii reflective, perechi de subcategorii conjugate, produsul de dreapta a două subcategorii.

Let $\mathcal{K}$ be a coreflective subcategory, and $\mathcal{R}$ a reflective subcategory of the category of locally convex topological vector Hausdorff spaces $\mathcal{C}_2\mathcal{V}$ with respective functors $k : \mathcal{C}_2\mathcal{V} \longrightarrow \mathcal{K}$ and $r : \mathcal{C}_2\mathcal{V} \longrightarrow \mathcal{R}$.

Concerning the terminology and notation see [1]. Note by $\mu\mathcal{K} = \{m \in \mathcal{M}ono \mid k(m) \in \mathcal{I}so\}$, $\varepsilon\mathcal{R} = \{e \in \mathcal{E}pi \mid r(e) \in \mathcal{I}so\}$. Further for an arbitrary object X of the category $\mathcal{C}_2\mathcal{V}$ we examine the following construction: let $k^X : kX \longrightarrow X$ is $\mathcal{K}$-coreplique, and $r^{kX} : kX \longrightarrow rkX$-replique of the respective objects. On the morphisms k^X and r^{kX} we construct the cocartesian square

$$\overline{v}^X \cdot k^X = u^X \cdot r^{kX}. \tag{1}$$

Definition 1. *1. The full subcategory of all isomorphic objects with the type of objects is called $\overline{v}X$ cartesian product of the subcategories $\mathcal{K}$ and $\mathcal{R}$, noted by $\overline{v} = \mathcal{K} *_{dc} \mathcal{R}$.*
2.The diagram of cartesian product is called the diagram of cartesian product of the pair of conjugate subcategories $(\mathcal{K}, \mathcal{R})$ (Diagram (RCP)).

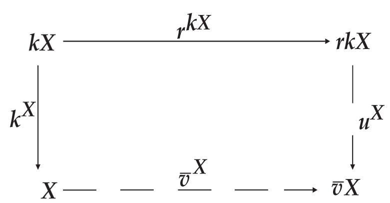


Diagram (RCP)

Reciprocally. Let $\mathcal{R}$ be a reflective subcategory, and $\mathcal{K}$ be a coreflective subcategory of the category $\mathcal{C}_2\mathcal{V}$. Let X be an object of the category $\mathcal{C}_2\mathcal{V}$, $r^X : X \longrightarrow rX$-$\mathcal{R}$-replique and $k^{rX} : krX \longrightarrow rX$ be $\mathcal{K}$-coreplique of the respective objects. On the morphisms r^X and k^{rX} we construct the cartesian product

$$r^X \cdot \overline{w}^X = k^{rX} \cdot t^X \tag{2}$$

Definition 2. *1. The full subcategory of all isomorphic objects with the objects of type $\overline{v}X$ is called cartesian product of the subcategories $\mathcal{K}$ and $\mathcal{R}$, noted $\overline{\mathcal{W}} = \mathcal{K} *_{sc} \mathcal{R}$.*

2. The diagram of the cartesian square (2) is called the diagram of the left cartesian product of the pair of conjugate subcategories $(\mathcal{K}, \mathcal{R})$ *(Diagram (LCP)).*

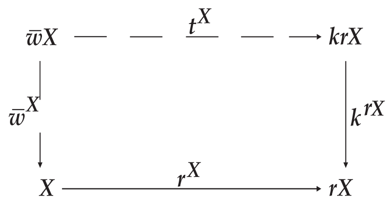
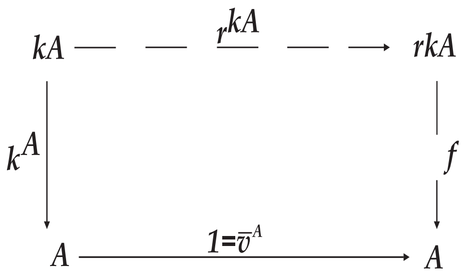


Diagram (LCP)

Lemma 1. $\mathcal{R} \subset \mathcal{K} *_{dc} \mathcal{R}$.

