ACTA ET COMMENTATIONES

Științe Exacte și ale Naturii

REVISTĂ ŞTIINŢIFICĂ

Nr.2 (14), 2022

SCIENTIA VINCET

ISSN 2537 - 6284
E-ISSN 2587-3644

ISSN 2537-6284
E-ISSN 2587-3644
Tip B

ACTA
ET
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REVISTĂ ŞTIINŢIFICĂ
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DOI: https://doi.org/10.36120/2587-3644.v14i2

Chişinău 2022

Fondator: UNIVERSITATEA DE STAT DIN TIRASPOL, CHIȘINĂU, REPUBLICA MOLDOVA

Redactor-şef: Dumitru COZMA, profesor universitar, doctor habilitat, Republica Moldova

Adresa redacției: str. Gh. Iablocikin 5, Mun. Chişinău, MD2069, Republica Moldova, tel (373) 22 754924, (373) 22 240084
Adresa web: revista.ust.md
e-mail: aec@ust.md

Tiparul: Tipografia Universităţii de Stat din Tiraspol, 100 ex.
ISSN 2537-6284
E-ISSN 2587-3644
Tip B

ISSN 2537-6284
E-ISSN 2587-3644
Type B

ACTA

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Exact and Natural Sciences

SCIENTIFIC JOURNAL
Nr. 2(14), 2022

DOI: https://doi.org/10.36120/2587-3644.v14i2

Chisinau 2022

Founder: TIRASPOL STATE UNIVERSITY, CHISINAU, REPUBLIC OF MOLDOVA

Editor-in-chief: Dumitru COZMA, professor, doctor habilitatus, Republic of Moldova

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tel (373) 22 754924, (373) 22 240084
Web address: revista.ust.md
e-mail: aec@ust.md

Printing: Typography of Tiraspol State University, 100 copies **ISSN 2537-6284**

E-ISSN 2587-3644
Type B

Cuprins

Table of Contents

Acta et Commentationes, Exact and Natural Sciences ISSN: 2537-6284
Volume 2(14), 2022, Pages 7–14 E-ISSN: 2587-3644
CZU 517.929 DOI: 10.36120/2587-3644.v14i2.7-14

Approximation of solutions of boundary value problems for integro-differential equations of the neutral type using a spline function method

Ihor Cherevko, Andrii Dorosh, Ivan Haiuk, and Andrii Pertsov

Abstract. Boundary value problems for nonlinear integro-differential equations of the neutral type are investigated. A scheme for approximating the boundary value problem solution using cubic splines of defect two is proposed and substantiated. A model example illustrating the proposed approximation scheme is considered.
2010 Mathematics Subject Classification: Primary 34K10; Secondary 34K28.
Keywords: boundary value problem, neutral type, cubic splines.

Aproximarea soluțiilor problemelor cu valori la limită pentru ecuațiile integro-diferențiale de tip neutru folosind metoda funcției spline

Rezumat. În lucrare sunt cercetate probleme cu valori la limită pentru ecuațiile integro-diferențiale neliniare de tip neutru. Este propusă și fundamentată o schemă de aproximare a soluției problemei cu valori la limită folosind spline cubice ale defectului doi. Se consideră un exemplu model care ilustrează schema de aproximare propusă.
Cuvinte-cheie: problemă cu valoarea la limită, tip neutru, spline cubice.

In mathematical modeling of physical and technical processes, the evolution of which depends on prehistory, we arrive at differential equations with a delay. With the help of such equations it was possible to identify and describe new effects and phenomena in physics, biology, technology [1].

Boundary value problems for integro-differential equations with a deviating argument are mathematical models of various applied processes in biology, immunology, and medicine. In particular, Volterra integro-differential equations with a delay play an important role in modeling many real phenomena in ecology [2]. An important task in their study is to establish convenient conditions that guarantee the existence of solutions of such problems [2, 3]. Finding solutions to boundary value problems with a time delay in analytical form is possible only in the simplest cases, so the real task is to develop efficient methods for finding their approximate solutions [4]. The application of the spline function method to the approximation of boundary value problems for integro-differential equations has been studied in [5]. Approximation schemes for differential-difference equations using systems of ordinary differential equations have been considered in [6, 7]. The aim of this work is to extend the approximation schemes using cubic splines of defect two for boundary value problems for integro-differential equations with many delays [8].

1. Problem statement. Solution existence

Let us denote

$$\begin{aligned} [y(x)] &= \Big(y\big(x-\tau_0(x)\big),\ldots,y\big(x-\tau_n(x)\big)\Big),\\ [y(x)]_1 &= \Big(y'\big(x-\tau_0(x)\big),\ldots,y'\big(x-\tau_n(x)\big)\Big),\\ [y(x)]_2 &= \Big(y''\big(x-\tau_0(x)\big),\ldots,y''\big(x-\tau_n(x)\big)\Big). \end{aligned} \tag{1}$$

Consider a boundary value problem

$$y''(x) = f\big(x,[y(x)],[y(x)]_1,[y(x)]_2\big)+ \tag{2}$$

$$+\int_a^b g\big(x,s,[y(s)],[y(s)]_1,[y(s)]_2\big)ds,\ x\in[a;b],$$

$$y^{(p)}(x)=\varphi^{(p)}(x),\ p=0,1,2,\ x\in[a^*;a],\ y(b)=\gamma, \tag{3}$$

where $\tau_0(x)=0$, $\tau_i(x)$, $i=\overline{1,n}$ are continuous nonnegative functions defined on $[a,b]$, $\varphi(x)$ is a twice continuously differentiable function on $[a^*;a]$, $\gamma\in R$,

$$a^*=\min_{0<i\le n}\left\{\inf_{x\in[a;b]}\left(x-\tau_i(x)\right)\right\}.$$

Let us introduce the sets of points defined by the delays $\tau_1(x),\ldots,\tau_n(x)$:

$$E_{i1}=\left\{x_j\in[a,b]:\ x_j-\tau_i\left(x_j\right)=a,\ j=1,2,\ldots\right\},$$

$$E_{i2}=\left\{x_j\in[a,b]:\ x_0=a,\ x_{j+1}-\tau_i\left(x_{j+1}\right)=x_j,\ j=0,1,2,\ldots\right\},$$

$$E_2=\bigcup_{i=1}^n\big(E_{i1}\cup E_{i2}\big).$$

Assume that the delays $\tau_i(x)$, $i=\overline{1,n}$ are such functions that the sets E_{i1},E_{i2}, $i=\overline{1,n}$ are finite. We will number the points of the set E_2 in ascending order.

Let us introduce the notation:

$$P = \sup\Bigg\{\Big|f\big(x, [y(x)], [y(x)]_1, [y(x)]_2\big)\Big| + \Bigg|\int\limits_a^b g\big(x, s, [y(s)], [y(s)]_1, [y(s)]_2\big)ds\Bigg| :$$

$$\Big|y\big(x - \tau_i(x)\big)\Big| \le P_1,\ \Big|y'\big(x - \tau_i(x)\big)\Big| \le P_2,$$

$$\Big|y''\big(x - \tau_i(x)\big)\Big| \le P_3,\ i = \overline{0,n},\ x, s \in [a;b]\Bigg\},$$

$$J = [a^*; a]\,,\ I = [a, b]\,,$$

$$I_1 = [a, x_1]\,,\ I_2 = [x_1, x_2]\,,\ \ldots,\ I_k = [x_{k-1}, x_k]\,,\ I_{k+1} = [x_k, b]\,,$$

$$B_2\left(J \cup I\right) = \Bigg\{y\left(x\right) : y\left(x\right) \in \Bigg(C(J \cup I) \cap \Big(C^1(J) \cup C^1(I)\Big) \cap$$

$$\cap\Bigg(\bigcup_{j=1}^{k+1} C^2\left(I_j\right)\Bigg)\Bigg),\ |y(x)| \le P_1,\ |y'(x)| \le P_2,\ |y''(x)| \le P_3\Bigg\},$$

where P_1, P_2, P_3 are positive constants.

A function $y = y\left(x\right)$ will be considered a solution of the boundary value problem (2)-(3) if it satisfies the equation (2) on $[a; b]$ (with the possible exception of the points of the set E_2) and the boundary conditions (3). We will find a solution of the problem (2)-(3) which belongs to the space $B_2(J \cup I)$.

The definition of the space $B_2(J \cup I)$ implies that the solution of (2)-(3) is continuously differentiable for any $x \in [a, b]$ where $y'\left(a\right)$ is the right derivative, and at points of E_2 there exist finite one-sided second derivatives of the solution which may not coincide.

Let us introduce a norm in the space $B_2(J \cup I)$:

$$\|y\|_{B_2} = \max\Bigg\{\frac{8}{(b-a)^2}\max_{x \in J \cup I}|y(x)|,\ \frac{2}{b-a}\max\Big(\max_{x \in J}|y'(x)|,\ \max_{x \in I}|y'(x)|\Big),$$

$$\max\Big(\max_{x \in J}|y''(x)|,\ \max_{x \in I_1}|y''(x)|,\ \ldots,\ \max_{x \in I_{k+1}}|y''(x)|\Big)\Bigg\}.$$

The space $B_2(J \cup I)$ with this norm is a Banach space.

The boundary value problem (2)-(3) is equivalent to the integral equation [9]

$$y\left(x\right) = \int\limits_{a^*}^{b}\Bigg[f\big(s, [y(s)], [y(s)]_1, [y(s)]_2\big) + \int\limits_a^b g\big(s, \xi, [y(\xi)], [y(\xi)]_1, [y(\xi)]_2\big)d\xi\Bigg]\times$$

$$\times\overline{G}\left(x, s\right) ds + l\left(x\right),\ x \in J \cup I, \tag{4}$$

$$\overline{G}(x,s)=\begin{cases}G(x,s), & x,s\in I,\\ 0, & \text{otherwise,}\end{cases}$$

$$l(x)=\begin{cases}\varphi(x), & x\in J,\\ \frac{\gamma-\varphi(a)}{b-a}(x-a)+\varphi(a), & x\in I,\end{cases}$$

where $G(x,s)$ is the Green's function of the boundary value problem

$$y''(x)=0,\ x\in I,\ y(a)=y(b)=0.$$

We define the operator T in the space $B_2(J\cup I)$ as follows

$$(Ty)(x)=\int_{a^*}^{b}\Big[f\big(s,[y(s)],[y(s)]_1,[y(s)]_2\big)+$$

$$+\int_a^b g\big(s,\xi,[y(\xi)],[y(\xi)]_1,[y(\xi)]_2\big)d\xi\Big]\overline{G}(x,s)\,ds+l(x),\ x\in J\cup I.$$

Hence

$$(Ty)'(x)=\int_{a^*}^{b}\Big[f\big(s,[y(s)],[y(s)]_1,[y(s)]_2\big)+ \tag{5}$$

$$+\int_a^b g\big(s,\xi,[y(\xi)],[y(\xi)]_1,[y(\xi)]_2\big)d\xi\Big]\overline{G}'_x(x,s)\,ds+\frac{\gamma-\varphi(a)}{b-a},$$

$$x\in J\cup I.$$

$$(Ty)''(x)=f\big(x,[y(x)],[y(x)]_1,[y(x)]_2\big)+ \tag{6}$$

$$+\int_a^b g\big(x,s,[y(s)],[y(s)]_1,[y(s)]_2\big)ds,\ x\in J\cup I.$$

Let the function $f\big(x,[y(x)],[y(x)]_1,[y(x)]_2\big)$ be continuous in $G=[a;b]\times G_1^{n+1}\times G_2^{n+1}\times G_3^{n+1}$ and let $g\big(x,s,[y(s)],[y(s)]_1,[y(s)]_2\big)$ be continuous in $Q=[a;b]\times G$ where $G_1=\{u\in R:\ |u|<P_1\}$, $G_2=\{v\in R:\ |v|\le P_2\}$, $G_3=\{w\in R:\ |w|\le P_3\}$, P_1,P_2,P_3 are positive constants that are included in the definition of the space $B_2(J\cup I)$.

The following theorem holds.

Theorem 1.1. *Let the conditions be met:*

1) $\max\left\{\max\limits_{x\in J}|\varphi(x)|,\ \frac{(b-a)^2}{8}P+\max\{|\varphi(a)|,\ |\gamma|\}\right\}\le P_1,$

2) $\max\left\{\max\limits_{x\in J}|\varphi'(x)|,\ \frac{b-a}{2}P+\left|\frac{\gamma-\varphi(a)}{b-a}\right|\right\}\le P_2,$

3) $\max\left\{\max\limits_{x\in J}|\varphi''(x)|,\ P\right\}\le P_3,$

4) the functions $f\big(x,[y(x)],[y(x)]_1,[y(x)]_2\big)$ *and* $g\big(x,s,[y(s)],[y(s)]_1,[y(s)]_2\big)$ *satisfy the Lipschitz condition in G on the variables* $[y(x)],[y(x)]_1,[y(x)]_2$ *with constants* L_i^1 *and* L_i^2 $(i=\overline{0,3n+2})$*, respectively,*

5) $\frac{(b-a)^2}{8}\sum\limits_{i=0}^{n}\Big(L_i^1+(b-a)L_i^2\Big)+\frac{b-a}{2}\sum\limits_{i=n+1}^{2n+1}\Big(L_i^1+(b-a)L_i^2\Big)+\sum\limits_{i=2n+2}^{3n+2}\Big(L_i^1+(b-a)L_i^2\Big)<1.$

Then there exists a unique solution of the problem (2)-(3) *in the space* $B_2(J\cup I)$.

The proof is carried out similarly to Theorem 1 [5] using the contraction mapping principle.

2. Computational scheme. Iterative process convergence

Choose an irregular grid $\Delta=\{a=x_0<x_1<\ldots<x_m=b\}$ on the interval $[a;b]$ such that $E_2\subset\Delta$. Let us denote by $S(x,y)$ an interpolation cubic spline of defect two on Δ for the function $y(x)$. $S(x,y)$ belongs to the space $B_2(J\cup I)$.

We introduce the notation $h_j=x_j-x_{j-1},\ j=1,\ldots,n,\ M_j^+=S''(x_j+0,y),\ j=0,\ldots,n-1,\ M_j^-=S''(x_j-0,y)\ j=1,\ldots,n.$ It is easy to obtain an image for the spline $S(x,y)$:

$$S(x,y)=M_{j-1}^+\frac{(x_j-x)^3}{6h_j}+M_j^-\frac{(x-x_{j-1})^3}{6h_j}+ \tag{7}$$
$$+\left(y_{j-1}-\frac{M_{j-1}^+h_j^2}{6}\right)\frac{x_j-x}{h_j}+\left(y_j-\frac{M_j^-h_j^2}{6}\right)\frac{x-x_{j-1}}{h_j},$$
$$x\in[x_{j-1};x_j],\ j=1,2,\ldots,m.$$

Taking into account the form of the spline (7) and the continuity of its first derivatives in the internal nodes of the grid Δ we obtain a system of linear algebraic equations satisfied by the values M_{j-1}^+ and M_j^- $(j=1,2,\ldots,m)$:

$$\begin{cases} h_{j+1}y_{j-1}-(h_j+h_{j+1})y_j+h_jy_{j+1}=\frac{h_jh_{j+1}}{6}\times\\ \times\left(h_jM_{j-1}^++2h_jM_j^-+2h_{j+1}M_j^++h_{j+1}M_{j+1}^-\right),\\ j=\overline{1,m-1}.\end{cases} \tag{8}$$

We will find a solution of the boundary value problem (2)-(3) in the form of a sequence of cubic splines with defect 2 according to the following scheme:

A) Choose an initial cubic spline $S\left(x,y^{(0)}\right)=\frac{\gamma-\varphi(a)}{b-a}(x-a)+\varphi(a)$ which satisfies the boundary conditions (3) at $x=a$ and $x=b$.

B) Using the original equation (2) and the spline $S\left(x, y^{(k)}\right)$, for $k = 0, 1, \ldots$ find:

$$M_j^{+(k+1)} = f\big(x_j, [S(x_j+0, y^{(k)})], [S(x_j+0, y^{(k)})]_1, [S(x_j+0, y^{(k)})]_2\big)+ \\ + \int_a^b g\big(x_j, s, [S(s, y^{(k)})], [S(s+0, y^{(k)})]_1, [S(s+0, y^{(k)})]_2\big)ds, \\ j = \overline{0, m-1}, \tag{9}$$

$$M_j^{-(k+1)} = f\big(x_j, [S(x_j-0, y^{(k)})], [S(x_j-0, y^{(k)})]_1, [S(x_j-0, y^{(k)})]_2\big), \\ + \int_a^b g\big(x_j, s, [S(s, y^{(k)})], [S(s-0, y^{(k)})]_1, [S(s-0, y^{(k)})]_2\big)ds, \\ j = \overline{1, m}. \tag{10}$$

In the correlations (9), (10) substitute $S^{(p)}\left(x, y^{(k)}\right) = \varphi^{(p)}(x), \ \ p = 0, 1, 2$ for $x < a$.

C) Calculate $y_j^{(k+1)}, \ j = \overline{0, m}$ by solving the system of equations (8).

D) Obtain the cubic spline $S\left(x, y^{(k+1)}\right)$ in the form (7) using the previously calculated values $y_j^{(k+1)}, \ j = \overline{0, m}, \ M_j^{+(k+1)}, \ j = \overline{0, m-1}, \ M_j^{-(k+1)}, \ j = \overline{1, m}$. This spline is the next iteration approximation.

Let us introduce the notation:

$$\lambda_1 = \sum_{i=0}^{n}\big(L_i^1 + (b-a)L_i^2\big), \tag{11}$$

$$\lambda_2 = \sum_{i=n+1}^{2n+1}\big(L_i^1 + (b-a)L_i^2\big), \ \lambda_3 = \sum_{i=2n+2}^{3n+2}\big(L_i^1 + (b-a)L_i^2\big),$$

$$u = \frac{K^5}{8}(b-a)^2 + \frac{H^2}{8}, \ v = \frac{K^5}{2}(b-a) + \frac{2H}{3},$$

$$\mu = 5\left(1 + \frac{1}{2}\lambda_1 H^2 + \lambda_2 H + \lambda_3\right).$$

Theorem 2.1. *Assume that there exists a solution of the boundary value problem* (2)-(3) *and it belongs to the space* $B_2(J \cup I)$. *When the following inequality is true*

$$\theta = u\lambda_1 + v\lambda_2 + \lambda_3 < 1, \tag{12}$$

then there exists H^* such that for all $0 < H < H^*$ the sequence of splines $\left\{S\left(x, y^{(k)}\right)\right\}$, $k = 0, 1, \ldots$ converges uniformly on $[a; b]$ and the following correlations hold

$$\left\|\lim_{k\to\infty} S^{(p)}(x, y^{(k)}) - y^{(p)}(x)\right\| \leq R_p\omega\left(y''(x), H\right), \; p = 0, 1, 2, \tag{13}$$

$$R_0 = \sup_{H\leq H^*}\left(\frac{u\mu}{1-\theta} + \frac{5H^2}{2}\right), \; R_1 = \sup_{H\leq H^*}\left(\frac{v\mu}{1-\theta} + 5H\right),$$

$$R_2 = \sup_{H\leq H^*}\left(\frac{\mu}{1-\theta} + 5\right),$$

$$\omega\left(y''(x), H\right) = \max_{1\leq r\leq l+1} \omega_r\left(y''(x), H\right),$$

where $\omega_r(f, H)$ is the continuity modulus of the function f on the interval δ_r.

3. Example

Consider the boundary value problem for the neutral type equation:

$$y''(x) = \frac{1}{4}y''(x-1) + 1, \; x \in [0; 2],$$

$$y(x) = x, \; y'(x) = 1, \; y''(x) = 0, \; x \in [-1; 0], \; y(2) = \frac{5}{2}.$$

The precise solution $y(x)$ was found using the method of steps. The approximate solution $y_S^{20}(x)$ and $y_S^{40}(x)$, according to the iterative scheme proposed in the work, was obtained on the 2nd iteration with a 20 and 40 segment grid respectively. Δ_S^{20} and Δ_S^{40} are the absolute errors of the approximate solutions.

Table 1. Precise and approximate solutions

x	$y(x)$	$y_S^{20}(x)$	Δ_S^{20}	$y_S^{40}(x)$	Δ_S^{40}
0.5	0.21875	0.21552	0.00323	0.21716	0.00159
1	0.6875	0.68146	0.00604	0.68443	0.00307
1.5	1.4375	1.43448	0.00302	1.43596	0.00154

When comparing the exact and approximate solutions, one can notice that the absolute error at 20 segments does not exceed 0.006, and the relative error – 0.8%. But at 40 segments the absolute error does not exceed 0.003, and the relative error – 0.4%.

4. Conclusion

In this paper we investigate boundary value problems for nonlinear integro-differential equations of neutral type. Sufficient conditions for the existence of solutions of such

problems are established. The iterative schemes for finding approximate solutions of these problems using cubic splines of defect two are constructed and substantiated, the convergence of the iterative process is investigated. The use of the apparatus of spline functions allows us to construct algorithms that are simple to implement and at the same time suitable for solving a wide class of boundary value problems.

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Received: October 17, 2022 *Accepted: December 15, 2022*

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Acta et Commentationes, Exact and Natural Sciences ISSN: 2537-6284
Volume 2(14), 2022, Pages 15–23 E-ISSN: 2587-3644
CZU 517.925 DOI: 10.36120/2587-3644.v14i2.15-23

Problems of the theory of invariants and Lie algebras applied in the qualitative theory of differential systems

Mihail Popa

Abstract. In this work there were formulated 18 problems from the theory of invariant processes, Lie algebras, commutative graded algebras, generating functions and Hilbert series, orbit theory and Lyapunov stability theory that are important to be solved. There was substantiated the necessity of using the solutions of these problems in the qualitative theory of differential systems.
2010 Mathematics Subject Classification: 34C14, 34C45, 30E201, 30E202.
Keywords: differential system, comitants and invariants, Lie algebras and comutative graded algebras, generating functions and Hilbert series, orbit theory, stability of unperturbed motion.

Probleme din teoria invarianților şi algebrelor Lie pentru aplicaţii în teoria calitativă a sistemelor diferenţiale

Rezumat. În lucrare au fost formulate 18 probleme importante din teoria proceselor invariante, algebrelor Lie, algebrelor graduate comutative, funcţiilor generatoare şi seriilor Hilbert, teoria orbitelor şi teoria stabilităţii după Lyapunov ce se cer rezolvate. A fost argumentată necesitatea utilizării soluţiilor acestor probleme în teoria caliativă a sistemelor diferenţiale.
Cuvinte-cheie: sistem diferenţial, comitanţi şi invarianţi, algebre Lie şi algebre graduate comutative, funcţii generatoare şi serii Hilbert, teoria orbitelor, stabilitatea mişcării neperturbate.

1. Introduction

Since 1963 in the school of differential equations from Chisinau, Republic of Moldova, under the leadership of the academician C. Sibirsky (1928-1990), there has been founded a new research direction, which later it was formed as "The Method of Algebraic Invariants in the Theory of Differential Equations". This direction was based on the results of the monographs [1]-[4] after which there were published a lot of works of the academician C. Sibirsky and his disciples. Among them we can mention the works of N. Vulpe, M. Popa, Iu. Calin, V. Baltag as well as their students'.

The main results of these works concerned the construction of polynomial bases of invariants and comitants of some classical linear groups (the centro-affine $GL(2,\mathbb{R})$, the rotation $SO(2,\mathbb{R})$ and the orthogonal group $O(2,\mathbb{R})$) with the help of which there were determined some qualitative properties of autonomous polynomial differential systems as well as the geometric behavior of their solutions.

The mentioned direction of investigation has been recognized and has aroused the interest of specialists from Canada, USA, Brazil, Spain, Slovenia, Belarus, France, Algeria and some scientific centers of other countries. This is confirmed by the monograph [5] recently published on this topic by a group of authors from the scientific centers of three countries (Spain, Canada, Moldova).

„*The Method of Algebraic Invariants in the Theory of Differential Equations*" is developed even today quite effectively in the research realized in the Republic of Moldova. The above mentioned direction made it possible since the 90-s of the last century in the course of the next decades to appear and develop together with this direction, researches in the fields of invariant processes, Lie algebras and commutative graded algebras, generating functions and Hilbert series, the theory of orbits, stability of unperturbed motion after Lyapunov, governed by autonomous polynomial differential systems. This direction was confirmed under the name „Differential Equations and Algebras".

The essential results of these researches together with „*The Method of Algebraic Invariants in the Theory of Differential Equations*" were brought in the monographs [6]-[13]. M. Popa, P. Macari, A. Braicov, S. Port, E. Bâcova, E. Staruş (Naidenova), N. Gerştega, O. Cerba (Diaconescu) , V. Orlov, V. Pricop, N. Neagu, V. Repeşco, D. Cozma, contributed to the mentioned research and the ones that followed.

Let us examine the system of autonomous polynomial differential equations (PDS) of the first order in the general form, which contains the maximum possible number of non-zero coefficients

$$\frac{dx^j}{dt} = \sum_{m_i \in \Gamma} P^j_{m_i}(x) \ (j = 1, 2, \dots, n; i = 1, 2, \dots, l), \tag{1}$$

where $\Gamma = \{m_1, m_2, \dots, m_l\}$ is a finite set of non-negative integers and $x = (x^1, x^2, \dots, x^n)$ is the vector of phase variables with n coordinates. We denote by N the maximum number of non-zero coefficients of system (1) and by m_i the degree of homogeneity of the polynomial $P^j_{m_i}(x)$ of system (1) with respect to the coordinates of the phase vector x. Such systems, we will denote by $s^n(\Gamma)$. In the case, when $n = 2$, we will write them simply $s(\Gamma)$. The coefficients and phase variables of PDS (1) take values from the field of real numbers $\mathbb{R}$.

2. The problem of the minimal polynomial basis of centro-affine comitants and invariants

1. Find the minimal polynomial basis of centro-affine comitants and invariants of systems $s(1,2,3)$ *and* $s(0,1,2,3)$ *by tensor method.*

Comments to Problem 1: The number of elements in this basis was considered in [15], where the types and the number of comitants (1170) and invariants (652) of system $s(1,2,3)$ were brought. The expressions for comitants and invariants were constructed by the classical method of transvectants [4], taken from [16], without expressions of mentioned comitants and invariants being published anywhere. A part of expressions of comitants and invariants referred in [15] were constructed earlier in tensorial form by other authors and were brought in the works [1]-[4], [16]-[18]. If we know the tensor expressions from the basis of comitants and invariants of system $s(1,2,3)$, then using the method described in [19], it is easy to build this basis for system $s(0,1,2,3)$.

The necessity to know the elements of the basis of centro-affine comitants and invariants of systems $s(1,2,3)$ and $s(0,1,2,3)$ results from the importance of investigating these systems both theoretically and practically in various scientific centers of the world. The apparatus of the theory of invariants allows us to obtain answer to some problems from the qualitative theory of PDS, which cannot be obtained by other known methods.

2. Find the minimal polynomial bases of centro-affine comitants and invariants of other systems $s(\Gamma)$.

Comments to Problem 2: Until now, the minimal polynomial bases of centro-affine comitants and invariants for the systems $s(0)$, $s(1)$, $s(2)$, $s(3)$, $s(0,1)$, $s(0,2)$, $s(0,3)$, $s(1,2)$, $s(1,3)$, $s(0,1,2)$, $s(0,1,3)$ are known from the papers [1]-[4], [17], [19], [20]. Using the elements of these bases, there were obtained complete, important and surprising results for the mentioned systems.

Remark 2.1. According to [6], [7], [11], [13] the minimal polynomial basis of centro-affine comitants and invariants for PDS forms finitely determined commutative graded algebras of these elements in relation to the unimodular group $SL(2,\mathbb{R})$, which in [11], [13] are called the Sibirsky graded algebras or simply Sibirsky algebras.

3. Construction problems of Hilbert series for Sibirsky graded algebras of unimodular comitants and invariants of systems $s(\Gamma)$

3. Determine a more effective method for solving Cayley's functional equation [6], [7], [11], [13] *for the generalized and ordinary Hilbert series of Sibirsky graded algebras* [6], [7], [11], [13] *of unimodular comitants and invariants of the systems* $s(\Gamma)$ *from* (1).

4. Determine the maximal degree of the generators of Sibirsky graded algebras of comitants and invariants for any differential system $s(\Gamma)$ *from* (1) *with respect to the coefficients and phase variables of this system, or to indicate a fairly reasonable upper bound for this degree of generators for any system* $s(\Gamma)$ *from* (1).

5. Determine the formula for the number of generators or indicate a fairly reasonable upper bound of this number for Sibirsky algebras of comitants and invariants for all differential systems $s(\Gamma)$ *from* (1).

6. Determine if the Krull dimension [6], [7], [11], [13] *of Sibirsky algebras of comitants (invariants) of the system* $s(\Gamma)$ *from* (1) *denoted by* $N-1$ $(N-3)$ *can be the upper bound of maximal number of algebraic limit cycles of this system.*

7. *Prove that the projection of Sibirschi algebra of comitants (invariants) of system* $s(\Gamma)$ *with* $1 \in \Gamma\backslash\{0\}$ *form a graded algebras on the invariant variety of system* $s(\Gamma)$ *from* (1), *when the matrix elements of the linear part on the main diagonal are equal to zero and the elements on the secondary diagonal are equal to* 1, *and they are with opposite signs. What is the number of algebraically independent elements of this set ?*

Comments to Problems 3-6: In the papers [6], [7] it is shown the tight connection in the construction of centro-affine comitants and invariants of systems $s(\Gamma)$ from (1) with the study of generating functions of generalized and ordinary Hilbert series of Sibirsky graded algebras of unimodular comitants and invariants of the mentioned systems. Here an important role is due to the solution of Cayley's functional equation, for which in the papers [6], [7] it is used the generalized method of J. Silvester. But this method is connected with cumbersome computations and application of supercomputers, which for $s(\Gamma)$ systems (1) with Γ more complicated cannot be realized.

8. Prove the formula

$$H(SI_{1,2k+1,b} = H(SI_1, b)H(S_{2k+1}, u, z)|_{u^2=b, z=b},$$

for system $s(1, 2k+1)$ $(k \geq 1)$. *This formula has been applied to* $s(1,3)$ *and* $s(1,5)$ *systems, for which the generalized Hilbert series are known from* [11], [13].

9. Find the generalized Hilbert series of Sibirsky algebra $S_{1,2,3}$ *of comitants for system* $s(1,2,3)$ *from* (1).

Comments to Problem 9: We mention that using the residue method in [11], [13] the ordinary Hilbert series of Sibirsky algebras of comitants $S_{1,2,3}$ and invariants $SI_{1,2,3}$ for system $s(1,2,3)$ were constructed, as well as for other systems $s(\Gamma)$ from (1). But for the construction of generalized Hilbert series, this method couldn't be used.

10. Suppose that the Hilbert series $H(S_m, u, z_1)$ and $H(S_n, u, z_2)$ $(m \neq n)$ are known. Can the Hilbert series $H(S_{m,n}, u, z_1, z_2)$ be constructed using these series without solving Cayley's equation

$$\varphi_\Gamma(u) - u^{-2}\varphi_\Gamma(u^{-1}) = \varphi_\Gamma^{(0)}(u)$$

known from [6], [7], [11], [13] *?*

Comments to Problem 10: The problem was formulated in [6], [7], but until now no positive or negative answer has been given to this problem.

4. Problems of construction the Hammond's functions for syzygies (definition relations), related to generators of Sibirsky algebras

11. a) Determine the Hammond series of differential systems for the generators of Sibirsky algebras of invariants $SI_{1,2,3}$ and comitants $S_{1,2,3}$.

b) Determine the type of syzygies. Carry out their construction and show their irreducibility.

Comments to Problem 11: It is known from [11], [12], [13] that any finitely determined algebra A can be written as follows

$$A =< a_1, a_2, \dots, a_m | f_1 = 0, f_2 = 0, \dots, f_n = 0 > (m, n < \infty), \tag{2}$$

where m is the number of generators, and n is the number of definition relations (syzygies). These numbers are related by the formula

$$n = m - \varrho(A). \tag{3}$$

where $\varrho(A)$ is the Krull dimension of algebra A.

Using the formula of Hammond series from [6], [7] and the generators from [17], [20], Problem 11 can be solved.

5. Problems of constructing the generating functions for centro-affine comitants and invariants of the systems $s^n(\Gamma)$ $(n \geq 3)$

12. Determine the generating functions of comitants and invariants for systems $s^3(1)$ and $s^3(0, 1)$. Generalize this result for any system $s^3(\Gamma)$.

Comments to Problem 12: In papers [21]-[23] some centro-affine comitants and invariants necessary for the research carried out within the systems $s^n(\Gamma)$ were constructed for various values of $n \geq 3$. However, the generating functions, which determine the dimensions of linear spaces of these invariant polynomials according to their type, are not known. These dimensions play an important role in the construction of the polynomial bases of comitants and invariants for the mentioned systems, which are very important in the qualitative investigation of these systems. In the two-dimensional case, these functions were constructed in [6], [7], [11], [13]. The idea of constructing the generating functions for this case could also help us for systems $s^n(\Gamma)$ with $n \geq 3$. Here, the work of the famous German mathematician Paul Gordan of the 19th-20th centuries, who constructed generating functions for the comitants and invariants of ternary forms, could be useful. This work could serve as a point of inspiration for the formulated problem.

6. Lie algebras admitted by PDS, which govern the comitants and invariants of PDS

13. Determine the Lie algebra of operators with or without representations admitted by the Lorenz generalized system:

$$\dot{x} = gx + hy + kz + ayz,$$
$$\dot{y} = px + qy + rz + bxz,$$
$$\dot{z} = sx + my + nz + cxy.$$

Investigate the integrability of this system and the behavior of its solutions.

Does Problem 13 *remain open, when the system can be generalized by adding or excluding some terms from the right-hand sides of the system, provided that the classical Lorenz system indicated below can be obtained ?*

Comments to Problem 13: If we denote $g = -\sigma$, $h = \sigma$, $a = 0$, $p = r$, $q = -1$, $b = -1$, $s = m = 0$, $n = -\beta$, $c = 1$, then we obtain the classical form of Lorenz's system [24]. The determining equations of Lie algebras and the formula of the Lie integrating factor can be found in papers [21], [22].

14. a) Determine the form of polynomial systems of type (1) $(j = 2, 3, 4, 5)$*, which admit Lie operators with degree coordinates* ≥ 2 *besides the partial derivatives of the phase variables. Determine the comitants of these systems with respect to the Lie algebra admitted by examined system. Carry out the qualitative investigation of these systems.*

b) Determine the form of polynomial systems of type (1) $(j = 2, 3, 4, 5)$*, which admit Lie operators with rational function coordinates besides the partial derivatives of the phase variables. Determine the invariants and comitants of this Lie algebra and carry out the qualitative investigation of these systems.*

Comments to Problem 14: The author did not know any examples that would give a positive or negative answer to this problem.

15. a) Study the proprieties of the factorized systems $s(1, 2)/GL(2, \mathbb{R})$ *and* $s(0, 1, 2, 3)/GL(2, \mathbb{R})$ *from* [20].

b) Complete the classification of the dimension of the orbits for system $s(0,1,2,3)$ *determining more successfully the invariant elements, which form the Krull dimension of Sibirsky algebra of comitants for this system, which can separate all non-singular invariant variety* [6]-[8], [10].

Comments to Problem 15: The factorized systems [20] belong to non-singular invariant varieties that contain $GL(2,\mathbb{R})$ - orbits of the maximal dimension [6]-[8], [10]. These varieties are closed sets and the differential systems, which are on these orbits, are the richest in qualitative properties.

16. a) Obtain the classification of $GL(3,\mathbb{R})$ *– orbits for system* (1) *of the Darboux type* $s^3(1,2)$ *şi* $s^3(1,3)$ *and determine their factored systems.*

b) Carry out investigation on the stability of the Lyapunov unperturbed motion governed by systems 16 a).

Comments to Problem 16: The factorized systems defined in [20] for the two-dimensional case can also be extended to the ternary case. But here it is necessary to build the algebraic base of $GL(3,\mathbb{R})$ - comitants and invariants for these systems.

17. *Let the ternary linear system be given*

$$\frac{dx^j}{dt} = a^j_\alpha x^\alpha (j, \alpha = 1, 2, 3). \tag{4}$$

a) *Prove that the centro-affine invariants*

$$\theta_1 = a^\alpha_\alpha,\ \theta_2 = a^\alpha_\beta a^\beta_\alpha,\ \theta_3 = a^\alpha_\gamma a^\beta_\alpha a^\gamma_\beta \tag{5}$$

form the polynomial base of system (4).

b) *Prove that the centro-affine comitants*

$$\sigma_1 = a^\alpha_\mu a^\delta_\beta a^\gamma_\alpha x^\delta x^\mu x^\nu \varepsilon_{\beta\gamma\nu},\ \chi_1 = x^\alpha u_\alpha,\ \chi_2 = a^\alpha_\beta x^\beta u_\alpha,$$

$$\chi_3 = a^\alpha_\gamma a^\beta_\alpha x^\gamma u_\beta,\ \delta_4 = a^\alpha_\gamma a^\beta_p a^\gamma_q u_\alpha u\beta u_r \varepsilon_{pqr} \tag{6}$$

together with the invariants (5) *forms the polynomial base of comitants and invariants for system* (4).

Comments to Problem 17: The centro-affine invariants (5) and comitants (6) of system (4) were studied in [21]. Here is brought the syzygy

$$\chi_1(l\chi_1 + m\chi_2)^2 + [n(l\chi_1 + m\chi_2) - m\chi_3](\chi_2^2 + \chi_1\chi_3) + l\chi_2(\chi_2^2 -$$

$$-3\chi_1\chi_3) + \chi_3(n\chi_2 - \chi_3)^2 + \delta_4\sigma_1 = 0,$$

where

$$l = \frac{1}{6}(\theta_1^3 - 3\theta_1\theta_2 + 2\theta_3),\ m = \frac{1}{2}(\theta_2 - \theta_1^2),\ n = \theta_1.$$

But, until now, the answer to Problem 17 is not known.

7. The Center and Focus Problem

18. *Prove that the number of essential Poincaré-Lyapunov quantities* [11], [13] *for system* (1) *with* $\Gamma = \{1, m_1, m_2, \ldots, m_l\}$ *is equal to the Krull dimension of Sibirsky algebra of invariants of system* (1) *with* $\Gamma = \{m_1, m_2, \ldots, m_l\}$.

Comments to Problem 18: This statement is true for systems $s(1,2)$ and $s(1,3)$. We suppose that the proof of this hypothesis came from the study of many invariants and comitants of systems (1) with $\Gamma = \{1, m_1, m_2, \ldots, m_l\}$ and $\Gamma = \{m_1, m_2, \ldots, m_l\}$ with respect to the groups $GL(2,\mathbb{R}) \supset SL(2,\mathbb{R}) \supset SO(2,\mathbb{R})$.

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Received: September 30, 2022 *Accepted: November 28, 2022*

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Acta et Commentationes, Exact and Natural Sciences ISSN: 2537-6284
Volume 2(14), 2022, Pages 24–37 E-ISSN: 2587-3644
CZU 517.925 DOI: 10.36120/2587-3644.v14i2.24-37

Extension of linear operators with applications

Vasile Neagu and Diana Bîclea

Abstract. The article presents a method for solving characteristic singular integral equations perturbed with compact operators. The method consists in reducing the solution of these equations to the solution of the systems of singular (unperturbed) equations, which are solved by factoring the coefficients of the obtained systems. The method presented concerns some results of Gohberg and Krupnik and can be used in solving other classes of functional equations with composite operators that fit into the scheme described by Theorem 1.1.
2010 Mathematics Subject Classification: Primary 45E05.
Keywords: singular integral equation, compact operator, factorization.

Extensii ale operatorilor liniari cu aplicaţii

Rezumat. În lucrare este prezentată o metodă de rezolvare a unor ecuaţii integrale singulare caracteristice perturbate cu operatori compacţi. Metoda constă în reducerea soluţionării acestor ecuaţii la soluţionarea unor sisteme de ecuaţii singulare (neperturbate), care se rezolvă prin factorizarea coeficienţilor sistemelor obţinute. Metoda prezentată are tangenţă cu unele rezultate ale lui Gohberg şi Krupnik şi ar putea fi folosită la rezolvarea altor clase de ecuaţii funcţionale cu operatori compuşi, care se încadrează în schema descrisă de Teorema 1.1.
Cuvinte-cheie: ecuaţii integrale singulare, operator compact, factorizare.

