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on the occasion of his 70th birthday

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PROFESSOR ALEXANDRU ȘUBĂ ON HIS 70TH BIRTHDAY

Alexandru Șubă is University Professor, Doctor Habilitatus in Mathematical and Physical Sciences. He is a Moldovan mathematician and a remarkable leader of the Moldovan School of Differential Equations, who contributed a lot to the qualitative theory of differential equations and to the education of new generations of highly-qualified specialists. On December 2, 2023, Professor Alexandru Șubă celebrated his 70th anniversary.

He was born in the village of Dănceni from the district of Ialoveni, Republic of Moldova. In 1969 he finished the elementary school from the village of Dănceni; then, in 1971 he finished the secondary school from the town of Ialoveni and in 1976 he graduated from the Faculty of Physics and Mathematics of Moldova State University from Chișinău. At the same time, in 1976 he started his Candidate Degree (equivalent of PhD Degree) at the Institute of Mathematics and Computer Sciences of the Academy of Sciences of Moldova (specialty 01.01.02 – Differential Equations).

In 1982, Alexandru Șubă defended his Candidate Degree thesis in Mathematical and Physical Sciences at the State University of Sankt-Petersburg, Russia. He did it under the supervision of the well-known mathematician Academician Constantin Sibirschi. In 1999 he defended his Doctor Habilitatus Degree thesis (2nd PhD thesis) in Chișinău at

Moldova State University, Institute of Mathematics and Informatics of the Academy of Science of Moldova.

The professional activity of Professor Alexandru Șubă belongs to three institutions: Institute of Mathematics and Informatics of the Academy of Sciences of Moldova (1976–1990, 2010–present) which merged with State University of Moldova in 2022, State University of Moldova (1990 – 2010) and from 1997 to Tiraspol State University, located in Chișinău, which merged with "Ion Creangă" State Pedagogical University in 2022.

Within the Institute of Mathematics and Informatics of the Academy of Sciences of Moldova, the professional activity of Professor Alexandru Șubă evolved as follows: Collaborator of the Laboratory (1976–1981), Scientific Researcher (1981–1985), Senior Scientific Researcher (1985–1990), Deputy Director (2010–2015), head of the Laboratory of Differential Equations (2015–2019), Principal Scientific Researcher (2006–present), while at Moldova State University (Chair of Differential Equations) his career took place as follows: Associate Professor (1990–2000) and University Professor (2001–2010).

Since 1991, Professor Alexandru Șubă has been working fruitfully at Tiraspol State University / "Ion Creangă" State Pedagogical University and lectures to Bachelor, Master and PhD Degree students. In 2006 he won by competition the position of Professor at the Department of Mathematical Analysis and Algebra. The contribution of Professor Alexandru Șubă to the education of new generations of highly-qualified mathematicians is enormous. He supervised scientifically one Doctor Habilitatus thesis in mathematics, 6 doctoral theses in physical and mathematical sciences and about 40 Bachelor and Master Degree theses – all of them defended. In 2015 he was awarded Doctor Honoris Causa of Tiraspol State University.

The scientific activity of Professor Alexandru Șubă is related to dynamic systems: topological theory, integrability and special orbits. Within this research direction he studied the following problems: the development and systematization of the topological theory of dispersed and semi-dynamic systems; the Dulac integrability problem of dynamical systems; GL(2,R)-orbits problem; the problem of distinguishing between a center and the focus; the problem of classifying differential systems with invariant straight lines.

The main results concerning the topological theory of dynamical systems were published in the monograph [1]. There were elaborated the axioms of dynamic systems without uniqueness (dispersed systems) and systematized their topological theory. For planar semi-dynamical systems, the existence of singular points in the presence of non-wandering points is proved. The structure of minimal pseudo-invariant sets, periodic dot sheets and the semi-dynamical systems of characteristic 0 was studied.

Another research direction of Professor Alexandru Şubă concerns the problem of distinguishing between a center and a focus (the problem of the center) for polynomial differential systems

$$\dot{x} = P(x, y), \quad \dot{y} = Q(x, y) \tag{1}$$

having a singular point $O(0, 0)$ with pure imaginary eigenvalues (of a center or a focus type), where $P(x, y)$ and $Q(x, y)$ are real and coprime polynomials in the variables x and y of degree n, $n = \max\{\deg P, \deg Q\}$. The importance of this problem in the qualitative theory of differential equations arose as part of the still unsolved 16th Hilbert problem and it remains to be one of the most difficult problems to be solved of the list given by Hilbert [2] at the beginning of the past century.