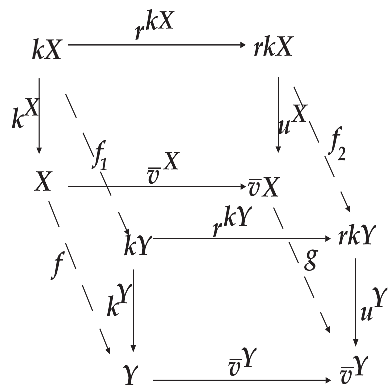
Proof. Let $\mid \mathcal{A} \mid \in \mid \mathcal{R} \mid$ and $k^A : kA \longrightarrow A$ be $\mathcal{K}$-coreplique, $r^{kA} : kA \longrightarrow rkA$, $\mathcal{R}$-replique of the respective objects. Then $k^A = f \cdot r^{kA}$ for an morphism f. It is obvious that $f \cdot r^{kA} = 1 \cdot k^A$ is cocartesian square construct on the morphisms k^A and r^{kA}. So $\overline{v}^A = 1$.



Theorem 1. *The application* $X \mapsto \overline{v}X$ *define a functor*

$$\overline{v} : \mathcal{C}_2\mathcal{V} \longrightarrow \mathcal{K} *_{dc} \mathcal{R}.$$

Proof. We define the functor $\overline{v}$ on the morphism. Let $f : X \longrightarrow Y \in \mathcal{C}_2\mathcal{V}$. We examine the diagram (RCP) constructed for the objects X and Y.



For the morphism $f \cdot k^X$ exists one single morphism $f_1 : kX \longrightarrow kY$ so that

$$f \cdot k^X = k^Y \cdot f_1. \tag{3}$$

The same for the morphism $r^{kY} \cdot f_1$ exists one single morphism $f_2 : rkX \longrightarrow rkY$. It follows that

$$r^{kY} \cdot f_1 = f_2 \cdot r^{kX}. \tag{4}$$

Then we have

$$\overline{v}^Y \cdot f \cdot k^X = (from3) = \overline{v}^Y \cdot k^Y \cdot f_1 = (from1) = u^Y \cdot r^{kY} \cdot f_1 = (from4) = u^Y \cdot f_2 \cdot r^{kX}$$

or

$$(\overline{v}^Y \cdot f) \cdot k^X = (u^Y \cdot f_2) \cdot r^{kX}. \tag{5}$$

From equality (5) concerning that (3) is cocartesian square, it results the existence of a single morphism g, such that

$$\overline{v}^Y \cdot f = g \cdot \overline{v}^X, \tag{6}$$

$$u^Y \cdot f_2 = g \cdot u^X. \tag{7}$$

Define $g = t(f)$. In equality (6) $\overline{v}^Y$ is an epimorphism. Thus, we deduce that the morphism g verifing equality (6), is unique. And here we come out with the result $\overline{v}(1) = 1$ and $\overline{v}(f \cdot h) = \overline{v}(f) \cdot \overline{v}(h)$.

Concerning the functor $\overline{v} : \mathcal{C}_2\mathcal{V} \longrightarrow \mathcal{K} *_{dc} \mathcal{R}$ appears the following problem: When $\overline{v}$ is a reflector functor?

We examine the following condition:

(RCP) For any object X of the category $\mathcal{C}_2\mathcal{V}$ in the diagram (RCP) the morphism u^X belongs to the class $\mu\mathcal{K}$.

Theorem 2. *Let it be a pairs of the subcategories* $(\mathcal{K}, \mathcal{R})$ *verify the condition (RCP). Then* $\overline{v}$ *it is a reflector functor.*

Proof. We examine the diagram (RCP) constructed for objects X and Y of the category $\mathcal{C}_2\mathcal{V}$. Let $f : X \longrightarrow \overline{v}Y$. Since $u^Y \in \mu\mathcal{K}$, it follows that

$$f \cdot k^X = u^Y \cdot g \tag{8}$$

for a morphism g. Further, r^{kX} is $\mathcal{R}$-replique of object kX. So

$$g = h \cdot r^{kX} \tag{9}$$

for a morphism h. We have

$$f \cdot k^X = (from8) = u^Y \cdot g = (from9) = u^Y \cdot h \cdot r^{kX}$$

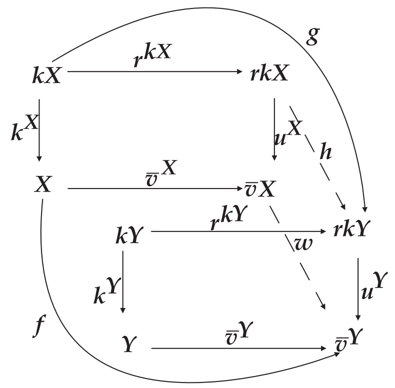
or

$$f \cdot k^X = (u^Y \cdot h) \cdot r^{kX}. \tag{10}$$

I mean that square (1) is cocartesian, we deduce that:

$$f = w \cdot \overline{v}^X, \tag{11}$$

$$u^Y \cdot h = w \cdot u^X. \tag{12}$$



So morphism f extends through morphism $\overline{v}^X$. The uniqueness of this extension results from the fact that $\overline{v}^X$ it is like r^{kX} an epimorphism.

Theorem 3. *Let $\mathcal{K}$ be a coreflective subcategory, but $\mathcal{R}$ is a reflective subcategory of the category $\mathcal{C}_2\mathcal{V}$, $\widetilde{\mathcal{M}}$ - the subcategory of the spaces with Mackey topology, $\mathcal{S}$ is the subcategory of the spaces with weak topology. If $\mathcal{K} \subset \widetilde{\mathcal{M}}$, but $\mathcal{S} \subset \mathcal{R}$, then the pair of subcategories $(\mathcal{K}, \mathcal{R})$ verify condition (RCP) the cartesian product is a reflective subcategory.*

Proof. Since $\mathcal{S} \subset \mathcal{R}$, it follows that $\varepsilon\mathcal{R} \subset \varepsilon\mathcal{S} = \mathcal{E}_u \cap \mathcal{M}_u = \mu\widetilde{\mathcal{M}} \subset \mu\mathcal{K}$. We examine the diagram (RCP) for an arbitrary object X of the category $\mathcal{C}_2\mathcal{V}$. We have $r^{kX} \in \varepsilon\mathcal{R}$. So, and $\overline{v}^X \in \varepsilon\mathcal{R}$. Thus $\overline{v}^X$, $k^X \in \mu\mathcal{K}$. On the other hand $\overline{v}^X \cdot k^X \in \mu\mathcal{K}$. In equality

$$\overline{v}^X \cdot k^X = u^X \cdot r^{kX},$$

where r^{kX}, k^X, $\overline{v}^X$ are bijective application. In other words u^X is a bijective application. Thus $u^X \in \mu\mathcal{K}$.

Example. **1.** *For any coreflective subcategory $\mathcal{K}$ we have $\mathcal{K} *_{dc} \Pi = \Pi$, Π-reflective subcategory of the complete space with weak topology.*

2. *For any coreflective subcategory $\mathcal{K}$ we have $\mathcal{K} *_{dc} \mathcal{S} = \mathcal{S}$, $\mathcal{S}$-reflective subcategory of the space with weak topology.*

Proof. We construct the (RCP) diagram for an arbitrary object X of the category $\mathcal{C}_2\mathcal{V}$ in relation to the pair of subcategories $(\mathcal{K}, \Pi)$. We represent the reflector functor $\pi : \mathcal{C}_2\mathcal{V} \longrightarrow \Pi$ as a composition

$$\pi = g_0 \cdot s.$$

So either $s^{kX} : kX \longrightarrow skX$ $\mathcal{S}$-replique of the object kX, but $g_0^{skX} : skX \longrightarrow g_0skX$ is a Γ_0-replique of the object skX, where Γ_0 is subcategory of the complete space.

Thus $g_0^{skX} \cdot s^{kX}$is a replique of the object kX. We construct the cocartesian square on the morphism k^X and s^{kX}:

$$u_1^X \cdot s^{kX} = \overline{v}_1^X,$$

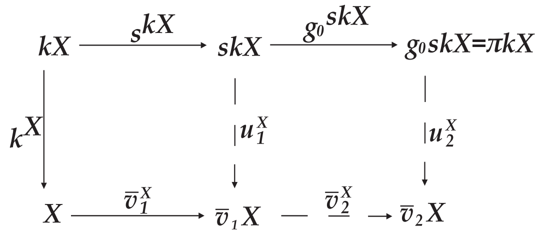
on the morphisms u_1^X and g_0^{skX}:

$$\overline{v}_2^X \cdot u_1^X = u_2^X \cdot g_0^{skX}.$$

Then

$$(\overline{v}_{1^X} \cdot \overline{v}_2^X) \cdot k^X = u_2^X \cdot (g_0^{skX} \cdot s^{kX})$$

is a cocartesian square construct on morphisms k^X and $g_0^{skX} \cdot s^{kX}$ or morphisms k^X and π^{kX}.