Introduction

In the monographs of Muskhelishvili [1] and Gakhov [2] and in other works it is indicated that the solution of singular integral equations can be found in rare cases. Even in these cases finding the exact solution requires complicated calculations of singular integrals accompanied with combersome theoretical and computational difficulties. The content of this article, as well as the studies of other authors [3], [4], [5], [6], [7], [8], [9] once again confirms the statement of academicians Muskhelishvili and Gakhov.

In this paper we study the problem of solving singular integral equations containing compact terms

$$A\varphi \equiv a(t)\varphi(t) + \frac{b(t)}{\pi i}\int_\Gamma \frac{\varphi(\tau)}{\tau - t}d\tau + \int_\Gamma k(t,\tau)\varphi(\tau) = f(t), \tag{1}$$

where function $k(t,\tau)$ is continuous or with weak singularities. To each operator A, defined by the left hand side of (1), according to the rule described in Theorem 1.1, we associate a matrix operator $\widetilde{A}$

$$\widetilde{A}\psi = C(t)\,\psi(t) + \frac{D(t)}{\pi i}\int_{\Gamma}\frac{\psi(\tau)}{\tau - t}d\tau, \tag{2}$$

which has the property that both operators A and $\widetilde{A}$ are or are not invertible in the respective spaces. Thus, the solution of the considered equation is reduced to a similar problem for a system of equations, which turns out to be a system of "ordinary" singular integral equations, without compact terms. The obtained system of singular integral equations is solved by the method of factoring the coefficients, a method developed in the monograph [3] etc. An explicit expression of the solution of the considered equation is obtained through the solution of the system of equations. The method presented in this paper is based on the results of the works of Gohberg and Krupnik [10], and can be used for solving other classes of functional equations with composite operators that fit into the scheme described by Theorem 1.1.

To invert operators of the form (2), where $C(t)$ and $D(t)$ are matrices of continuous functions satisfying the conditions $\det(C(t) \mp D(t)) \neq 0$, it is necessary (see [3]) to factorize the matrix

$$G(t) = (C(t) - D(t))^{-1}(C(t) + D(t)).$$

This means that the matrix $G(t)$ must be represented in the form

$$G(t) = G_{-}(t) \cdot diag\left(t^{k_1}, t^{k_2}, \ldots, t^{k_n}\right) \cdot G_{+}(t),$$

where $G_{+}(z)$ $(G_{-}(z))$ are matrices of functions with analytic elements in the domains $F_{+} = \{z|\ |z| < 1\}$ $(F_{-} = \{z|\ |z| > 1\})$, and $k_1, k_2, \ldots, k_n$ are integers called *partial indices* of the operator $\widetilde{A}$. Depending on the numbers $k_1, k_2, \ldots, k_n$, the operator $\widetilde{A}$ can be invertible, left invertible or right invertible. In particular, if all numbers $k_1, k_2, \ldots, k_n$ are positive, then the operator $\widetilde{A}$ is left invertible, if all are negative, then $\widetilde{A}$ is right invertible, and finally, if all numbers are equal to zero, then $\widetilde{A}$ is invertible. We will apply these results to the inversion of the operator $\widetilde{A}$.

1. Extension of linear operators

Let V be some Banach algebra of linear bounded operators acting in a Banach space B, and $V^{(m)}$ be a Banach algebra of elements of the form $\left\|A_{jk}\right\|_{j,k=1}^{m}$, where $A_{jk} \in V$. If $B^{(m)}$ is a Banach space of vectors $X = \lfloor x_1, \ldots, x_m \rfloor$ with elements $x_j \in B$ and with the norm $\|X\| = \max_k \|x_k\|$, then $V^{(m)}$ is a Banach algebra of linear bounded operators in the space $B^{(m)}$. Denote by I and I_m the unit operators acting in the spaces V and, respectively, $V^{(m)}$. Suppose also that $I \in V$ and $I_m \in V^{(m)}$. Assume that

$$A = \sum_{j=1}^{r} A_{j1}A_{j2}\cdots A_{js}, \tag{3}$$

where $A_{jk} \in V$. The operator $\widetilde{A} \in V^{(m)}$ is called a *linear extension o*f the operator A (of order m) if:

1) the elements of the matrix $\widetilde{A}$ are linear combinations of the elements A_{jk} and the unit operator;

2) there exist invertible operators X and Z from the algebra $V^{(m)}$ such that

$$\widetilde{A} = Y \cdot \begin{pmatrix} I_{m-1} & 0 \\ 0 & A \end{pmatrix} \cdot Z. \tag{4}$$

It is easy to see that the operator $A = \sum_{j=1}^{r} A_{j1}A_{j2}\cdots A_{js}$ and its linear extension $\widetilde{A}$ (if it exists) are Noetherian (or are not Noetherian) simultaneously in the spaces B and $B^{(m)}$, respectively, and

$$dimker A = dimker\widetilde{A} \quad \text{and} \quad dimcoker A = dimcoker\widetilde{A}.$$

The following Theorem holds

Theorem 1.1. *Each element A from the algebra V of the form $A = \sum_{j=1}^{r} A_{j1}A_{j2}\cdots A_{js}$ $(A_{jk} \in V)$ admits the linear expansion (of order $m \le r\,(s+1)+1$).*

Proof. Let us compose the following matrix of order $r\,(s+1)$

$$M = \begin{pmatrix} I_r & B_1 & 0 & \cdots & 0 \\ 0 & I_r & B_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & B_s \\ 0 & 0 & 0 & \cdots & I_r \end{pmatrix},$$

where

$$B_k = \begin{pmatrix} A_{1k} & 0 & \cdots & 0 \\ 0 & A_{2k} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & A_{kk} \end{pmatrix}.$$

Denote by F a column of the length $r(s+1)$, whose top rs elements are equal to zero and the bottom r elements are equal to the identity operator. Let also $G = \| \underbrace{I, \ldots, I}_{r}, \underbrace{0, \cdots, 0}_{rs} \|$.

It is easy to verify the validity of the expansion

$$\begin{pmatrix} M & F \\ G & 0 \end{pmatrix} = \begin{pmatrix} I_{m-1} & 0 \\ H & I \end{pmatrix} \cdot \begin{pmatrix} I_{m-1} & 0 \\ 0 & A \end{pmatrix} \cdot \begin{pmatrix} I_{m-1} & F \\ 0 & I \end{pmatrix}, \tag{5}$$

with $m = r\,(s+1)+1$, $H = \|M_0, M_1, \ldots, M_s\|$, where $M_0 = \|\underbrace{I, \ldots, I}_{r}\|$ and

$$M_k = \|A_{11}A_{12}\ldots A_{1j}, A_{21}A_{22}\ldots A_{2j}, \ldots, A_{k1}A_{k2}\ldots A_{rk}\| \quad (k = 1, 2, \ldots, s).$$

Note that the operators

$$Y = \begin{pmatrix} I_{m-1} & 0 \\ H & I \end{pmatrix}, \quad Z = \begin{pmatrix} I_{m-1} & F \\ 0 & I \end{pmatrix}$$

are invertible in the space $B^{(m)}$ and their inverse operators are of the form

$$Y^{-1} = \begin{pmatrix} I_{m-1} & 0 \\ -H & I \end{pmatrix}, \quad Z^{-1} = \begin{pmatrix} I_{m-1} & -F \\ 0 & I \end{pmatrix},$$

respectively. Therefore, the operator

$$\widetilde{A} = \begin{pmatrix} M & F \\ G & 0 \end{pmatrix}$$

is a linear extension of the operator A. Theorem 1.1 is proved. □

Note that the extreme factors on the right hand side of equality (5) are triangular matrices with unity on the main diagonal, therefore, they are invertible. This implies that the operator A is normally solvable. It is Noetherian or invertible if and only if the operator $\widetilde{A}$ is of such type.

Corollary 1.1. *The operator A is invertible in the space B if and only if the operator*

$$\widetilde{A} = \begin{pmatrix} M & F \\ G & 0 \end{pmatrix}$$

is invertible in the space $B^{n(N+1)+1}$.

Corollary 1.2. *Let A_0, C_k, $D_k \in L(B)$ $(k = 1, 2, \ldots, n)$ and $\widetilde{A}$ be an operator defined by the equality*

$$\widetilde{A} = \begin{pmatrix} I & 0 & \ldots & 0 & D_1 \\ 0 & I & \ldots & 0 & D_2 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & I & D_n \\ C_1 & C_2 & \ldots & C_n & A_0 \end{pmatrix}. \tag{6}$$

In this case, the following statements are true:

$$\widetilde{A} \in GL\left(B^{n+1}\right) \Leftrightarrow A = A_0 - \sum_{k=1}^{n} C_k D_k \in GL(B).$$

Indeed, we note that

$$A_0 - \sum_{k=1}^{n} B_k C_k = det\widehat{A}$$

and the validity of the following equality is directly verified

$$\begin{pmatrix} I & 0 & \dots & 0 & D_1 \\ 0 & I & \dots & 0 & D_2 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & I & D_n \\ C_1 & C_2 & \dots & C_n & A_0 \end{pmatrix} = \begin{pmatrix} I & 0 & \dots & 0 & 0 \\ 0 & I & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & I & 0 \\ C_1 & C_2 & \dots & C_n & I \end{pmatrix} \times$$

$$\times \begin{pmatrix} I & 0 & \dots & 0 & D_1 \\ 0 & I & \dots & 0 & D_2 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & I & D_n \\ 0 & 0 & \dots & 0 & A_0 - \Delta \end{pmatrix} \times \begin{pmatrix} I & 0 & \dots & 0 & D_1 \\ 0 & I & \dots & 0 & D_2 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & I & D_n \\ 0 & 0 & \dots & 0 & I \end{pmatrix}, \qquad (7)$$

where $\Delta = \sum_{k=1}^{n} C_k D_k$. Since the left and right factors of equality (7) are invertible operators in the space B^{n+1}, then $\widetilde{A} \in GL\left(B^{n+1}\right) \Leftrightarrow A_0 - \sum_{k=1}^{n} C_k D_k$.

Corollary 1.3. *If the vector* $\varphi = (\varphi_1, \dots, \varphi_{n+1}) \in B^{n+1}$ *is a solution of the equation* $\widetilde{A}\varphi = \widetilde{\psi}$ *with the right hand side* $\widetilde{\psi} = (0, 0, \dots, \psi)$, *then the equality* $A\varphi_{n+1} = \psi$ *holds. That is, the coordinate standing on* $n+1$ *place of the solution of the equation* $\widetilde{A}\varphi = \widetilde{\psi}$ *with the right hand side* $\widetilde{\psi} = (0, 0, \dots, \psi)$ *is the solution of the equation* $Af = \psi$. *Solutions of this type exhaust all solutions of the equation* $Af = \psi$.

Indeed, from equality (6) it follows that the equation $\widetilde{A}\varphi = \widetilde{\psi}$ is equivalent to the system of equations:

$$\begin{cases} \varphi_1 + D_1\varphi_{n+1} = 0, \\ \varphi_2 + D_2\varphi_{n+1} = 0, \\ \qquad \dots \\ \varphi_n + D_n\varphi_{n+1} = 0, \\ A\varphi_{n+1} = \psi. \end{cases}$$

This implies the assertion of Corollary 1.3.

Remark 1.1. It is clear that Theorem 1.1 and Corollary 1.3 can be effectively applied only in the cases when the solvability criteria for the operators of the form (4) are known. This is done in the cases when the operator $\widetilde{A}$ is a singular integral operator.

2. Application to the solution of singular equations

We apply Theorem 1.1 and Corollary 1.3 to solve singular integral equations perturbed by compact operators. Such equations are also called *complete singular integral equations* (see [3]). We noted above that singular integral equations are solved in rather rare cases. This problem becomes more complicated (see [1]) in the case of systems of singular equations being related to the problem of the factorization of functional matrices and the solution of the corresponding Riemann problem.Taking into account these difficulties, we will study equations that can be reduced to systems of equations whose coefficients can be effectively factorized.

Before we pass to solving the proposed equations, we pay our attention to an unexpected result that is obtained by means of Theorem 1.1. It is known that the theory of singular integral equations

$$a(t)\,\varphi(t) + \frac{b(t)}{\pi i}\int_{\Gamma}\frac{\varphi(\tau)}{\tau - t}d\tau = f(t) \tag{8}$$

is well developed and presented in monographs [1]-[3] and others. Usually, the contour of integration Γ is assumed to be of Lyapunov type, and in the case of a contour with angular points, certain difficulties appear. Let Γ be a contour having an angular point of size $\frac{\pi}{2}$ and consider the equation (8). After certain integral (equivalent) transformations, the operator

$$V\varphi \equiv a(t)\,\varphi(t) + \frac{b(t)}{\pi i}\int_{\Gamma}\frac{\varphi(\tau)}{\tau - t}d\tau,$$

determined by the left hand side of equation (8), turns into the operator

$$W\psi = \tilde{a}(t)\,\psi(t) + \frac{\tilde{b}(t)}{\pi i}\int_{\widetilde{\Gamma}}\frac{\psi(\tau)}{\tau - t}d\tau + \frac{\tilde{b}(t)}{2\pi i}\left[\int_{\widetilde{\Gamma}}\left(\frac{\sqrt{t+1}}{(\tau - t)\sqrt{\tau+1}} - \frac{1}{\tau - t}\right)\psi(\tau)\,d\tau\right],$$

where $\tilde{a}(t) = a\sqrt{t+1}$, $\tilde{b}(t) = b\sqrt{t+1}$ and $\widetilde{\Gamma}$ is already a Lyapunov contour! The operator W satisfies the conditions of Theorem 1.1 and Corollary 1.3 by means of which (we do not dwell on the details) we have that the operator V is Noetherian if and only if the following operator is Noetherian

$$W_0\psi = \tilde{a}(t)\,\psi(t) + \frac{\tilde{b}(t)}{\pi i}\int_{\widetilde{\Gamma}}\frac{\psi(\tau)}{\tau - t}d\tau.$$

Thus, the study of a singular operator, in the case of the contour with an angular point, is reduced to the study of a similar operator on a Lyapunov type contour. From these results it follows that the Noetherian conditions of the operator W_0 do not change being perturbed by the operators of the form

$$H\psi = \frac{\tilde{b}(t)}{2\pi i}\left[\int_{\widetilde{\Gamma}}\left(\frac{\sqrt{t+1}}{(\tau-t)\sqrt{\tau+1}} - \frac{1}{\tau-t}\right)\right]\psi(\tau)\,d\tau,$$

which is not compact!

Let $\Gamma = \{t \in \mathbb{C} : \ |t| = 1\}$. In space $B = L_p(\Gamma)$ $(p > 1)$, we consider the equation

$$\frac{1}{\pi i}\int_{\Gamma}\frac{\tau^3 - t^3}{(\tau-t)^2}\varphi(\tau)d\tau = \psi(t). \tag{9}$$

The left hand side of equation (9) corresponds to the operator, which can be written in the following form

$$(A\varphi)(t) = \frac{3t^2}{\pi i}\int_{\Gamma}\frac{\varphi(\tau)}{\tau-t}d\tau + (T\varphi)(t),$$

where

$$(T\varphi)(t) = \frac{1}{\pi i}\int_{\Gamma}(\tau+2t)\varphi(\tau)d\tau.$$

The operator T, being an integral operator with a continuous kernel, is compact in $L_p(\Gamma)$. In the case of studying the Noetherian properties and the index of the operator A, the operator T can be neglected, i.e. the operator T does not affect the Noetherian properties of the operator A. However, this does not happen if the operator A is inverted or in the case of solving the equation $A\varphi = \psi$.

Let

$$(S\varphi)(t) = \frac{1}{\pi i}\int_{\Gamma}(\tau-t)^{-1}\varphi(\tau)d\tau, \quad (B\varphi)(t) = t\varphi(t), \tag{10}$$

then the operator A can be written as follows

$$A = SB^2 + BSB + B^2S,$$

and the corresponding operator $\widetilde{A}$, defined by equality (6), of the operator A has the form

$$\widetilde{A} = \begin{pmatrix} I & 0 & B^2 \\ 0 & I & B \\ -S & -BS & B^2S \end{pmatrix}.$$

By virtue of Corollary 1.3, any solution of equation (9) can be obtained as the last coordinate φ_3 of the solution of equation $\widetilde{A}\widetilde{\varphi} = \widetilde{\psi}$ $(\widetilde{\varphi} = (\varphi_1, \ \varphi_2, \varphi_3), \ \widetilde{\psi} = (0, 0, \psi))$.

The operator $\widetilde{A}$ represents the characteristic singular operator with matrix coefficients:

$$\widetilde{A} = \begin{pmatrix} 1 & 0 & t^2 \\ 0 & 1 & t \\ -1 & -t & t^2 \end{pmatrix} P + \begin{pmatrix} 1 & 0 & t^2 \\ 0 & 1 & t \\ 1 & t & -t^2 \end{pmatrix} Q =$$

$$= \begin{pmatrix} 1 & 0 & t^2 \\ 0 & 1 & t \\ 1 & t & -t^2 \end{pmatrix} \left[\frac{1}{3} \begin{pmatrix} 1 & -2t & 2t^2 \\ -2t^{-1} & 1 & 2t \\ 2t^{-2} & 2t^{-1} & 1 \end{pmatrix} P + Q \right],$$

where $P = \frac{1}{2} diag(I+S)$ and $Q = \frac{1}{2} diag\,(I-S)$.

The matrix that is the coefficient of the operator P can be factorized:

$$\frac{1}{3} \begin{pmatrix} 1 & -2t & 2t^2 \\ -2t^{-1} & 1 & 2t \\ 2t^{-2} & 2t^{-1} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -2t^{-1} & 1 & 0 \\ 2t^{-2} & 2t^{-1} & 1 \end{pmatrix} \begin{pmatrix} 1/3 & -2t/3 & 2t^2/3 \\ 0 & -1 & 2t \\ 0 & 0 & 3 \end{pmatrix} = c_- \cdot c_+.$$

Since the partial indices under this factorization are equal to zero, the operator $\widetilde{A}$ is invertible in B^3 [1] and its inverse operator is defined by the following equality:

$$\widetilde{A}^{-1} = \left[\begin{pmatrix} 1/3 & -2t/3 & 2t^{-2}/3 \\ 0 & -1 & 2t \\ 0 & 0 & 3 \end{pmatrix}^{-1} P + \begin{pmatrix} 1 & 0 & 0 \\ -2t^{-1} & 1 & 0 \\ 2t^{-2} & -2t^{-1} & 1 \end{pmatrix} Q \right] \times$$

$$\times \begin{pmatrix} 1 & 0 & 0 \\ -2t^{-1} & 1 & 0 \\ 2t^{-2} & -2t^{-1} & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 & t^2 \\ 0 & 1 & t \\ 1 & t & t^2 \end{pmatrix} =$$

$$= \left[\begin{pmatrix} 3 & -2t & 2t^2/3 \\ 0 & -1 & 2t/3 \\ 0 & 0 & 1/3 \end{pmatrix} P + \begin{pmatrix} 1 & 0 & 0 \\ -2t^{-1} & 1 & 0 \\ 2t^{-2} & -2t^{-1} & 1 \end{pmatrix} Q \right] \cdot \begin{pmatrix} 2/3 & -t/3 & 1/3 \\ t^{-1} & 0 & t^{-1} \\ t^{-2} & t^{-1} & t^{-2} \end{pmatrix}.$$

According to the scheme of inversion of the singular operator A, given in Corollary 1.3, we find

$$A^{-1} \begin{pmatrix} 0 \\ 0 \\ \psi \end{pmatrix} = \begin{pmatrix} 3P+Q & -2tP & 2t^2/3P \\ -2/tQ & -P+Q & 2t/3P \\ 2/t^2Q & -2/tQ & 1/3P+Q \end{pmatrix} \begin{pmatrix} 1/3\psi \\ 1/t\psi \\ 1/t^2\psi \end{pmatrix}.$$

Hence

$$A^{-1}\psi = \begin{pmatrix} \frac{2}{t^2}Q & \frac{-2}{t}Q & \frac{1}{3}P+Q \end{pmatrix} \begin{pmatrix} 1/3\psi \\ 1/t\psi \\ 1/t^2\psi \end{pmatrix} =$$

$$= \left(-\frac{1}{3} S B^{-2} + B^{-1} S B^{-1} - \frac{2}{3} B^{-2} S \right) \psi.$$

Thus, equation (9) is uniquely solvable and its solution is found by the formula

$$\varphi(t) = \frac{1}{3\pi i} \int_{\Gamma} \frac{3\tau t - 2\tau^2 - t^2}{\tau^2 t^2 (\tau - t)} \psi(\tau)\, d\tau. \tag{11}$$

Consider two more equations

$$\frac{t^2+1}{t} \varphi(t) + \frac{1}{\pi i} \int_{\Gamma} \frac{\tau t - 1}{\tau(\tau - t)} \varphi(\tau)\, d\tau = \psi(t) \tag{12}$$

and

$$\frac{t^2+1}{t} f(t) + \frac{1}{\pi i} \int_{\Gamma} \frac{1 - \tau t}{\tau(\tau - t)} f(\tau)\, d\tau = \psi(t). \tag{13}$$

Let A and C be operators defined by the left hand sides of equalities (12) and (13), respectively. It is directly verified that in this case the operators A and C differ from the characteristic singular operators by compact terms, i.e., equations (12) and (13) are complete singular equations. With the notation (10), the operators A and B can be written in the following form

$$A = B + B^{-1} + BS - SB^{-1}.$$

Since $S^* = S$ and $C^* = C^{-1}$, then $C = A^*$. As operators $\widetilde{A}$ and $\widetilde{C}$, appearing in Corollary 1.2, we can take

$$\widetilde{A} = \begin{pmatrix} I & B^{-1} \\ S & B + B^{-1} + BS \end{pmatrix}, \quad \tilde{C} = \begin{pmatrix} I & B^{-1} \\ -S & B + B^{-1} - BS \end{pmatrix}.$$

The operators $\widetilde{A}$ and $\widetilde{C}$ (as in the previous example) are characteristic singular operators with matrix coefficients:

$$\widetilde{A} = \begin{pmatrix} 1 & t^{-1} \\ -1 & t^{-1} \end{pmatrix} \left[\begin{pmatrix} 0 & -t \\ t & 1+t^2 \end{pmatrix} P + Q \right],$$

$$\widetilde{C} = \begin{pmatrix} 1 & t^{-1} \\ 1 & 2t + t^{-1} \end{pmatrix} \left[\begin{pmatrix} 1+t^{-2} & t^{-1} \\ -t^{-1} & 0 \end{pmatrix} P + Q \right].$$

However, unlike the previous example, the matrices-coefficients of P have non-zero partial indices. In the case of the operator $\widetilde{A}$ this index is equal to 2, and in the case of $\widetilde{C}$ it is equal to -2. This results from the factorization of the coefficients of the operator P :

$$\begin{pmatrix} 0 & -t \\ t & 1+t^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -t^{-1} & 1 \end{pmatrix} \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & t \end{pmatrix},$$

$$\begin{pmatrix} 1+t^{-2} & t^{-1} \\ -t^{-1} & 0 \end{pmatrix} = \begin{pmatrix} t^{-1} & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} t^{-1} & 0 \\ 0 & t^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}.$$

By virtue of well-known results from the theory of singular equations with matrix coefficients, the operator $\widetilde{A}$ is left invertible, while the operator $\widetilde{C}$ is left invertible. This implies the operator A to be left invertible and the operator C to be right invertible. The general solution of the equation $\widetilde{C}\varphi = 0$ is of the form (see [1]):

$$\begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = \begin{pmatrix} \alpha\left(1 - t^{-1} - t^{-2}\right) \\ \beta(1 - t + t^{-1}) \end{pmatrix}$$

and the particular solution of the equation $\widetilde{C}\varphi = \psi$ has de form

$$\begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = \begin{pmatrix} 0 \\ (t^2 P + Q)t^{-1}\psi(t) \end{pmatrix}.$$

Thus, equation (13) is solvable for any right hand side and its general solution is of the form

$$f(t) = \beta\left(1 - t + t^{-1}\right) + \frac{t^2+1}{4}\psi(t) + \frac{t^2-1}{4\pi i}\int_\Gamma \frac{\psi(\tau)}{\tau(\tau - t)}d\tau,$$

where $\beta \in \mathbb{C}$. Equation (12) is not solvable for any right hand side. Since the operator A is left invertible, it is normally solvable. For its solvability it is necessary and sufficient that the right hand side of ψ be orthogonal to each solution of the equation $C\varphi = 0$, i.e. to fulfill the condition

$$\int_\Gamma (1 - t + t^{-1})\psi(t)|dt| = 0.$$

If this condition is satisfied, then equation (12) has a unique solution, which can be found by formula

$$\varphi(t) = \frac{t+1}{4t}\psi(t) + \frac{1}{4\pi i}\int_\Gamma \frac{\tau + t + \tau t - \tau^2 t}{\tau^2 t(\tau - t)}\psi(\tau)\,d\tau.$$

This solution is obtained according to the scheme proposed in Corollaries 1.2 and 1.3.

3. Solution of integral equations by the regularization method

Let A be some Noetherian operator. If the regularizing operator M for A is known, then the solution of the equation

$$A\varphi = f \tag{14}$$

can be reduced to solving the equation

$$MA\varphi = Mf, \tag{15}$$

in which the operator $MA - I$ is completely continuous. To equation (15) may be applied many methods developed for inverting operators of the form $I - T$, where T is a completely continuous operator.

A special interest is represented by the case when equations (14) and (15) are equivalent for any vector f, i.e., equations (14) and (15) are simultaneously solvable or unsolvable, and in the case of solvability, they have the same solutions. This happens to be if and only if $KerM = 0$. Indeed, if $MA\varphi = 0$, then $A\varphi = h$, where $h \in KerM$.

Assume that equations (14) and (15) are equivalent, then either $KerM = \{0\}$, or $dim\, KerM > 0$ and $KerM \bigcap ImA = \{0\}$. The last assertion is impossible, since in this case the equations $A\varphi = f \ \ (f \in KerM)$ and $MA\varphi = Mf = 0$ are not equivalent. Conversely, if $KerM = \{0\}$, then it is obvious that equations (14) and (15) are equivalent.

We say that an operator A admits equivalent regularization if it has a regularizing operator M for which equations (14) and (15) are equivalent for any vector f. In this case, the operator M is called an *equivalent regularizing operator* for A.

It follows from the above that an operator M is an equivalent regularizer for A if it is a regularizer for A and also is left invertible.

Theorem 3.1. *(see* [11]*) Operator A admits an equivalent regularization if and only if*

$$IndA \geq 0. \tag{16}$$

Indeed, if M is an equivalent regularizer for A, then it is left invertible and, therefore, $IndM \leq 0$. Since $IndMA = IndM + IndA = 0$, then $IndA \geq 0$. Let condition (16) be satisfied and M_1 be a regularizer for A. Then M_1 is Noetherian and $IndM_1 + IndA = 0$. Hence, $IndM_1 \leq 0$. According to the results of [12], the operator M_1 can be represented as $M_1 = M + T$, where M is left invertible. Obviously, M is an equivalent regularizing operator for A. Theorem 3.1 is proved.

We now consider the case when the Noetherian operator A does not admit equivalent regularization, that is, the following condition holds

$$IndA < 0. \tag{17}$$

Let M_1 be a regularizer for A. Since $IndM_1 > 0$, then according to the results of work [12] the operator M_1 can be represented as $M_1 = M + T$, where M is right invertible. The operator M is also a regularizing operator for A and all solutions of the equation

$$A\varphi = f \quad (f \in ImA)$$

can be obtained by formula $\varphi = M\psi$, where ψ runs through all solutions of the equation

$$AM\psi = f.$$

As an example, to illustrate the stated theory, let us regularize (see [2]) the following singular integral equation

$$A\varphi \equiv \left(t+t^{-1}\right)\varphi(t)+\frac{t-t^{-1}}{\pi i}\int_\Gamma \frac{\varphi(\tau)}{\tau-t}d\tau-\frac{1}{2\pi i}\int_\Gamma\left(t+t^{-1}\right)\left(\tau+\tau^{-1}\right)\varphi(\tau)d\tau=2t^2, \tag{18}$$

in various ways, where Γ is the unit circle.

The regular part of the kernel is degenerate. Therefore, in the same way that is used in solving the Fredholm equations with the degenerate kernel, the equation (17) can be reduced to a combination of the characteristic equation and a linear algebraic equation and, it can be solved in the closed form. Thus, there is no necessity for regularization here, but the equation under consideration is convenient for illustrating general methods on it. Here all the calculations can be carried out to the end.

For further reasoning, we first solve this equation denoting

$$\frac{1}{2\pi i}\int_\Gamma\left(\tau+\tau^{-1}\right)\varphi(\tau)d\tau=C, \tag{19}$$

We write it in the characteristic form:

$$\left(t+t^{-1}\right)\varphi(t)+\frac{t-t^{-1}}{\pi i}\int_\Gamma\frac{\varphi(\tau)}{\tau-t}d\tau=2t^2+C\left(t+t^{-1}\right).$$

For the corresponding Riemann boundary value problem [2]

$$\Phi^+(t)=t^{-2}\Phi^-(t)+t+\frac{C}{2}(1+t^{-2})$$

the index $\kappa=-2$ and the solvability conditions will be satisfied only for $C=0$. In this case

$$\Phi^+(z)=z,\quad \Phi^-(z)=0.$$

From here we obtain the solution of equation (18):

$$\varphi(t)=\Phi^+(t)-\Phi^-(t)=t.$$

Putting the last expression into the equality (19), we make sure that it is satisfied at $C=0$. Therefore, this equation is solvable and has a unique solution $\varphi(t)=t$.

Regularization on the left. Since the index of the equation is $\kappa=-2<0$, then any of its regularizing operators will have eigenfunctions (at least two). Therefore, regularization on the left leads, generally speaking, to an equation that is not equivalent to the original one (regularization is not equivalent).

Consider first the regularization on the left using the regularizer R:

$$(Rh)(t)\equiv\left(t+t^{-1}\right)h(t)-\frac{t-t^{-1}}{\pi i}\int_\Gamma\frac{h(\tau)}{\tau-t}d\tau. \tag{20}$$

The corresponding Riemann boundary value problem

$$H^{+}(t) = t^{2} H^{-}(t)$$

has now the index $\kappa = 2$. Finding the eigenfunctions of the operator R, we obtain that

$$\lambda_1(t) = 1 - t^{-2}, \quad \lambda_2(t) = t - t^{-1}.$$

Based on the general theory, the regular equation $RA\varphi = Rf$ will be equivalent to the singular equation

$$A\varphi = f + \alpha_1\lambda_1 + \alpha_2\lambda_2, \tag{21}$$

where α_1, α_2 are some constants, which can be either arbitrary or defined. Taking into account (19), we write equation (21) in the characteristic form:

$$\left(t + t^{-1}\right)\varphi(t) + \frac{t - t^{-1}}{\pi i}\int_{\Gamma}\frac{\varphi(\tau)}{\tau - t}d\tau = 2t^{2} + C\left(t + t^{-1}\right) + \alpha_1\left(1 - t^{-2}\right) + \alpha_2\left(t - t^{-1}\right).$$

The corresponding Riemann boundary value problem for this equation is

$$\Phi^{+}(t) = t^{-2}\Phi^{-}(t) + t + \frac{C}{2}\left(1 + t^{-2}\right) + \frac{\alpha_1}{2}\left(t^{-1} - t^{-3}\right) + \frac{\alpha_2}{2}\left(1 - t^{-2}\right).$$

Its solution can be presented in the form

$$\Phi^{+}(z) = z + \frac{1}{2}C + \frac{1}{2}\alpha_2, \quad \Phi^{-}(t) = \frac{1}{2}z^{2}\left[\alpha_1 z^{-3} + (\alpha_2 - C)z^{-2} - \alpha_1 z^{-1}\right].$$

The solvability condition will give $\alpha_1 = 0,\ \alpha_2 = C$. Then, the solution of equation (21) is determined by formula $\varphi(t) = \Phi^{+}(t) - \Phi^{-}(t) = t + C$.

Substituting the found value of φ into equality (19), we obtain the identity $C = C$. Therefore, the constant $\alpha_2 = C$ remains to be arbitrary and the regularized equation is not equivalent to the original equation, but to the equation

$$A\varphi = f + \alpha_2\lambda_2,$$

having the solution $\varphi(t) = t + C$, where C is an arbitrary constant. The last solution satisfies the original equation only at $C = 0$.

Regularization on the right. As a regularizer on the right, we take the operator R, defined by equality (20). Assuming that

$$\varphi(t) = (Rh)(t) \equiv \left(t + t^{-1}\right)h(t) - \frac{t - t^{-1}}{\pi i}\int_{\Gamma}\frac{h(\tau)}{\tau - t}d\tau \tag{22}$$

we obtain the Fredholm equation for the function $h(t)$:

$$(ARh)(t) \equiv$$

$$h(t) - \frac{1}{4\pi i}\int_{\Gamma}\left[t\left(\tau^{2} - 1 + \tau^{-2}\right) + 2\tau^{-1} + t^{-1}\left(\tau^{2} + 3 + \tau^{-2}\right) - 2t^{-2}\tau^{-1}\right)\Big]h(\tau)\,d\tau = \frac{t^{2}}{2}.$$

Solving the last equation as a degenerate one, we have

$$h(t) = \frac{t^2}{2} + \alpha\left(t - t^{-1}\right) + \beta\left(1 - t^{-2}\right),$$

where $\alpha,\ \beta$ are arbitrary constants.

Thus, the regularized equation has two linearly independent solutions with respect to $h(t)$ while the original equation (18) was solved uniquely. Substituting the found value $h(t)$ in formula (22), we obtain that

$$\varphi(t) = R\left[\frac{t^2}{2} + \alpha\left(t - t^{-1}\right) + \beta\left(1 - t^{-2}\right)\right] = t$$

is the solution of the original singular equation. The result agrees with the general theory, since the regularization on the right is equivalent for a negative index.

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Received: October 04, 2022 *Accepted: November 30, 2022*

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Acta et Commentationes, Exact and Natural Sciences ISSN: 2537-6284
Volume 2(14), 2022, Pages 38–46 E-ISSN: 2587-3644
CZU 517.925 DOI: 10.36120/2587-3644.v14i2.38-46

Centers of cubic differential systems with the line at infinity of maximal multiplicity

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Abstract. We classify all cubic differential systems with a center-focus critical point and the line at infinity of maximal multiplicity. It is proved that the critical point is of the center type if and only if the divergence of the vector field associated to differential system vanishes.

2010 Mathematics Subject Classification: 34C05.
Keywords: cubic differential system, multiple invariant line, center problem.

Centre în sistemele diferenţiale cubice ce au linia de la infinit de multiplicitate maximală

Rezumat. Sunt clasificate sistemele diferenţiale cubice ce au puncte critice de tip centru-focar şi infinitul e de multiplicitate maximală. Se arată că în punctul critic avem centru, dacă şi numai dacă divergenţa câmpului vectorial asociat sistemului diferenţial se anulează.

Cuvinte-cheie: sistem diferenţial cubic, linii invariante multiple, problema centrului.

1. Introduction

Consider the real cubic system of differential equations

$$\begin{cases} \dot{x} = y + ax^2 + cxy + fy^2 + kx^3 + mx^2y + pxy^2 + ry^3 \equiv p(x,y), \\ \dot{y} = -(x + gx^2 + dxy + by^2 + sx^3 + qx^2y + nxy^2 + ly^3) \equiv q(x,y), \\ \gcd(p,q) = 1, sx^4 + (k+q)x^3y + (m+n)x^2y^2 + (l+p)xy^3 + ry^4 \not\equiv 0. \end{cases} \tag{1}$$

The critical point $(0,0)$ of the system (1) is either a focus or a center. The problem of distinguishing between a center and a focus is called *the center problem*. It is well known that $(0,0)$ is a center if and only if the Lyapunov quantities $L_1, L_2, ..., L_j, ...$ vanish (see, for example, [2], [6], [7]). Also, the critical point $(0,0)$ is a center if the system (1) has an analytic in $(0,0)$ first integral $F(x,y)$.

The homogeneous system associated to the system (1) has the form

$$\begin{cases} \dot{x} = yZ^2 + (ax^2 + cxy + fy^2)Z + kx^3 + mx^2y + pxy^2 + ry^3 \equiv P(x,y,Z), \\ \dot{y} = -(xZ^2 + (gx^2 + dxy + by^2)Z + sx^3 + qx^2y + nxy^2 + ly^3) \equiv Q(x,y,Z). \end{cases}$$

Denote $\mathbb{X} = p\,(x,y)\,\frac{\partial}{\partial x} + q\,(x,y)\,\frac{\partial}{\partial y}$, $\mathbb{X}_\infty = P\,(x,y,Z)\,\frac{\partial}{\partial x} + Q\,(x,y,Z)\,\frac{\partial}{\partial y}$ and $E_\infty = P \cdot \mathbb{X}_\infty(Q) - Q \cdot \mathbb{X}_\infty(P)$. The polynomial E_∞ has the form $E_\infty = C_2(x,y) + C_3(x,y)Z + C_4(x,y)Z^2 + C_5(x,y)Z^3 + C_6(x,y)Z^4 + C_7(x,Y)Z^5 + C_8(x,y)Z^6$, where $C_j(x,y),\ j = 2,...,8$, are polynomial in x and y. For example,

$$\begin{array}{l} C_j(x,y) = D_j(a,b,c,d,f,g,k,l,m,n,p,q,r,s,x,y) \\ \qquad\qquad +D_j(b,a,d,c,g,f,l,k,n,m,q,p,s,r,y,x),\ \ j = 5,6, \end{array} \tag{2}$$

where

$$\begin{array}{l} D_5(a,b,c,d,f,g,k,l,m,n,p,q,r,s,x,y) = \\ \quad (adg - cg^2 + 3ak + dk - 2gm + aq - 2cs - 2gs)x^5 \\ \quad +(a^2d + ad^2 + 2abg - acg - cdg - 2fg^2 + 2bk + 2ck - gk \\ \quad +2am - dm \quad + 2an - 4gp - cq - gq - 4as - ds - 4fs)x^4y \\ \quad +(2a^2b + 3abd + acd - c^2g - 2afg - 3dfg + dk + fk + 3al \\ \quad +cm - 2gm + ap - 3dp - 2aq - 3fq - 6gr - 5cs)x^3y^2, \\ D_6(a,b,c,d,f,g,k,l,m,n,p,q,r,s,x,y) = \\ (a^2 + ad - 2cg - g^2 - m - 2s)x^4 \\ +(2ab + ac - cd - 2ag - dg - 4fg + k - 2p - q)x^3y - 3(cg + r)x^2y^2 \end{array}$$

We say that the line at infinity $Z = 0$ has *multiplicity* ν if $C_2(x,y) \equiv 0, ..., C_\nu(x,y) \equiv 0$, $C_{\nu+1}(x,y) \not\equiv 0$, i.e. $\nu - 1$ is the greatest positive integer such that $Z^{\nu-1}$ divides E_∞. In particular, $Z = 0$ has multiplicity five if the following identity and non-identity in Z:

$$C_2(x,y) + C_3(x,y)Z + C_4(x,y)Z^2 + C_5(x,y)Z^3 \equiv 0,\ \ C_6(x,y) \not\equiv 0, \tag{3}$$

holds, i.e. $C_2(x,y) \equiv 0$, $C_3(x,y) \equiv 0$, $C_4(x,y) \equiv 0$, $C_5(x,y) \equiv 0$ and $C_6(x,y) \not\equiv 0$. If $C_2(x,y) \not\equiv 0$, then we say that $Z = 0$ has the multiplicity one. Denote by $m(Z_\infty)$ the multiplicity of the line at infinity $Z = 0$.

The algebraic line $f(x,y) = 0$ is called *invariant* for (1) if there exists a polynomial $K_f \in \mathbb{C}[x,y]$ such that the identity $\mathbb{X}(f) \equiv f \cdot K_f(x,y)$ holds. In particular, *a straight line* $\mathcal{L} \equiv \alpha x + \beta y + \gamma = 0$, $\alpha,\beta,\gamma \in \mathbb{C}$ is called *invariant* for the system (1) if there exists a polynomial $K_\mathcal{L} \in \mathbb{C}[x,y]$ such that the identity $\alpha p(x,y) + \beta q(x,y) \equiv (\alpha x + \beta y + \gamma)K_\mathcal{L}(x,y)$, $(x,y) \in \mathbb{R}^2$, i.e. $\mathbb{X}(\mathcal{L}) \equiv \mathcal{L}(x,y)K_\mathcal{L}(x,y)$, $(x,y) \in \mathbb{R}^2$, holds. Some notions on multiplicity (algebraic, integrable, infinitesimal, geometric) of an invariant algebraic line and its equivalence for polynomial differential systems are given in [1].

The cubic differential systems with multiple invariant straight lines (including the line at infinity) were studied in [11], [14], and the center problem for (1) with invariant straight lines was considered in [2], [3], [4], [5], [8], [10], [12], [13], [15].

2. Cubic systems (1) with the line at infinity of maximal multiplicity

Let $\mathcal{X} = (x, y)$, $\mathcal{A}_2 = (a, b, c, d, f, g)$, $\mathcal{A}_3 = (k, l, m, n, p, q, r, s)$, $\mathcal{U} = (u, v)$, $\mathcal{B}_2 = (A, B, C, D, F, G)$, $\mathcal{B}_3 = (K, L, M, N, P, Q, R, S)$ and $\mathcal{X} = 2^{-1}M_1\mathcal{U}$,

$$\mathcal{A}_2 = 2^{-3}\mathcal{M}_2\mathcal{B}_2,\ \mathcal{A}_3 = 2^{-4}\mathcal{M}_3\mathcal{B}_3, \tag{4}$$

where

$$\mathcal{M}_1 = \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}, \quad \mathcal{M}_2 = \begin{pmatrix} -i & i & -i & i & -i & i \\ -1 & -1 & 1 & 1 & -1 & -1 \\ -2 & -2 & 0 & 0 & 2 & 2 \\ -2i & 2i & 0 & 0 & 2i & -2i \\ i & -i & -i & i & i & -i \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix},$$

$$\mathcal{M}_3 = \begin{pmatrix} -i & i & -i & i & -i & i & -i & i \\ i & -i & -i & i & i & -i & -i & i \\ -3 & -3 & -1 & -1 & 1 & 1 & 3 & 3 \\ -3 & -3 & 1 & 1 & 1 & 1 & -3 & -3 \\ 3i & -3i & -i & i & -i & i & 3i & -3i \\ -3i & 3i & -i & i & i & -i & 3i & -3i \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix},$$

$\det\mathcal{M}_1 = -2i$, $\det\mathcal{M}_2 = -2^9 i$, $\det\mathcal{M}_3 = 2^{16}$, $i^2 = -1$.

We remark that, in general, the elements $\mathcal{U}$, $\mathcal{B}_2$, $\mathcal{B}_3$ are complex and $v = \overline{u}$,

$$B = \overline{A},\ D = \overline{C},\ G = \overline{F},\ L = \overline{K},\ N = \overline{M},\ Q = \overline{P},\ S = \overline{R}. \tag{5}$$

In $u, v, A, B, ..., R, S$ the identity (3), up to a non zero factor, looks as

$$M_2(u, v) + M_3(u, v)Z + M_4(u, v)Z^2 + M_5(u, v)Z^3 \equiv 0,$$

where

$$\begin{aligned}
&M_j(u, v){=}2^{j-12}(N_j(u, v) + \overline{N_j(u, v)}),\ j{=}2, 3, 4, 5,\\
&N_5(u, v) = u^3((A^2D - ACG - 2CK + 4GK - 2AM - 2AS)u^2\\
&+(2A^2B + ACD - C^2G + ADG - 2AFG - CG^2 + 10DK - 4FK\\
&-4CM - 2GM - 4AP + 4AQ - 8CS - 10GS)uv + (3ABC + AD^2\\
&+2ABG - CDG - 3CFG - 2FG^2 + 16BK + 4DM - 6FM\\
&+10AN - 6CP - 8GP - 2CQ - 8GQ - 6AR - 8DS - 14FS)v^2),
\end{aligned}$$

and for $C_6(x,y)$ we have $M_6(u,v) = 2^{-5}(N_6(u,v) + \overline{N_6(u,v)})$,

$$N_6(u,v) = u^2((AC-2K)u^2 + (C^2 - 3AD + 2AF + 3CG + 2G^2 + 2M + 10S)uv \\ +3(CF - AB + FG + 2P)v^2).$$

Solving the series of identities

$$\{M_2(u,v) \equiv 0,\ M_3(u,v) \equiv 0,\ M_4(u,v) \equiv 0\}$$

the following Theorem is obtained in [9]:

Theorem 2.1. *The line at infinity has for cubic system* (1) *the multiplicity:*

- at least two ($m(Z_\infty) \geq 2$: $M_2(u,v) \equiv 0$) if and only if the coefficients of (1) *verify one of the following three sets of conditions:*

$$2.1) K = L = R = S = 0,\ P = \alpha M, Q = N/\alpha,\ MN \neq 0,\ \alpha \in \mathbb{C},\ \alpha\overline{\alpha} = 1;$$

$$2.2) M = N = P = Q = 0,\ R = \beta K,\ S = L/\beta,\ KL \neq 0, \beta \in \mathbb{C},\ \beta\overline{\beta} = 1;$$

$$2.3) P = \gamma N,\ Q = M/\gamma,\ R = \gamma L,\ S = K/\gamma,\ KLMN \neq 0,\ \gamma \in \mathbb{C},\ \gamma\overline{\gamma} = 1;$$

-at least three ($m(Z_\infty) \geq 3$: $\{M_2(u,v) \equiv 0, M_3(u,v) \equiv 0\}$) iff

$$3.1) K = L = R = S = 0, F = B/\alpha, G = \alpha A, N = \alpha^2 M, \\ P = \alpha M, Q = \alpha M, M \neq 0, \alpha\overline{\alpha} = 1;$$

$$3.2) K = L = R = S = 0,\ D = CN/(\alpha M),\ F = \alpha BM/N,\ G = AN/(\alpha M), \\ P = \alpha M, Q = N/\alpha, M(N - \alpha^2 M) \neq 0, \alpha\overline{\alpha} = 1;$$

$$3.3) M = N = P = Q = 0,\ C = \beta DK/L,\ F = \beta BK/L, \\ G = AL/(\beta K),\ R = \beta K,\ S = L/\beta,\ \beta\overline{\beta} = 1;$$

$$3.4) M = N = P = Q = 0,\ F = D + (\beta^2 BK^2 - CL^2)/(\beta KL), \\ G = C + (AL^2 - \beta^2 DK^2)/(\beta KL),\ R = \beta K,\ S = L/\beta, \\ L^3 - \beta^4 K^3 = 0,\ \beta\overline{\beta} = 1;$$

$$3.5) D = C/\gamma, F = B\gamma, G = A/\gamma, P = \gamma N, R = \gamma L, \\ Q = M/\gamma, S = K/\gamma, KM \neq 0, \gamma\overline{\gamma} = 1;$$

$$3.6) D = (CL\gamma^3 + (F - B\gamma)(K - M\gamma))/(L\gamma^4),\ Q = M/\gamma, S = K/\gamma, \\ G = (K(B\gamma - F) + AL\gamma^2)/(L\gamma^3),\ N = (-K + M\gamma + L\gamma^4)/\gamma^3, \\ P = (-K + M\gamma + L\gamma^4)/\gamma^2, R = L\gamma, M(F - B\gamma) \neq 0, \gamma\overline{\gamma} = 1.$$

(In the cases 3.1) − 3.4) the multiplicity is exactly three);

- *at least four* ($m(Z_\infty) \geq 4: \{M_2(u,v) \equiv 0, M_3(u,v) \equiv 0, M_4(u,v) \equiv 0\}$) *iff*

4.1) $D = CS/K, F = BK/S, G = AS/K, L = -S^4/K^3,$
$M = S, N = R = -S^3/K^2, Q = -P = S^2/K;$

4.2) $A = 2(K^3L + S^4)/(S^2(BK - FS)) - S(BK - 2FS)/(KL),$
$R = KL/S,\ C = 2(K^3L + S^4)/(KS(BK - FS))$
$-(BK^4L - 2FK^3LS - FS^5)/(K^2LS^2),$
$D = (FK^2L + BS^3)/(K^2L) + 2(K^3L + S^4)/(K^2(BK - FS)),$
$G = FS^3/(K^2L) + 2(K^3L + S^4)/(KS(BK - FS)),$
$M = (K^3L + 2S^4)/S^3, N = (2K^3L + S^4)/(K^2S),$
$P = (2K^3L + S^4)/(KS^2), Q = (K^3L + 2S^4)/(KS^2).$

Solving in each of conditions 4.1) and 4.2) the identity $M_5(u,v) \equiv 0$, we obtain

Theorem 2.2. *The system* (1) *has the line at infinity of multiplicity five if and only if its coefficients verify one of the following three sets of conditions:*

$$\begin{gathered} B = -AS^3/K^3, C = D = 0, F = -AS^2/K^2, G = AS/K, L = -S^4/K^3, \\ M = S, N = -S^3/K^2, P = -S^2/K, R = -S^3/K^2, Q = S^2/K; \end{gathered} \tag{6}$$

$$\begin{gathered} A = 5F^3/B^2, C = -6F^2/B,\ D = 2F,\ G = -3F^2/B,\ K = F^5/B^3, \\ L = BF,\ M = -3F^4/B^2,\ N = -3F^2,\ P = 3F^3/B,\ Q = 3F^3/B, \\ R = -F^2,\ S = -F^4/B^2,\ F \neq 0; \end{gathered} \tag{7}$$

$$\begin{gathered} A = (F^3K^2 + 8BKS^2 + 4FS^3)/(B^2K^2),\ D = 2F,\ L = S^4/K^3, \\ C = 2(F^2K^2 + 4S^3)/(BK^2),\ G = (F^2K^2 + 4S^3)/(BK^2),\ M = 3S, \\ N = 3S^3/K^2,\ P = 3S^2/K,\ Q = 3S^2/K,\ R = S^3/K^2, \\ K^2(BK - FS)^2 + 4S^5 = 0. \end{gathered} \tag{8}$$

Theorem 2.3. *In the class of cubic differential systems of the form* (1) *the maximal multiplicity of the line at infinity is five.*

Indeed, under the conditions (6), (7) and (8) the polynomial $M_6(u,v)$ becomes, respectively:

$$(Ku - Sv)(Ku + Sv)(K^2u^2 - 6KSuv + S^2v^2)/K^3 \not\equiv 0;$$
$$F^2u(Bv - 4Fu)(Bv - Fu)^2/B^3 \not\equiv 0;$$
$$\begin{gathered} (B^3K^5 - F^5K^4 - 8BF^2K^3S^2 - 8F^3K^2S^3 - 32BKS^5 - 16FS^6)u^4 \\ -4B(F^4K^4 + 2B^2K^4S - 4BFK^3S^2 + 10F^2K^2S^3 + 24S^6)u^3v \\ +6B^2K^2(BKS^2 - F^3K^2 - 4FS^3)u^2v^2 - 4B^3F^2K^4uv^3 \\ -B^3K(BFK^3 - S^4)v^4 \not\equiv 0. \end{gathered}$$

In the expressions of $M_6(u,v)$ we have neglected non-zero numerical factors.

Taking into account (4) and (5), the equalities (6) give us the following four series of conditions in the real coefficient of system (1):

$$a=b=c=f=g=k=l=0,\ m=n=p=r=s=0,\ q\neq 0; \tag{9}$$

$$\begin{gathered} b=-as/k,\ c=a(k^2-s^2)/(ks),\ d=a(k^2-s^2)/k^2,\\ f=-a, g=as/k,\ l=-k,\ m=(2k^2-s^2)/s,\ n=(k^2-2s^2)/s,\\ p=k(k^2-2s^2)/s^2,\ q=(2k^2-s^2)/k,\ r=-k^2/s; \end{gathered} \tag{10}$$

$$a=b=d=f=g=k=l=0,\ m=n=q=r=s=0,\ p\neq 0; \tag{11}$$

$$\begin{gathered} a=-br/l,\ c=b(l^2-r^2)/l^2,\ d=b(l^2-r^2)/(lr),\\ f=br/l,\ g=-b, k=-l,\ m=(l^2-2r^2)/r,\ n=(2l^2-r^2)/r,\\ p=(2l^2-r^2)/l,\ q=l(l^2-2r^2)/r^2,\ s=-l^2/r, \end{gathered} \tag{12}$$

and the equalities (8) give us eight real series of conditions:

$$b=c=0,\ d=2a,\ f=k=l=m=n=p=q=r=0,\ s=a^2,\ a\neq 0; \tag{13}$$

$$\begin{gathered} a=0,\ b=-gk^2/s^2,\ c=-2gk^2/s^2,\ d=0,\ f=-2gk^3/s^3,\ l=k^3/s^2,\\ m=n=3k^2/s,\ p=3k^3/s^2,\ q=3k,\ r=k^4/s^3,\ g^2k^2-k^2s-s^3=0; \end{gathered} \tag{14}$$

$$\begin{gathered} c=2b=-2as/k,\ f=-a(k^2+2s^2)/s^2,\ g=(k^2+a^2s)/(ak),\\ m=n=3k^2/s,\ p=3l=3k^3/s^2,\ q=3k,\ r=k^4/s^3,\ d=2a,\\ k^4-a^2k^2s-a^2s^3=0; \end{gathered} \tag{15}$$

$$\begin{gathered} b=k(-agk+k^2+a^2s+s^2)/(s(as-gk)),\ c=2k(2as-gk)/s^2,\\ d=2a,\ f=k^2(3as-2gk)/s^3,\ l=k^3/s^2,\ m=3k^2/s,\ n=3k^2/s,\\ p=3k^3/s^2,\ q=3k,\ r=k^4/s^3,\ g^2k^2-2agks-k^2s+a^2s^2-s^3=0. \end{gathered} \tag{16}$$

$$a=d=0,\ c=2b,\ g=k=l=m=n=p=q=s=0,\ r=a^2,\ b\neq 0; \tag{17}$$

$$\begin{gathered} b=c=0,\ a=-fl^2/r^2,\ d=-2fl^2/r^2,\ g=-2fl^3/r^3,\ k=l^3/r^2,\\ m=n=3l^2/r,\ p=3l,\ q=3l^3/r^2,\ s=l^4/r^3,\ f^2l^2-l^2r-r^3=0; \end{gathered} \tag{18}$$

$$\begin{gathered} d=2a=-2br/l,\ f=(l^2+b^2r)/(bl),\ g=-b(l^2+2r^2)/r^2,\\ m=n=3l^2/r,\ p=3l,\ q=3k=3l^3/r^2,\ s=l^4/r^3,\ c=2b,\\ l^4-b^2l^2r-b^2r^3=0; \end{gathered} \tag{19}$$

$$\begin{gathered} a=l(-bfl+l^2+b^2r+r^2)/(r(br-fl)),\ d=2l(2br-fl)/r^2,\\ c=2b,\ g=l^2(3br-2fl)/r^3,\ k=l^3/r^2,\ m=3l^2/r,\ n=3l^2/r,\\ p=3l,\ q=3l^3/r^2,\ s=l^4/r^3,\ f^2l^2-2bflr-l^2r+b^2r^2-r^3=0. \end{gathered} \tag{20}$$

Remark 2.1. 1) The set of equalities (7) is not satisfied in the real coefficients of cubic system (1).

2) The transformation $x \leftrightarrow y$, $t \to -t$ reduce the system $\{(1), (11)\}$ (respectively, $\{(1), (12)\}$ $\{(1), (17)\}$ $\{(1), (18)\}$ $\{(1), (19)\}$ $\{(1), (20)\}$ to the system $\{(1), (9)\}$ (respectively, $\{(1), (10)$ $\{(1),(13)\}$ $\{(1), (14)\}$ $\{(1), (15)\}$ $\{(1), (16)\}$).

Theorem 2.4. *The real cubic system* (1) *has the line at infinity of multiplicity five if and only if one of the following twelve sets of conditions* (13) − (20) *holds.*

3. Solution of the center problem for cubic systems with the line at infinity of maximal multiplicity.

In each of the series of conditions (6), (7) and (8) we calculate the first Lyapunov quantity L_1. In the cases (7) and (8) this quantity vanishes and the divergence of vector field $\mathbb{X}$ associated to system (1) also vanishes.

In the case (6) we have $L_1 = 2iS^2/K \neq 0$ (see, (5), (6)) and therefore, $(0,0)$ is a focus.

In this way we prove the statements of the following two theorem.

Theorem 3.1. *The cubic system* (1) *with the line at infinity of maximal multiplicity has a center at the origin* $(0,0)$ *if and only if the first Lyapunov quantity vanishes* $L_1 = 0$.

Theorem 3.2. *The cubic system* (1) *with the line at infinity of maximal multiplicity has a center at the origin* $(0,0)$ *if and only if the divergence of the vector field* $\mathbb{X}$ *associated to system* (1) *vanishes, i.e. iff* (1) *has a polynomial first integral.*

In the cases of real conditions (9)-(12) we have, respectively, $L_1 = q \neq 0$, $L_1 = -(k^2+s^2)^2/(ks^2) \neq 0, L_2 = -p \neq 0$, $L_1 = (l^2+r^2)^2/(lr^2) \neq 0$. Therefore, in each of the cases (9)-(12) the origin is a focus for (1).

In each of the cases (13)-(20) the first Lyapunov quantity and the divergence vanishes.

The first integrals $\mathcal{F}$ of the systems $\{(1),(13)\} - \{(1),(20)\}$ are, respectively,

$$\mathcal{F} = 6(x^2+y^2) + 4gx^3 + 12ax^2y + 3a^2x^4;$$

$$\mathcal{F} = 6s^3(x^2+y^2) + (sx+ky)^2(4gsx - 8gky + 3s^2x^2 + 6ksxy + 3k^2y^2);$$

$$\mathcal{F} = 6aks^3(x^2+y^2) + 4s^3(k^2+a^2s)x^3 + 12a^2s^3xy(kx-sy)$$
$$-4a^2ks(k^2+2s^2)y^3 + 3ak(sx+ky)^4;$$

$$\mathcal{F} = 6s^3(x^2+y^2) + (sx+ky)^2(4(gsx - 2gky + 3asy) + 3(sx+ky)^2);$$

$$\mathcal{F} = 6(x^2+y^2) + 12bxy^2 + 4fy^3 + 3b^2y^4;$$

$$\mathcal{F} = 6r^3(x^2+y^2) - (lx+ry)^2(8flx - 4fry - 3l^2x^2 - 6lrxy - 3r^2y^2);$$

$$\mathcal{F} = 6blr^3(x^2+y^2) - 4b^2lr(l^2+2r^2)x^3 - 12b^2r^3xy(rx-ly) \\ +4r^3(l^2+b^2r)y^3 + 3bl(lx+ry)^4;$$

$$\mathcal{F} = 6r^3(x^2+y^2) + (lx+ry)^2(4(3brx - 2flx + fry) + 3(lx+ry)^2).$$

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Received: October 15, 2022 *Accepted: December 06, 2022*

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Acta et Commentationes, Exact and Natural Sciences ISSN: 2537-6284
Volume 2(14), 2022, Pages 47–52 E-ISSN: 2587-3644
CZU 512.548 (043.3) DOI: 10.36120/2587-3644.v14i2.47-52

Schröder T-quasigroups of generalized associativity

Victor Shcherbacov

Abstract. We prolong research of Schröder quasigroups and Schröder T-quasigroups [14].

2010 Mathematics Subject Classification: 20N05.

Keywords: quasigroup, loop, groupoid, Schröder quasigroups, Schröder identity.

Schröder T-cvasigrupuri de asociativitate generalizată

Rezumat. Se extinde cercetarea quasigrupurilor de tip Schröder şi a T-quasigrupurilor de tip Schröder [14].

Cuvinte-cheie: quasigrup, buclă, grupoid, quasigrupuri de tip Schröder, identitatea Schröder.

1. Introduction

Necessary definitions can be found in [1, 3, 2, 7, 10, 15].

Definition 1.1. Binary groupoid $(Q, \circ)$ is called a left quasigroup if for any ordered pair $(a, b) \in Q^2$ there exist the unique solution $x \in Q$ to the equation $a \circ x = b$ [1].

Definition 1.2. Binary groupoid $(Q, \circ)$ is called a right quasigroup if for any ordered pair $(a, b) \in Q^2$ there exist the unique solution $y \in Q$ to the equation $y \circ a = b$ [1].

Definition 1.3. A quasigroup $(Q, \cdot)$ with an element $1 \in Q$, such that $1 \cdot x = x \cdot 1 = x$ for all $x \in Q$, is called a *loop*.

Definition 1.4. Binary groupoid $(Q, \cdot)$ is called medial if this groupoid satisfies the following medial identity:

$$xy \cdot uv = xu \cdot yv \tag{1}$$

for all $x, y, u, v \in Q$ [1].

We recall

Definition 1.5. Quasigroup $(Q, \cdot)$ is a T-quasigroup if and only if there exists an abelian group $(Q, +)$, its automorphisms φ and ψ, and a fixed element $a \in Q$ such that $x \cdot y = \varphi x + \psi y + a$ for all $x, y \in Q$ [8].

A T-quasigroup with the additional condition $\varphi\psi = \psi\varphi$ is medial.

Definition 1.6. Garrett Birkhoff [2] has defined an equational quasigroup as an algebra with three binary operations $(Q, \cdot, /, \backslash)$ that satisfies the following six identities:

$$x \cdot (x\backslash y) = y, \tag{2}$$

$$(y/x) \cdot x = y, \tag{3}$$

$$x\backslash(x \cdot y) = y, \tag{4}$$

$$(y \cdot x)/x = y, \tag{5}$$

$$x/(y\backslash x) = y, \tag{6}$$

$$(x/y)\backslash x = y. \tag{7}$$

Ernst Schröder (a German mathematician mainly known for his work on algebraic logic) introduced and studied the following identity of generalized associativity [13]:

$$(y \cdot z)\backslash x = z(x \cdot y). \tag{8}$$

In the quasigroup case the identity (8) is equivalent to the following identity [11]:

$$(y \cdot z) \cdot (z \cdot (x \cdot y)) = x \tag{9}$$

If in the idempotent quasigroup $(Q; \cdot)$, the identity (9), we put $x = y$, then we obtain the following standard Schröder's identity:

$$(x \cdot y) \cdot (y \cdot x) = x. \tag{10}$$

Definition 1.7. Any quasigroup with the identity (10) is called a Schröder quasigroup.

So we have different objects that have name Schröder. Namely,

(i) the following identity of generalized associativity on groupoids [13]:

$(y \cdot z)\backslash x = z(x \cdot y)$ (8);

(ii) the Schröder's identity of generalized associativity in quasigroups (9);

(iii) the Schröder's identity (Schröder's 2-nd identity [12]) $(x \cdot y) \cdot (y \cdot x) = x$ (10);

(iv) identity

$$(x \cdot y) \cdot (y \cdot x) = y \tag{11}$$

is named by Albert Sade [12] as Stein's 3-rd identity.

Many information about these identities is given in articles [4, 5]. We tried do not repeat information from these articles here.

Each of these identities deserves a separate study in the class of groupoids, left (right) quasigroups; in the classes of quasigroups and of T-quasigroups.

1.1. Schröder's identity of generalized associativity in quasigroups

It is convenient to call this identity the Schröder's identity of generalized associativity.

Often various variants of associative identity, which are true in a quasigroup, guarantee that this quasigroup is a loop.

It is not so in the case with the identity. We give an example of quasigroup which is not a loop with the identity (9) [11]. See also [15]. A quasigroup from this example does not have left and right identity element.

Quasigroups with Schröder's identity of generalized associativity are not necessary idempotent and associative. See the following example [11].

$\cdot$	0	1	2	3	4	5	6	7
0	1	4	7	0	6	5	2	3
1	5	2	3	6	0	1	4	7
2	0	7	4	1	5	6	3	2
3	6	3	2	5	1	0	7	4
4	4	1	0	7	3	2	5	6
5	3	6	5	2	4	7	0	1
6	7	0	1	4	2	3	6	5
7	2	5	6	3	7	4	1	0

The left cancellation (left division) groupoid with the identity (9) and with the identity $(x/x = y/y)$ (in a quasigroup this identity guarantees existence of the left identity element) is a commutative group of exponent two [11].

The similar results are true for the right case [11]. In this case we use the identity

$$(x\backslash x = y\backslash y).$$

It is clear that this result is true for any quasigroup with the left or right identity element.

Notice, any 2-group $(G, +)$ (in such group $x + x = 0$ for any $x \in G$) satisfies Schröder's identity of generalized associativity.

2. Schröder's identity of generalized associativity in T-quasigroups

Theorem 2.1. *In T-quasigroup $(Q, \cdot)$ of the form $x \cdot y = \varphi x + \psi y$ Schröder's identity of generalized associativity is true if and only if $\varphi x = \psi^{-2}x$, $\varepsilon = \varphi^7$, $\varepsilon = \psi^{14}$, $\varphi\psi z + \psi\varphi z = 0$.*

Proof. We rewrite identity (9) in the following form:

$$\varphi^2 y + \psi^3 y + \varphi\psi z + \psi\varphi z + \psi^2\varphi x = x. \tag{12}$$

If we substitute in equality (12) $y = z = 0$, then we have

$$\varphi x = \psi^{-2} x. \tag{13}$$

If we substitute in equality (12) $x = z = 0$, then we have

$$\varphi^2 y + \psi^3 y = 0. \tag{14}$$

Taking into consideration equality (13), we can re-write equality (14) in the form

$$\psi^{-4} y + \psi^3 y = 0, \tag{15}$$

or in the form

$$\psi^3 = I\psi^{-4}, \tag{16}$$

where $Ix = -x$ for all $x \in Q$. Notice, the permutation I is an automorphism of the group $(Q, +)$ here. Therefore, we can rewrite previous equalities in the form

$$\varepsilon = I\psi^{-7}, I = \psi^{-7}, \varepsilon = \psi^{-14}, \varepsilon = \psi^{14}, \varepsilon = \varphi^7. \tag{17}$$

If we substitute in equality (12) $x = y = 0$ then we have

$$\varphi\psi z + \psi\varphi z = 0. \tag{18}$$

Converse. If we substitute in identity (9) the expression $x \cdot y = \varphi x + \psi y$, then we obtain equality (12), which is true taking into consideration the equalities (13), (14), (18). Then we obtain, that identity (9) is true in this case. □

Corollary 2.1. *In medial quasigroup $(Q, \cdot)$ of the form $x \cdot y = \varphi x + \psi y$ Schröder's identity of generalized associativity is true if and only if the group $(Q, +)$ is an abelian 2-group (i.e. $x + x = 0$ for any $x \in Q$), $\varphi x = \psi^{-2} x$, $\varepsilon = \varphi^7$, $\varepsilon = \psi^{14}$.*

Proof. From the identity of mediality it follows that $\varphi\psi z + \psi\varphi z = 2 \cdot \varphi\psi z = 0$ for all $z \in Q$, i.e., the group $(Q, +)$ is an abelian 2-group. □

Example 2.1. We present elements of the group $(Z_2^3, +)$ in the following form: $1 = (000), 2 = (001), 3 = (010), 4 = (011), 5 = (100), 6 = (101), 7 = (110), 8 = (111)$.

+	1	2	3	4	5	6	7	8
1	1	2	3	4	5	6	7	8
2	2	1	4	3	6	5	8	7
3	3	4	1	2	7	8	5	6
4	4	3	1	1	8	7	6	5
5	5	6	7	8	1	2	3	4
6	6	5	8	7	2	1	4	3
7	7	8	5	6	3	4	1	2
8	8	7	6	5	4	3	2	1

We can see on the group $Aut(Z_2^3, +)$ as on the group $GL(3, 2)$. This group is the group of non-degenerate matrices of size 3×3 over the field of order 2 relatively to standard multiplication of matrices [7].

The group $PSL(2, 7)$ is the group of non-degenerate matrices of size 2×2 over the field of order 7. These groups are isomorphic, i.e., $Aut(Z_2^3, +) \cong GL(3, 2) \cong PSL(2, 7)$. Notice $|(GL(3, 2)| = 168 = 3 \times 7 \times 8$ [7].

We have the following automorphisms of the group $Aut(Z_2^3, +) \cong GL(3, 2)$:

$$\varphi = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \ \psi = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

Notice that $\varphi^7 = \psi^7 = \varepsilon, \varphi = \psi^{-2}, \varphi\psi = \psi\varphi$. Therefore, Schröder's medial quasigroup $(Q, \circ)$ of generalized associativity can have the form $x \circ y = \varphi x + \psi y$:

$\circ$	1	2	3	4	5	6	7	8
1	1	4	8	5	3	2	6	7
2	3	2	6	7	1	4	8	5
3	6	7	3	2	8	5	1	4
4	8	5	1	4	6	7	3	2
5	7	6	2	3	5	8	4	1
6	5	8	4	1	7	6	2	3
7	4	1	5	8	2	3	7	6
8	2	3	7	6	4	1	5	8

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Received: October 26, 2022 *Accepted: December 08, 2022*

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Acta et Commentationes, Exact and Natural Sciences ISSN: 2537-6284
Volume 2(14), 2022, Pages 53–56 E-ISSN: 2587-3644
CZU 517.98 DOI: 10.36120/2587-3644.v14i2.53-56

On regular operators on Banach Lattices

Omer Gok

Abstract. Let E and F be Banach lattices and X and Y be Banach spaces. A linear operator $T : E \to F$ is called regular if it is the difference of two positive operators. $L_r(E, F)$ denotes the vector space of all regular operators from E into F. A continuous linear operator $T : E \to X$ is called M-weakly compact operator if for every disjoint bounded sequence (x_n) in E, we have $lim_{n\to\infty} \|Tx_n\| = 0$. $W^r_M(E, F)$ denotes the regular M-weakly compact operators from E into F. This paper is devoted to the study of regular operators and M-weakly compact operators on Banach lattices. We show that F has a b-property if and only if $L_r(E, F)$ has b-property. Also, $W^r_M(E, F)$ is a KB-space if and only if F is a KB-space.

2010 Mathematics Subject Classification: 46B25, 46B42, 47B60, 47B65.
Keywords: Banach lattice, regular operators, M-weakly compact operators, order continuous norm.

Operatori regulari pe latice Banach

Rezumat. Fie E şi F latice Banach, iar X şi Y spaţii Banach. Operatorul linear $T : E \to F$ se numeşte regular dacă reprezintă diferenţa a doi operatori pozitivi. $L_r(E, F)$ este spaţiul vectorial al operatorilor regulari din E în F. Operatorul linear şi continuu $T : E \to X$ se numeşte operator M-slab compact dacă pentru orice şiruri mărginite şi disjuncte (x_n) din E, urmează că $lim_{n\to\infty} \|Tx_n\| = 0$. $W^r_M(E, F)$ reprezintă operatorii regulari M-slab compacţi din E în F. Lucrarea este dedicată studiului operatorilor regulari şi operatorilor M-slab compacţi pe latice Banach. Se demonstrează că F posedă b-proprietate dacă şi numai dacă $L_r(E, F)$ are b-proprietate. La fel, $W^r_M(E, F)$ este KB-spaţiu dacă şi numai dacă F este KB-spaţiu.

Cuvinte-cheie: latice Banach, operatori regulari, operatori M-slab compacţi, norma continue de ordine.

1. Introduction

Let X be a Banach space and E be a Banach lattice. E^+ denotes the positive cone of E. That is, $E^+ = \{x : 0 \leq x\}$. We denote $E^\sim$ by the set of all order bounded linear functionals on E, and $E^{\sim\sim}$ by the set of all second order dual of E, [7]. By X' we denote the set of all continuous linear functionals on X. Since E is a Banach lattice, order dual and continuous dual coincide [1, 5].

A set $[x, y] = \{z \in E : x \leq z \leq y\}$ in a Banach lattice E is called an order interval. Let A be a subset of E. The set A is called order bounded if $A \subseteq [x, y]$ for some $x, y \in E$. A is called a b-order bounded if A is an order bounded in the second order dual E'' of E.

A Banach lattice is said to have b-property if every b-order bounded set is an order bounded set in E [2, 3]. Order dual of a Banach lattice has b-property. The space $C(K)$ of all continuous real valued functions defined on a compact Hausdorff space K has b-property.

A Banach lattice E is called a KB-space if every positive increasing norm bounded sequence in E converges. A Banach lattice E is a KB space if and only if it has an order continuous norm and with property (b) [2]. Reflexive Banach lattice, AL spaces are examples of KB spaces. There are a lot of KB spaces in Banach lattices[1, 5].

A Banach lattice E is said to have an order continuous norm if $x_n \downarrow 0$ in E implies $\| x_n \| \to 0$ as $n \to \infty$. For example, Banach space c_0 of all sequences converging to zero has an order continuous norm. Let E be a Banach lattice. E' is a KB space if and only if E has an order continuous norm.

A Banach lattice E is called Dedekind complete if every non empty subset of E, which is bounded from above, has a supremum. Alternatively, every non empty subset of E, which is bounded from below, has an infimum.

2. KB space of M-weakly compact operators

Definition 2.1. [1, 5] Let X be a Banach space and E be a Banach lattice. A continuous linear operator $T : X \to E$ is called L-weakly compact if $T(ball(X))$ is an L-weakly compact set. A subset A of E is called L-weakly compact if $\| x_n \| \to 0$ as $n \to \infty$ for every disjoint sequence (x_n) in the solid hull of A, where $ball(X)$ denotes the clesed unit ball of X.

By $W_L(E, F)$ we denote the set of all L-weakly compact operators.

Definition 2.2. ([1, 5]) A continuous linear operator $T : E \to X$ is called M-weakly compact if $lim_{n\to\infty} \| Tx_n \| = 0$ for every disjoint sequence (x_n) in the closed unit ball of E.

By $W_M(E, F)$ we denote the set of all M-weakly compact operators from a Banach lattice E into a Banach lattice F. If F is a Dedekind complete Banach lattice, then it is a Banach lattice. We denote the linear span of the positive operators in $W_M(E, F)$ by $W^r_M(E, F)$. If F is a Dedekind complete Banach lattice, then $W^r_M(E, F)$ is a Dedekind complete Banach lattice under the regular norm.

Adjoint of an M-weakly compact operator is an L-weakly compact and adjoint of an L-weakly compact operator is an M-weakly compact. Every L-weakly compact and M-weakly compact operators are weakly compact.

Definition 2.3. A Banach lattice E is called an AL space if the norm

$$\| x + y \| = \| x \| + \| y \|$$

holds for every $x, y \in E$.

A linear operator T from a E Banach lattice to a Banach lattice F is called regular if it is the difference of two positive operators from E into F. By $L_r(E, F)$ we denote the vector space of all regular operators from E into F, [6]. Every positive linear operator from a Banach lattice E into a Banach lattice F is continuous. By $L(E, F)$ we denote the vector space of all linear continuous operators from E into F.

A linear operator T from a Banach lattice E into a Banach lattice F is called order bounded if it sends an order bounded set in E to an order bounded set in F. $L_b(E, F)$ denotes the vector space of all order bounded linear operators from E into F. The following inclusions hold: $L_r(E, F) \subseteq L_b(E, F) \subseteq L(E, F)$.

Let $T \in L_r(E, F)$. The regular operator norm of T is given by

$$\|T\| = inf\{\|S\| : |T| \leq S \quad \text{for} \quad 0 \leq S \in L_r(E, F)\}.$$

Theorem 2.1. ([2, 3]) *Let E, F be Banach lattices with F Dedekind complete. Then, $L_r(E, F)$ has b-property if and only if F has b-property.*

Proof. Take a sequence (x_n) in F with the property $x_n \uparrow y$ in F''. Let us choose $0 \neq f \in E'$. We define the map $\psi : F \to L_r(E, F)$, $\psi(y) = f \otimes y$, which is given by $(f \otimes y)(x) = f(x)y$ for all $x \in E$. $\psi(x_n)$ is b-order bounded in $L_r(E, F)$. So, there is a $T \in L_r(E, F)$ such that $0 \leq f \otimes x_n \leq T$. There is an $x \in E$ such that $0 \leq x_n \leq T(x)$ in F. It means F has b-property.

Suppose that F has b-property. Assume that (T_n) in $L_r(E, F)$ such that $0 \leq T_n \uparrow T$ in $L_r(E, F)''$. Let us choose $0 \neq x \in E^+$. We define a map $\varphi : F' \to L_r(E, F)'$ which is defined by $\varphi(f)T = f(Tx)$ for $T \in L_r(E, F)$.This mapping is one-one and positive. From here, for every $x \in E^+$, we have that $T_n(x)$ is b-order bounded in F. That is, $T_n(x)$ is order bounded in F. By Kantorovich lemma, we extend the mapping defined by $T(x) = sup\{T_n(x) : n = 1, 2, 3, ...\}$. Therefore, $L_r(E, F)$ has b-property.

Let E be a Banach lattice. By E^a, we denote the maximal ideal space on which the norm is order continuous. Equivalently,

$$E^a = \{x : any \quad monotone \quad sequence \quad (x_n) \quad in \quad [0, | x |] \quad converges\}.$$

Theorem 2.2. *([4]) Let E, F be Banach lattices with $(E')^a \neq \{0\}$. Then, $W_M^r(E,F)$ has order continuous norm if and only if F has an order continuous norm.*

Let X and Y be Banach spaces and $T : X \to Y$ be a continuous linear operator.The adjoint operator T' of T is defined from Y' into X' by $T'(f)(x) = f(Tx)$ for every $f \in Y'$ and for every $x \in X$.

Theorem 2.3. *Let E, F be Banach lattices. Then, $W_M^r(E,F)$ has b-property if and only if F has b-property.*

Proof. Proof is similar to the proof of Theorem 2.4. So, it is omitted.

Theorem 2.4. *Let E, F be Banach lattices. Then, $W_M^r(E,F)$ is a KB space if and only if F is a KB space.*

Proof. It is proved this result by using the fact that a Banach lattice is a KB space if and only if it has an order continuous norm and it has b-property [2].

$W_L^r(E,F)$ denotes the vector space of all regular L-weakly compact operators. It is a Banach lattice.

Theorem 2.5. *Let E, F be Banach lattices and $(E')^a \neq \{0\}$ and $F^a \neq \{0\}$. Then, the following claims are equivalent:*

(i) $W_L^r(E,F)$ has order continuous norm.

(ii) E' has order continuous norm.

(iii) $W_M^r(F',E')$ is a KB space.

Proof. Proof is done by using [4].

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Received: July 20, 2022 *Accepted: December 10, 2022*

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Acta et Commentationes, Exact and Natural Sciences ISSN: 2537-6284
Volume 2(14), 2022, Pages 57–67 E-ISSN: 2587-3644
CZU 517.925 DOI: 10.36120/2587-3644.v14i2.57-67

Stability of unperturbed motion governed by the ternary differential system of Lyapunov-Darboux type with nonlinearities of degree four

Natalia Neagu, Mihail Popa, and Victor Orlov

Abstract. For the ternary differential system of Lyapunov-Darboux type with nonlinearities of degree four, using the Lie algebra admitted by this system, was obtained the analytic first integral, determined the Lyapunov function and the conditions of stability of the unperturbed motion.
2010 Mathematics Subject Classification: Primary 30E201; Secondary 30E202.
Keywords: differential systems, stability of unperturbed motion, center-affine comitant and invariant, Lia algebra, first integral, Lyapunov function.

Stabilitatea mişcării neperturbate guvernate de sistemul diferenţial ternar de tip Lyapunov-Darboux cu nelinearităţi de gradul patru

Rezumat. Pentru sistemul difereţial ternar de tip Lyapunov-Darboux cu nelinearităţi de gradul patru, utilizând algebra Lie admisă de acest sistem, s-a obţinut integrala primă analitică, determinată funcţia Lyapunov şi condiţiile de stabilitate a mişcării neperturbate.
Cuvinte-cheie: sistem diferenţial, stabilitatea mişcării neperturbate, comitant şi invariant centroafin, algebră Lie, integrală primă, funcţie Lyapunov.

1. Notion of comitant and invariant for ternary differential system

We examine the differential system of the unperturbed motion [1, 2] with nonlinearities of degree four $s^3(1,4)$, written in the tensorial form [3, 4]

$$\frac{dx^j}{dt} = a^j_\alpha x^\alpha + a^j_{\alpha\beta\gamma\delta} x^\alpha x^\beta x^\gamma x^\delta \quad (j,\alpha,\beta,\gamma,\delta = 1,2,3) \tag{1}$$

where $a^j_{\alpha\beta\gamma\delta}$ is a symmetric tensor in lower indices in which the total convolution is done. The centro-affine group $GL(3,\mathbb{R})$ is given by transformations q:

$$\bar{x}^j = q^j_\alpha x^\alpha \quad (\Delta = det(q^j_\alpha) \neq 0) \quad (j,\alpha = 1,2,3). \tag{2}$$

In the theory of invariants [5] the vector $x = (x^1, x^2, x^3)$, which is changed by formulas (2), is usually called *contravariant*. The vector $u = (u_1, u_2, u_3)$, which is changed by formulas $\bar{u}_r = p^j_r u_j$ $(r, j = 1, 2, 3)$, where $p^r_j q^j_s = \delta^r_s$ is the Kroniker's symbol, is call *covariant*. Any other vector $y = (y^1, y^2, y^3)$, different from x, which is changed by formulas (2) $\bar{y}^j = q^j_\alpha y^\alpha$ $(j, \alpha = 1, 2, 3)$, is call *cogradient* with the vector x. The coefficients of the system (1) and the coordinates of the vectors x, u, y take values from the field of real numbers $\mathbb{R}$.