Since k^X is an epimorphism it results as well u_1^X and u_2^X are epimorphisms. Therefore u_2^X is retractable, but $\overline{v}_2^X \in\mid \Pi \mid$. Further $g_0^{skX} \in \varepsilon\Gamma_0$. So $\overline{v}_2^X \in \varepsilon\Gamma_0$, but $\overline{v}_1^X \in\mid \mathcal{S} \mid$. Thus we have proved that $\mathcal{K} *_{dc} \Pi = \Pi$ and $\mathcal{K} *_{dc} \mathcal{S} = \mathcal{S}$.

Return to previous diagram. If for any object $X \in\mid \mathcal{C}_2\mathcal{V} \mid$ we have $u_2^X \in \mu\mathcal{K}$, then u_2^X is an isomorphism, and from equality

$$g_0^{skX} \cdot s^{kX} = (u_2^X)^{-1} \cdot \overline{v}_2^X \cdot \overline{v}_1^X \cdot k^X$$

it results that $k^X \in \mathcal{M}_u$, and $\widetilde{\mathcal{M}} \subset \mathcal{K}$.

Remark. 1. May it be $\widetilde{\mathcal{M}} \notin \mathcal{K}$. Then the pair $(\mathcal{K}, \Pi)$ do not check the condition (RCP), but $\mathcal{K} *_{dc} \Pi = \Pi$. So the condition (RCP) is sufficient, but not necessary that the respective product is a reflective subcategory.

2. Lemma 1 indicates inclusion $\mathcal{R} \subset \mathcal{K} *_{dc} \mathcal{R}$, and the preceding examples indicate the equality of these subcategories.

Definition 3 (see [1]). *Let $\mathcal{K}$ a coreflective subcategory and $\mathcal{L}$ a reflective subcategory of the category $\mathcal{C}_2\mathcal{V}$ with those functors $k : \mathcal{C}_2\mathcal{V} \longrightarrow \mathcal{K}$ and $l : \mathcal{C}_2\mathcal{V} \longrightarrow \mathcal{L}$. Pair $(\mathcal{K}, \mathcal{L})$ is called a pair of conjugate subcategories if*

$$\mu\mathcal{K} = \varepsilon\mathcal{L}.$$

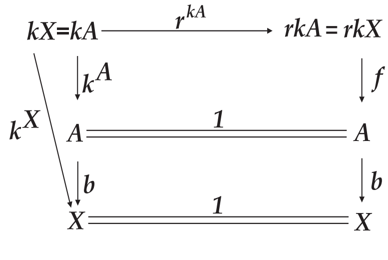
Theorem 4. *Let $(\mathcal{K}, \mathcal{L})$ a pair of conjugate subcategories, and $\mathcal{R}$ a reflective subcategory of the category $\mathcal{C}_2\mathcal{V}$. Then:*

1. *$\mathcal{K} *_{dc} \mathcal{R} = \mathcal{Q}_{\varepsilon\mathcal{L}}(\mathcal{R})$, where $\mathcal{Q}_{\varepsilon\mathcal{L}}(\mathcal{R})$ is the full subcategory of all $\varepsilon\mathcal{L}$-factorobjects of objects of the subcategory $\mathcal{R}$.*
2. *$\mathcal{K} *_{dc} \mathcal{R}$ is a reflective subcategory of the category $\mathcal{C}_2\mathcal{V}$.*
3. *The subcategory $\mathcal{K} *_{dc} \mathcal{R}$ is closed in relation to $\varepsilon\mathcal{L}$-factorobjects.*
4. *$\overline{v} \cdot k = r \cdot k$.*
5. *If $r(\mathcal{K}) \subset \mathcal{K}$, then the coreflector functor $k : \mathcal{C}_2\mathcal{V} \longrightarrow \mathcal{K}$ and the reflector $\overline{v} : \mathcal{C}_2\mathcal{V} \longrightarrow \mathcal{K} *_{dc} \mathcal{R}$ commute: $k \cdot \overline{v} = \overline{v} \cdot k$.*