Observe that the transformation (2) preserves the form of the system (1)

$$\frac{d\bar{x}^j}{dt} = \bar{a}^j_\alpha \bar{x}^\alpha + \bar{a}^j_{\alpha\beta\gamma\delta}\bar{x}^\alpha \bar{x}^\beta \bar{x}^\gamma \bar{x}^\delta \quad (j, \alpha, \beta, \gamma, \delta = 1, 2, 3), \tag{3}$$

where the coordinates of the vector $\bar{x} = (\bar{x}^1, \bar{x}^2, \bar{x}^3)$ are determined by the relations (2). The coefficients $\bar{a}^j_\alpha$ şi $\bar{a}^j_{\alpha\beta\gamma\delta}$ from the right–hand sides of (3) are some linear functions in the coefficients of system (1) and rational in the parameters q^j_α of transformations (2).

We will denote the set of coefficients (1) by a and of the system (3) by $\bar{a}$.

Definition 1.1. According to [3, 4, 5], we say that the polynomial $\varkappa(x, y, u, a)$ of the coefficients of system (1) and of the coordinates of vectors x, y and u is called center-affine *mixt comitant* of the system (1) with respect to $GL(3, \mathbb{R})$ group, if the following identity holds

$$\varkappa(\bar{x}, \bar{y}, \bar{u}, \bar{a}) = \Delta^{-g}\varkappa(x, y, u, a) \tag{4}$$

for all q from $GL(3, \mathbb{R})$ and for every coordinates of vectors x, y and u, as well as all the coefficients of the system (1).

Size g is an integer number called *the weight of comitant*.

If the mixt comitant $\varkappa$ does not depend of the coordinates of the vector u, then according to [3, 4, 5], we call it simply *comitant*; but if $\varkappa$ does not depend of the coordinates of the vectors x and y, we call it *contravariant* according to [5]. If $\varkappa$ does not depend of the coordinates of the vectors x, y and u, then we will call it *invariant* of the system (1) with respect to $GL(3, \mathbb{R})$ group.

For simplicity, in some cases, we will omit the words "center-affine" or "with respect to $GL(3, \mathbb{R})$ group" for comitants (invariants).

From [5] it is known that the alternation operation, in the case of ternary tensors, is performed by means of the unit trivector ε^{pqr} $(\varepsilon_{\alpha\beta\gamma})$ with coordinates $\varepsilon^{123} = -\varepsilon^{132} = \varepsilon^{312} = -\varepsilon^{321} = \varepsilon^{231} = -\varepsilon^{213} = 1$ $(\varepsilon_{123} = -\varepsilon_{132} = \varepsilon_{312} = -\varepsilon_{321} = \varepsilon_{231} = -\varepsilon_{213} = 1)$ and $\varepsilon^{pqr} = 0$ $(\varepsilon_{\alpha\beta\gamma} = 0)$ $(p, q, r = 1, 2, 3)$ $((\alpha, \beta, \gamma = 1, 2, 3))$ in the other cases.

From [3, 4, 5] results the following assertion

Theorem 1.1. *The expressions obtained by the product of the coefficients of the tensors* a^j_α *and* $a^j_{\alpha\beta\gamma\delta}$, *of system* (1), *as well as the coordinates* x^i, y^j, u_r *of the vectors* x, y, u, *using* *the alternation operation followed by the total convolution, form the basis of the comitants (mixed), contravariants and invariants of the system* (1) *with respect to* $GL(3,\mathbb{R})$ *group.*

Using Theorem 1.1 it is easy to see that the expressions

$$\varkappa_1 = x^\alpha u_\alpha,\ \ \varkappa_2 = a^\alpha_\beta x^\beta u_\alpha,\ \ \varkappa_3 = a^\alpha_\gamma a^\beta_\alpha x^\gamma u_\beta \tag{5}$$

form *the mixed comitants*, and

$$\delta_1 = a^\alpha_\gamma a^\beta_p a^\gamma_q u_\alpha u_\beta u_r \varepsilon^{pqr} \tag{6}$$

is a *contravariant* of the system (1) with respect to $GL(3,\mathbb{R})$ group.

Likewise the expressions

$$\begin{aligned} \sigma_1 &= a^\alpha_\mu a^\beta_\delta a^\gamma_\alpha x^\delta x^\mu x^\nu \varepsilon_{\beta\gamma\nu}, \\ \eta_1 &= a^\alpha_{\beta\gamma\delta\mu} x^\beta x^\gamma x^\delta x^\mu x^\nu y^\theta \varepsilon_{\alpha\nu\theta}, \end{aligned} \tag{7}$$

are *comitants* of the system (1) with respect to $GL(3,\mathbb{R})$ group.

Some of *the invariants* of the system (1), with respect to $GL(3,\mathbb{R})$ group, are the expressions

$$\theta_1 = a^\alpha_\alpha, \qquad \theta_2 = a^\alpha_\beta a^\beta_\alpha, \qquad \theta_3 = a^\alpha_\gamma a^\beta_\alpha a^\gamma_\beta. \tag{8}$$

We will mention that the expressions $\varkappa_i\ (i = 1,2,3), \delta_1, \sigma_1$ and $\theta_i\ (i = 1,2,3)$ are known from [6, 7].

If we examine the differential system of the first approximation [1, 2] for system (1), written in the expanded form

$$\frac{dx^1}{dt} = a^1_1 x^1 + a^1_2 x^2 + a^1_3 x^3,\ \ \frac{dx^2}{dt} = a^2_1 x^1 + a^2_2 x^2 + a^2_3 x^3,\ \ \frac{dx^3}{dt} = a^3_1 x^1 + a^3_2 x^2 + a^3_3 x^3, \tag{9}$$

then it can be easily verified the following assertion

Lemma 1.1. *The expression* $\sigma_1 = 0$ *forms a* $GL(3,\mathbb{R})$ *particular invariant integral for system* (9)*.*

The proof follows directly from the equality

$$(a^1_1 x^1 + a^1_2 x^2 + a^1_3 x^3)\frac{\partial \sigma_1}{\partial x^1} + (a^2_1 x^1 + a^2_2 x^2 + a^2_3 x^3)\frac{\partial \sigma_1}{\partial x^2} + (a^3_1 x^1 + a^3_2 x^2 + a^3_3 x^3)\frac{\partial \sigma_1}{\partial x^3} = \theta_1 \sigma_1.$$

Lemma 1.2. *Let* $\delta_1 \equiv 0$ *in* (6)*. Then we obtain the following relations between the coefficients of the system* (9)*:*

$$a)\ a^2_1 = a^3_1 = 0;\ \ a^3_2 \neq 0;\ \ a^1_3 = \frac{a^1_2(a^3_3 - a^1_1)}{a^3_2};\ \ a^2_3 = \frac{(a^1_1 - a^2_2)(a^1_1 - a^3_3)}{a^3_2};$$

$$b)\ a_1^3 = a_2^3 = 0;\ \ a_1^2 \neq 0;\ \ a_2^1 = \frac{(a_1^1 - a_3^3)(a_2^2 - a_3^3)}{a_1^2};\ \ a_3^1 = \frac{a_3^2(a_1^1 - a_3^3)}{a_1^2};$$

$$c)\ a_1^2 = a_1^3 = a_2^3 = 0;\ \ a_2^1 \neq 0;\ \ a_1^1 = a_3^3;;\ \ a_3^2 = \frac{a_3^1(a_2^2 - a_3^3)}{a_2^1}$$

$$d)\ a_1^2 = a_3^2 = a_1^3 = a_2^3 = 0;\ \ a_2^1 \neq 0;\ \ a_2^2 = a_3^3;$$

$$e)\ a_2^1 = a_1^2 = a_1^3 = a_2^3 = 0;\ \ a_3^1 \neq 0;\ \ a_1^1 = a_2^2;$$

$$f)\ a_2^1 = a_1^2 = a_3^2 = a_1^3 = a_2^3 = 0;\ \ a_3^1 \neq 0;\ \ a_2^2 = a_3^3;$$

$$g)\ a_2^1 = a_3^1 = a_1^2 = a_1^3 = a_2^3 = 0;\ \ a_3^2 \neq 0;\ \ a_1^1 = a_2^2;$$

$$h)\ a_2^1 = a_3^1 = a_1^2 = a_1^3 = a_2^3 = 0;\ \ a_3^2 \neq 0;\ \ a_1^1 = a_3^3;$$

$$i)\ a_2^1 = a_3^1 = a_1^2 = a_3^2 = a_1^3 = a_2^3 = 0;\ \ a_1^1 = a_2^2;$$

$$j)\ a_2^1 = a_3^1 = a_1^2 = a_3^2 = a_1^3 = a_2^3 = 0;\ \ a_1^1 = a_3^3;$$

$$k)\ a_2^1 = a_3^1 = a_1^2 = a_3^2 = a_1^3 = a_2^3 = 0;\ \ a_2^2 = a_3^3. \tag{10}$$

Proof. From $\delta_1 \equiv 0$ we get the equalities:

$$(a_2^1)^2 a_3^2 - a_2^1 a_3^1 a_2^2 + a_2^1 a_3^1 a_3^3 - (a_3^1)^2 a_2^3 = 0,$$

$$-2a_1^1 a_2^1 a_3^2 + a_1^1 a_3^1 a_2^2 - a_1^1 a_3^1 a_3^3 + a_2^1 a_3^1 a_1^2 + a_2^1 a_2^2 a_3^2 + a_2^1 a_3^2 a_3^3 + (a_3^1)^2 a_1^3 - a_3^1 (a_2^2)^2 +$$
$$+ a_3^1 a_2^2 a_3^3 - 2a_3^1 a_3^2 a_2^3 = 0,$$

$$(a_1^1)^2 a_3^2 - a_1^1 a_3^1 a_1^2 - a_1^1 a_2^2 a_3^2 - a_1^1 a_3^2 a_3^3 - a_2^1 a_1^2 a_3^2 + 2a_3^1 a_1^2 a_2^2 - a_3^1 a_1^2 a_3^3 + 2a_3^1 a_3^2 a_1^3 +$$
$$+ a_2^2 a_3^2 a_3^3 - (a_3^2)^2 a_2^3 = 0,$$

$$a_1^1 a_2^1 a_2^2 - a_1^1 a_2^1 a_3^3 + 2a_1^1 a_3^1 a_2^3 - (a_2^1)^2 a_1^2 - a_2^1 a_3^1 a_1^3 - a_2^1 a_2^2 a_3^3 + 2a_2^1 a_3^2 a_2^3 + a_2^1 (a_3^3)^2 -$$
$$- a_3^1 a_2^2 a_2^3 - a_3^1 a_2^3 a_3^3 = 0,$$

$$-(a_1^1)^2 a_2^2 + (a_1^1)^2 a_3^3 + a_1^1 a_2^1 a_1^2 - a_1^1 a_3^1 a_1^3 + a_1^1 (a_2^2)^2 - a_1^1 (a_3^3)^2 - a_2^1 a_1^2 a_2^2 - 3a_2^1 a_3^2 a_1^3 + 3a_3^1 a_1^2 a_2^2 +$$
$$+ a_3^1 a_1^3 a_3^3 - (a_2^2)^2 a_3^3 + a_2^2 a_3^2 a_2^3 + a_2^2 (a_3^3)^2 - a_3^2 a_2^3 a_3^3 = 0,$$

$$-(a_1^1)^2 a_2^3 + a_1^1 a_2^1 a_1^3 + a_1^1 a_2^2 a_2^3 + a_1^1 a_2^3 a_3^3 - 2a_2^1 a_1^2 a_2^3 + a_2^1 a_2^2 a_1^3 - 2a_2^1 a_1^3 a_3^3 + a_3^1 a_1^3 a_2^3 -$$
$$- a_2^2 a_2^3 a_3^3 + a_3^2 (a_2^3)^2 = 0,$$

$$a_1^1 a_1^2 a_3^2 - a_3^1 (a_1^2)^2 - a_1^2 a_3^2 a_3^3 + (a_3^2)^2 a_1^3 = 0,$$

$$-a_1^1 a_1^2 a_2^2 + a_1^1 a_1^2 a_3^3 + a_1^1 a_3^2 a_1^3 + a_2^1 (a_1^2)^2 - 2a_3^1 a_1^2 a_1^3 + a_1^2 a_2^2 a_3^3 + a_1^2 a_3^2 a_2^3 - a_1^2 (a_3^3)^2 -$$
$$- 2a_2^2 a_3^2 a_1^3 + a_3^2 a_1^3 a_3^3 = 0,$$

$$-a_1^1 a_1^2 a_2^3 - a_1^1 a_2^2 a_1^3 + a_1^1 a_1^3 a_3^3 + 2a_2^1 a_1^2 a_1^3 - a_3^1 (a_1^3)^2 - a_1^2 a_2^2 a_2^3 + 2a_1^2 a_2^3 a_3^3 + (a_2^2)^2 a_1^3 -$$
$$- a_2^2 a_1^3 a_3^3 - a_3^2 a_1^3 a_2^3 = 0,$$

$$-a_1^1 a_1^3 a_2^3 + a_2^1 (a_1^3)^2 - a_1^2 (a_2^3)^2 + a_2^2 a_1^3 a_2^3 = 0. \tag{11}$$

Without loss of generality we can assume that

$$a_1^3 = 0, \tag{12}$$

because, otherwise, we can obtain this equality by transformation

$$\bar{x}^1 = x^2, \ \ \bar{x}^2 = x^1 + \frac{a_2^3}{a_1^3}x^2, \ \ \bar{x}^3 = x^3. \tag{13}$$

Substituting $a_1^3 = 0$ in (11), from the last equality, we get $a_1^2 a_2^3 = 0$. This implies the following cases: 1) $a_1^3 = a_1^2 = 0,\ a_2^3 \neq 0$; 2) $a_1^3 = a_2^3 = 0,\ a_1^2 \neq 0$; 3) $a_1^3 = a_1^2 = a_2^3 = 0$. Calculating the other coefficients by means of the equalities (11) from 1), we obtain the case a) from (10). From 3) we get the cases $c) - k$), from (10).

Lemma 1.2 is proved. □

Lemma 1.3. *Assume that* $\sigma_1 \equiv 0$ *in* (7). *Then we get the relation* (10).

The proof of Lemma 1.3 is analogous to the proof of Lemma 1.2.

Using Lemmas 1.2 and 1.3, it is obtained

Theorem 1.2.

$$\sigma_1(x) \equiv 0 \Leftrightarrow \delta_1(u) \equiv 0 \tag{14}$$

and conversely

$$\sigma_1(x) \neq 0 \Leftrightarrow \delta_1(u) \neq 0. \tag{15}$$

2. Notions of stability of unperturbed motion and the Lyapunov function

Let the differential system of the perturbed motion [2] be given in the form (1). Then, according to [2], the zero values of the variables x^j $(j = 1, 2, 3)$ correspond to the unperturbed motion.

Definition of stability by Lyapunov [2]**.** Let for any small number ε, there exists a positive number δ such that for any perturbation $x^j(t_0)$ is satisfied the condition

$$\sum_{j=1}^{3}(x^j(t_0))^2 \leq \delta, \tag{16}$$

and for any $t \geq t_0$ is satisfied the condition

$$\sum_{j=1}^{3}(x^j(t))^2 < \varepsilon.$$

Then the unperturbed motion $x^j = 0$ $(j = 1, 2, 3)$ is called *stable*, otherwise *unstable*.

If the unperturbed motion is stable and the value δ can be found however small such that for any perturbed motions satisfying (16) the condition

$$\lim_{t\to+\infty}\sum_{j=1}^{3}(x^j(t))^2=0$$

is valid, then the unperturbed motion is called *asymptotically stable*.

We will examine the system (9). The characteristic equation of this system is

$$\varrho^3+L_1\varrho^2+L_2\varrho+L_3=0, \tag{17}$$

where the coefficients of this equation are expressed by center-affine invariants (8), and have the form

$$L_1=-\theta_1,\quad L_2=\frac{1}{2}(\theta_2-\theta_1^2),\quad L_3=-\frac{1}{6}(\theta_1^3-3\theta_1\theta_2+2\theta_3). \tag{18}$$

Using the Lyapunov's theorems on stability of unperturbed and perturbed motion in the first approximation [2], and the Hurwitz's theorem [2], we obtain the following theorems:

Theorem 2.1. *Assume that the center-affine invariants* (18) *of the system* (1) *satisfy the inequalities*

$$L_1>0,\quad L_2>0,\quad L_3>0,\quad L_1L_2-L_3>0,$$

then the unperturbed motion $x^1=x^2=x^3=0$ *of the system* (1) *is asymptotically stable.*

Theorem 2.2. *If at least one of the center-affine invariant expressions* (18) *of system* (1) *has the sign less than zero, then the unperturbed motion* $x^1=x^2=x^3=0$*, of the system* (1)*, is unstable.*

Following [2], we consider the real function $V(x)=V(x^1,x^2,x^3)$, which is defined in the domain

$$\sum_{j=1}^{3}(x^j)^2\le\mu, \tag{19}$$

where μ is a positive numerical constant.

In this domain, the function $V(x)$ is unique and continuous and is vanishing for $x^1=x^2=x^3=0$, i.e.

$$V(0)=0. \tag{20}$$

If in the domain (19) this function takes values of the same sign, then it is called of *constant sign* (respectively *positive* or *negative*). If the function of constant sign vanishes only when x^1,x^2,x^3 are zero, then V is called of *determined sign*. The introduction of such functions V, in the research of the stability of motion, are called *Lyapunov functions*.

Later on, we will use the following Lyapunov Theorem:

Theorem 2.3. [1, 2] *Let for equations of the perturbed motion can be found a function $V(x) = V(x^1, x^2, x^3)$ of the determined sign such that its derivative $\dot{V}$, by virtue of the system (45) from* [1]*, with $s = 1$, would be of constant sign, opposite to the sign of the function V or identically zero. Then the unperturbed motion is stable.*

3. Invariant conditions for obtaining the Lyapunov form of differential system (1)

Lemma 3.1. *Suppose that $\sigma_1 \not\equiv 0$ in* (7)*. Then system* (1)*, by means of a centro-affine transformation, can be brought to the form*

$$\begin{aligned}\frac{dx^1}{dt} &= x^2 + a^1_{\alpha\beta\gamma\delta}x^\alpha x^\beta x^\gamma x^\delta,\\ \frac{dx^2}{dt} &= x^3 + a^2_{\alpha\beta\gamma\delta}x^\alpha x^\beta x^\gamma x^\delta,\\ \frac{dx^3}{dt} &= -L_3x^1 - L_2x^2 - L_1x^3 + a^3_{\alpha\beta\gamma\delta}x^\alpha x^\beta x^\gamma x^\delta,\end{aligned} \tag{21}$$

where L_i $(i = 1, 2, 3)$ are from (18).

Proof. Consider the substitution

$$\bar{x}^1 = \varkappa_1, \quad \bar{x}^2 = \varkappa_2, \quad \bar{x}^3 = \varkappa_3, \tag{22}$$

where $\varkappa_i$ $(i = 1, 2, 3)$ are given in (5). From (22), by means of expressions $\varkappa_i$, it is obtained (in $\varkappa_2$ the index α is renotated by α_1)

$$\Delta \equiv det(\varkappa_1, \varkappa_2, \varkappa_3) = \begin{vmatrix} u_1 & u_2 & u_3 \\ a^{\alpha_1}_1 u_{\alpha_1} & a^{\alpha_1}_2 u_{\alpha_1} & a^{\alpha_1}_3 u_{\alpha_1} \\ a^\alpha_1 a^\beta_\alpha u_\beta & a^\alpha_2 a^\beta_\alpha u_\beta & a^\alpha_3 a^\beta_\alpha u_\beta \end{vmatrix} = \delta_1, \tag{23}$$

where δ_1 is from (6) and

$$\begin{aligned} x^1 &= \frac{1}{\delta_1}[(a^{\alpha_1}_2 a^\alpha_3 a^\beta_\alpha u_{\alpha_1} u_\beta - a^{\alpha_1}_3 a^\alpha_2 a^\beta_\alpha u_{\alpha_1})\bar{x}^1 + (a^\alpha_2 a^\beta_\alpha u_\alpha u_\beta - a^\alpha_3 a^\beta_\alpha u_\beta u_2)\bar{x}^2 + \\ &\quad +(a^{\alpha_1}_3 u_{\alpha_1} u_2 - a^{\alpha_1}_2 u_{\alpha_1} u_3)\bar{x}^3], \\ x^2 &= \frac{1}{\delta_1}[(a^{\alpha_1}_3 a^\alpha_1 a^\beta_\alpha u_{\alpha_1} u_\beta - a^{\alpha_1}_1 a^\alpha_3 a^\beta_\alpha u_{\alpha_1} u_\beta)\bar{x}^1 + (a^\alpha_3 a^\beta_\alpha u_\beta u_1 - a^\alpha_1 a^\beta_\alpha u_\beta u_3)\bar{x}^2 + \\ &\quad +(a^{\alpha_1}_1 u_{\alpha_1} u_3 - a^{\alpha_1}_3 u_{\alpha_1} u_1)\bar{x}^3], \\ x^3 &= \frac{1}{\delta_1}[(a^{\alpha_1}_1 a^\alpha_2 a^\beta_\alpha u_{\alpha_1} u_\beta - a^{\alpha_1}_2 a^\alpha_1 a^\beta_\alpha u_{\alpha_1} u_\beta)\bar{x}^1 + (a^\alpha_1 a^\beta_\alpha u_\beta u_2 - a^\alpha_2 a^\beta_\alpha u_\beta u_1)\bar{x}^2 + \\ &\quad +(a^{\alpha_1}_2 u_{\alpha_1} u_1 - a^{\alpha_1}_1 u_{\alpha_1} u_2)\bar{x}^3]. \end{aligned} \tag{24}$$

Considering (5) and substitutions (22)-(24), then from the system (1) we obtain the system (21) with $\delta_1 \not\equiv 0$, which according to Theorem 1.2 is equivalent to $\sigma_1 \not\equiv 0$. Lemma 3.1 is proved. □

Using Lemma 1.3, it can easily be verified that the following assertion is proved:

Remark 3.1. If for ystem (9) of the first approximation, the condition $\sigma_1 \equiv 0$ holds from (7), then the characteristic equation (17) has only real roots.

Taking into consideration Remark 3.1, it can easily be verified that the following assertion is proved:

Lemma 3.2. *The characteristic equation of system* (21)*, with* $\sigma_1 \not\equiv 0$*, has purely imaginary eigenvalues if and only if the system has the form*

$$\begin{aligned} \frac{dx^1}{dt} &= x^2 + a^1_{\alpha\beta\gamma\delta}x^\alpha x^\beta x^\gamma x^\delta, \\ \frac{dx^2}{dt} &= x^3 + a^2_{\alpha\beta\gamma\delta}x^\alpha x^\beta x^\gamma x^\delta, \\ \frac{dx^3}{dt} &= -L_1L_2x^1 - L_2x^2 - L_1x^3 + a^3_{\alpha\beta\gamma\delta}x^\alpha x^\beta x^\gamma x^\delta, \quad (L_1, L_2 > 0), \end{aligned} \tag{25}$$

where L_i $(i = 1, 2)$ *are of the form* (18)*.*

Theorem 3.1. *Let* $\sigma_1 \not\equiv 0$ *in* (7)*. Then, by a centro-affine transformation, the system* (21) *can be brought to the form*

$$\begin{aligned} \frac{dx^1}{dt} &= -\lambda x^2 + a^1_{\alpha\beta\gamma\delta}x^\alpha x^\beta x^\gamma x^\delta, \\ \frac{dx^2}{dt} &= \lambda x^1 + a^2_{\alpha\beta\gamma\delta}x^\alpha x^\beta x^\gamma x^\delta, \\ \frac{dx^3}{dt} &= x^2 - L_1x^3 + a^3_{\alpha\beta\gamma\delta}x^\alpha x^\beta x^\gamma x^\delta, \end{aligned} \tag{26}$$

with the linear parts of the first two equations in the Lyapunov form, where L_1, L_3 *are from* (18) *and* $\lambda^2 = L_3$ $(L_1, L_3 > 0)$*.*

Proof. We will examine the system (21). According to Lyapunov system (45) from [1], the linear part of this system must have the form

$$\frac{dX^1}{dt} = -\lambda X^2 + \dots, \quad \frac{dX^2}{dt} = \lambda X^1 + \dots, \quad \frac{dX^3}{dt} = aX^1 + bX^2 + cX^3 + \dots, \tag{27}$$

where by dots we mean the homogeneities of the fourth order with respect to X^1, X^2, X^3. The coefficients λ, a, b, c are expressions in L_i $(i = 1, 2, 3)$ and the new variables

X^1, X^2, X^3 have the form

$$X^1 = \alpha_1 x^1 + \alpha_2 x^2 + \alpha_3 x^3,\ X^2 = \beta_1 x^1 + \beta_2 x^2 + \beta_3 x^3,\ X^3 = \gamma_1 x^1 + \gamma_2 x^2 + \gamma_3 x^3, \quad (28)$$

where

$$\Delta = \begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{vmatrix} \neq 0. \quad (29)$$

Unde the conditions (29), we observe that the substitution (28) forms a centro-affine transformation. Substituting (28) in the linear part of the system (27) and comparing with the system (25), we obtain a system of nine algebraic equation in 12 unknowns $a, b, c, \alpha_i, \beta_j, \gamma_k\ \ (i, j, k = 1, 2, 3)$. Solving this system, we have

$$X^1 = -L_1^2 \lambda x^1 + \lambda x^3,\ X^2 = L_1 L_2 x^1 + (L_1^2 + L_2) x^2 + L_1 x^3,\ X^3 = 2L_2 x^1 + L_1 x^2 + x^3,$$

where $\lambda^2 = L_2$, and the determinant of this transformation is

$$\Delta = -2L_2 \lambda (L_1^2 + L_2) \neq 0 \quad (L_2 > 0).$$

This transformation brings the system (25) to a system with the linear part in the Lyapunov form (26) for which the initial notations of the phase variables are preserved. The form of the fourth-degree homogeneity does not change, apart from the coefficients and the phase variables. Theorem 3.1 is proved. □

4. Lyapunov-Darboux form of system (1) and stability conditions of unperturbed motion

Remark 4.1. For $\eta_1 \equiv 0$, from (7), the system (1) will get the following Darboux form

$$\frac{dx^j}{dt} = a^j_\alpha x^\alpha + 4x^j R(x) \ \ (j = 1, 2, 3), \quad (30)$$

where $R(x)$ is a homogeneous polynomial of the third degree with respect to the vector coordinates $x = (x^1, x^2, x^3)$.

Remark 4.2. The system (30) has as $GL(3, \mathbb{R})$-invariant integral the expression $\sigma_1 \not\equiv 0$.

This affirmation results from the identity

$$[a^1_\alpha x^\alpha + 4x^1 R(x)] \frac{\partial \sigma_1}{\partial x^1} + [a^2_\alpha x^\alpha + 4x^2 R(x)] \frac{\partial \sigma_1}{\partial x^2} + [a^3_\alpha x^\alpha + 4x^3 R(x)] \frac{\partial \sigma_1}{\partial x^3} = [\theta_1 + 12R(x)]\sigma_1,$$

where θ_1 is from (8).

Taking into consideration that the system (26) was obtained using the invariant condition $\sigma_1 \not\equiv 0$ by means of the centro-affine transformations (22), and the Darboux system (30) is governed by the invariant condition $\eta_1 \equiv 0$, we obtain

Lemma 4.1. *Let* $\sigma_1 \not\equiv 0,\ \eta_1 \equiv 0$ *in* (7) *and* $L_1, L_2 > 0$ *in* (18). *Then system* (1), *by the centro-affine transformations, can be brought to the following Lyapunov-Darboux form*

$$\begin{aligned} \frac{dx}{dt} &= -\lambda y + 4xR(x,y,z), \\ \frac{dy}{dt} &= \lambda x + 4yR(x,y,z), \\ \frac{dz}{dt} &= y - L_1 z + 4zR(x,y,z), \end{aligned} \tag{31}$$

where $x = x^1, y = x^2, z = x^3$, *and*

$$\begin{gathered} R(x,y,z) = a_1x^3 + a_2y^3 + a_3z^3 + 3a_4x^2y + 3a_5x^2z + 3a_6xy^2 + 3a_7xz^2 + \\ +3a_8xyz + 3a_9y^2z + 3a_{10}yz^2. \end{gathered} \tag{32}$$

By means of the determining equations, from [7], we construct the Lie algebra admitted by the system (31)-(32). Using Lie algebra for the mentioned system, we obtain the analytic first integral of the form

$$F(x,y,z) \equiv \frac{h_1^3}{(J+h_2)^2} = 0 \tag{33}$$

governed by the condition

$$J(J+h_2) \neq 0, \tag{34}$$

where

$$h_1 = x^2 + y^2, \quad J = -L_1\lambda^2(4L_1^2 + \lambda^2)(L_1^2 + 4\lambda^2),$$

$$\begin{gathered} h_2 = \lambda[4(8a_3L_1^2 + 24a_{10}L_1^3 + 12a_5L_1^4 + 24a_9L_1^4 + 8a_2L_1^5 + 12a_4L_1^5 - 24a_7L_1^2\lambda + 22a_3\lambda^2 - \\ -12a_8L_1^3\lambda + 66a_{10}L_1\lambda^2 + 75a_5L_1^2\lambda^2 + 78a_9L_1^2\lambda^2 + 34a_2L_1^3\lambda^2 + 51a_4L_1^3\lambda^2 - 36a_7\lambda^3 - \\ -3a_8L_1\lambda^3 + 18a_5\lambda^4 + 18a_9\lambda^4 + 8a_2L_1\lambda^4 + 12a_4L_1\lambda^4)x^3 - 4L_1(12a_7L_1^2 + 12a_8L_1^3 + \\ +8a_1L_1^4 + 12a_6L_1^4 + 10a_3\lambda + 30a_{10}L_1\lambda - 24a_5L_1^2\lambda + 24a_9L_1^2\lambda - 12a_7\lambda^2 + \\ +3a_8L_1\lambda^2 + 34a_1L_1^2\lambda^2 + 51a_6L_1^2\lambda^2 - 6a_5\lambda^3 + 6a_9\lambda^3 + 8a_1\lambda^4 + 12a_6\lambda^4)y^3 + \\ +4a_3\lambda(4L_1^2 + \lambda^2)(L_1^2 + 4\lambda^2)z^3 - 12a_1L_1(4L_1^2 + \lambda^2)(L_1^2 + 4\lambda^2)x^2y + \\ +12\lambda(12a_5L_1^4 + 12a_7L_1^2\lambda + 12a_8L_1^3\lambda + 10a_3\lambda^2 + 30a_{10}L_1\lambda^2 + 27a_5L_1^2\lambda^2 + \\ +24a_9L_1^2\lambda^2 - 12a_7\lambda^3 + 3a_8L_1\lambda^3 + 6a_5\lambda^4 + 6a_9\lambda^4)x^2z + 12(4a_3L_1^2 + 12a_{10}L_1^3 + \\ +12a_9L_1^4 + 4a_2L_1^5 - 18a_7L_1^2\lambda - 12a_8L_1^3\lambda + 6a_3\lambda^2 + 18a_{10}L_1\lambda^2 + 24a_5L_1^2\lambda^2 + \\ +27a_9L_1^2\lambda^2 + 17a_2L_1^3\lambda^2 - 12a_7\lambda^3 - 3a_8L_1\lambda^3 + 6a_5\lambda^4 + 6a_9\lambda^4 + 4a_2L_1\lambda^4)xy^2 + \\ +12\lambda(6a_7L_1^2 + a_3\lambda + 3a_{10}L_1\lambda)(L_1^2 + 4\lambda^2)xz^2 + 12L_1\lambda(12a_7L_1^2 + 12a_8L_1^3 + \\ +10a_3\lambda + 30a_{10}L_1\lambda - 24a_5L_1^2\lambda + 24a_9L_1^2\lambda - 12a_7\lambda^2 + 3a_8L_1\lambda^2 - 6a_5\lambda^3 + 6a_9\lambda^3)xyz + \end{gathered}$$

$$+12\lambda(4a_3L_1^2 + 12a_{10}L_1^3 + 12a_9L_1^4 - 18a_7L_1^2\lambda - 12a_8L_1^3\lambda + 6a_3\lambda^2 + 18a_{10}L_1\lambda^2 +$$
$$+24a_5L_1^2\lambda^2 + 27a_9L_1^2\lambda^2 - 12a_7\lambda^3 - 3a_8L_1\lambda^3 + 6a_5\lambda^4 + 6a_9\lambda^4)y^2z +$$
$$+12L_1\lambda(2a_3 + 6a_{10}L_1 - 3a_7\lambda)(L_1^2 + 4\lambda^2)yz^2].$$

Analyzing the first integral (33), we notice that if the inequality (34) holds, then the function $F(x, y, z)$ forms *the Lyapunov function*. According to Theorem 2.3, we have

Theorem 4.1. *Let for system of the Lyapunov-Darboux type* (31)-(32) *the inequality* (34) *holds. Then the unperturbed motion $x = y = z = 0$, governed by this system, is stable.*

Remark 4.3. For the first time, a problem analogous to that examined in this paper was investigated for ternary system with quadratic nonlinearities in [8]. Here, the invariant centro-affine conditions of stability or instability of unperturbed motion were obtained.

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Received: September 30, 2022 *Accepted: December 10, 2022*

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Acta et Commentationes, Exact and Natural Sciences ISSN: 2537-6284
Volume 2(14), 2022, Pages 68–80 E-ISSN: 2587-3644
CZU 517.925 DOI: 10.36120/2587-3644.v14i2.68-80

Phase portraits of some polynomial differential systems with maximal multiplicity of the line at the infinity

Vadim Repeşco

Abstract. The present study delves into the investigation of phase portraits of polynomial differential systems, which are systems of differential equations of the form $\frac{dx}{dt} = P(x, y)$, $\frac{dy}{dt} = Q(x, y)$, where x and y are the dependent variables and t is the independent variable. The functions $P(x, y)$ and $Q(x, y)$ are polynomials in x and y. The main objective of this research is to obtain the phase portraits of polynomial differential systems of degree $n \in \{3, 4, 5\}$ and having an invariant straight line at the infinity of maximal multiplicity.

2010 Mathematics Subject Classification: 34C05.
Keywords: phase portrait, singular point, Poincaré transformation.

Tablouri fazice ale unor sisteme diferențiale polinomiale cu dreapta de la infinit de multiplicitate maximală

Rezumat. Prezentul studiu se aprofundează în investigarea portretelor de fază ale sistemelor diferențiale polinomiale, care sunt sisteme de ecuații diferențiale de forma $\frac{dx}{dt} = P(x, y)$, $\frac{dy}{dt} = Q(x, y)$, unde x și y sunt variabilele dependente și t este variabila independentă. Funcțiile $P(x, y)$ și $Q(x, y)$ au polinoame în x și y. Obiectivul principal al acestei cercetări este obținerea portretelor de fază ale sistemelor diferențiale polinomiale de grad $n \in \{3, 4, 5\}$ și având o dreaptă invariantă la infinit de multiplicitate maximală.

Cuvinte-cheie: portret fazic, punct singular, transformarea Poincaré.

1. Introduction

Phase portraits are graphical representations of the behaviour of a system of differential equations over time and they can be used to visualize the long-term behaviour of a system. Overall, the phase portrait of a polynomial differential system with maximal multiplicity of the line at infinity can be quite complex and may exhibit a variety of different behaviours.

The study of invariant algebraic curves plays a crucial role in the qualitative analysis of dynamical systems. The problem of determining the maximum number of invariant straight lines present in a polynomial differential system is explored in [1]. Additionally, the utilization of invariant straight lines in the calculation of Darboux first integrals is a significant area of study, as outlined in [2], where it is demonstrated that a Darboux first integral can be calculated for a polynomial differential system if it possesses a sufficient number of invariant straight lines, taking into account their multiplicities.

In this article, we will focus on phase portraits of polynomial differential systems, which are systems of differential equations of the form:

$$\frac{dx}{dt} = P(x, y), \ \frac{dy}{dt} = Q(x, y),$$

where x and y are the dependent variables and t is the independent variable. The functions $P(x, y)$ and $Q(x, y)$ are polynomials in x and y. We will obtain the phase portraits of polynomial differential systems of degree $n \in \{3, 4, 5\}$ and having the invariant straight line at the infinity of the maximal multiplicity.

2. Cubic polynomial differential systems

According to [3], the maximal multiplicity of the line at infinity for the cubic systems is equal to seven and the systems can be brought to the following two forms:

$$\begin{cases} \dot{x} = 1, \\ \dot{y} = x^3 + ax, \end{cases} \qquad a \in \mathbb{R}; \tag{1}$$

and

$$\begin{cases} \dot{x} = -x, \\ \dot{y} = x^3 + 2y. \end{cases} \tag{2}$$

The system (1) does not possess any singular points within the finite region of the phase plane. However, at infinity, there exist $I_{1,2}^{\infty}(0, \pm 1, 0)$, which are multiple singular points. In order to analyze the behavior of trajectories in proximity to these points, we shall employ the first Poincaré transformation. Prior to this, we shall utilize the transformation $x \to y$, $y \to x$ to relocate these points to opposite sides of the Ox axis:

$$\begin{cases} \dot{x} = y^3 + ay, \\ \dot{y} = 1, \end{cases} \qquad a \in \mathbb{R}. \tag{3}$$

By effecting the transformation $x \to 1/y$, $y \to x/y$, the system represented by equation (3) is transformed into the following form:

$$\begin{cases} \dot{x} = -x^4 + ax^2y^2 + y^3, \\ \dot{y} = -xy(x^2 + ay^2). \end{cases} \tag{4}$$

This system possesses a single singular point $I_1(0, 0)$ which corresponds to the singular points $I_{1,2}^{\infty}(\pm 1, 0, 0)$ of the original system represented by equation (3).

The singular point I_1 is a multiple one, thus we shall utilize the blow-up method. This results in the following differential system:

$$\begin{cases} \dot{x} = -\frac{1}{4}x\sin y\,(2(a+1)x + 2x(a-1)\cos(2y) - 3\cos y - \cos(3y)), \\ \dot{y} = \cos^4 y. \end{cases} \tag{5}$$

The system at hand possesses the singular points $S_1(0, \frac{\pi}{2})$ and $S_2(0, \frac{3\pi}{2})$, both of which are multiple. Thus, we shall utilize the transformation $x \to x$, $y \to y - \pi/2$ to relocate the point S_1 to the origin. Then by expanding the right-hand sides in a Taylor series about $y = 0$, and retaining only a subset of the first monomials, we obtain the following system:

$$\begin{cases} \dot{x} = -xy^3 + x^2\left(-ay^2 + \frac{3y^2}{2} - 1\right), \\ \dot{y} = y^4 - \frac{2y^6}{3}. \end{cases} \tag{6}$$

Using the blow-up method we get that the singular point S_1 decompose in 4 singular points $N_1(0,0)$, $N_2(0, \frac{\pi}{2})$, $N_3(0,\pi)$ and $N_4(0, \frac{3\pi}{2})$ of the following system:

$$\begin{cases} \dot{x} = -\frac{1}{6}x\Big(6ax^2\sin^2 y\cos^3 y + 4x^4\sin^7 y - 6x^2\sin^5 y + 6x^2\sin^3 y\cos^2 y - \\ \quad -9x^2\sin^2 y\cos^3 y + 6\cos^3 y\Big), \\ \dot{y} = \frac{1}{6}\sin y\cos y\Big(6ax^2\sin^2 y\cos y - 4x^4\sin^5 y + 12x^2\sin^3 y - \\ \quad -9x^2\sin^2 y\cos y + 6\cos y\Big). \end{cases} \tag{7}$$

The singular points N_1 and N_3 are of saddle type, while the singular points N_2 and N_4 are multiple. By utilizing the transformation $x \to x$, $y \to y - \pi/2$, we reposition the point N_2 to the origin of coordinates. Subsequently, by expanding the right-hand sides in a Taylor series in the vicinity of $y = 0$, and discarding all terms of higher order, we arrive at the system:

$$\begin{cases} \dot{x} = xy^3 + x^3\left(1 - \frac{7y^2}{2}\right), \\ \dot{y} = \frac{1}{6}y\left(6ax^2y - 9x^2y + 4x^4 - 12x^2 + 6y\right). \end{cases} \tag{8}$$

By utilizing the blow-up method, we obtain the following system:

$$\begin{cases} \dot{x} = -\frac{1}{24}x\Big(-6ax^2\sin y\sin^2(2y) + 3x^2\sin y\sin^2(2y) + 68x^3\sin^2 y\cos^4 y + \\ \quad +12x\sin^2(2y) - 24x\cos^4 y - 24\sin^3 y\Big), \\ \dot{y} = \frac{1}{6}\sin y\cos y\Big(6ax^2\sin y\cos^2 y - 6x^2\sin^3 y + 4x^3\cos^4 y + 21x^3\sin^2 y \cdot \\ \quad \cdot\cos^2 y - 9x^2\sin y\cos^2 y - 18x\cos^2 y + 6\sin y\Big). \end{cases} \tag{9}$$

By solving the equation $Q(0, y) = 0$, it follows that the point N_2 decomposes into the following four points $P_1(0,0)$, $P_2(0, \frac{\pi}{2})$, $P_3(0,\pi)$, $P_4(0, \frac{3\pi}{2})$, wherein P_1 and P_3 are multiple, while P_2 and P_4 are saddles.

Once more, we shall expand the right-hand sides in a Taylor series in the vicinity of the point $y = 0$, and subsequently apply the blow-up method. This results in the following system:

$$\begin{cases} \dot{x} = -\frac{1}{6}x\left(-6ax^2\sin^3 y\cos^2 y + 20x^2\sin^2 y\cos^3 y + 3x^2\sin^3 y\cos^2 y - \right.\\ \quad \left. -6\sin^3 y - 6\cos^3 y + 18\sin^2 y\cos y\right) \\ \dot{y} = \frac{1}{6}\sin y\cos y\left(6ax^2\sin y\cos^2 y - 6x^2\sin^3 y + 4x^2\cos^3 y - 9x^2\sin y\cdot\right.\\ \quad \left.\cdot\cos^2 y + 24x^2\sin^2 y\cos y + 6\sin y - 24\cos y\right), \end{cases} \tag{10}$$

which possesses 6 singular points. The coordinates and types of these singular points are listed in Table 1.

Table 1. Blow up for point P_1

S.P.	$O_1(0,0)$	$O_2(0, \operatorname{arctg} 4)$	$O_3(0, \frac{\pi}{2})$	$O_4(0, \pi)$	$O_5(0, \pi + \operatorname{arctg} 4)$	$O_6(0, \frac{3\pi}{2}$
$\lambda_{1,2}$	$-4; 1$	$\frac{1}{\sqrt{17}}; \frac{4}{\sqrt{17}}$	± 1	$-1; 4$	$-\frac{4}{\sqrt{17}}; -\frac{1}{\sqrt{17}}$	± 1
Type	S	N^u	S	S	N^s	S

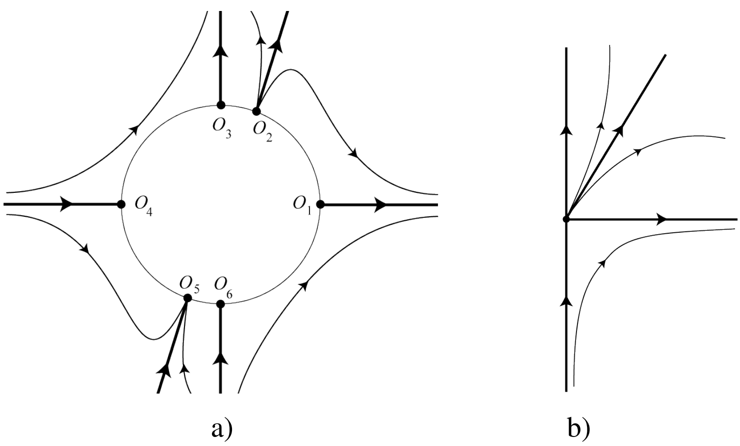


Figure 1. Blow-up for the point P_1 $(0, 0)$.

By constructing all these points on a circle, and plotting their behaviour in proximity to them (Figure 1 a)), followed by compressing the circle into a single point, we can obtain the behaviour of the trajectories in the vicinity of the singular point P_1 of the system represented by equation (9). To determine the type of the singular point N_2 of the system represented by equation (7), we shall only utilize the portion of the phase plane corresponding to $x > 0$ (as depicted in Figure 1 b)).

By utilizing P_1, we can construct the phase portrait for the singular point N_2 (Figure 2 a), b)). The phase portrait of the singular point $N_4(0, \frac{3\pi}{2})$ is also depicted in Figure 2

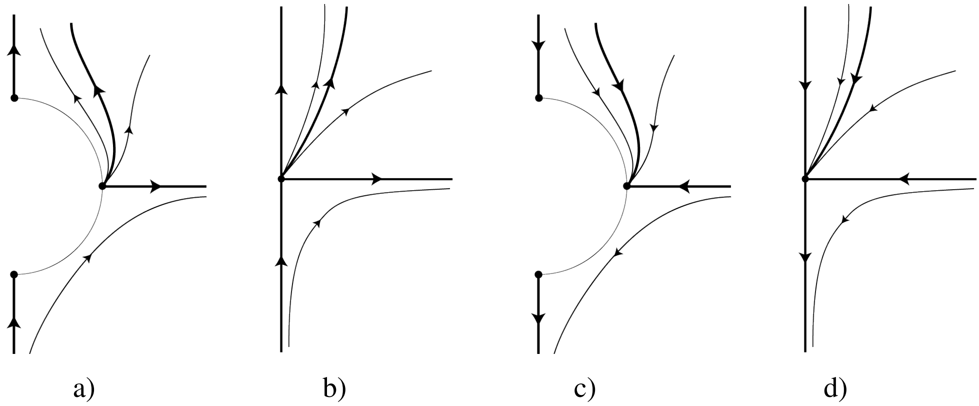


Figure 2. Blow-up for the points $N_2\left(0, \frac{\pi}{2}\right)$ and $N_4(0, \frac{3\pi}{2})$.

c) d), which is obtained in an analogous manner, with the only difference being that the direction of the trajectories is inverted.

We can now construct the trajectories for the singular point S_1, which was decomposed into the points P_1, P_2, P_3 and P_4 (see Figure 3 a)). Since we only require the half-plane $x > 0$, we obtain Figure 3 b).

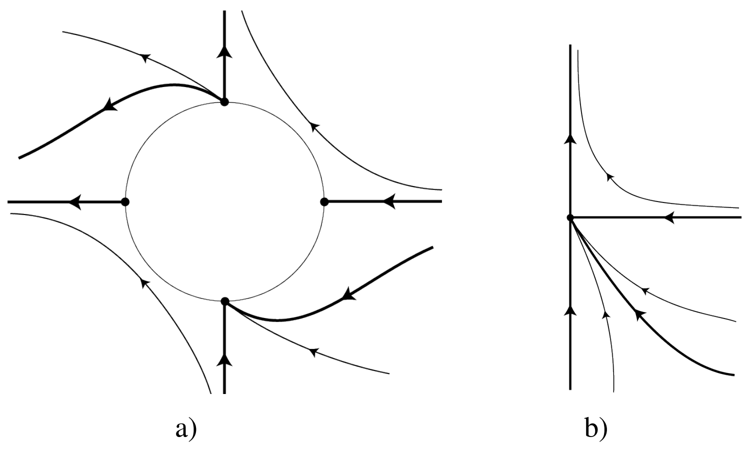
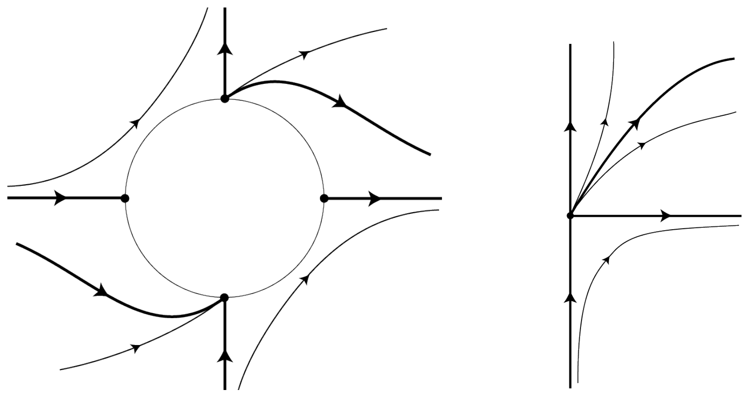


Figure 3. Blow-up for the point $S_1\left(0, \frac{\pi}{2}\right)$.

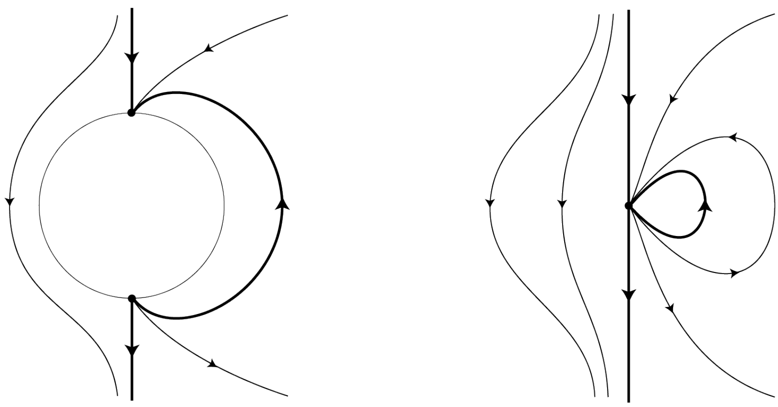
By executing the same procedure (applying the blow-up method three times) for the point $S_2(0, \frac{3\pi}{2})$, we obtain the representation illustrated in Figure 4.

Ultimately, by utilizing S_1 and S_2, we can construct the phase portrait for the point $I_1(0, 0)$ of the system represented by equation (4) (see Figure 5).



a) b)

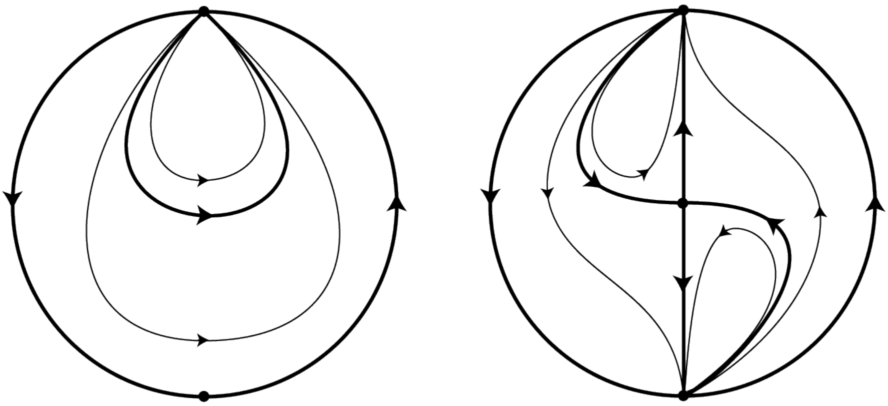
Figure 4. Blow-up for the point $S_2\left(0, \frac{3\pi}{2}\right)$.



a) b)

Figure 5. Blow-up for the point $I_1\ (0,0)$.

Having now determined the behaviour of trajectories in the proximity of $I^{\infty}_{1,2}$, we can construct the phase portrait for the polynomial differential system represented by equation (3). By applying the transformation $x \to y$, $y \to x$, we can obtain the phase portrait for the system represented by equation (1) (see Figure 6 a)). By following a similar procedure, we can construct the phase portrait for the system represented by equation (2) (see Figure 6 b)).



a) b)

Figure 6. The phase portraits for the cubic differential systems (1) and (2).

3. Quartic polynomial differential systems

According to [4], a quartic polynomial differential system with maximal multiplicity can be brought into the following form:

$$\begin{cases} \dot{x} = -3x + ay^4, \\ \dot{y} = y, \quad a > 0. \end{cases} \tag{11}$$

This system has an invariant line at infinity with multiplicity equal to 10. By referring to [5], we can construct its phase portrait. However, we first need to relocate the singular points at infinity to be situated at the ends of the Oy axis by applying the transformation $x \to y$, $y \to x$ (see Figure 7).

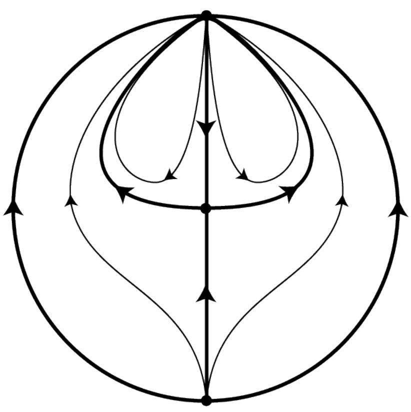


Figure 7. The phase portrait for the quartic differential system (11).

4. Quintic polynomial differential systems

As stated in [6], a quintic polynomial differential system with the line at infinity of maximal multiplicity can be transformed into the following form:

$$\begin{cases} \dot{x} = x, \quad a \neq 0, \\ \dot{y} = -4y + ax^5. \end{cases} \tag{12}$$

The transformations $x \to x$, $y \to -y$, $a \to -a$ do not alter the form of the system, in order to maintain its generality, the condition $a > 0$ is imposed. Furthermore, it is apparent that the transformations $x \to -x$, $y \to -y$ does not affect the form of the system, thus the trajectories of the system are symmetric with respect to the origin of coordinates.

By applying the transformation $x \to y$, $y \to x$, the system (12) can be transformed into the following system:

$$\begin{cases} \dot{x} = -4x + ay^5, \\ \dot{y} = y, \quad a \neq 0. \end{cases} \tag{13}$$

The system (13) can be transformed into the following form by applying the Poincaré transformation $x \to \frac{1}{x}$, $y \to \frac{y}{x}$:

$$\begin{cases} \dot{x} = x(4x^4 - ay^5), \\ \dot{y} = y(5x^4 - ay^5) \end{cases} \tag{14}$$

By utilizing a blow-up transformation on the system (13), we are able to perform a detailed analysis of the phase space behaviour in the vicinity of the multiple singular point at the origin for the system (14), which corresponds to the points $I^{\infty}_{1,2}(\pm 1, 0, 0)$ of the system (13).

$$\begin{cases} \dot{x} = x\left(ax\sin^7 y + ax\sin^5 y\cos^2 y - 4\cos^6 y - 5\sin^2 y\cos^4 y\right), \\ \dot{y} = \sin y\cos^5 y \end{cases} \tag{15}$$

Specifically, by solving the equation $Q(0, y) = 0$, we can identify the coordinates and topological classification of the resulting singular points M_1, M_2, M_3, and M_4. Of these, M_1 and M_3 are classified as nodal singularities, with M_1 being an unstable node and M_3 being a stable node. On the other hand, M_2 and M_4 are multiple singular points.

We translate the singular point M_2 to the origin of coordinates, then we expand the right-hand sides in a Taylor series in the neighbourhood of $y = 0$, and retain only the terms of low degree, we obtain the system:

$$\begin{cases} \dot{x} = \frac{5}{2}ax^2y^2 - ax^2 + xy^4\left(5 - \frac{65ax}{24}\right), \\ \dot{y} = \frac{4y^7}{3} - y^5, \end{cases} \tag{16}$$

Subsequently, by utilizing the blow-up method, we derive the following system:

$$\begin{cases} \dot{x} = \frac{1}{24}x\left(-65ax^4\sin^4 y\cos^3 y + 60ax^2\sin^2 y\cos^3 y - 24a\cos^3 y+\right.\\ \quad\left.+32x^5\sin^8 y - 24x^3\sin^6 y + 120x^3\sin^4 y\cos^2 y\right),\\ \dot{y} = \frac{1}{24}\sin y\cos y\left(65ax^4\sin^4 y\cos y - 60ax^2\sin^2 y\cos y+\right.\\ \quad\left.+\,24a\cos y + 32x^5\sin^6 y - 144x^3\sin^4 y\right), \end{cases} \tag{17}$$

We resolve the equation $Q(0, y) = 0$, resulting in the identification of the singular points $N_1(0,0)$, $N_2(0, \frac{\pi}{2})$, $N_3(0,\pi)$ and $N_4(0, \frac{3\pi}{2})$. The points N_1 and N_3 are classified as saddle singularities, while N_2 and N_4 are classified as compound singularities.

Subsequently, we effect a translation of N_2 to the origin of coordinates, followed by an expansion in a Taylor series in the vicinity of $y = 0$, and a blow-up transformation. This results in the decomposition of the singularity into four distinct points: $R_1(0,0)$, $R_2(0, \frac{\pi}{2})$, $R_3(0,\pi)$ and $R_4(0, \frac{3\pi}{2})$. The points R_2 and R_4 are classified as saddle singularities, while the points R_1 and R_3 are compound singularities.

Employing the blow-up technique on R_1, results in the decomposition of this point into four distinct singularities: $S_1(0,0)$, $S_2(0, \frac{\pi}{2})$, $S_3(0,\pi)$ and $S_4(0, \frac{3\pi}{2})$. The points S_2 and S_4 are classified as hyperbolic saddle singularities, while the singularities S_1 and S_3 are classified as non-hyperbolic multiple singularities.

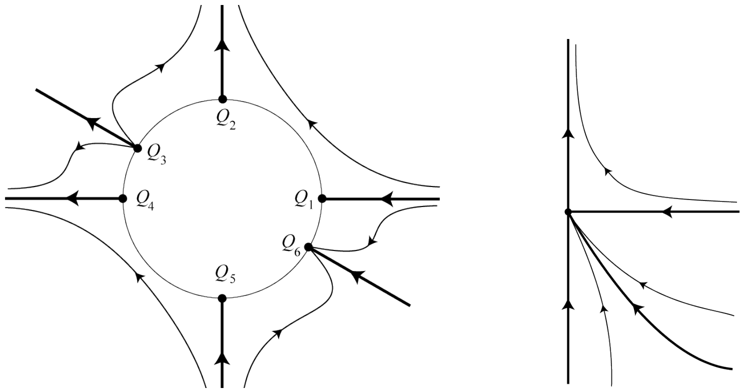
Finally, by utilizing the blow-up method on S_1, we obtain a further decomposition of this point into six distinct singularities: $Q_1(0,0)$, $Q_2(0, \frac{\pi}{2})$, $Q_3(0, \pi - arctg\frac{9}{a})$, $Q_4(0,\pi)$, $Q_5(0, \frac{3\pi}{2})$ and $Q_6(0, arctg\frac{9}{a})$, their eigenvalues and types are tabulated in Table 2.

Table 2. Blow up for point S_1

S.P.	Q_1	Q_2	Q_3	Q_4	Q_5	Q_6
$\lambda_{1,2}$	$-1, 9$	$\pm a$	$\frac{a}{\sqrt{81+a^2}}, \frac{9a}{\sqrt{81+a^2}}$	$-9, 1$	$-a, a$	$-\frac{a}{\sqrt{81+a^2}}, -\frac{9a}{\sqrt{81+a^2}},$
Type	S	S	N^u	S	S	N^s

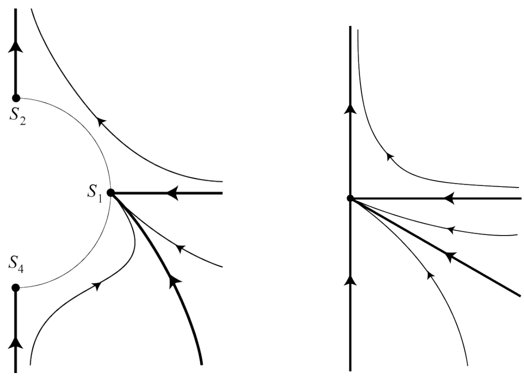
By leveraging the data presented in Table 2, we are able to construct the phase portrait in the vicinity of the singular point S_1 (Figure 8). Utilizing this information, we can then graphically depict the qualitative behaviour of the trajectories in the immediate vicinity of R_1 (as illustrated in Figure 9).

Utilizing the information obtained from the analysis of the type of singularity at R_1, we are able to construct the local phase portrait for the point N_2 (as depicted in Figure 10 a), b)). By applying a similar procedure, we are able to construct the phase portrait for the point N_4 (depicted in Figure 10 c), d)).



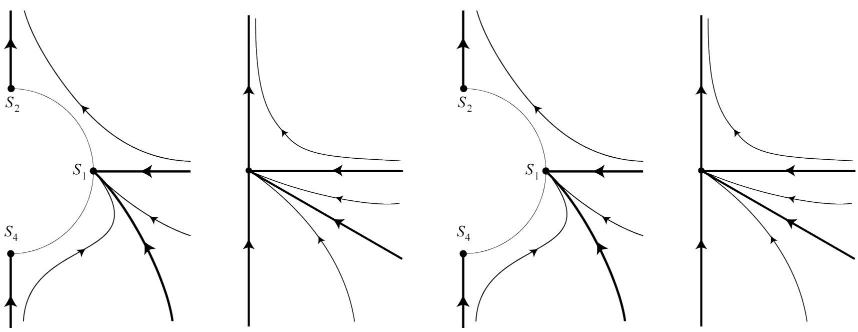
a) b)

Figure 8. Blow-up for the point S_1 $(0, 0)$.



a) b)

Figure 9. Blow-up for the point R_1 $(0, 0)$.



a) b) c) d)

Figure 10. Blow-up for the points N_2 $\left(0, \frac{\pi}{2}\right)$ and N_4 $\left(0, \frac{3\pi}{2}\right)$.

Through the utilization of blow-up techniques, we previously decomposed the singular point $M_2(0, \frac{\pi}{2})$ into the points N_1, N_2, N_3 and N_4. Given that N_1 and N_4 are classified as saddle singularities, we are able to construct the phase portrait for M_2 as depicted in Figure 11.

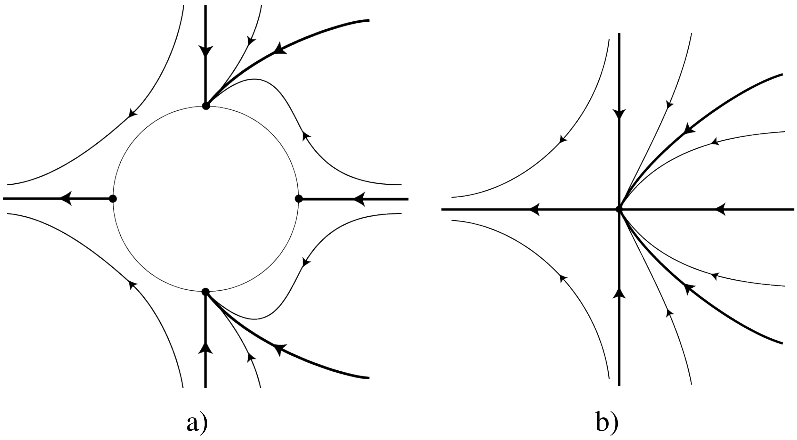


Figure 11. Blow-up for the points $M_2\left(0, \frac{\pi}{2}\right)$.

By applying similar techniques, we can construct the phase portrait for the point M_4 (as illustrated in Figure 12).

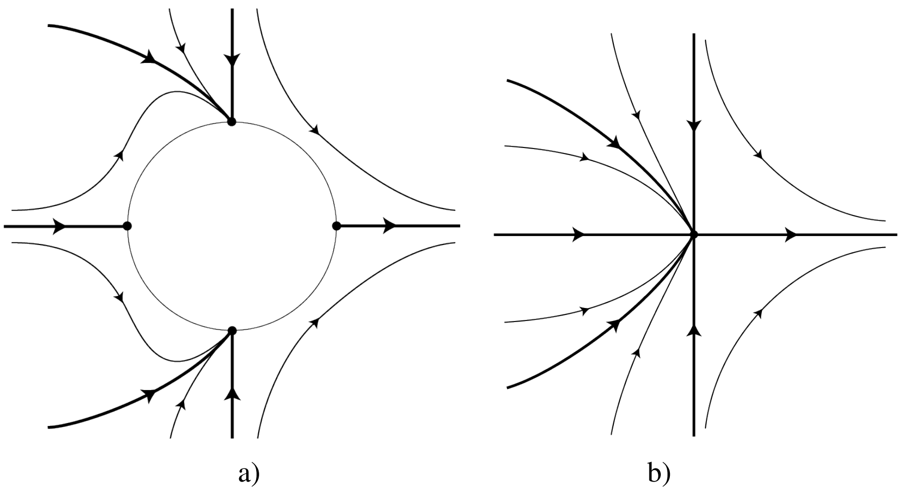


Figure 12. Blow-up for the point $M_4\left(0, \frac{3\pi}{2}\right)$.

By utilizing the information obtained from the phase portraits of M_2 and M_4 in conjunction with the fact that M_1 is an unstable node and M_2 is a stable node, we can construct the local phase portrait (as depicted in Figure 13) in the vicinity of the origin of coordinates for the system (14), which corresponds to the points $I^{\infty}_{1,2}(\pm 1, 0, 0)$ of the system (13).

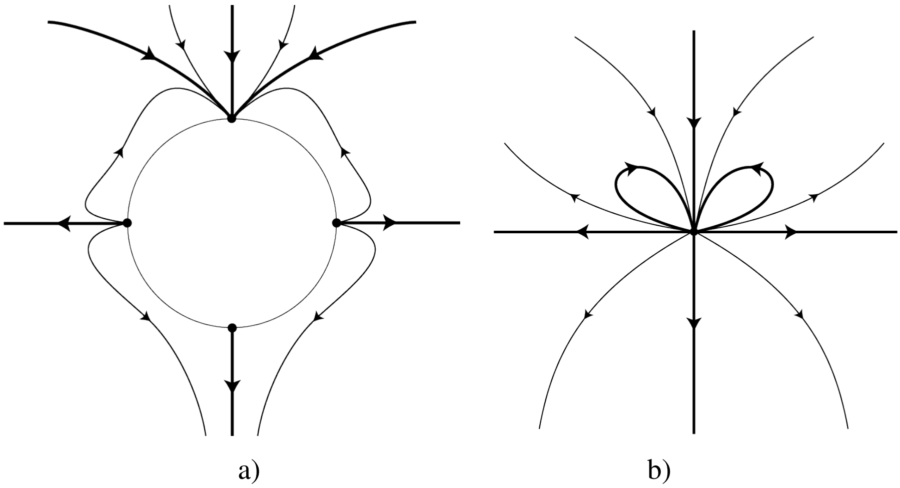


Figure 13. Blow-up for the points $I^{\infty}_{1,2}$ $(\pm 1, 0, 0)$.

Taking into account that the singular point in the finite portion of the phase space is of saddle type, upon application of the transformation $x \to y$, $y \to x$, we obtain the phase portrait (Figure 14) for the quintic differential system (12) which possesses a line of maximal multiplicity at infinity.

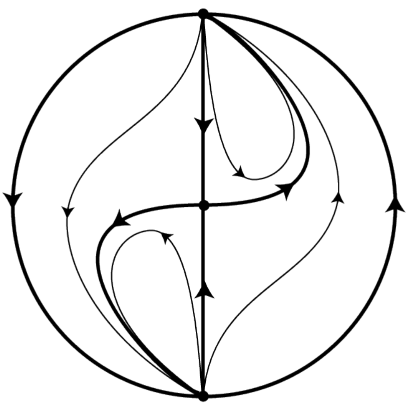


Figure 14. The phase portrait for the quartic differential system (11).

5. Conclusion

This article has provided an in-depth analysis of the phase portraits of polynomial differential systems of degree at most five and having the invariant straight line at the infinity of the maximal multiplicity. By using the blow-up method, we were able to decompose singular points, transform the systems, and obtain the phase portraits for the systems. This understanding of polynomial differential systems is valuable in many fields including physics, engineering, and biology.

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Received: October 28, 2022 *Accepted: December 14, 2022*

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Acta et Commentationes, Exact and Natural Sciences ISSN: 2537-6284
Volume 2(14), 2022, Pages 81–87 E-ISSN: 2587-3644
CZU 512.57 DOI: 10.36120/2587-3644.v14i2.81-87

The left product, the right product and the theories of relative torsion

Alina Turcanu

Abstract. It is demonstrated that any theory of relative torsion is defined by the left and the right products.

2010 Mathematics Subject Classification: 46M15, 18B30.

Keywords: reflexive and coreflective subcategories, the right and left product of two subcategories, the relative torsions theories.

Produsul de stânga, produsul de dreapta şi teorii de torsiune relative

Rezumat. Se demonstrează că orice teorie de torsiuni relative este descrisă de produsele de stânga şi de dreapta.

Cuvinte-cheie: subcategorii reflexive şi coreflexive, produsul de dreapta şi de stânga a două subcategorii, teorii de torsiune relative.

1. Introduction

The paper is a continuation (with notations and terminology) of the article [6] (see also [4]). Note by $C_2\mathcal{V}$ the category of topological vector locally convex Hausdorff spaces (see [9]), where you can also find all the notions referred totopologies. We will use the following notation.

Factorization structures (see [4]):

$\mathbb{B}$ the class of factorization structures;

$(\mathcal{E}pi, \mathcal{M}_f)$ - (the class of epimorphisms, the class of kernels) = (the class of morphisms with dense image, the class of topological inclusions with closed images);

$(\mathcal{E}_u, \mathcal{M}_p)$=(the class of universal epimorphisms, the class of exact monomorphisms)=(the class of surjective morphisms, the class of topological inclusions);

$(\mathcal{E}_p, \mathcal{M}_u)$=(the class of exact epimorphisms, the class of universal monomorphisms);

$(\mathcal{E}_f, \mathcal{M}ono)$=(the class of cokernels, the class of monomorphisms)=(the class of factorial morphisms, the class of injective morphisms);

The properties of factorization structures $(\mathcal{E}_f, \mathcal{M}ono)$ and $(\mathcal{E}pi, \mathcal{M}_f)$ characterize the category $C_2\mathcal{V}$ as a semiabelian category. The factorization structures $(\mathcal{E}_u, \mathcal{M}_p)$

and $(\mathcal{E}_p, \mathcal{M}_u)$ play an important role in the study of the reflective and coreflective subcategories. We need some notions and results from [3], [4] and [6].

We use the following notations for some subcategories of the category $C_2\mathcal{V}$.

$\mathbb{R}$ - the class of non-zero reflective subcategories;

$\mathbb{K}$ - the class of nonzero coreflective subcategories;

Π - the subcategory of complete spaces with a weak topology and with respective functor $\pi : C_2\mathcal{V} \to \Pi$;

$\mathcal{S}$ - the subcategory of spaces endowed with a weak topology, $s : C_2\mathcal{V} \to \mathcal{S}$;

Γ_0 - the subcategory of complete spaces, $g_o : C_2\mathcal{V} \to \Gamma_o$;

Σ - the coreflective subcategory of spaces with the strongest locally convex topology, $\sigma : C_2\mathcal{V} \to \Sigma$;

$\widetilde{\mathcal{M}}$ - the subcategory of spaces endowed with the Mackey topology, $m : C_2\mathcal{V} \to \widetilde{\mathcal{M}}$.

Let $\mathcal{A}$ and $\mathcal{B}$ be two classes of morphisms of the category $C_2\mathcal{V}$. We will use the notations:

1. $\mathcal{A} \circ \mathcal{B} = \{a \cdot b |\ a \in \mathcal{A}, b \in \mathcal{B}$ and there is the composition $a \cdot b\}$.

2. The class $\mathcal{A}$ is called $\mathcal{B}$-hereditary if from the fact that $f \cdot g \in \mathcal{A}$ and $f \in \mathcal{B}$, it follows that $g \in \mathcal{A}$. The class $\mathcal{E}pi$ is $\mathcal{M}_u$-hereditary ([4], Lemma 2.6);

2^*. The class $\mathcal{A}$ is called $\mathcal{B}$-cohereditary if from the fact that $f \cdot g \in \mathcal{A}$ and $g \in \mathcal{B}$, it follows that $f \in \mathcal{A}$.

If $\mathcal{R} \in \mathbb{R}$, then $(\mathcal{P}''(\mathcal{R}), \mathcal{I}''(\mathcal{R})) = ((\varepsilon\mathcal{R}) \circ \mathcal{E}_p, (\varepsilon\mathcal{R})^{\perp} \cap \mathcal{M}_u)$.

If $\mathcal{K} \in \mathbb{K}$, then $(\mathcal{P}'(\mathcal{K}), \mathcal{I}'(\mathcal{K})) = ((\mu\mathcal{K})^{\top} \cap \mathcal{E}_u, \mathcal{M}_p \circ (\mu\mathcal{K}))$ (see [5]).

We will show the application of left and right products to the description of relative torsion theories.

2. The right and left product of two subcategories

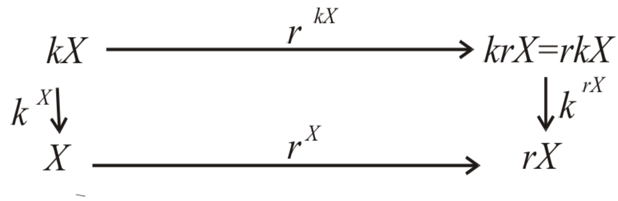
Definition 2.1 ([1]). Let $\mathcal{K}$ be a coreflective subcategory, and $\mathcal{R}$ a reflective subcategory of category C. The pair $(\mathcal{K}, \mathcal{R})$ is called relative torsion theory (TTR), that is, relative to the subcategory $\mathcal{K} \cap \mathcal{R}$, if the functors $k : C \to \mathcal{K}$ and $r : C \to \mathcal{R}$ verify the following two relations:

1. The functors k and r commute: $k \cdot r = r \cdot k$;

2. For any object X of category C the square

$$r^X \cdot k^X = k^{rX} \cdot r^{kX} \quad (1)$$

is puschout and pullback.



Remark 2.1. In abelian categories a theory of torsion $(\mathcal{T}, \mathcal{F})$ is a TTR relative to intersections $\mathcal{T} \cap \mathcal{F} = 0$ [2].

Theorem 2.1. *([3], Theorem 2.1). Let $\mathcal{K}$ be a non-zero coreflective subcategories, and $\mathcal{R}$ be a non-zero reflective subcategories of category $C_2\mathcal{V}$ and $\mathcal{R} \in \mathbb{R}(\mathcal{M}_p)$. The pair $(\mathcal{K}, \mathcal{R})$ forms a TTR if and only if the coreflector functor $k : C_2\mathcal{V} \longrightarrow \mathcal{K}$ and reflector $r : C_2\mathcal{V} \longrightarrow \mathcal{R}$ commute: $k \cdot r = r \cdot k$.*

In the work [3] this theorem is without proof, therefore, for completeness, the proof will be included here.

Proof. Let the respective functors commute: $k \cdot r = r \cdot k$ and we will prove that for any object X of the category $C_2\mathcal{V}$ the square

$$r^X \cdot k^X = k^{rX} \cdot r^{kX} \tag{2}$$

is puschout and pullback. Indeed, either

$$u^X \cdot k^X = v^X \cdot r^{kX} \tag{3}$$

the puschout built on the morphisms k^X and r^{kX}. Then

$$r^X = t^X \cdot u^X, \tag{4}$$

$$k^{rX} = t^X \cdot v^X \tag{5}$$

for a morphism t^X. Since r^{kX} is an epi, according to construction, we deduce that u^X is also an epi. Moreover, $r^X \in \mathcal{M}_p$, $u^X \in \mathcal{E}pi$ and the class $\mathcal{M}_p$ is $\mathcal{E}pi$-cohereditary. So from equality (5) it turns out that $t^X \in \mathcal{M}_p$. Also $k^{rX} \in \mathcal{E}_u$. Thus from equality (5) we deduce as $t^X \in \mathcal{E}_u$. Finally $v^X \in \mathcal{E}_u \cap \mathcal{M}_p = \mathcal{I}so$.

This is how we proved that the square (2) is puschout. The class $\mathcal{A}$ is called $\mathcal{B}$-hereditary if from the fact that $f \cdot g \in \mathcal{A}$ and $f \in \mathcal{B}$, it follows that $g \in \mathcal{A}$. Let's prove that it is also pullback. Let

$$r^X \cdot l^X = k^{rX} \cdot m^X \tag{6}$$

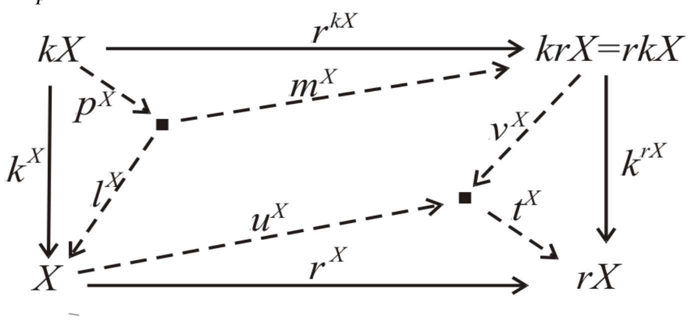
the pullback built on morphisms r^X and k^{rX}. Then

$$k^X = l^X \cdot p^X, \tag{7}$$

$$r^{kX} = m^X \cdot p^X \tag{8}$$

for un morphism p^X. Since $k^{rX} \in \mathcal{E}_u \cap \mathcal{M}ono$, it turns out that $l^X \in \mathcal{E}_u \cap \mathcal{M}ono$.

Thus in equality (7) the morphisms k^X and l^X belong to the class $\mathcal{E}_u \cap \mathcal{M}ono$. So also p^X belongs to this class. From equality (8) it follows that $p^X \in \mathcal{M}_p$, because $r^{kX} \in \mathcal{M}_p$. So $p^X \in \mathcal{E}_u \cap \mathcal{M}_p = \mathcal{I}so$.



Remark 2.2. Regarding examples of TTR (see [1-3]).

Since $\mathcal{K}$-coreplica for any object of the category $C_2\mathcal{V}$ is a bijective application, we get:

Lemma 2.1. *Let $\mathcal{R} \in \mathbb{R}^s(\mu\mathcal{K})$. Then for any object (E, u) of it and any locally convex topology v with the property $u \leq v \leq k(u)$, where $(E, k(u))$ is $\mathcal{K}$-core replica of the object (E, u), the object (E, v) also belongs to the subcategory $\mathcal{R}$.*

Lemma 2.2. *For the subcategories $\mathcal{K}$ and $\mathcal{R}$ of the category $C_2\mathcal{V}$ the following statements are equivalent:*

1. $\mathcal{K} *_s \mathcal{R} = \mathcal{K}$.

2. $\mathcal{K} \in \mathbb{K}_f(\varepsilon\mathcal{R})$.

If the subcategory $\mathcal{K}$ contains the subcategory $\widetilde{\mathcal{M}}$ of spaces with Mackey topology, then the previous conditions are equivalent to the condition:

3. The subcategory $\mathcal{K}$ is $\mathcal{I}''(\mathcal{R})$-coreflective.

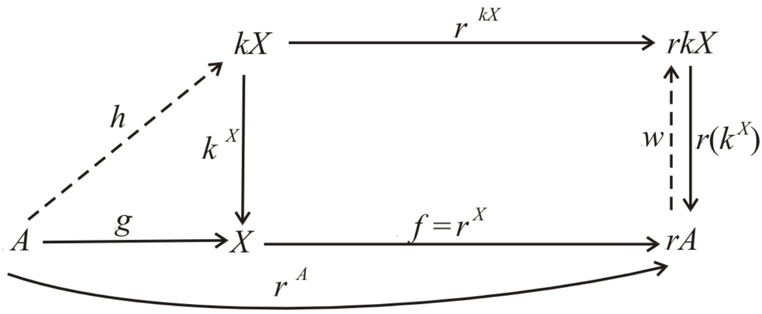
Proof. $1 \Rightarrow 2$. Let $A \in |\mathcal{K}|$ and

$$r^A = f \cdot g \tag{9}$$

be a decomposition of the morphism r^A with g as an epi. We will prove that $X \in |\mathcal{K}|$. Since g is an epi, we deduce that f is the $\mathcal{R}$-replica of object X. Let k^X be the $\mathcal{K}$-coreplica of the object X. We have $A \in |\mathcal{K}|$, so

$$g = k^X \cdot h \tag{10}$$

for a morphism h. We examine the left product diagram for the object X.