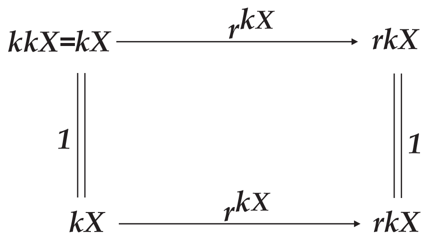
Proof. 1. In the (RCP) diagram $k^X \in \mu\mathcal{K} = \varepsilon\mathcal{L}$. So $k^X \in \varepsilon\mathcal{L} = \mu\mathcal{K}$. Thus $\mathcal{K} *_{dc} \mathcal{R} \subset \mathcal{Q}_{\varepsilon\mathcal{L}}(\mathcal{R})$.
Reciprocally: Let $b : A \longrightarrow X \in \varepsilon\mathcal{L}$ and $A \in\mid \mathcal{R} \mid$. Then $b \cdot k^A : kA \longrightarrow X$ is $\mathcal{K}$-coreplique of the object X, and

$$k^A = f \cdot r^{kA} \tag{13}$$

is an cocartesian square construct on the morphisms k^X and r^{kX}. So $X \in\mid \mathcal{K} *_{dc} \mathcal{R} \mid$.



2. Result from 1. and the Theorem 2.
3. Result from 1.
4. For an object of form kX, diagram (RCP) is the next one



Thus $\overline{v}kX = rkX$.

5. Examine the diagram (RCP) construct for an arbitrary object X of the category $\mathcal{C}_2\mathcal{V}$. Then k^{rX} it is also $\mathcal{V}$-replique of the object kX. Further, $u^X \in \mu\mathcal{K}$ and $rkX \in \mid \mathcal{K} \mid$, according to the hypothesis $r(\mathcal{K}) \subset \mathcal{K}$. So

$$k\overline{v}X = rkX = \overline{vkX}$$

or

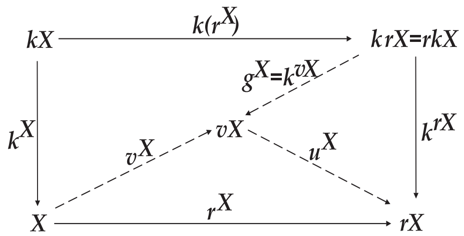
$$k \cdot \overline{v} = \overline{v} \cdot k.$$

In the paper [2] was introduced the right product of the product $\mathcal{K}*_d\mathcal{R}$ of the coreflective subcategory $\mathcal{K}$ and of the reflective subcategory $\mathcal{R}$, the properties of this product have been examined and examples have been construct.

Theorem 5. *Let $\mathcal{K}$ (respective $\mathcal{R}$) a coreflective subcategory (respective: reflective)of the category $\mathcal{C}_2\mathcal{V}$, those functors $k : \mathcal{C}_2\mathcal{V} \longrightarrow \mathcal{K}$ and $r : \mathcal{C}_2\mathcal{V} \longrightarrow \mathcal{R}$ commute: $k \cdot r = r \cdot k$. Then*

$$\mathcal{K} *_{dc} \mathcal{R} = \mathcal{K} *_d \mathcal{R}.$$

Proof. Let's examine the diagram of the right product constructed for an arbitrary object X of the category $\mathcal{C}_2\mathcal{V}$ in relation to the subcategories $\mathcal{K}$ and $\mathcal{L}$.



Because functors k and r commute, be sure to verify that $k(r^X) = r^{kX}$. Thus, the right product is obtained by constructing the cocartesian square on morphisms k^X and $k(r^X)$, and the right cocartesian product is obtained by constructing the cocartesian square on morphisms k^X and r^{kX}. So these products coincide.

References

1. Botnaru D. Structures bicatégorielles complémentaires. ROMAI Journal, v.5, N.2, 2009, p. 5-27.
2. Botnaru D., Turcanu A. The factorization of the right product of two subcategories. ROMAI Journal, v.6, N.2, 2010, p. 41-53.