Thus we have equality

$$r^X \cdot k^X = r(k^X) \cdot r^{kX} \tag{11}$$

For the morphism $r^{kX} \cdot h$ there is a morphism w as follows

$$w \cdot r^A = r^{kX} \cdot h \tag{12}$$

We have

$$r(k^X) \cdot w \cdot r^A = (din(14)) = r(k^X) \cdot r^{kX} \cdot h = (din(13)) = r^X \cdot k^X \cdot h =$$

$$= (din(12)) = r^X \cdot g = r^A$$

i.e.

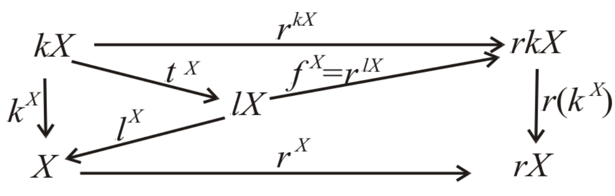
$$r(k^X) \cdot w \cdot r^A = r^A \tag{13}$$

Since r^A is an epi, it follows that

$$r(k^X) \cdot w = 1 \tag{14}$$

According to the first hypothesis, the square (10) is pullback, and the morphism $r(k^X)$ is a retraction, it turns out that k^X is the same. But k^X is also a mono. Thus we proved that $X \in |\mathcal{K}|$.

$2 \Rightarrow 1$. Let X be an arbitrary object of the category $C_2\mathcal{V}$. We construct the left product diagram for it.

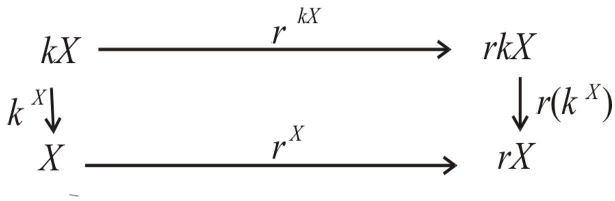


We examine the equality

$$r^{kX} = f^X \cdot t^X. \tag{15}$$

Because the class $\mathcal{E}pi$ is $\mathcal{M}_u$-hereditary ([4], Lemma 2.6), the morphism t^X is an epi. Thus according to hypothesis (2) lX is an object of the subcategory $\mathcal{K}$. Therefore, t^X is an iso, and $\mathcal{K} *_s \mathcal{R} = \mathcal{K}$.

$3 \Longleftrightarrow 1$. For an arbitrary object of the category $C_2\mathcal{V}$ we examine the commutative square:



$$r^X \cdot k^X = r(k^X) \cdot r^{kX} \tag{16}$$

Since $\widetilde{\mathcal{M}} \subset \mathcal{K}$, it follows that $\mathcal{K}$ is a $\mathcal{M}_u$-coreflective subcategory. Thus $k^X \in \mathcal{M}_u$. According to Theorem 2.12 [4] the square (16) is pullback if and only if $k^X \in \mathcal{I}''(\mathcal{R})$.

We formulate the dual statement.

Lemma 2.3. *For the subcategories $\mathcal{K}$ and $\mathcal{R}$ of the category $C_2\mathcal{V}$ the following statements are equivalent:*

1. $\mathcal{K} *_d \mathcal{R} = \mathcal{R}$.

2. $\mathcal{R} \in \mathbb{R}^s(\mu\mathcal{K})$.

If the subcategory $\mathcal{R}$ contains the subcategory $\mathcal{S}$ of spaces with weak topology, then the previous conditions are equivalent to the condition:

3. The subcategory $\mathcal{R}$ is $\mathcal{E}'(\mathcal{K})$-reflective.

The proven Lemmas allow us to formulate the following result. From Theorem 2.1 and Lemmas 2.2, 2.3 we obtain:

Theorem 2.2. *Let $\mathcal{K} \in \mathbb{K}(\mathcal{M}_u)$, i.e. $\mathcal{K}$ is a $\mathcal{M}_u$- coreflective subcategory of the category $C_2\mathcal{V}$ ($\widetilde{\mathcal{M}} \subset \mathcal{K}$), and $\mathcal{R} \in \mathbb{R}(\mathcal{M}_p)$, i.e. $\mathcal{R}$ is a $\mathcal{M}_p$-reflective subcategory ($\Gamma_0 \subset \mathcal{R}$). Then:*

1. The subcategory $\mathcal{K}$ is closed in relation to $(\mathcal{E}pi \cap \mathcal{M}_p)$-factorobjects. In other words, the subcategory $\mathcal{K}$ is closed in relation to the extensions.

2. $\mathcal{R} \in \mathbb{R}^s(\mu\mathcal{K})$.

Remark 2.3. For some subcategories $\mathcal{K}$ of the class $\mathbb{K}(\mathcal{M}_u)$, in particular, for the subcategory $\widetilde{\mathcal{M}}$, it is well known that they are closed in relation to extensions ([9], Assertion IV.3.5.)

2. Any fully convex local space (E, t) remains complete in any topology u finer than t and compatible with the same duality:

$$t \leq u \leq m(t),$$

910], VI, Proposition 5). This result was generalized for any $\mathcal{M}_p$-reflective subcategory by D. Botnaru and O. Cerbu [6], Theorem 1.12.

3. The theories of relative torsion

Theorem 3.1. *Let $\mathcal{K}$ be a coreflective subcategory, and $\mathcal{R}$ - a nonzero reflective subcategory of the category $C_2\mathcal{V}$. The following statements are equivalent:*

1. The pair $(\mathcal{K}, \mathcal{R})$ forms a TTR.

2. a) The coreflector function $k : C_2\mathcal{V} \longrightarrow \mathcal{K}$ and reflector $r : C_2\mathcal{V} \longrightarrow \mathcal{R}$ commute $k \cdot r = r \cdot k$; b) $\mathcal{K} *_s \mathcal{R} = \mathcal{K}$; c) $\mathcal{K} *_d \mathcal{R} = \mathcal{R}$.

3. a) The functors k and r commute $k \cdot r = r \cdot k$; b) $\mathcal{K} \in \mathbb{K}_f(\varepsilon\mathcal{R})$; c) $\mathcal{R} \in \mathbb{R}^s(\mu\mathcal{K})$.

If $\widetilde{\mathcal{M}} \subset \mathcal{K}$ and $\mathcal{S} \subset \mathcal{R}$ then the preceding conditions are equivalent to the following:

4. a) The functors k and r commute $k \cdot r = r \cdot k$; b) The subcategory $\mathcal{K}$ is $\mathcal{I}''(\mathcal{R})$-coreflective; c) The subcategory $\mathcal{R}$ is $\mathcal{E}'(\mathcal{K})$-reflective.

Remark 3.1. In the previous Theorem p.2 and p.3 condition a) is not a consequence of conditions b) and c).

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Received: November 01, 2022 *Accepted: December 16, 2022*

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Acta et Commentationes, Exact and Natural Sciences ISSN: 2537-6284
Volume 2(14), 2022, Pages 88–100 E-ISSN: 2587-3644
CZU 517.925 DOI: 10.36120/2587-3644.v14i2.88-100

Center conditions for a cubic system with two homogeneous invariant straight lines and exponential factors

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Abstract. In this paper for a cubic differential system with a singular point $O(0,0)$ of a center or a focus type, having two homogeneous invariant straight lines and exponential factors, we determine conditions under which the singular point is a center.
2010 Mathematics Subject Classification: 34C05.
Keywords: cubic differential system, the problem of the center, invariant algebraic curve, exponential factor.

Condiţii de existenţă a centrului pentru un sistem cubic cu două drepte invariante omogene şi factori exponenţiali

Rezumat. În această lucrare pentru un sistem diferenţial cubic cu punctul singular $O(0,0)$ de tip centru sau focar, care are două drepte invariante omogene şi factori exponenţiali, sunt determinate condiţiile încât punctul singular să fie centru.
Cuvinte-cheie: sistem diferenţial cubic, problema centrului şi focarului, curbă algebrică invariantă, factor exponenţial.

1. Introduction

We consider the cubic differential system of the form

$$\begin{cases} \dot{x} = y + ax^2 + cxy + fy^2 + kx^3 + mx^2y + pxy^2 + ry^3 \equiv P(x,y), \\ \dot{y} = -(x + gx^2 + dxy + by^2 + sx^3 + qx^2y + nxy^2 + ly^3) \equiv Q(x,y), \end{cases} \quad (1)$$

in which $P(x,y)$ and $Q(x,y)$ are real and coprime polynomials in the variables x and y. The origin $O(0,0)$ is a singular point of a center or a focus type for (1). The problem arises of distinguishing between a center and a focus, i.e. of finding the coefficient conditions under which $O(0,0)$ is, for example, a center. These conditions are called *the center conditions* and the problem - *the problem of the center*. When the cubic system (1) contains both quadratic and cubic nonlinearities, the problem of finding a finite number of necessary and sufficient conditions for the center is still open.

It is well known that $O(0,0)$ is a center for system (1) if and only if the Lyapunov quanities $L_1, L_2, \ldots, L_k, \ldots$ vanish [5], [18].

The problem of finding the center conditions for system (1) has a long history and a variety of methods have been developed. An approach to the problem of the center is to study the local integrability of the system (1) in some neighborhood of the singular point $O(0,0)$. It is known that a singular point $O(0,0)$ is a center for system (1) if and only if it has a holomorphic first integral of the form $F(x,y) = C$ in some neighborhood of $O(0,0)$ [17]. Also, $O(0,0)$ is a center if and only if the system (1) has a holomorphic integrating factor of the form $\mu = 1 + \sum \mu_j(x,y)$ in some neighborhood of $O(0,0)$ [1].

The problem of the center was solved for some families of cubic differential systems having invariant algebraic curves (invariant straight lines, invariant conics, invariant cubics) in [5], [7], [9], [10], [12], [13], [16], [19], [20], [21]. Center conditions were determined for some cubic systems having integrating factors in [8], [11], [14], for some reversible cubic systems in [2] and for a few families of the complex cubic system in [15].

In this paper we determine the center conditions for cubic differential system (1) assuming that the system has invariant straight lines and exponential factors. The paper is organized as follows. In Section 2 we present the results concerning the existence of invariant straight lines and exponential factors. In Section 3 we find conditions under which the cubic system has exponential factors. In Section 4 we obtain center conditions for system (1) with two homogeneous invariant straight lines and one exponential factor.

2. Invariant straight lines and exponential factors

We study the problem of the center for cubic differential system (1) assuming that the system has invariant algebraic curves and exponential factors.

Definition 2.1. An algebraic curve $\Phi(x,y) = 0$ in $\mathbb{C}^2$ with $\Phi \in \mathbb{C}[x,y]$ is said to be an *invariant algebraic curve* of system (1) if

$$\frac{\partial \Phi}{\partial x} P(x,y) + \frac{\partial \Phi}{\partial y} Q(x,y) = \Phi(x,y) K(x,y), \tag{2}$$

for some polynomial $K(x,y) \in \mathbb{C}[x,y]$, called the *cofactor* of the invariant algebraic curve $\Phi(x,y) = 0$.

By the above definition, a straight line

$$C + Ax + By = 0, \quad A, B, C \in \mathbb{C}, \quad (A,B) \neq (0,0), \tag{3}$$

is an invariant straight line for system (1) if and only if there exists a polynomial $K(x,y)$ such that the following identity holds

$$A \cdot P(x,y) + B \cdot Q(x,y) \equiv (C + Ax + By) \cdot K(x,y). \tag{4}$$

If the cubic system (1) has complex invariant straight lines then obviously they occur in complex conjugated pairs [5]

$$C + Ax + By = 0 \quad \text{and} \quad \overline{C} + \overline{A}x + \overline{B}y = 0.$$

According to [6] the cubic system (1) cannot have more than four nonhomogeneous invariant straight lines, i.e. invariant straight lines of the form

$$1 + Ax + By = 0, \quad (A, B) \neq (0, 0). \tag{5}$$

As homogeneous invariant straight lines $Ax + By = 0$, the system (1) can have only the lines $x \mp iy = 0, i^2 = -1$.

Lemma 2.1. *The cubic system* (1) *has the invariant straight lines* $x \mp iy = 0$ *if and only if the following set of conditions holds*

$$d = f - a, \quad c = g - b, \quad k - l = p - q, \quad r + s = m + n. \tag{6}$$

Proof. By Definition 2.1, the straight lines $x \mp iy = 0$ are invariant for (1) if and only if

$$P(x, y) \mp iQ(x, y) \equiv (x \mp iy)(c_{00} + c_{10}x + c_{01}y + c_{20}x^2 + c_{11}xy + c_{02}y^2). \tag{7}$$

Identifying the coefficients of the monomials $x^j y^h$ in (7), we find that

$$\begin{aligned} &c_{00} = \pm i,\ c_{10} = a \pm ig,\ c_{02} = p - k - q \pm i(m + n - s), \\ &c_{01} = c - g \pm i(a + d),\ c_{20} = k \pm is,\ c_{11} = m - s \pm i(k + q) \end{aligned}$$

and

$$f - a - d \pm i(b + c - g) = 0, \quad r + s - m - n \pm i(l - k + p - q) = 0.$$

The last identities yield the set of conditions (6). □

The cofactors of the invariant straight lines $x \mp iy = 0$ are

$$\begin{aligned} K_1(x, y) = {} & i + (a + i(b + c))x + (-b + i(a + d))y + (k + is)x^2 + \\ & + (m - s + i(k + q))xy + (p - k - q + i(m + n - s))y^2, \\ K_2(x, y) = {} & \overline{K_1(x, y)}. \end{aligned} \tag{8}$$

Denote $k = u + l,\ p = u + q,\ s = v - r,\ n = v - m$, where u, v are some real parameters. Assume that the conditions (6) are fulfilled, then system (1) can be written as follows

$$\begin{aligned} \dot{x} = {} & y + ax^2 + (g - b)xy + fy^2 + (u + l)x^3 + mx^2y + \\ & + (u + q)xy^2 + ry^3 \equiv P(x, y), \\ \dot{y} = {} & -(x + gx^2 + (f - a)xy + by^2 + (v - r)x^3 + qx^2y + \\ & + (v - m)xy^2 + ly^3) \equiv Q(x, y). \end{aligned} \tag{9}$$

The problem of the center was solved for system (9) with: one invariant straight line $1 + Ax + By = 0$ in [20], two invariant straight lines of the form (5) in [5], one invariant conic $a_{20}x^2 + a_{11}xy + a_{02}y^2 + a_{10}x + a_{01}y + 1 = 0$ in [7]. By using the method of Darboux integrability and rational reversibility, the center conditions were found for (9) in [8].

In this Section, we investigate the problem of the existence of exponential factors for cubic differential system (9).

Definition 2.2. Let $h,\ g \in \mathbb{C}[x, y]$ be relatively prime in the ring $\mathbb{C}[x, y]$. The function $\Phi = \exp(g/h)$ is called an *exponential factor* of a system (1) if for some polynomial $K \in \mathbb{C}[x, y]$ of degree at most two it satisfies the equation

$$\frac{\partial \Phi}{\partial x} P(x, y) + \frac{\partial \Phi}{\partial y} Q(x, y) \equiv \Phi K(x, y). \tag{10}$$

As before, we say that K is the cofactor of the exponential factor $\exp(g/h)$.

This means that if we have a cubic differential system (1) with an exponential factor of the form $\exp(g/h)$, then there is a 1-parameter perturbation of system (1), given by a small ε, with two invariant algebraic curves, namely $h = 0$ and $h + \varepsilon g = 0$. Hence, when $\varepsilon = 0$, these two curves coalesce giving the exponential factor $\exp(g/h)$ for the system with $\varepsilon = 0$ (the invariant algebraic curve $h = 0$ has geometric multiplicity larger than one), as well as the invariant algebraic curve $h = 0$ which does not disappear [3].

Since the exponential factor cannot vanish, it does not define invariant curves of the cubic system (1). The next theorem, proved in [3], gives the relationship between the notion of invariant algebraic curve and exponential factor.

Theorem 2.1. *If* $\exp(g/h)$ *is an exponential factor with cofactor* K *for a cubic system* (1) *and if* h *is not a constant, then* $h = 0$ *is an invariant algebraic curve with cofactor* K_h*, and* g *satisfies the equation* $\mathcal{X}(g) = gK_h + hK$.

Eventually $\Phi = \exp(g)$ can be an exponential factor coming from the multiplicity of the infinite invariant straight line.

3. Cubic differential systems with exponential factors

In this Section, we consider the cubic differential system (9) with two homogeneous invariant straight lines $x \mp iy = 0$. We determine the conditions under which the system (9) has exponential factors of the form

$$\Phi = \exp\left(g(x, y)\right), \quad \Phi = \exp\left(\frac{g(x, y)}{x^2 + y^2}\right), \tag{11}$$

where $g(x, y)$ is a real polynomial with degree$(g) \le 2$.

Lemma 3.1. *The cubic differential system* (9) *has an exponential factor of the form* $\Phi = \exp(a_{10}x + a_{01}y)$ *if and only if one the following two sets of conditions holds:*

(i_1) $m = r,\ q = l,\ l = (ra_{10})/a_{01},\ v = (ra_{01}^2 + ua_{01}a_{10} + ra_{10}^2)/a_{01}^2$;
(i_2) $m = r = 0,\ q = l,\ u = -l.$

Proof. By Definition 2.2, the function $\Phi = \exp(a_{10}x + a_{01}y)$ is an exponential factor for system (9) if there exists numbers $c_{10}, c_{01}, c_{20}, c_{11}, c_{02}$ such that

$$a_{10}P(x,y) + a_{01}Q(x,y) = c_{10}x + c_{01}y + c_{20}x^2 + c_{11}xy + c_{02}y^2. \tag{12}$$

Substituting $P(x,y), Q(x,y)$ in (12) and identifying the coefficients of the monomials $x^i y^j$, $i + j = 1, 2, 3$, we find that $c_{01} = a_{10},\ c_{10} = -a_{01},\ c_{20} = aa_{10} - ga_{01},\ c_{11} = (a - f)a_{01} + (g - b)a_{10},\ c_{02} = fa_{10} - ba_{01}$ and $a_{10},\ a_{01}$ satisfy the system of equations:

$$\begin{aligned} U_{30} &\equiv (l + u)a_{10} + (r - v)a_{01} = 0,\\ U_{21} &\equiv ma_{10} - qa_{01} = 0,\\ U_{12} &\equiv (q + u)a_{10} + (m - v)a_{01} = 0,\\ U_{03} &\equiv ra_{10} - la_{01} = 0. \end{aligned} \tag{13}$$

Assume that $a_{01} \neq 0$. Then the equations of (13) yield

$$m = r,\ q = l,\ l = (ra_{10})/a_{01},\ v = (ra_{01}^2 + ua_{01}a_{10} + ra_{10}^2)/a_{01}^2.$$

We obtain the set of conditions (i_1) of Lemma 3.1. The system (9) has the exponential factor $\Phi = \exp(a_{10}x + a_{01}y)$ with cofactor $K(x,y) = (aa_{10} - ga_{01})x^2 + (aa_{01} - fa_{01} - ba_{10} + ga_{10})xy + (fa_{10} - ba_{01})y^2 - a_{01}x + a_{10}y$.

Assume that $a_{01} = 0$, then $a_{10} \neq 0$. In this case the equations of (13) imply $m = r = 0,\ q = l,\ u = -l$. We obtain the set of conditions (i_2) of Lemma 3.1. The system (9) has the exponential factor $\Phi = \exp(x)$ with cofactor $K(x,y) = ax^2 - bxy + gxy + fy^2 + y$. □

Lemma 3.2. *The cubic system* (9) *has an exponential factor of the form*

$$\Phi = \exp(a_{20}x^2 + a_{11}xy + a_{02}y^2 + a_{10}x + a_{01}y + a_{00}) \tag{14}$$

if and only if one the following three sets of conditions holds:

(i_1) $c = d = 0, f = a, g = b, k = p = q = l, r = m, s = n$;
(i_2) $c = d = 0,\ f = a,\ g = b,\ k = p,\ p = l + u,\ r = m,\ m = (al)/b,\ s = n,\ n = ((l + u)b)/a, q = l$;
(i_3) $c = g - b, d = f - a, k = l = p = q = 0, m = n = s = r.$

Proof. By Definition 2.2, the function (14) is an exponential factor for cubic system (9) if there exist numbers $c_{20}, c_{11}, c_{02}, c_{10}, c_{01}$ such that

$$\begin{aligned}&(2a_{20}x + a_{11}y + a_{10})P(x, y) + (2a_{02}y + a_{11}x + a_{01})Q(x, y) = \\ &= c_{20}x^2 + c_{11}xy + c_{02}y^2 + c_{10}x + c_{01}y.\end{aligned} \tag{15}$$

Substituting $P(x, y)$ and $Q(x, y)$ in (15) and identifying the coefficients of the monomials $x^i y^j$, we reduce this identity to a system of fourteen equations

$$\{U_{ij} = 0, \ \ i + j = 1, 2, 3, 4\} \tag{16}$$

for the unknowns a_{ij}, c_{ij} and the coefficient of system (9).

When $i + j = 1, 2$, from the equations of (16) we get

$$c_{01} = a_{10}, c_{10} = -a_{01}, c_{02} = a_{11} - ba_{01} + fa_{10},$$
$$c_{20} = aa_{10} - ga_{01} - a_{11}, c_{11} = (a - f)a_{01} + 2a_{20} - 2a_{02} + (g - b)a_{10}.$$

1. Assume that $a_{11} \neq 0$. In this case the equations $U_{40} = 0, U_{22} = 0$ and $U_{04} = 0$ of (16) yield $v = (ra_{11} + 2la_{20} + 2ua_{20})/a_{11}, r = (2la_{02})/a_{11}, m = ((l + q)a_{02} + (l - q)a_{20})/a_{11}$. The resultant of the polynomials U_{31} and U_{13} with respect to q is $Res(U_{31}, U_{13}, q) = 2uf_1f_2$, where $f_1 = 4a_{02}a_{20} - a_{11}^2$, $f_2 = (a_{02} - a_{20})^2 + a_{11}^2 \neq 0$.

Let $u = 0$. Then $U_{31} = 0$ and $U_{13} = 0$ imply $q = l$. From the equations $U_{30} = 0, U_{03} = 0$ of (16) we express f, g and calculate the resultant of the polynomials U_{21} and U_{12} with respect to b. We obtain that $Res(U_{21}, U_{12}, b) = -4a_{11}g_1g_2g_3$, where $g_1 = aa_{11} - la_{01}$, $g_2 = 4a_{02}a_{20} - a_{11}^2$, $g_3 = (a_{02} - a_{20})^2 + a_{11}^2 \neq 0$.

If $g_1 = 0$, then $a_{01} = (aa_{11})/l$ and $a_{10} = (ba_{11})/l$. In this case we obtain the set of conditions (i_1) of Lemma 3.2. The exponential factor is

$$\Phi = \exp(2bx + 2ay + 2lxy + nx^2 + my^2)$$

having the cofactor $K(x, y) = 2(-ax + by - lx^2 + ly^2 - mxy + nxy)$.

If $g_1 \neq 0$ and $g_2 = 0$, then $a_{20} = a_{11}^2/(4a_{02})$, $a_{10} = [(la_{01} - aa_{11} + 2ba_{02})a_{11}]/(2la_{02})$. This case is contained in Lemma 3.2, (i_1) $(n = l^2/m)$.

Assume that $u \neq 0$ and let $f_1 = 0$. Then $U_{31} = 0$ and $U_{13} = 0$ yield $q = l$. From the equations $U_{30} = 0, U_{03} = 0$ we express a, f and calculate the resultant of the polynomials U_{21} and U_{12} with respect to b. We obtain that $Res(U_{21}, U_{12}, b) = a_{11}uh_1h_2$, where $h_1 = a_{01}a_{11} - 2a_{02}a_{10}$, $h_2 = 4a_{02}^2 + a_{11}^2 \neq 0$, $a_{11}u \neq 0$.

Let $h_1 = 0$. Then $a_{10} = (a_{01}a_{11})/(2a_{02})$ and $g = b$. In this case we get the set of conditions (i_2) of Lemma 3.2. The exponential factor is

$$\Phi = \exp((bx + ay)(2ba_{01} + ba_{11}x + aa_{11}y)/(2ab))$$

having the cofactor $K(x, y) = (by - ax)(ba_{01} + ba_{11}x + aa_{11}y)/(ab)$.

2. Assume that $a_{11} = 0$ and let $a_{02} = 0$. This case is contained in Lemma 3.2, (i_1) $(l = n = 0)$.

3. Assume that $a_{11} = 0$ and let $a_{02} \neq 0$. In this case $l = 0$ and $U_{40} \equiv ua_{20} = 0$.

If $u = 0$, then $a_{20} = a_{02}$, $v = m + r$. The equations $U_{ij} = 0, i + j = 3$ of (16) yield $m = r$, $q = 0$, $a_{01} = (2aa_{02})/r$, $a_{10} = (2ba_{02})/r$. We obtain the set of conditions (i_3) of Lemma 3.2. The exponential factor is

$$\Phi = \exp(2bx + 2ay + rx^2 + ry^2)$$

with cofactor $K(x, y) = 2(ax - by)(ay + bx - fy - gx - 1)$.

If $a_{20} = 0$ and $u \neq 0$, then $q = 0, m = v = r$. The equations $U_{ij} = 0, i + j = 3$ of (16) yield $a_{10} = 0$, $b = g = 0$, $f = a$. In this case $P(x, y) = -x$ is not a cubic polynomial. □

Lemma 3.3. *The cubic system* (9) *has an exponential factor of the form*

$$\Phi = \exp((a_{10}x + a_{01}y + a_{00})/(x^2 + y^2)) \tag{17}$$

if and only if the following set of conditions holds

$$l = b(a+f), m = 2a^2+2af-2b^2-2bg+r, q = -3ab-2ag-bf, u = ag-bf,$$
$$v = m + r - a^2 - af + b^2 + bg.$$

Proof. By Definition 2.2, the function (17) is an exponential factor for cubic system (9) if there exist numbers c_{10}, c_{01} such that

$$\begin{aligned}(-a_{10}x^2 + a_{10}y^2 - 2a_{01}xy - 2a_{00}x)P(x, y)+\\ +(a_{01}x^2 - a_{01}y^2 - 2a_{10}xy - 2a_{00}y)Q(x, y) = (x^2 + y^2)^2(c_{10}x + c_{01}y).\end{aligned} \tag{18}$$

Substituting $P(x, y), Q(x, y)$ in (18) and identifying the coefficients of the monomials $x^i y^j$, we reduce this identity to a system of nine equations

$$\{U_{ij} = 0, \quad i + j = 1,\ 2,\ 3\} \tag{19}$$

for the unknowns $a_{10}, a_{01}, a_{00}, c_{10}, c_{01}$ and the coefficient of system (9).

From the equations $U_{30} = 0$, $U_{03} = 0$ of (19), we find that

$$c_{01} = la_{01} + ra_{10}, c_{10} = (r - v)a_{01} - (l + u)a_{10}.$$

When $i + j = 1$, we obtain that $a_{10} = -2ba_{00}$ and $a_{01} = -2aa_{00}$. Then the equations $U_{20} = 0, U_{11} = 0, U_{02} = 0$ of (19) yield

$$l = b(a + f), u = ag - bf, v = m + r - a^2 - af + b^2 + bg.$$

The equations $U_{21} = 0$, $U_{12} = 0$ of (19) imply

$$m = 2a^2 + 2af - 2b^2 - 2bg + r, q = -3ab - 2ag - bf.$$

In this case we determine the exponential factor

$$\Phi = \exp((1 - 2bx - 2ay)/(x^2 + y^2))$$

with cofactor $K_3(x, y) = 2(a^2 + af + r)(ax - by)$. □

Lemma 3.4. *The cubic system* (9) *has an exponential factor of the form*

$$\Phi = \exp((a_{20}x^2 + a_{11}xy + a_{02}y^2 + a_{10}x + a_{01}y + a_{00})/(x^2 + y^2)) \tag{20}$$

if and only if one of the following two sets of conditions holds:

(i_1) $l = bf, m = r + af - bg, q = -ag, v = 2r, u = ag - bf$;
(i_2) $f = -a, g = -b, q = -l, v = 2r, u = 0$.

Proof. By Definition 2.2, the function (20) is an exponential factor for cubic differential system (9) if there exist numbers $c_{20}, c_{11}, c_{02}, c_{10}, c_{01}$ such that

$$\begin{aligned}&(2a_{20}xy^2 - 2a_{02}xy^2 - a_{11}x^2y + a_{11}y^3 - a_{10}x^2 - 2a_{01}xy +\\ &+ a_{10}y^2 - 2a_{00}x)P(x, y) + (a_{11}x^3 + 2a_{02}x^2y - 2a_{20}x^2y - a_{11}xy^2 +\\ &+ a_{01}x^2 - a_{01}y^2 - 2a_{10}xy - 2a_{00}y)Q(x, y) =\\ &= (x^2 + y^2)(c_{20}x^2 + c_{11}xy + c_{02}y^2 + c_{10}x + c_{01}y).\end{aligned} \tag{21}$$

Substituting the polynomials $P(x, y)$, $Q(x, y)$ from (9) in (21) and identifying the coefficients of the monomials $x^i y^j$, we reduce (21) to a system of fourteen equations

$$\{U_{ij} = 0, \;\; i + j = 1, 2, 3, 4\} \tag{22}$$

for the unknowns $a_{20}, a_{11}, a_{02}, a_{10}, a_{01}, a_{00}, c_{20}, c_{11}, c_{02}, c_{10}, c_{01}$ and the coefficient of system (9). From the equations $U_{04} = 0, U_{13} = 0, U_{40} = 0$ of (22), we obtain that

$$c_{02} = ra_{11}, \;\; c_{11} = 2r(a_{20} - a_{02}) + (l + q + u)a_{11}, \;\; c_{20} = (r - v)a_{11}$$

and from the equations $U_{03} = 0, U_{30} = 0$, we get that

$$c_{01} = la_{01} + ra_{10} + fa_{11}, \;\; c_{10} = (r - v)a_{01} - (l + u)a_{10} - ga_{11}.$$

The equations $U_{10} = 0, U_{01} = 0$ and $U_{20} = 0$ yield

$$a_{01} = -2aa_{00}, \;\; a_{10} = -2ba_{00}, \;\; a_{11} = 2a_{00}(ab + ag - l - u)$$

and the equations $U_{02} = 0, U_{11} = 0$ imply

$$u = ag - bf, \;\; a_{02} = (a^2 + af - b^2 - bg - m - r + v)a_{00} + a_{20}.$$

Then the system of equations (22) becomes

$$U_{21} = 0, U_{12} = 0, U_{31} = 0, U_{22} = 0. \tag{23}$$

The resultant of the polynomials U_{22}, U_{31} with respect to m is $Res(U_{22}, U_{31}, m) = f_1 f_2$, where $f_1 = ab + bf - l, \;\; f_2 = (ag - bf + l + q)^2 + (2r - v)^2$.

1. Assume that $f_1 = 0$, then $l = b(a + f)$ and $U_{31} \equiv g_1 g_2 = 0$, where

$$g_1 = a^2 + af - b^2 - bg - m - r + v, g_2 = 2r - v.$$

When $g_1 = 0$ we obtain that $U_{31} \equiv 0$, $U_{22} \equiv 0$. The resultant of the polynomials U_{21}, U_{12} with respect to q is $Res(U_{21}, U_{12}, q) = h_1 h_2$, where

$$h_1 = 2a^2 + 2af - 2b^2 - 2bg - m + r, \;\; h_2 = a^2 + b^2.$$

If $h_1 = 0$, then we have the exponential factor $\Phi = \exp((1 - 2bx - 2ay)/(x^2 + y^2))$ obtained in Lemma 3.3. If $h_1 \neq 0$ and $a = b = 0$, then the right hand sides of (9) have a common factor $rx^2 + ry^2 + qxy + gx + fy + 1$.

Assume that $g_1 \neq 0$ and let $g_2 = 0$. Then $v = 2r$, $U_{31} \equiv 0$ and the equation $U_{22} = 0$ yields $q = -a(b + g)$. The resultant of the polynomials U_{21}, U_{12} with respect to m is

$$Res(U_{21}, U_{12}, m) = -2ab((a + f)^2 + (b + g)^2).$$

If $f = -a$ and $g = -b$, then we obtain the set of conditions (i_2) $(l = 0)$, Lemma 3.4.

Suppose that $(a + f)^2 + (b + g)^2 \neq 0$. If $a = 0$, then $U_{21} = 0, U_{12} = 0$ imply $m = r - bg$ and we get set of conditions (i_1) $(a = 0)$. If $a \neq 0$ and $b = 0$, then we have the set of conditions (i_1) $(b = f = 0)$.

2. Assume that $f_1 \neq 0$ and let $f_2 = 0$. Then $q = bf - l - ag$ and $v = 2r$. In this case $U_{22} \equiv 0$, $U_{31} \equiv 0$ and the resultant of the polynomials U_{21}, U_{12} with respect to m is $Res(U_{21}, U_{12}, m) = e_1 e_2$, where $e_1 = l - bf$, $e_2 = (a + f)^2 + (b + g)^2$.

If $e_1 = 0$, then $m = r + af - bg$. We get the condition (i_1). The exponential factor is

$$\Phi = \exp((b^2x^2 - a^2x^2 + 2abxy - 2bx - 2ay + 1)/(x^2 + y^2))$$

and have the cofactor $K_3(x, y) = 2r(by - ax)(ay + bx - 1)$.

If $e_1 \neq 0$ and $e_2 = 0$, then $f = -a$, $g = -b$. We obtain the set of conditions (i_2), Lemma 3.4. The exponential factor is

$$\Phi = \exp((mx^2 - rx^2 - 2lxy - 2bx - 2ay + 1)/(x^2 + y^2))$$

and have the cofactor $K_3(x, y) = 2r(ax - by + lx^2 - ly^2 + mxy - rxy)$. □

4. The problem of the center

We are interested in the algebraic integrability of a cubic differential system (1) with two homogeneous invariant straight lines $x \mp iy = 0$ and an exponential factor of the form (11), called *the Darboux integrability* [4], [22]. It consists in constructing of a first integral or an integrating factor of the Darboux form

$$f_1^{\alpha_1} f_2^{\alpha_2} \Phi^{\alpha_3}, \tag{24}$$

where $\alpha_j \in \mathbb{C}$, $f_1 = x - iy$, $f_2 = x + iy$ and Φ is of the form (11).

By [18, pag. 141], if for the cubic system (1) we can construct an integrating factor (a first integral) of the form (24), then $O(0, 0)$ is a center.

Definition 4.1. An integrating factor for system (1) on some open set U of $\mathbb{R}^2$ is a C^1 function μ defined on U, not identically zero on U such that

$$P(x, y)\frac{\partial \mu}{\partial x} + Q(x, y)\frac{\partial \mu}{\partial y} + \mu \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) \equiv 0. \tag{25}$$

Lemma 4.1. *The following three sets of conditions are sufficient conditions for the origin to be a center for system* (1)*:*

(i) $c = g - b,\ d = l = m = q = r = 0,\ f = a,\ k = p = a(g - b),\ s = n$;
(ii) $c = g - b,\ d = f - a,\ g = (abf - bf^2 + fl - al + br)/(a^2 - af + r),\ s = n,\ n = (pl)/r,\ k = p = [r(ab - bf + l)]/(a^2 - af + r),\ q = l,\ m = r$;
(iii) $c = g - b,\ d = f - a,\ k = m = p = r = 0,\ q = l,\ l = bf - ag,\ s = n,\ (ag - bf)(b - g) + (a - f)n = 0$.

Proof. Let the conditions (i_1) and (i_2) of Lemma 3.1 be fulfilled. By Definition 4.1, the cubic system (1) has an integrating factor of the form (24) if and only if the identity (25) holds. Identifying the coefficients of the monomials $x^i y^j$ in (25), we obtain that $\alpha_1 = \alpha_2$ and α_2, α_3 are the solutions of the system

$$\{F_{ij} = 0, \quad i + j = 1, 2\}. \tag{26}$$

In Case (i_1), from the equations $F_{10} = 0$ and $F_{02} = 0$ of (26) we express α_3 and u. Then we reduce the equations $F_{20} = 0, F_{11} = 0$ by g from $F_{01} = 0$. If $r = a(f - a)$, then $f = a$ and we get the condition (i) of Lemma 4.1. We obtain the exponential factor $\Phi = \exp\left((aby - agy - nx)/(a(b - g))\right)$ and the system (1) has the integrating factor

$$\mu = (x^2 + y^2)^{(4bg - 3b^2 - g^2 - 2n)/(2(b^2 - bg + n))} \exp\left(\frac{(b - g)(nx - aby + agy)}{b^2 - bg + n}\right).$$

If $r \neq a(f - a)$, then $\alpha_2 = (4af - 3a^2 - f^2 - 2r)/[(a^2 - af + r)]$. In this case we determine the condition (ii) of Lemma 4.1. We have the exponential factor $\Phi = \exp\left((lx + ry)/r\right)$ and the system (1) has the integrating factor

$$\mu = (x^2 + y^2)^{(4af - 3a^2 - f^2 - 2r)/(2(a^2 - af + r))} \exp\left(\frac{(a - f)(lx + ry)}{a^2 - af + r}\right).$$

In Case (i_2), the equations $F_{20} = 0,\ F_{02} = 0,\ F_{01} = 0$ of (26) yield $\alpha_3 = l/a$, $\alpha_2 = (f - 3a)/(2a)$, $l = f(b - a)$. In this case we get the condition (iii) of Lemma 4.1. The system (1) has the exponential factor $\Phi = \exp(x)$ and the functon

$$\mu = (x^2 + y^2)^{(f - 3a)/(2a)} \exp\left(\frac{(bf - ag)x}{a}\right).$$

is an integrating factor for system (1). □

Lemma 4.2. *The following four sets of conditions are sufficient condition for the origin to be a center for system* (1).

(i) $c = d = 0,\ f = a,\ g = b,\ k = p = q = l,\ r = m,\ s = n$;

(ii) $c = g - b$, $d = f - a$, $k = l$, $q = p$, $n = 2r - s$, $m = 2s - r$, $p = -3l$, $l = b(a + f)$, $s = a^2 + af - b^2 - bg + r$, $ag - bf = 0$;

(iii) $c = g - b$, $d = f - a$, $k = ag$, $l = bf$, $m = af - bg + r$, $n = -m$, $p = -bf$, $q = -ag$, $s = r$;

(iv) $c = -2b$, $d = -2a$, $f = -a$, $g = -b$, $k = l$, $n = 2r - m$, $p = -l$, $q = -l$, $s = r$.

Proof. In each of the cases (i) – (iv) the cubic differential system (1) has two homogeneous invariant straight lines $x \mp iy = 0$ and an exponential factor Φ.

In Case (i), we find the exponential factor $\Phi = \exp(2bx + 2ay + 2lxy + nx^2 + my^2)$ and the system (1) has the first integral

$$(x^2 + y^2) \exp(2bx + 2ay + 2lxy + nx^2 + my^2) = C.$$

In Case (ii), we have $\Phi = \exp((1 - 2bx - 2ay)/(x^2 + y^2))$. We can construct an integrating factor of the form

$$\mu = (x^2 + y^2)^{-3} \exp\left(\frac{(3b + g)(1 - 2bx - 2ay)}{2b(a^2 + af + r)(x^2 + y^2)}\right).$$

In Case (iii), we determine $\Phi = \exp((b^2x^2 - a^2x^2 + 2abxy - 2bx - 2ay + 1)/(x^2 + y^2))$. We can construct an integrating factor of the form

$$\mu = (x^2 + y^2)^{-3/2} \exp\left(\frac{g(b^2x^2 - a^2x^2 + 2abxy - 2bx - 2ay + 1)}{2br(x^2 + y^2)}\right).$$

In Case (iv), we have $\Phi = \exp((mx^2 - rx^2 - 2lxy - 2bx - 2ay + 1)/(x^2 + y^2))$. We can construct the first integral

$$(x^2 + y^2)^{-r} \exp\left(\frac{mx^2 - rx^2 - 2lxy - 2bx - 2ay + 1}{x^2 + y^2}\right) = C.$$

□

Theorem 4.1. *The cubic system* (1) *with two invariant straight lines* $x \mp iy = 0$ *and an exponential factor of the form* (14) *has a center at the origin* $O(0, 0)$ *if and only if the first Lyapunov quantity vanishes.*

Proof. We compute the first Lyapunov quantities L_1 for cubic system (9) assuming that the conditions of Lemma 3.2 hold.

In Case (i_1) the first Lyapunov quantity vanishes. We have Lemma 4.2, (i).

In Case (i_2) we find that $L_1 = u \neq 0$. Therefore, the origin is a focus.

In Case (i_3) the first Lyapunov quantity is $L_1 = ag - bf$. If $L_1 = 0$, then we have Lemma 2.2.2, (iv) $(l = 0, n = r)$ from [5] and the origin is a center. □

Theorem 4.2. *The cubic system* (1) *with two invariant straight lines* $x \mp iy = 0$ *and an exponential factor* $\Phi = \exp((1-2bx-2ay)/(x^2+y^2))$ *has a center at the origin* $O(0,0)$ *if and only if the first two Lyapunov quantities vanish.*

Proof. We compute the first two Lyapunov quantities L_1, L_2 for cubic system (9) assuming that the set of conditions of Lemma 3.3 is fulfilled. The vanishing of the first Lyapunov quantity gives $u = ag - bf$. The second Lyapunov quantity looks

$$L_2 = 48(a^2 + af + r)(ag - bf).$$

Let $r = -a^2 - af$. Then the right hand sides of (1) have a common factor $h(x,y) = ay + bx + fy + gx + 1$. Assume that $r \neq -a^2 - af$ and let $ag - bf = 0$. In this case $L_2 = 0$ and we have Lemma 4.2, (ii). □

Theorem 4.3. *The cubic system* (1) *with two invariant straight lines* $x \mp iy = 0$ *and an exponential factor of the form* (20) *has a center at the origin* $O(0,0)$ *if and only if the first two Lyapunov quantities vanish.*

Proof. We compute the first two Lyapunov quantities L_1, L_2 for cubic system (9) assuming that the conditions of Lemma 3.4 hold.

In Case (i_1) we have $L_1 = 0$ and the second Lyapunov quanity is $L_2 = 48r(ag - bf)$.

If $r = 0$, then the right hand sides of (1) have a common factor $h(x,y) = gx + fy + 1$. Assume that $r \neq 0$ and let $ag - bf = 0$. In this case $L_2 = 0$ and we have Lemma 4.2, (iii).

In Case (i_2) we find that $L_1 = L_2 = 0$. Then Lemma 4.2, (iv). □

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Received: October 07, 2022 *Accepted: December 19, 2022*

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Acta et Commentationes, Exact and Natural Sciences ISSN: 2537-6284
Volume 2(14), 2022, Pages 101–105 E-ISSN: 2587-3644
CZU 004.627:519.61 DOI: 10.36120/2587-3644.v14i2.101-105

Finding N bits using $O(\frac{N}{\log N})$ sums

Sergiu Corlat, Veaceslav Guzun, and Victor Vorona

Abstract. The problem we are trying to solve sounds as follows: You are given N bits. Find the value of each bit. We will show a technique which enables finding the values of N bits using $O(\frac{N}{\log N})$ subsequence-sum queries. The algorithm consists of two phases: Constructing the queries for each layer and using the queries for a particular layer to get the value of every bit. We described the following technique in this blog [1], which was inspired by this article [2]. It should be noted that this number of queries is indeed the optimal one for finding all N bits of a binary array, since each subsequence-sum queries offers us at most $\log_2 N$ bits of information.
Keywords: binary array, query problem, divide et impera, optimization.

Aflarea a N biţi folosind $O(\frac{N}{\log N})$ sume

Rezumat. Problema pe care încercăm să o rezolvăm sună astfel: vi se oferă N biți. Găsiți valoarea fiecărui bit. Vom prezenta o tehnică care permite aflarea valorilor a N biți folosind $O(\frac{N}{\log N})$ interogări de sume pe subsecvențe. Algoritmul se divide în două faze: construirea interogărilor pentru fiecare strat și utlizarea acestora pentru un strat particular pentru a obține valoarea fiecărui bit. Am descris această tehnică în articolul de blog [1], care dezvoltă rezultatele din articolul lui Zhenting Zhu din universitatea Tsinghua [2]. Trebuie să mentionăm faptul că acest număr de interogări este într-adevăr optimal pentru găsirea tuturor N biților unui șir binar, deoarece fiecare interogare de sumă de subsecvențe ne oferă cel mult $\log_2 N$ biți de informație.
Cuvinte-cheie: șir binar, problemă de interogare, divide et impera, optimizare.

1. Introduction

1.1. Core of the Problem

Finding the whole array of elements by knowing some information about some of its subsequences is a popular problem in computer science and can be found in many forms. In this case, we will explain how to find each element in a binary array (an array consisting only of zeroes and ones) by only being able to query the sums of some subsequences of it and try to minimize the number of queries we perform.

We will treat the problem as an interactive one. Initially, the only information about the binary array b we have is its size, and we can ask the interactor several questions in the following format:

What is the sum of the subsequence of the values on positions: $p_1, p_2, ..., p_k$. In other words, you will give the sequence $p_1, p_2, ..., p_k$ to the interactor and it will return you the value of $\sum_{i=1}^{k} b_{p_i}$.

After querying some number of sums, we should be able to tell the value of the element on each position.

1.2. Main Idea

The main idea of the algorithm involves a divide-and-conquer-like approach [3] which will work in two phases. In the first phase, the set of queries will be constructed, and the second phase will reconstruct the array values. We will show that it is possible to reconstruct the whole array using $O(\frac{N}{\log N})$ [4] well-built queries, and will also explain how the queries should be constructed.

2. Overview

2.1. Notations

In the coming explanation we will use the following notations:

- x_i – refers to the position i
- A – any capital letter (except S) refers to a set of points x_i
- $v_A = \sum b_{x_i}$ for $x_i \in A$
- k – the layer we are currently considering
- S_i – A set of queries

We will also use 0-indexing when talking about the array's elements' positions.

2.2. Explanation

The idea is to use a divide-and-conquer-like approach but in two phases. The first phase will be the construction of the queries we will ask at the end and the second phase will reconstruct the array of elements by having the answers to the relevant subsequences already obtained after the first phase.

The first phase

As it is a divide-and-conquer-like idea, we are going to work with layers. Let's say that for the k^{th} layer we use 2^k queries and that by using them we can find out the value of f_k elements.

Using the idea that is described below we will be able to make the following recurrence possible: $f_{k+1} = 2 \cdot f_k + 2^k - 1$.

Firstly, we will need to set our base case, which is $k = 0$. So, for $k = 0$, we will have $f_0 = 1$ and the query set will be 1. This means we will find out the value of a single element using a query.

Now, what we are trying to achieve in order to make the recurrence possible is to form the new block (f_{k+1}), using two blocks of size f_k, and find $2^k - 1$ additional elements in the process. Let's say k_1 will denote the first block of the f_k elements we will use, and k_2 the second such block.

The first query is used to get the sum on $[f_k, 2 \cdot f_k)$ – the sum of the second block. Then we add two new queries for each **non-last** query in S_{k_1} and S_{k_2}. First one is $S_{k_1}[i] \cup S_{k_2}[i]$. Second one is $S_{k_1}[i] \cup ([f_k, 2 \cdot f_k)/S_{k_2}[i]) \cup x_{(2 \cdot f_k + i)}$.

The last query is for the entire range $[0, f_{k+1})$. It's easy to see that now, we have used exactly 2^{k+1} queries. Now, why don't we lose any value in the process? And how will we be able to recursively [5] obtain the elements back? This will be clear in the second phase of the algorithm.

The second phase

Having answered all the S_{k+1} queries, we can calculate all the $v_{S_{k_1}[i]}$ and $v_{S_{k_2}[i]}$.

Now, when we reach a k with a value of $f_k >= n$ we can stop there. Let's assume $n = f_k$ since it will be easier to work with it (when n is smaller than f_k we can just think of it as appending $f_k - n$ zeroes at the end since they won't influence the sum at all). Using the set of queries responsible for the k^{th} layer we can in fact now reconstruct the whole sequence, recursively going from the k^{th} layer to the $(k-1)^{th}$ one (but consider each layer can have multiple blocks).

First of all, the only information relevant for the k^{th} layer are:

- The query set for the corresponding block of the corresponding layer.
- The block we are currently at (can be dealt with using an offset value in the recursion).

So we will store them when going recursively.

Firstly, let's set our base case: $k = 0$. We are now sure that only one element is responsible for this block from this layer, so we can just set the value of the b_x-th bit (where x is some offset value we use to keep track of the block) to v_{S_0} (since that's the sum for a single element which we've seen at the build-up).

Now, since the base case is already dealt with, here's how we will go to the $(k-1)^{th}$ layer:

We will need to reconstruct the previous query sets for the first block and the second block of size f_{k-1}, and set the values for the other $2^{(k-1)} - 1$ values respectively (since they aren't part of any of the blocks they shouldn't be part of the recursion either).

Let's denote the numbers of 1-s in $[f_k, 2 \cdot f_k)$ with c. It's obvious that $c = v_{S[0]}$. We will now go through every pair of queries, starting from 1. That means we will be analyzing queries $S_1[i] \cup S_2[i]$ and $S_1[i] \cup ([f_{k-1}, 2 \cdot f_{k-1})/S_2[i]) \cup x_{(2 \cdot f_{k-1}+i)}$.

- $v_{S[2 \cdot i+1]} = v_{S_1[i]} + v_{S_2[i]}$
- $v_{S[2 \cdot i+2]} = v_{S_1[i]} + c - v_{S_2[i]} + b_{2 \cdot f_{k-1}+i}$

In this case we will calculate 3 values: $v_{S_1[i]}, v_{S_2[i]}, b_{2 \cdot f_{k-1}+i}$.

- $v_{S_1[i]} = \lfloor \frac{v_{S[2 \cdot i+1]} + v_{S[2 \cdot i+2]} - c}{2} \rfloor$
- $v_{S_2[i]} = \lceil \frac{v_{S[2 \cdot i+1]} - v_{S[2 \cdot i+2]} + c}{2} \rceil$
- $b_{2 \cdot f_{k-1}+i} = (v_{S[2 \cdot i+1]} + v_{S[2 \cdot i+2]} - c) \wedge 1$

The only remaining queries to answer are $v_{S_1[2^{k-1}]}$ and $v_{S_2[2^{k-1}]}$ as they were not added in S.

- $v_{S_2[2^{k-1}]} = c = v_{S[0]}$
- $v_{S_1[2^{k-1}]} = v_{S[2^k]} - c - \sum_{i=0}^{2^{k-1}-1} b_{2 \cdot f_{k-1}+i}$

After calculating this, we could use the divide-and-conquer property specified earlier and go down a layer. We are going to do this recursively from layer k til layer 0. The last layer will consist of only 1 bit and only 1 sum, the value of said bit.

3. Visual Representation

Representation of how the algorithm works for $k = 3$ (Figure 1). Here is the color coding we used:

(1) The green squares represent the queries responsible for the first block of the previous layer.
(2) The orange squares represent the queries responsible for the second block of the previous layer.
(3) The blue squares represent the queries that query the whole second block excluding the elements from the query of the second block.
(4) The red squares represent the last $2^k - 1$ bits.

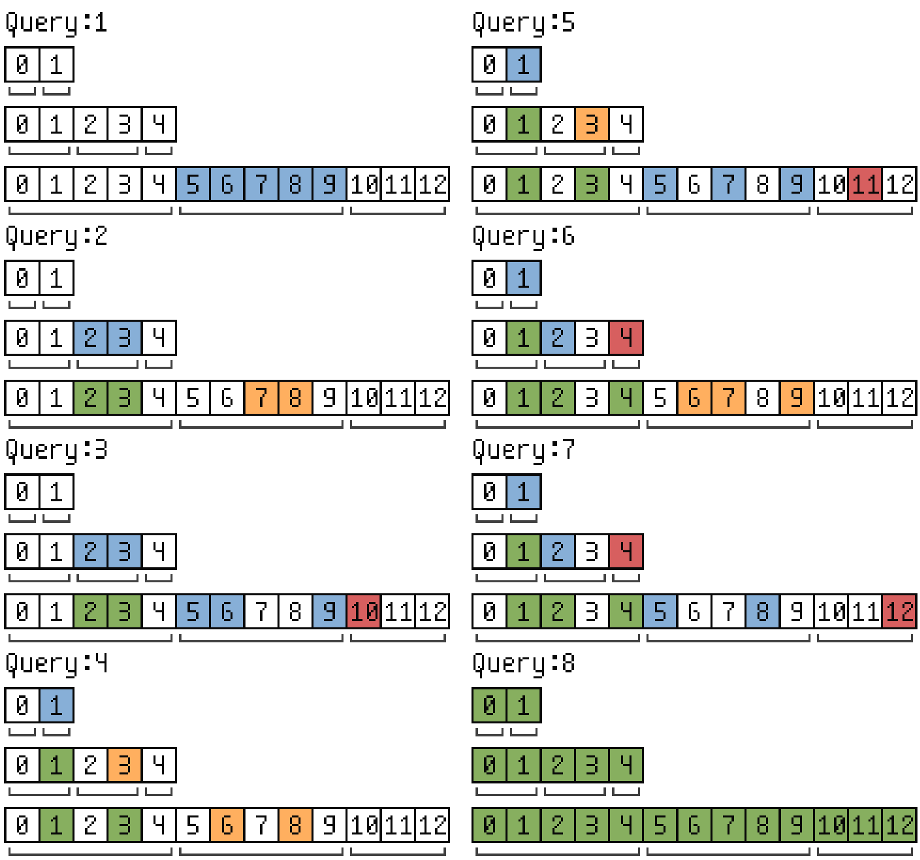

Figure 1. The second phase for $k = 3$

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Received: November 03, 2022 *Accepted: December 21, 2022*

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Acta et Commentationes, Exact and Natural Sciences ISSN: 2537-6284
Volume 2(14), 2022, Pages 106–116 E-ISSN: 2587-3644
CZU 517.95 DOI: 10.36120/2587-3644.v14i2.106-116

On determination of some exact solutions of the stationary Navier-Stokes equations

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Abstract. In this paper, various exact solutions of the stationary Navier-Stokes equations, which describe the planar flow of an incompressible liquid (or gas), are determined, i.e., solutions containing the components of the velocity of flow - the functions u, v and the created pressure P. We mention that in the paper a series of exact solutions is obtained, in which the viscosity coefficient λ participates explicitly.

2010 Mathematics Subject Classification: 35Q30.

Keywords: stationary two-dimensional Navier-Stokes equations, exact solutions, method of separation of variables, viscosity, pressure, velocity of plane flow of a liquid or gas.

Determinarea unor soluții exacte ale ecuaţiilor staţionare Navier-Stokes

Rezumat. În această lucrare se determină diverse soluții exacte ale ecuațiilor staționare Navier-Stokes, care descriu curgerea plană a unui lichid (sau gaz) incompresibil, și anume soluții ce conțin componentele vitezei fluxului de curgere - funcțiile u, v şi presiunea creată P. Menționăm, că în lucrare sunt obținute un șir de soluții exacte, în care participă în mod explicit coeficientul vâscozității λ.

Cuvinte-cheie: ecuaţii staţionare bidimensionale Navier-Stokes, soluţii exacte, metoda separării variabilelor, vâscozitate, presiune, viteza fluxului de curgere plană a unui lichid sau gaz.

1. Introduction

In the present paper, the Navier-Stokes equations are studied in the two-dimensional case. In this case the Navier-Stokes equations represent a system, containing three differential equations with partial derivatives with three unknown functions. Until today, the examined problem has not been definitively solved even in the case of stationary equations, that is, equations that describe the processes of the planar flow of a liquid or gas that do not vary in time. The complexity of the problem lies in the fact that the first two equations in the system are non-linear. It has been developed a method that would allow us to determine all the solutions of this system. Determining the solutions of the system of Navier-Stokes equations is an important mathematical problem and has various applications in fluid and gas mechanics.

In this paper it is examined the following system of partial differential equations:

$$\begin{cases} \frac{P_x}{\mu} + uu_x + vu_y = \lambda(u_{xx} + u_{yy}) + F_x, \\ \frac{P_y}{\mu} + uv_x + vv_y = \lambda(v_{xx} + v_{yy}) + F_y, \\ u_x + v_y = 0, \end{cases} \tag{1}$$

where $x, y \in R; P = P\,(x, y)\,; F = F\,(x; y)\,; u = u\,(x, y)\,, v = v\,(x, y)\,; u_x = \frac{\partial u}{\partial x}$.

The system (1) describes the stationary processes of planar flow of an incompressible liquid or gas. Regarding the derivation of the equations of system (1) and the meaning of the physical processes described by this system, consult the works [1], [2], [3].

The unknowns of the system (1) are the following three functions: P, which represents the created pressure; u and v, which represent the components of the flow velocity of a liquid or gas. The given exterior force is F and has a potential nature, that is, its components are equal to the partial derivatives of this force - F_x and F_y. The constants $\lambda > 0$ and $\mu > 0$ are the parameters determined by the viscosity and density of the examined liquid or gas. We mention here, that the viscosity parameter has the form $\lambda = C_0/R_e$, where R_e is the Reinolds number and C_0 is a constant.

Some exact solutions of the system (1) are obtained in the papers [4] - [7]. In [8] a series of solutions of the examined system are indicated only for the components of the flow velocity, without determining the pressure.

2. Equations for determining the velocity and pressure components. Solutions that do not depend on the viscosity parameter

The system (1) is equivalent to the following system:

$$\begin{cases} \frac{P_x}{\mu} - F_x + uu_x + vv_x = \lambda\Delta u - v\left(u_y - v_x\right), \\ \frac{P_y}{\mu} - F_y + uu_y + vv_y = \lambda\Delta v + u\left(u_y - v_x\right), \\ u_x + v_y = 0, \end{cases} \tag{2}$$

where $\Delta u = u_{xx} + u_{yy},\ \Delta v = v_{xx} + v_{yy}$. Denote

$$G = \frac{1}{\mu}P - F + 0,5\left(u^2 + v^2\right). \tag{3}$$

Then from (2) it follows that

$$\begin{cases} G_x = \lambda\Delta u - v\left(u_y - v_x\right), \\ G_y = \lambda\Delta v + u\left(u_y - v_x\right). \end{cases} \tag{4}$$

Since $G_{xy} = G_{yx}$, we differentiate the first equation from (4) with respect to y and the second one with respect to x. Then we equalize the right hand sides of the obtained equations. As a result, we obtain the following equation for determining the functions u and v:

$$\lambda\Delta(u_y - v_x) - u(u_y - v_x)_x - v(u_y - v_x)_y = 0. \tag{5}$$

In addition, u and v must also check the last equation in the system (2):

$$u_x + v_y = 0. \tag{6}$$

Thus, firstly the components of the flow velocity from the system formed by equations (5) and (6) are determined, then the function G from the system (4), and finally the pressure P from the equation (3).

The following Theorem generates a series of solutions of the system (1).

Theorem 2.1. *Let D be a connected domain in the coordinate plane OXY, and let u, v and P be functions that in this domain admit continuous partial derivatives up to and including the second order. If $f(z)$ is a function of complex variable $z = x + iy$, differentiable at any interior point $(x; y)$ of the domain D, then system (1) admits solutions of the following form in this domain:*

$$u = Imf, v = Ref; P = \left[F - 0,5\left(u^2 + v^2\right) + C\right]\mu, \tag{7}$$

where C is an arbitrary constant.

Proof. May it be $u = Imf, v = Ref, f = v(x; y) + iu(x; y)$, where $f(z)$ is a function of complex variable $z = x + iy$, differentiable at any interior point $(x; y)$ of the domain D. Then from Cauchy – Riemann conditions [9] we obtain:

$$\begin{cases} v_x = u_y, \\ v_y = -u_x, \end{cases} \Leftrightarrow \begin{cases} u_y - v_x = 0, \\ u_x + v_y = 0. \end{cases} \tag{8}$$

The second equation in (8) coincides with (6), and from the first it follows that these functions verify the equation (5). It remains to determine the pressure P. Since the functions u and v admit continuous derivatives up to the second order, inclusively in D, we have that

$$u_{xy} = u_{yx}, v_{xy} = v_{yx}.$$

Differentiating the first equation from (8) with respect to y and the second one with respect to x, and adding the results, we will obtain that $\Delta u = u_{xx} + u_{yy} = 0$. Then, differentiating the second equation from (8) with respect to y and the first one with respect to x, and subtracting the results, we obtain that $\Delta v = 0$.

Then from (4) we obtain that $\begin{cases} G_x = 0, \\ G_y = 0, \end{cases} \Rightarrow G(x;y) = C - const.$

We substitute this result in (3) and express the pressure P. Theorem 2.1 is proved.

Below we will give one example of determining the solutions of the system (1) according to Theorem 2.1. If $f(z) = (z - z_0)^{-1}, z_0 = x_0 + iy_0,$ then

$$\begin{cases} u = \frac{C_0(y_0-y)}{(x-x_0)^2+(y-y_0)^2}; v = \frac{C_0(x-x_0)}{(x-x_0)^2+(y-y_0)^2}, \\ P = \left[F - \frac{0{,}5C_0{}^2}{(x-x_0)^2+(y-y_0)^2} + C\right]\mu, \end{cases} \quad D = OXY \setminus \{M(x_0;y_0)\}. \tag{9}$$

In the solutions (9), C and C_0 are arbitrary constants.

Next, to determine the solutions of the system (1), we will apply the method of separation of variables.

3. Method of Separation of Variables

We look for the velocity components in the following form:

$$u = g(x) f_1(y); v = f(y) g_1(x), \tag{10}$$

where the functions f and g are differentiable up to the fourth order while the functions f_1 and g_1 up to and including the third order.

From the equation (6) we deduce that

$$g'(x) f_1(y) + g_1(x) f'(y) = 0 \Rightarrow \frac{g'}{g_1} = \frac{-f'}{f_1} = \frac{1}{C} \Rightarrow g_1 = Cg', f_1 = -Cf'.$$

From here we obtain, that:

$$u = -Cg(x) f'(y); v = Cf(y) g'(x); u_y - v_x = -C(gf'' + fg''). \tag{11}$$

In the equations (11) C is an arbitrary non-zero constant.

We will consider the case when the functions $g(x)$ and $f(y)$ are not constant because, if one of these functions is constant, then from (10) it follows that one of the functions u or v is equal to zero. In this case, the well-known Poiseuille or Couette ([2], [10]) type flows are obtained.

By replacing (10) into equation (5), we get:

$$\lambda\left(g^{(4)}f + 2g''f'' + gf^{(4)}\right) + Cgf'\left(g'f'' + g^{(3)}f\right) - Cg'f(gf^{(3)} + g''f') = 0 \Rightarrow$$

$$\lambda\left[\frac{g^{(4)}}{g} + \frac{2g''f''}{gf} + \frac{f^{(4)}}{f}\right] + C\left[g'\left(\frac{f'f''}{f} - f^{(3)}\right) + f'\left(g^{(3)} - \frac{g'g''}{g}\right)\right] = 0. \tag{12}$$

We will study firstly the case when the expressions from the square brackets in the equation (12) are equal to zero. After we equalize to zero the expression next to C and applying the method of separation of variables, we get:

$$g'\left(\frac{f'f''}{f}-f^{(3)}\right)+f'\left(g^{(3)}-\frac{g'g''}{g}\right)=0 \qquad \Rightarrow$$

$$\frac{f}{f'}\left(\frac{f'f''-ff^{(3)}}{f^2}\right)+\frac{g}{g'}\left(\frac{gg^{(3)}-g'g''}{g^2}\right)=0 \qquad \Rightarrow$$

$$\frac{g}{g'}\left(\frac{g''}{g}\right)'=\frac{f}{f'}\left(\frac{f''}{f}\right)'=k-\text{ const.} \tag{13}$$

From (13) it follows that the functions $g(x)$ and $f(y)$ are determined in the same way. Let us first examine the case $k=0$. From (13) we deduce, that

$$\begin{cases} g''=ag, \\ f''=bf, \end{cases} \quad \Rightarrow \quad \begin{cases} g^{(4)}=ag'', \\ f^{(4)}=bf'', \end{cases} \tag{14}$$

where a and b are arbitrary constants.

We replace (14) in the equality (12) while the expression in the second square bracket cancels, and in the first square bracket in (12) we obtain $a^2+2ab+b^2$. Thus, the equation (12) takes the following form: $\lambda(a^2+2ab+b^2)=0$. Because $\lambda>0$, it follows that $b=-a$. Therefore, the following three situations are possible:

(1) $a=c^2, b=-c^2, c\neq 0.$ Then $\begin{cases} g''=c^2g, \\ f''=-c^2f, \end{cases} \Rightarrow \begin{cases} g=a_1e^{cx}+a_2e^{-cx}, \\ f=b_1\cos(cy)+b_2\sin(cy). \end{cases}$

(2) $a=-c^2, b=c^2, c\neq 0.$ Then $\begin{cases} g''=-c^2g, \\ f''=c^2f, \end{cases} \Rightarrow \begin{cases} g=a_1\cos(cx)+a_2\sin(cx), \\ f=b_1e^{cy}+b_2e^{-cy}. \end{cases}$

(3) $a=0, b=0.$ Then $\begin{cases} g''=0, \\ f''=0, \end{cases} \Rightarrow \begin{cases} g=a_1x+a_2, \\ f=b_1y+b_2. \end{cases}$

In the all three cases c, a_1, a_2, b_1, b_2 are arbitrary constants.

According to the equalities (10), we have that

$$\begin{cases} u=-Cg(x)f'(y), \\ v=Cg'(x)f(y), \end{cases} \tag{15}$$

with $g(x)$ and $f(y)$ determined in cases (1), (2) and (3).

From (15) and (14) we obtain in these cases, that $u_y-v_x=0, \Delta u=0, \Delta v=0$. Then, from (4) we easily determine that $G=C_0$ – constant and then, from relation (3) we find that the pressure is

$$P=\left(F-0,5C^2\left[(gf')^2+(g'f)\,2\right]+C_0\right)\mu. \tag{16}$$

Thus, in the case $k = 0$ the solutions of the system (1) are given in the formulas (15) and (16).

Therefore, the following theorem is proved:

Theorem 3.1. *If the functions f and g are differentiable up to the fourth order on any interval of the real axis, then system (1) admits solutions of form (15), (16) where the functions $g(x)$ and $f(y)$ are determined according to cases (1), (2) or (3) above.*

We will investigate the case $k \neq 0$. Let us find $g(x)$. Regarding the solution of different ordinary differential equations, consult [11]. From (13) we deduce that

$$\left(\frac{g''}{g}\right)' = k\frac{g'}{g} \quad \Rightarrow \quad \frac{g''}{g} = k \ln g + c.$$

Denote $g' = p(g)$, then $g'' = p'p$. As a result we obtain a first order differential equation:

$$p'p = k\cdot g \ln g + cg \Rightarrow 0,5p^2 = 0,5k\left(g^2 \ln g - \frac{g^2}{2}\right) + 0,5cg^2 + c_1.$$

If $c_1 = 0$, then $g' = \pm g\sqrt{k(\ln g - 0,5) + c} \Rightarrow g = e^{[\frac{k}{4}(c_2 \pm x)^2 + c_3]}$.

Analogously, we obtain that $f = e^{[\frac{k}{4}(c_4 \pm y)^2 + c_5]}$.

But in this case the equality (12) in not fulfilled because in the equality (12) the expression in the second square bracket cancels and the expression in the parenthesis next to λ is different of zero. Thus, in the case $k \neq 0$ the solutions of the equation (12) cannot be determined.

4. The case when one of the functions $g(x)$ or $f(y)$ is linear. Solutions in which the viscosity parameter participates explicitly

We return to the equality (12) and now we are studying the situation when the expressions in the square brackets are not equal to zero. We examine the case when the derivative of the function f or the derivative of the function g is constant. That is, we will examine the case when one of the functions $g(x)$ or $f(y)$ is linear.

Let $f(y) = by + m$, with $b \neq 0$ and m arbitrary constants, then $f'(y) = b$. From the equality (12) we obtain a fourth-order non-linear differential equation, containing only the function $g(x)$:

$$\lambda g^{(4)} + Cb\left(gg^{(3)} - g'g''\right) = 0. \tag{17}$$

We integrate the equation (17), taking into account that $(gg'')' = g'g'' + gg^{(3)}$, and we obtain the following third-order differential equation:

$$\lambda g^{(3)} + Cb\left(gg'' - 2\int g'g''dx\right) = C_1 \Rightarrow$$

$$\lambda g^{(3)} + Cb\left(gg'' - (g')^2\right) = C_1 \tag{18}$$

where C_1 is an arbitrary constant.

We notice that in the case $C_1 = -Cba^2$ this equation admits solutions of the form $g = ax + d$ for any reals constants a and b. Then, according to the formulas (15), the solutions of system (1) will be:

$$u = -Cb\,(ax + d)\,; \quad v = Ca\,(by + m)\,, \tag{19}$$

with a, b, C, m, d – arbitrary constants. The pressure P is determined from the relation (16):

$$P = \left(F - 0,5C^2\left[b^2(ax + d)^2 + a^2(by + m)^2\right] + C_0\right)\mu. \tag{20}$$

In the formula (20) and in all formulas that follow, C_0 is an arbitrary constant. Thus, we obtain solutions of the system (1) given by (19) and (20).

Next, we will look for solutions of the equation (18) of the form $g = a(x + d)^n$ with the constants a, d, n, $n \neq 0$, $n \neq 1$. Substituting in (18), we find that

$$n = -1, C_1 = 0, a = \frac{6\lambda}{Cb} \Rightarrow g\,(x) = \frac{6\lambda}{Cb\,(x + d)}.$$

Substituting the obtained function $g(x)$ and $f(y) = by + m$ into (15), we get the following solutions of the system (1):

$$\begin{cases} u = -\frac{6\lambda}{x+d}, \\ v = -\frac{6\lambda(by+m)}{b(x+d)^2}, \end{cases} \tag{21}$$

with the arbitrary constants b, d, m. We determine the function G from the system (4):

$$G = \frac{18\lambda^2(by + m)^2}{b^2(x + d)^4} + \frac{6\lambda^2}{(x + d)^2} + C_0.$$

Then, we substitute the determined function G in (3) and find the pressure in the case of the solutions (21):

$$P = \left(F - \frac{12\lambda^2}{(x + d)^2} + C_0\right)\mu \tag{22}$$

As a result, we obtain solutions of the system (1) in the form of the formulas (21), (22). Next, we will look for solutions of the equation (14) of the form $g = a + ne^{kx}$ with the constants a, n, k. Substituting into (18), we obtain

$$C_1 = 0, a = -\frac{k\lambda}{Cb} \Rightarrow g\,(x) = -\frac{k\lambda}{Cb} + ne^{kx},$$

with k, b, C, n – arbitrary real constants.

Substituting into (15), we obtain the following solutions of the system (1):

$$\begin{cases} u = \lambda k - Cbne^{kx}, \\ v = C(by+m)nke^{kx}, \end{cases} \tag{23}$$

C, b, m, n, k - arbitrary constants. We find the function G from the system (4):

$$G = \frac{(Cnk)^2(by+m)^2e^{2kx}}{2} - \lambda Cbnke^{kx} + C_0.$$

Then, we substitute the determined function G in (3) and find the pressure corresponding to the case of the solutions (23):

$$P = \left[F - 0,5\left[(Cbn)^2e^{2kx} + (\lambda k)^2\right] + C_0\right]\mu \tag{24}$$

Thus, we obtain solutions for the system (1) given by (23) and (24).

Now let $g(x) = bx+m$, with $b \neq 0$ and m arbitrary constants; then $g'(x) = b$. From (12) we obtain a forth-order nonlinear differential equation which contains only the function $f(y)$:

$$\lambda f^{(4)} - Cb\left(ff^{(3)} - f'f''\right) = 0. \tag{25}$$

We integrate the equation (25) and we obtain the following equation of order 3:

$$\lambda f^{(3)} - Cb\left(ff'' - (f')^2\right) = C_1, \tag{26}$$

where C_1 is an arbitrary constant.

We notice that in the case $C_1 = Cba^2$ the equation (25) admits solutions of the form $f = ay + d$ for any real constants a and d. According to the formulas (15), the solutions of the system (1) are:

$$u = -Ca\,(bx+m)\,; \quad v = Cb\,(ay+d) \tag{27}$$

with the arbitrary constants a, b, C, m, d. The pressure P in this case is:

$$P = \left(F - 0,5C^2\left[a^2(bx+m)^2 + b^2(ay+d)^2\right] + C_0\right)\mu. \tag{28}$$

Looking for solutions of this equation of the form $f = a(y+d)^n$ with constants a, d and n, we will obtain that

$$n = -1, \quad a = -\frac{6\lambda}{Cb} \quad \Rightarrow \quad f\,(y) = -\frac{6\lambda}{Cb\,(y+d)}.$$

Substituting the determined function $f(y)$ and $g(x) = bx + m$ into (15), we obtain the following solutions:

$$\begin{cases} u = -\frac{6\lambda(bx+m)}{b(y+d)^2}, \\ v = -\frac{6\lambda}{y+d}, \end{cases} \tag{29}$$

with the arbitrary constants b, d, m. The pressure, corresponding to the solution (26), is

$$P = \left(F - \frac{12\lambda^2}{(y+d)^2} + C_0\right)\mu. \tag{30}$$

Thus, we obtain solutions for the system (1) given by (29) and (30).

Looking further for solutions of the equation (11) of the form $f = a + ne^{ky}$ with the constants a, n, k, we will find out that

$$C_1 = 0, a = \frac{k\lambda}{Cb} \Rightarrow f(y) = \frac{k\lambda}{Cb} + ne^{ky}.$$

Substituting into (15), we obtain the following solutions for the system (1):

$$\begin{cases} u = -C(bx+m)\,kne^{ky}, \\ v = \lambda k + Cbne^{ky}, \end{cases} \tag{31}$$

where C, b, m, n, k are arbitrary constants.

The pressure corresponding to the solution (30) is

$$P = \left[F - 0,5\left[(Cbn)^2 e^{2ky} + (\lambda k)^2\right] + C_0\right]\mu. \tag{32}$$

As a result, we obtain solutions of the system (1) in the form of the formulas (31), (32). Based on what has been proved in this section, the following theorem results:

Theorem 4.1. *If the function $f(y)$ is linear, i.e. $f(y) = by + m$, $b \neq 0$, then the function $g(x)$ is the solution of equation (18). In this case, the system (1) admits the exact solutions (19), (20); (21), (22) and (23), (24). If $g(x)$ is linear, i.e. $g(x) = ax + d$, $a \neq 0$, then the function $f(y)$ is the solution of equation (26). In this case, the system (1) admits the exact solutions (27), (28); (29), (30) and (31), (32).*

Remark 4.1. Unlike the solutions obtained in Theorems 2.1 and 3.1, Theorem 4.1 mentions solutions for the system (1) in which the viscosity parameter λ is explicitly indicated.

Remark 4.2. Equations (18) and (26) can be reduced to second-order differential equations.

Let us illustrate what was said, for example, for the equation (18). Then, making the substitution $g' = p(g)$, $g'' = p'p$, we arrive at a second-order nonlinear differential equation for determining the function $p(g)$:

$$\lambda\left(pp'' + (p')^2\right) + Cb\,(gp' - p) = c_1 p^{-1}. \tag{33}$$

However, the problem of determining the solutions of equation (33) is not simpler than the problem of determining the solutions of equation (18). We observe in the case of

$c_1 \neq 0$, that particular solutions of equation (33) are the following constants:

$$p = a, C_1 = -Cba^2; \Rightarrow g(x) = ax + d.$$

In this case, we obtain solutions for the system (1) in the form of (19), (20).

In the case $c_1 = 0$, we will look for particular solutions for the equation (33) of the form $p(g) = ag^2 + mg + k$, where a, m, k are constants and $a \neq 0$ or $m \neq 0$. Substituting into (33), we get:

$$\lambda \left[2a\left(ag^2 + mg + k\right) + (2ag + m)^2\right] + cb\left[g(2ag + m) - ag^2 - mg - k\right] = 0. \quad \Rightarrow$$

$$g^2\left[6\lambda a^2 + Cba\right] + g\left[\lambda am\right] + \left[\lambda\left(2ak + m^2\right) - Cbk\right] = 0 \tag{34}$$

Because the function $g(x)$ is not constant, the equality (34) can be fulfilled only when all the expressions in the square brackets cancel, i.e. the following equalities are true:

$$6\lambda a^2 + Cba = 0 \quad \text{and} \quad \lambda am = 0 \quad \text{and} \quad \lambda\left(2ak + m^2\right) - Cbk = 0. \tag{35}$$

The equalities (35) take place simultaneously in the following two cases:

(1) $a = 0, m \neq 0$. Then the first two equalities are satisfied and from the third one we have that $k = \dfrac{\lambda m^2}{Cb} \Rightarrow p(g) = g' = mg + k$. From here we get

$$g' - mg = k \Rightarrow g(x) = ne^{mx} - \frac{k}{m} \Rightarrow g(x) = ne^{mx} - \frac{\lambda m}{Cb}.$$

In this case we obtain the solutions of the form (23), (24).

(2) $m = 0$, $a \neq 0$. The second equality in (35) is satisfied. Then from the first and the third equalities we find that $a = -\frac{Cb}{6\lambda}$, $k = 0 \Rightarrow p(g) = ag^2$. In this case we have that

$$p = g' = ag^2 \Rightarrow g(x) = \frac{6\lambda}{Cb(C_0 + x)}.$$

Thus, we obtain the solutions of the form (21), (22).

Analogously reducing the equation (26) to a second-order differential equation and studying it in the same way as the equation (33), we will obtain the solutions of the system (1) of the form (27), (28); (29), (30) and (31), (32).

5. Conclusions

In the current article are determined a lot of exact solutions for the stationary bidimensional Navier-Stokes equations. We mention, that the solutions obtained in Theorems 2.1 and 3.1 do not explicitly depend on the viscosity. This result occurs because the expression $u_y - v_x$ (the rotor), corresponding to these solutions, is constant and equal to zero.

In the other solutions obtained in Theorem 4.1 it is indicated the explicit dependence on the viscosity coefficient λ. In the case of these solutions, the rotor is not constant. We also mention that several arbitrary constants participate in the expressions containing the found solutions. The values of these constants can be determined based on initial conditions and boundary conditions of the given physical problems.

Most of the results obtained in this work were presented and discussed during the 29th International Conference on Applied and Industrial Mathematics held on August 25-27, 2022 at Tiraspol State University, Chisinau [12].

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Received: October 17, 2022 *Accepted: December 21, 2022*

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Acta et Commentationes, Exact and Natural Sciences ISSN: 2537-6284
Volume 2(14), 2022, Pages 117–131 E-ISSN: 2587-3644
CZU 512.548 (043.3) DOI: 10.36120/2587-3644.v14i2.117-131

Dedicated to the memory of Professor Alexandru Basarab

Encryption and decryption algorithm based on the Latin groupoid isotopes

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Abstract. This paper studies encryption and decryption algorithm, using isotopes of Latin groupoid. Cryptographic algorithms are computationally intensive processes which consume large amount of CPU time and space during the process of encryption and decryption. The goal of this paper is to study the encryption and decryption algorithm with the help of the concept of Latin groupoid and notion of isotopes. The proposed algorithm is safe in the implementation process and can be verified without much difficulty. An example of encryption and decryption based on the Latin groupoid and the concept of isotopy is examined.

2010 Mathematics Subject Classification: 34C14, 34C40.
Keywords: Latin groupoid, isotopes, encryption and decryption algorithm.

Algoritmul de criptare și decriptare bazat pe izotopii grupoidului latin

Rezumat. În lucrarea de faţă este dezvoltat un algoritm de criptare şi decriptare care se bazează pe utilizarea grupoidului latin, concept care a fost introdus de autori, şi a izotopilor grupoidului examinat. Implementarea algoritmilor criptografici reprezintă procese intensive din punct de vedere computaţional şi presupune consumul unei cantităti mari de timp pentru funţionarea procesorului, cât şi un volum important de spaţiu pentru memoria calculatorului pe durata procesului de criptare şi decriptare. Scopul lucrării este de a studia algoritmul de criptare şi decriptare conceput cu ajutorul conceptului de grupoid latin şi noţiunii de izotop. Algoritmul propus de autori este sigur în procesul de implementare şi poate fi verificat fără prea multe dificultăţi. Este soluţionat un exemplu practic privind utilizarea algoritmului de criptare şi decriptare dezvoltat în baza grupoidului latin şi a izotopiilor de grupoid.

Cuvinte-cheie: grupoid latin, izotopi, algoritm de criptare si decriptare.

1. Introduction

In cryptography the encryption and decryption procedures consist of a set of algorithms and mathematical concepts and formulas that indicate the rules of conversion of plain text to cipher text and vice versa combined with the secured key. In some encryption and decryption algorithms sender and receiver use the same key. While in other encryption and decryption procedures sender and receiver use different keys. The major goal is to develop any algorithmic encryption and decryption procedure to improve the level of security. Therefore, this paper aims to propose a new encryption and decryption algorithm to improve the secure level using, the concept of the Latin groupoid and notion of isotopes.

Our main results can be summarized as follows. In Section 2 we give the basic algebraic notions. In Section 3 we propose an Algorithm to Encrypt and Decrypt message, using isotopes of Latin groupoids. Finally, in Section 4 we give one example of Encryption and Decryption algorithm based on the concept of Latin groupoid and isotopes.

We dedicate this paper to the memory of Professor Alexandru Basarab, who worked for more than 50 years at the Faculty of Physics and Mathematics of Tiraspol State University, Republic of Moldova and made many important contributions to theory of loops and quasigroups.

2. Basic notions

In this section we recall some fundamental definitions and notations.

A non-empty set G is said to be a *groupoid* with respect to a binary operation denoted by $\{\cdot\}$, if for every ordered pair (a, b) of elements of G there is a unique element $ab \in G$.

A quasigroup is a binary algebraic structure in which one-sided multiplication is a bijection in that all equations of the form $ax = b$ and $ya = b$ have unique solutions.

An element $e \in G$ is called an *identity* if $ex = xe = x$ every $x \in G$.

A quasigroup with an identity is called a *loop*. The notion of quasigroup is hence a generalization of the notion of group, in that it does not require the associativity law, nor the existence of an identity element.

A groupoid G is called *medial* if it satisfies the law $xy \cdot zt = xz \cdot yt$ for all $x, y, z, t \in G$.

If a guasigroup G contains an element e such that $e \cdot x = x$ $(x \cdot e = x)$ for all x in G, then e is called a *left (right) identity* element of G and G is called a *left (right) loop.*

In mathematical terms, a permutation of a set is defined as a bijective function $p : X \rightarrow X$. For example, there are six permutations of the set $\{1, 2, 3\}$, namely $(1, 2, 3), (1, 3, 2), (2, 1, 3), (2, 3, 1), (3, 1, 2)$ and $(3, 2, 1)$.

It is called permutation of degree n bijective function $p : N^* \rightarrow N^*$ and is written in the following form:

$$p = \begin{pmatrix} 1 & 2 & ... & n \\ p(1) & p(2) & ... & p(n) \end{pmatrix}.$$

Denote by S_n the set of permutations of degree n and card$(S_n) = n!$ Permutations can be defined as bijections from a set S onto itself. All permutations of a set with n elements form a symmetric group, denoted S_n, where the group operation is function composition. Thus, for two permutations, α and β in the group S_n, the four group axioms hold: closure, associativity, identity and invertibility. For every permutation α, there exists an inverse permutation α^{-1}, so that $\alpha \cdot \alpha^{-1} = \alpha^{-1} \cdot \alpha = e$, where e is identity permutation. In general, composition of two permutations is not commutative.

Let E be a non-empty set. Then the set $S(E) = \{f : E \longrightarrow E : f-bijective\}$, together with the function composition operation, is a group, called the permutation group of the set E (or the symmetric group associated with the set E). If F is a set with the property that there is a bijection between F and E, then the groups $S(F)$ and $S(E)$ are isomorphic.

Permutations are used in almost every branch of mathematics and many other areas of science. In computer science, they are used to analyze sorting algorithms; in quantum physics, for describing the states of particles; and in biology, for example, for describing RNA sequences.

Let $(G, \star)$, $(H, \circ)$ be groupoids. An isotopy from $(G, \star)$ to $(H, \circ)$ is an ordered triple: $\phi = (f, g, h)$, of bijections from $(G, \star)$ to $(H, \circ)$, such that $f(a) \circ g(b) = h(a \star b)$ or $h^{-1}(f(a) \circ g(b)) = a \star b$ for all $a, b \in G$.

An $(H, \circ)$ is called an isotope of $(G, \star)$, or $(H, \circ)$ is isotopic to $(G, \star)$ if there is an isotopy $\phi = (f, g, h){:}(G, \star) \to (H, \circ)$.

Hereafter, we share some examples of isotopies. If $f : G \to H$ is an isomorphism, then $(f, f, f) : G \to H$ is an isotopy. We can write $f = (f, f, f) : G \to H$. If all 3 permutations coincide: $f = g = h$, then isotopy turns into isomorphism. In this case we will write $f(x) \circ f(y) = f(x * y)$. In particular $(1_G, 1_G, 1_G) : G \to G$ is an isotopy where 1_G is the identity function on G.

If $\phi = (f, g, h){:}(G, \star) \to (H, \circ)$ is an isotopy, then so is

$$\phi^{-1} = (f^{-1}, g^{-1}, h^{-1}) : (H, \circ) \to (G, \star),$$

for if $f^{-1}(a) = c$ and $g^{-1}(b) = d$, then $ab = f(c)g(d) = h(cd)$, so that

$$f^{-1}(a)g^{-1}(b) = cd = h^{-1}(ab).$$

Let $\mathbb{N} = \{1, 2, ...\}$ and $\mathbb{Z} = \{..., -3, -2, -1, 0, 1, 2, 3, ...\}$. We shall use the notations and terminology from [1, 2, 3, 4, 5, 7]. The results established here are related to the work in [8, 6, 9, 10, 11, 12].

Example 1. Let $(Q, \star)$ be a quasigroup, determined by the following Cayley table:

⋆	1	2	3	4
1	1	3	2	4
2	2	1	4	3
3	3	4	1	2
4	4	2	3	1

Let α, β, and γ be three arbitrary permutations of the set Q. Then, applying the permutation α of the elements on the border line, the permutation β of the elements on the border column and the permutation γ of the elements inside the table, one obtains a new law of composition $(\circ)$ on Q and it is clear that $(Q, \circ)$ is isotopic to the quasigroup $(Q, \star)$.

Thus, we consider:
$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}, \beta = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}, \gamma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}.$

Applying the permutations α, β, and γ, it is obtained the following:

⋆	1	2	3	4
1	1	3	2	4
2	2	1	4	3
3	3	4	1	2
4	4	2	3	1

$\overrightarrow{\alpha}$

·	1	2	3	4
1	2	1	4	3
2	3	4	1	2
3	4	2	3	1
4	1	3	2	4

$\overrightarrow{\beta}$

∗	1	2	3	4
1	3	4	1	2
2	2	1	4	3
3	1	3	2	4
4	4	2	3	1

$\overrightarrow{\gamma}$

∘	1	2	3	4
1	4	3	2	1
2	1	2	3	4
3	2	4	1	3
4	3	1	4	2

We note that the quasigroup $(Q, \star)$ is medial, non-associative, since $(3 \star 4) \star 2 \neq 3 \star (4 \star 2)$ and $e = 1$ is the right identity because $x * 1 = x$ for every $x \in (Q, \star)$. Isotop quasigroup $(Q, \circ)$ is medial, non associative, but $e = 2$ is the left identity because $2 * x = x$ for every $x \in (Q, \circ)$. We conclude from this that, unlike isomorphism which preserves all properties of an algebraic operation, an isotopism does not preserve all properties.

A *Latin groupoid* of order n is a $n \times n$ array filled with s, distinct symbols (by convention $\{al_1, ..., al_s\}$),where $s \leq n^2$, such that there are symbols which are repeated twice or more times, in rows or columns.

It should be mentioned that a Latin groupoid is a Latin square of order n, is a $n \times n$ array filled with $n = s$ distinct symbols, such that no symbol is repeated twice in any row or column.

Two Latin groupoids are isotopic if each can be turned into the other by permuting the rows and columns. This isotopy relation is an equivalence relation; the equivalence classes are the isotopy classes.

During the exposition of the material we will use also and another definition.

A non-empty couple of sets (G, Al), where $|G| = n$ and $|Al| = s$, is said to be a *Latin groupoid* with respect to a composition or operation $(\bullet)$ that sends any two elements $a, b \in G$ to another element, $a \bullet b = al_i \in Al$, where $i = \{1, ..., s\}$ and the number of all elements of the type $al_i \in Al$, which some of them can be repeated several times in rows or columns, is equal to n^2.

Denote a *Latin groupoid* by $(G, Al, \bullet)$.

Example 2. Let be $Q = \{1, 2, 3, 4, 5, 6\}$ and Al = {space, S, !, B, V, R, H, G, O, D, H, E, L, T, I, A, W,}. Let $(\bullet)$ be defined by the following Cayley table:

•	1	2	3	4	5	6
1	M	G	O	D	H	E
2	L	T	I	S	B	V
3	W	A	R		O	D
4	H	E	R	L	T	I
5	S	B	!	W	A	M
6	O	E	A	H		T

Then $(Q, Al, \bullet)$ is a Latin groupoid.

3. Encryption and Decryption Algorithm based on the Latin Groupoid and Isotopes

Below we describe the respective algorithm.

3.1. Steps to Encrypt the message

1. Define the alphabet $Al = \{al_1, al_2, ..., al_t\}$, where t is dimension of the set Al and $t \in N$.

2. Define a set of n ordered elements $Q = \{1, 2, ..., n\}$, where $n \in N$ and $n^2 > t$.

3. Construct a Latin groupoid $(Q, Al, \bullet)$.

The construction of the Latin groupoid begins with the definition of the composition or operation on the set Q. Define the operation $(\bullet)$ on the couple of sets (Q, Al), taking into account the following conditions:

3.1. The result of the operation $a \bullet b$ with respect to a operation $(\bullet)$ is an element $a \bullet b = al_i \in Al$, for all $a, b \in Q$ and $i = \overline{1, t}$.

3.2. All results of the operations $a \bullet b = al_i \in Al$, for all $a, b \in Q$ and $i = \overline{1,t}$, made up of the elements of the alphabet Al, are placed in a Cayley table, which has the dimension $n \times n$.

3.3. The elements inside in the Cayley table $al_1, al_2, ..., al_t$ are placed randomly. The important thing is that each element of the alphabet $al_i \in Al$, $i = \overline{1,t}$ is found at least once as a result of the operation $a \bullet b$ for all $a, b \in G$. The number of all elements of the type $al_i \in Al$, where some of them can be repeated several times, is equal to n^2.

3.4. In this way we obtain a Latin groupoid $(G, Al, \bullet)$ in which the results of the operation $a \bullet b$ for all $a, b \in G$ are all the elements of the alphabet Al and some elements can be repeated several times. There is no maximum limit for how many times an element of the Al alphabet could be repeated as a result.

4. In the Cayley table, it is determined how many times each of the elements of the Latin groupoid $(Q, Al, \bullet)$ is repeated.

Denote by K the set of the number of repetition of all elements in the alphabet Al.

4.1. Let element al_1 be repeated by k_1 times, element al_2 be repeated by k_2 times,...,element al_t be repeated by k_t times. In this way we get the set $K = \{k_1, k_2, ..., k_t\}$, where k_s indicates the number of repetitions of the element al_s in the alphabet Al, and $s = \overline{1,t}$. By $r = max\{k_1, k_2, ..., k_t\}$ it is denoted the maximum number of repetitions of the elements $k_s \in K$, $s = \overline{1,t}$.

4.2. Then there are determined all the pairs $i \bullet j$, where $i, j = \overline{1,n}$, of the elements which give us the same result for each of the elements $al_s \in Al$, $s = \overline{1,t}$.

Denote by O_p, where $p = 1, 2, ..., r$, the p-set of the results of the operations $i \bullet j = al_s \in Al$ for all $i, j \in Q$ and $s = \overline{1,t}$.

First, the set O_1 is constructed by including only one result at a time of the operations $i \bullet j = al_s$ for all $i, j \in Q$ and for all elements $al_s \in Al$ and $s = \overline{1,t}$. For example, if $al_{s_\star} = i_1 \bullet j_1 = i_2 \bullet j_2 = ... = i_r \bullet j_r$ then will be include in the set O_1 only result of the operation $i_1 \bullet j_1$. It is obviously $|O_1| = m$, where m is the dimension of the secret message.

Afterwards, the set O_2 is constructed by counting and including only one result at a time of the operations $i \bullet j = al_s$ for all $i, j \in Q$ and for each element $al_s \in Al$,$s = \overline{1,t}$, with the exception of the results that were already included in the set O_1. For example, if $al_{s_\star} = i_1 \bullet j_1 = i_2 \bullet j_2 = ... = i_r \bullet j_r$ then will be include in the set O_2 only result of the operation $i_2 \bullet j_2$.

At the last stage the set O_r is constructed through identification and including all results of the operations (with the exception of the results that were already included in the sets $O_1, ..., O_{r-1}$) $i \bullet j = al_s \in Al$ for all $i, j \in Q$ and $s = \overline{1,t}$.

For example, if $al_{s_\star} = i_1 \bullet j_1 = i_2 \bullet j_2 = ... = i_r \bullet j_r$ then will be include in the set O_r only result of the operation $i_r \bullet j_r$.

It is important to note that $\bigcap_{p=1}^{r} O_p = \emptyset$.

5. Define a set of n_1 ordered elements $Q_1 = \{1, 2, ..., n_1\}$, where $n_1 \leq n$.

6. Define the permutations α and β on the set Q_1.

7. Get the message for Encryption.

Let the secret text be $M1 = \{al_1, al_2, ..., al_m\}$, where m is the dimension of the message and $al_s \in Al$, $s = \overline{1, t}$.

8. In the next table we will construct:

8.1. The set $M1$ which represents the message to be encrypted. The elements of the set $M1$ are the elements of the secret text and $|M1| = n_1^2 = m$.

8.2. The set K_1 which indicates the number of repetitions of the elements of the secret message $M1$. The number of elements in the set K_1 coincides to the dimension m of the secret message $M1$. All the elements of the set K_1 are one of the the numbers $\{1, ..., r\}$ which can be repeated several times.

8.3. The set P where the elements of the set P are formed by the numbers $p = 1, ..., r$. Each element of the set P indicates from which set O_p the corresponding element $i \bullet j = al_s \in Al$ for all $i, j \in Q$ and $s = \overline{1, t}$ is taken. The number of elements of the set P coincides with the dimension m of the secret message $M1$.

8.4. The set R that includes all the element $i \bullet j = al_s \in M1$ for all $i, j \in Q$ and $s = \overline{1, n_1^2}$ is taken.

The set $R = \{r_1, r_2, ..., r_m\}$ represents the secret message, where $r_i \in O_p$, $p = 1, ..., r$ is determined by all pairs $i \bullet j = al_s \in M1$ for all $i, j \in Q$ and $s = \overline{1, n_1^2}$.

9. Construct the Latin groupoid $(Q_1, R, \circ)$.

In order to increase the degree of protection of the message R we will construct the Latin groupoid $(Q_1, R, \circ)$. All results of the operations $i \bullet j = r_s \in R$ for all $i, j \in Q_1$ and $s = \overline{1, n_1^2}$, made up of the elements R, are placed in a Cayley table of a Latin groupoid $(Q_1, R, \circ)$ which has the dimension $n_1 \times n_1$, respects the order of the elements in the set R and places them one by one in the table, starting with the first row, the second one and so on until the last $n_1 - th$ row. In each row there will be exactly n_1-elements.

10. The permutation α is applied to the Latin groupoid $(Q_1, R, \circ)$. Get the Latin groupoid $(Q_1, R, \circ_\alpha)$.

11. The permutation β is applied to the Latin groupoid $(Q_1, R, \circ_\alpha)$. Get the Latin groupoid $(Q_1, R, \circ_\beta)$. The Latin groupoid $(Q_1, R, \circ_\beta)$ is an isotope of the Latin groupoid $(Q_1, R, \circ)$.

The secret key for encryption is Latin groupoid $(Q, Al, \bullet)$ and the permutations α and β on the set Q_1. The Latin groupoid $(Q_1, R, \circ_\beta)$ represents the secret message and will be send to receiver.

3.2. Steps to Decrypt the message

1. The secret key for decryption is the Latin groupoid $(Q, Al, \bullet)$ and the permutations α^{-1} and β^{-1} on the set Q_1.

2. Applying the permutation β^{-1} on the Latin groupoid $(Q_1, R, \circ_\beta)$, get the Latin groupoid $(Q_1, R, \circ_\beta^{-1})$.

3. Applying the permutation α^{-1} on the Latin groupoid $(Q_1, R, \circ_\beta^{-1})$, get the Latin groupoid $(Q_1, R, \circ_\alpha^{-1})$ which coincides to the Latin groupoid $(Q_1, R, \circ)$.

4. Using the Latin groupoid $(Q, Al, \bullet)$ that was constructed at step 3 in the Encryption Algorithm, we obtain the decrypted message.

4. Example of the use of the Encryption and Decryption Algorithm

Example. Interlocutor A needs to sent a secret message to interlocutor B. For this purpose, the following steps are undertaken:

Step 1. Interlocutor A decides to determine the number of symbols of the alphabet Al according to the secret message
$M1 = \{$GOOD HEALTH IS ABOVE WEALTH! REMEMBER!$\}$, where $m = 38$ is the number of all elements, inclusive empty space.

Thus, the alphabet, defined by interlocutor A for message $M1$, is
$Al = \{$space, G, O, D, H, E, L, T, I, A, W, S, !, B, V, R, H$\}$.

Since Al consists of $t = 17$ elements, then the set Q will have $n = 6$ elements. Hence, $Q = \{1, 2, 3, 4, 5, 6\}$ and $n^2 = 36 \geq 17 = t$.

Step 2. Interlocutor A, taking into the consideration the conditions 3.1 – 3.4., defines the operation $(\bullet)$ on the set Q and gets the Latin groupoid $(Q, Al, \bullet)$.

Interlocutor A defines the set $Q_1 = \{1, 2, 3, 4, 5, 6\}$ and the permutations α and β on the set Q_1. Let:

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 4 & 5 & 3 & 2 & 6 \end{pmatrix}, \beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 4 & 5 & 6 & 3 & 2 \end{pmatrix}.$$

The secret key for encryption is the Latin groupoid $(Q, Al, \bullet)$ and the permutations α and β.

Step 3. Interlocutor A defines the operation $(\circ)$ on the set Q_1 and gets the Latin groupoid $(Q_1, R, \circ)$, where R is set which represents secret message with the elements $i \bullet j = al_s \in M1$ for all $i, j \in Q$ and $s = \overline{1, 38}$.

Step 4. Interlocutor A applies the permutations α, β for encrypting the secret message and sends it to interlocutor B.

Step 5. Interlocutor B receives the secret message $(Q_1, R, \circ)$ and secret keys: Latin groupoid $(Q, Al, \bullet)$ and permutations α and β.

Step 6. Interlocutor B computes and applies the permutations α^{-1} and β^{-1} for decrypting the secret message $(Q_1, R, \circ)$ and read it.

It is required to describe more detailed Steps $1-5$ in accordance with the algorithm presented above.

Solve.

Steps to Encrypt the message.

1. Define the alphabet $Al = \{$space,G, O, D, H, E, L, T, I, A, W, S, !, B, V, R, H$\}$, where numbers of characters $t = 17$.

2. Define a set of n = 6 ordered elements $Q = \{1, 2, 3, 4, 5, 6\}$, where $36 = n^2 > t = 17$.

3. Construct a Latin groupoid $(Q, Al, \bullet)$.

Define the operation $(\bullet)$ on the couple of sets (Q, Al), taking into account the following conditions:

3.1. The result of the operation $i \bullet j$ with respect to an operation $(\bullet)$ is an element $i \bullet j = al_s \in Al$, for all $i, j \in Q$ and $s = \overline{1, 17}$, where $Al = \{$space, G, O, D, H, E, L, T, I, A, W, S, !, B, V, R, H$\}$.

3.2. All results of the operations $i \bullet j = al_s \in Al = \{$space, G, O, D, H, E, L, T, I, A, W, S, !, B, V, R, H$\}$ for all $i, j \in Q$ and $s = \overline{1, 17}$, made up of the elements Al, are placed in a Cayley table of a Latin groupoid $(Q, Al, \bullet)$, which has the dimension 6×6.

3.3. All elements $al_s \in Al$ inside the Cayley table are placed randomly. An important rule is that each element of the alphabet Al is found at least once as a result of the operation $i \bullet j$ for all $i, j \in Q$. Number of all elements of the type $al_i \in Al$, which some of them can be repeated several times, is equal to 6^2.

Below, can see the operation table of the Latin groupoid $(Q, Al, \bullet)$.

$\bullet$	1	2	3	4	5	6
1		G	O	D	H	E
2	L	T	I	S	B	V
3	W	A	R		O	D
4	H	E	R	L	T	I
5	S	B	!	W	A	M
6	O	E	A	H	T	M

4. In the Cayley table, it is determined how many times each of the elements of the Latin groupoid $(Q, Al, \bullet)$ is repeated.

In the tables below we will construct: the set K which indicates the number of repetition of all elements in the alphabet Al and the sets O_p, where $p = 1, 2, ..., r$, which indicate the results of the operations $i \bullet j = al_s \in Al$ for all $i, j \in Q$ and $s = \overline{1, 17}$.

As the maximum number of repetitions of the elements in Al is $r = 3$, then the set K is formed from the elements $1, 2, 3$. We will have the sets O_1, O_2 and O_3.

It should be mentioned that $\bigcap_{p=1}^{3} O_p = \emptyset$.

In this way we obtain the tables below.

Al	space	G	O	D	H	E	L	T	I	A
K	2	1	3	2	3	3	2	3	2	3
O_1	$1 \bullet 1$	$1 \bullet 2$	$1 \bullet 3$	$1 \bullet 4$	$1 \bullet 5$	$1 \bullet 6$	$2 \bullet 1$	$2 \bullet 2$	$2 \bullet 3$	$3 \bullet 2$
O_2	$3 \bullet 4$		$3 \bullet 5$	$3 \bullet 6$	$4 \bullet 1$	$4 \bullet 2$	$4 \bullet 4$	$4 \bullet 5$	$4 \bullet 6$	$5 \bullet 5$
O_3			$6 \bullet 1$		$6 \bullet 4$	$6 \bullet 2$		$6 \bullet 5$		$6 \bullet 3$

Al	W	S	!	B	V	R	M
K	2	2	1	2	2	2	2
O_1	$3 \bullet 1$	$2 \bullet 4$	$5 \bullet 3$	$2 \bullet 5$	$2 \bullet 6$	$3 \bullet 3$	$5 \bullet 6$
O_2	$5 \bullet 4$	$5 \bullet 1$		$5 \bullet 2$	$5 \bullet 3$	$4 \bullet 3$	$6 \bullet 6$

Each element $k_i \in K = \{2, 1, 3, 2, 3, 3, 2, 3, 2, 3, 2, 2, 1, 2, 2, 2, 2\}$ indicates the number of repetitions of the corresponding element $al_i \in Al = \{$space, G, O, D, H, E, L, T, I, A, W, S, !, B, V, R, H$\}$, where $i = \overline{1, 17}$.

5. Define a set of n_1=6 ordered elements $Q_1 = \{1, 2, 3, 4, 5, 6\}$.

6. Define the permutations α and β on the set Q_1 in the following way:

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 4 & 5 & 3 & 2 & 6 \end{pmatrix}, \beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 4 & 5 & 6 & 3 & 2 \end{pmatrix}.$$

7. Get the message for Encryption. Let the secret text be $M1 = \{$GOOD HEALTH IS ABOVE WEALTH! REMEMBER!$\}$, where $m = 38$ is the dimension of the message. In this message we have 5 empty spaces. To reduce the dimension of the message to 36 we will omit 2 empty spaces. Therefore, the secret message will be $M1 = \{$GOODHEALTH IS ABOVE WEALTH!REMEMBER!$\}$, where $m = 36$.

8. In the tables below we will construct the sets: $M1$, K_1, P and R.

$M1$	G	O	O	D	H	E	A	L	T	H
K_1	1	3	3	2	3	3	2	3	2	3
P	1	1	2	1	1	1	1	1	1	2
R	$1 \bullet 2$	$1 \bullet 3$	$3 \bullet 5$	$1 \bullet 4$	$1 \bullet 5$	$1 \bullet 6$	$3 \bullet 2$	$2 \bullet 1$	$2 \bullet 2$	$4 \bullet 1$

$M1$	space	I	S	space	A	B	O	V	E	space
K_1	2	2	2	2	3	2	3	2	3	2
P	1	1	1	1	2	1	3	1	1	1
R	$1 \bullet 1$	$2 \bullet 3$	$2 \bullet 4$	$1 \bullet 1$	$5 \bullet 5$	$2 \bullet 5$	$6 \bullet 1$	$2 \bullet 6$	$4 \bullet 2$	$3 \bullet 4$

$M1$	W	E	A	L	T	H	!	R	E	M
K_1	2	3	3	2	3	3	1	2	3	2
P	1	3	3	2	2	3	1	1	1	1
R	$3 \bullet 1$	$6 \bullet 2$	$6 \bullet 3$	$4 \bullet 4$	$4 \bullet 5$	$6 \bullet 4$	$5 \bullet 3$	$3 \bullet 3$	$1 \bullet 6$	$5 \bullet 6$

$M1$	E	M	B	E	R	!
K_1	3	2	3	3	2	1
P	2	2	3	3	2	1
R	$4 \bullet 2$	$6 \bullet 6$	$5 \bullet 2$	$6 \bullet 2$	$4 \bullet 3$	$5 \bullet 3$

In the above tables it is show that the result of the operation $i \bullet j \in R$, which determines the corresponding element $al_i \in M1$, is taken from the set O_r, where $r = 1, 2, 3$.

For example, the result of the binary operation $1 \bullet 3 \in R$, give us the element $O \in M1$ which is repeated 3 times in the text, because corresponding element in the set K_1 is $3 \in K_1$. The result $1 \bullet 3 = O$ is taken from the set O_1, because the corresponding element in the set P is $1 \in P$. The set

$R = \{1 \bullet 2, 1 \bullet 3, 3 \bullet 5, 1 \bullet 4, 1 \bullet 5, 1 \bullet 6, 3 \bullet 2, 2 \bullet 1, 2 \bullet 2, 4 \bullet 1, 1 \bullet 1, 2 \bullet 3, 2 \bullet 4, 1 \bullet 1, 5 \bullet 5, 2 \bullet 5, 6 \bullet 1, 2 \bullet 6, 4 \bullet 2, 3 \bullet 4, 3 \bullet 1, 6 \bullet 2, 6 \bullet 3, 4 \bullet 4, 4 \bullet 5, 6 \bullet 4, 5 \bullet 3, 3 \bullet 3, 1 \bullet 6, 5 \bullet 6, 4 \bullet 2, 6 \bullet 6, 5 \bullet 2, 6 \bullet 2, 4 \bullet 3, 5 \bullet 3\}$ determine the secret message. The dimension of the set R is $m = 36$.

9. Construct the Latin groupoid $(Q_1, R, \circ)$.

In order to increase the degree of protection of the message R we will construct the Latin groupoid $(Q_1, R, \circ)$. All results of the operations $i \bullet j = r_s \in R$ for all $i, j \in Q_1$ and $s = \overline{1, 36}$, made up of the elements R, are placed in a Cayley table of a Latin groupoid $(Q_1, R, \circ)$, which has the dimension 6×6.

$\circ$	1	2	3	4	5	6
1	$1 \bullet 2$	$1 \bullet 3$	$3 \bullet 5$	$1 \bullet 4$	$1 \bullet 5$	$1 \bullet 6$
2	$3 \bullet 2$	$2 \bullet 1$	$2 \bullet 2$	$4 \bullet 1$	$1 \bullet 1$	$2 \bullet 3$
3	$2 \bullet 4$	$1 \bullet 1$	$5 \bullet 5$	$2 \bullet 5$	$6 \bullet 1$	$2 \bullet 6$
4	$4 \bullet 2$	$3 \bullet 4$	$3 \bullet 1$	$6 \bullet 2$	$6 \bullet 3$	$4 \bullet 4$
5	$4 \bullet 5$	$6 \bullet 4$	$5 \bullet 3$	$3 \bullet 3$	$1 \bullet 6$	$5 \bullet 6$
6	$4 \bullet 2$	$6 \bullet 6$	$5 \bullet 2$	$6 \bullet 2$	$4 \bullet 3$	$5 \bullet 3$

10. The permutation:

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 4 & 5 & 3 & 2 & 6 \end{pmatrix}$$

is applied to the Latin groupoid $(Q_1, R, \circ)$. It is obtained the Latin groupoid $(Q_1, R, \circ_\alpha)$.

$\circ_\alpha$	1	2	3	4	5	6
1	$1 \bullet 2$	$1 \bullet 3$	$3 \bullet 5$	$1 \bullet 4$	$1 \bullet 5$	$1 \bullet 6$
2	$4 \bullet 2$	$3 \bullet 4$	$3 \bullet 1$	$6 \bullet 2$	$6 \bullet 3$	$4 \bullet 4$
3	$4 \bullet 5$	$6 \bullet 4$	$5 \bullet 3$	$3 \bullet 3$	$1 \bullet 6$	$5 \bullet 6$
4	$2 \bullet 4$	$1 \bullet 1$	$5 \bullet 5$	$2 \bullet 5$	$6 \bullet 1$	$2 \bullet 6$
5	$3 \bullet 2$	$2 \bullet 1$	$2 \bullet 2$	$4 \bullet 1$	$1 \bullet 1$	$2 \bullet 3$
6	$4 \bullet 2$	$6 \bullet 6$	$5 \bullet 2$	$6 \bullet 2$	$4 \bullet 3$	$5 \bullet 3$

11. The permutation

$$\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 4 & 5 & 6 & 3 & 2 \end{pmatrix}$$

is applied to the Latin groupoid $(Q_1, R, \circ_\alpha)$. It is obtained the Latin groupoid $(Q_1, R, \circ_\beta)$.

$\circ_\beta$	1	2	3	4	5	6
1	$1 \bullet 2$	$1 \bullet 4$	$1 \bullet 5$	$1 \bullet 6$	$3 \bullet 5$	$1 \bullet 3$
2	$4 \bullet 2$	$6 \bullet 2$	$6 \bullet 3$	$4 \bullet 4$	$3 \bullet 1$	$3 \bullet 4$
3	$4 \bullet 5$	$3 \bullet 3$	$1 \bullet 6$	$5 \bullet 6$	$5 \bullet 3$	$6 \bullet 4$
4	$2 \bullet 4$	$2 \bullet 5$	$6 \bullet 1$	$2 \bullet 6$	$5 \bullet 5$	$1 \bullet 1$
5	$3 \bullet 2$	$4 \bullet 1$	$1 \bullet 1$	$2 \bullet 3$	$2 \bullet 2$	$2 \bullet 1$
6	$4 \bullet 2$	$6 \bullet 2$	$4 \bullet 3$	$5 \bullet 3$	$5 \bullet 2$	$6 \bullet 6$

The Latin groupoid $(Q_1, R, \circ_\beta)$ is the isotope of the Latin groupoid $(Q_1, R, \circ)$. The Latin groupoid $(Q_1, R, \circ_\beta)$ represents the encryption of the message $M1$ and will be sent to receiver B.

The secret key for the encryption message $M1$ is the Latin groupoid $(Q, Al, \bullet)$ and the permutations α and β on the set Q_1.

Steps to Decrypt the message

1. The secret key for decryption is Latin groupoid $(Q, Al, \bullet)$ and the permutations α^{-1}, β^{-1} on the set Q_1.

2. We compute the permutation β^{-1} and we get:

$$\beta^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 6 & 5 & 2 & 3 & 4 \end{pmatrix}.$$

Applying the permutation β^{-1} on the Latin groupoid $(Q_1, R, \circ_\beta)$, which represent encryption of the message $M1$, it is got the Latin groupoid $(Q_1, R, \circ_\beta^{-1})$.

$\circ_{\beta^{-1}}$	1	2	3	4	5	6
1	$1\bullet 2$	$1\bullet 3$	$3\bullet 5$	$1\bullet 4$	$1\bullet 5$	$1\bullet 6$
2	$3\bullet 2$	$3\bullet 4$	$3\bullet 1$	$6\bullet 2$	$6\bullet 3$	$4\bullet 4$
3	$4\bullet 5$	$6\bullet 4$	$5\bullet 3$	$3\bullet 3$	$1\bullet 6$	$5\bullet 6$
4	$2\bullet 4$	$1\bullet 1$	$5\bullet 5$	$2\bullet 5$	$6\bullet 1$	$2\bullet 6$
5	$3\bullet 2$	$2\bullet 1$	$2\bullet 2$	$4\bullet 1$	$1\bullet 1$	$2\bullet 3$
6	$4\bullet 2$	$6\bullet 6$	$5\bullet 2$	$6\bullet 2$	$4\bullet 3$	$5\bullet 3$

3. We compute the permutation α^{-1} and we get:

$$\alpha^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 5 & 4 & 2 & 3 & 6 \end{pmatrix}.$$

Applying the permutation α^{-1} on the Latin groupoid $(Q_1, R, \circ_\beta^{-1})$, it is obtained the Latin groupoid $(Q_1, R, \circ_\alpha^{-1})$ which coincides to the groupoid $(Q_1, R, \circ)$.

$\circ$	1	2	3	4	5	6
1	$1\bullet 2$	$1\bullet 3$	$3\bullet 5$	$1\bullet 4$	$1\bullet 5$	$1\bullet 6$
2	$3\bullet 2$	$2\bullet 1$	$2\bullet 2$	$4\bullet 1$	$1\bullet 1$	$2\bullet 3$
3	$2\bullet 4$	$1\bullet 1$	$5\bullet 5$	$2\bullet 5$	$6\bullet 1$	$2\bullet 6$
4	$4\bullet 2$	$3\bullet 4$	$3\bullet 1$	$6\bullet 2$	$6\bullet 3$	$4\bullet 4$
5	$4\bullet 5$	$6\bullet 4$	$5\bullet 3$	$3\bullet 3$	$1\bullet 6$	$5\bullet 6$
6	$4\bullet 2$	$6\bullet 6$	$5\bullet 2$	$6\bullet 2$	$4\bullet 3$	$5\bullet 3$

4. Using the Latin groupoid $(Q, Al, \bullet)$ that was constructed at step 3 in the Encryption Algorithm, we obtain the decrypted message.

∘	1	2	3	4	5	6
1	G	O	O	D	H	E
2	A	L	T	H		I
3	S		A	B	O	V
4	E		W	E	A	L
5	T	H	!	R	E	M
6	E	M	B	E	R	!

In this way we obtained the secret message $M1 = \{$GOODHEALTH IS ABOVE WEALTH!REMEMBER!$\}$.

4. Conclusion

We have proposed a simply and efficient Encryption and Decryption Algorithm based on the Latin groupoid isotopes. Cryptographic developed algorithm does not consume large amount of CPU time and space during in the process of encryption and decryption. This cryptographic algorithm is safe in the process of the implementation and it is not complicated to develop a program for the developed algorithm. In this sense, the authors, based on the proposed algorithms, developed a program in the C++ programming language that works quickly and efficiently.

Acknowledgements. Research supported by “Methodology of ICT implementation in the process of study of the real sciences in the education system of the Republic of Moldova from the perspective of inter/transdisciplinarity (STEAM concept)”, Project 20.80009.0807.20, included in the “State Program” (2020-2023) of the National Agency for Development and Research.

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Received: November 04, 2022 *Accepted: December 23, 2022*

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Acta et Commentationes, Exact and Natural Sciences ISSN: 2537-6284
Volume 2(14), 2022, Pages 132–134 E-ISSN: 2587-3644
DOI: 10.36120/2587-3644.v14i2.132-134

Book Review

Study Of Useful Properties Of Some Coordination Compounds Containing Oximic Ligands, by E. Coropceanu, A. Ciloci, A. Ştefîrţă, I. Bulhac, ISBN 978-3-9402237-24-8, 266 p.

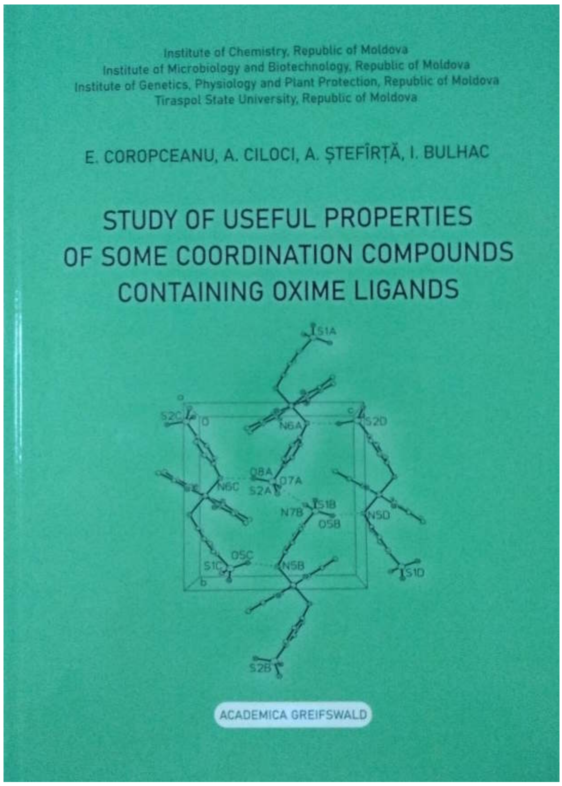
The monograph "STUDY OF USEFUL PROPERTIES OF SOME COORDINATION COMPOUNDS CONTAINING OXIMIC LIGANDS", ISBN 978-3-9402237-24-8, presents a study in the field of synthesis, determination of the composition and structure of new coordination compounds based on ligands with various functional groups, as well as the determination of the fields of practical utility of the new substances. The work includes four chapters on 266 pages.

In chapter I. AN OVERVIEW OF THE PERSPECTIVES OF USING COORDINATION COMPOUNDS BASED ON DIOXIME LIGANDS, studies in the field worldwide are described, which served as the basis for initiating the cycle of studies described by the authors. The most important research directions regarding the class of analyzed compounds and the properties that served to establish areas of practical use are reflected. The evolution of research in the field is briefly described, both from a chronological point of view and the increase in the degree of complexity of the molecules synthesized based on the ligands used for the assembly of metal-organic molecules.

In chapter II. THE INFLUENCE OF DIOXIME LIGANDS BASED COMPLEXES ON THE BIOLOGICAL ACTIVITY OF SOME ENZYME-PRODUCING FUNGI STRAINS describes the studies related to the use of coordinative compounds as an addition to the cultivation medium of some enzyme-producing microorganisms. As enzymes are of particular interest for the food, pharmaceutical and some branches of the agro-industrial complex, it is important to develop new innovative technologies to increase productivity and increase the profitability of economic processes. Studies have been carried out on the genera of fungi *Aspergillus, Rhizopus, Penicillium, Trichoderma, Fusarium*, etc., in which significant increases in enzyme genetic activity, biomass growth and reduction of the technological cycle have been recorded, a fact that increases the yield of cultivation methods and obtaining biologically active substances. Most of the results listed are patented. A number of inventions have been awarded at international salons with gold medals and other trophies.

In chapter III. THE EFFECT OF SOME COORDINATION COMPOUNDS ON PLANT PHYSIOLOGICAL PROCESSES UNDER THE IMPACT OF ECOLOGICAL STRESS describes the influence of new coordination compounds on physiological processes in higher crop plants. It was established that the analyzed compounds possess properties of bioactive substances with a positive impact on the growth, development, increasing resistance and productivity of plants. The treatment of the seeds for sowing and the leaf apparatus during the vegetation, with aqueous solutions of some compositions conditions the optimization of the functional state, growth and development of the plants of some agricultural crops, both in favorable humidity conditions and in a moderate water deficit. The coordinative compounds used have the property of activating vital processes already at the initial stages of individual plant development, stimulate the growth of the root system and the shoot.

In conditions of low humidity, some compositions with coordinating compounds have an influence of reducing the effect of drought on the formation of the assimilative apparatus, the accumulation of biomass and the harvest of plants. The use of physiologically active substances ensures a stabilization of the plant production process. Compositions based on coordinating compounds possess antioxidant properties that are manifested in increasing the antioxidant protection capacity of plants.



In chapter IV. PERSPECTIVES OF USING COORDINATION COMPOUNDS BASED ON DIOXIME LIGANDS IN INDUSTRIAL PROCESSES the fields of perspective for the use of new coordination compounds are presented. The compounds of some transition metals based on α-dioximes show different useful properties: catalysts of different industrial processes; compounds with dual function properties of catalysts and stabilizers in polyurethane production reactions; inhibitors of steel corrosion processes in the aquatic environment; materials with dielectric properties, etc.

The monograph is valuable in that it describes the achievements of the schools of coordinative chemistry, microbiology and plant physiology in the Republic of Moldova in the last three decades. The work is elaborated in an academic style, presenting scientific results obtained through the use of high-performance research equipment.

The monograph is recommended for students, master's students, doctoral students, but also for the general public of readers, who show interest in the subject addressed.

Received: June 21, 2022 *Accepted: October 28, 2022*

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