For the first time in the papers of Professor Alexandru Şubă it was proposed a new approach to the problem of the center by simultaneously taking into account the invariant algebraic curves (exponential factors), the focus quantities and Darboux integrability [3], [4]. This result is an improvement of the classical Darboux integrability theorem and leads to the notion of a *center sequence* or, in a more general form, a *center pair*.

We say that a pair of two numbers $(M; N)$ is a *center pair* for (1) if the existence of M invariant algebraic curves (exponential factors) and the vanishing of the focus quantities $L_k,\ k = 1, \ldots, N$ imply the singular point $O(0, 0)$ to be a center for (1).

Till present, for polynomial differential systems with irreducible non-homogeneous invariant algebraic curves (exponential factors) the following center sequences are known [3], [4]:

$$\left(\frac{n(n+1)}{2}; 0\right), \left(\frac{n(n+1)}{2} - 2; 1\right), \ldots, \left(\frac{n(n+1)}{2} - \left[\frac{n+1}{2}\right]; \left[\frac{n-1}{2}\right]\right)$$

The problem of center sequences was solved completely for some classes of cubic differential systems ($n = 3$) with a given number of invariant straight lines ($l_j \equiv a_j x + b_j y + c_j = 0$), four invariant straight lines ($j = 1, 2, 3, 4$), three invariant straight lines ($j = 1, 2, 3$). In this way, in the period 1992-2005 it is shown that ($l_j, j = 1, 2, 3, 4; N = 2$) and ($l_j, j = 1, 2, 3; N = 7$) are center sequences [5], [6].

In the last years, this direction of investigation was highly appreciated by many mathematicians and the obtained results were cited in several papers, see for instance Christopher and Llibre [9], Chavarriga, Giacomini and Giné [7], Chavarriga and Grau [8], Cozma [10], Garcia and Giné [11], Giné [12], Romanovski and Shafer [13].

Concerning the Dulac integrability problem of dynamical systems, the problem of the existence of a center in the sense of Dulac was completely solved by Professor Şubă for plane cubic differential systems having a singular point with one zero eigenvalue [14].

The investigation of the orbits of a differential system belongs to the theory elaborated by Professor Mihail Popa, Doctor Honoris Causa of Tiraspol State University (2013) and refers to the interaction of Lie algebras, systems of differential equations and their algebraic invariants [15]. Professor Alexandru Șubă proved that the $GL(2, R)$–orbit dimension of any polynomial differential system is different from one. He proposed a classification of polynomial differential systems with respect to the dimensions of $GL(2, R)$–orbits [16], [17].

At present, the activity of Professor Alexandru Șubă is focused on the study of polynomial differential systems with multiple invariant straight lines (see, for example, [18], [19]). In this direction, in addition to some important results, he also formulated some problems to be solved in the future. Here we bring only a few of them.

Denote by $M(n)$ (respectively, $M_\infty(n)$) the maximal multiplicity of affine invariant straight lines (respectively, the line at infinity) in the class of polynomial differential systems of degree n. For affine invariant straight lines we have $M(2) = 4$, $M(3) = 7$, $M(4) = 10$ and the evaluation [20]:

$$3n - 2 \leq M(n) \leq 3n - 1, \;\; n \geq 2$$

Problem 1 (Conjecture 1). $M(n) = 3n - 2$?

Problem 2. *Is it linear the equation of trajectories of each polynomial differential system (1) which has an affine invariant straight line of the maximal multiplicity $M(n)$?*

Problem 3. *Is it true the following equality $M_\infty(n) = 3n-2$? Are linear and has only one affine invariant straight line of multiplicity one the systems for which the line at infinity $\mathcal{L}_\infty$ is of maximal multiplicity ($M_\infty(n) \geq 3n - 2,\; n > 3$) ?*

Problem 4. *Is $M_\infty(n) = 2n + 1$ the maximal multiplicity of the line at infinity in the class of polynomial differential systems of degree n without affine invariant straight lines ?*

Professor Alexandru Șubă is the author of over 160 scientific publications, published in prestigious journals from the USA, Italy, Spain, Ukraine, Belarus, China, Romania, among them being one monograph, four text books for Bachelor and Master Degree students. He contributed to the organization and development of research in the field of differential equations, founding, by training highly qualified personnel, a scientific school related to the theory of integrability of systems of differential equations.

He worked within two international grants (Canada-France-Moldova, 1999–2001; USA-Moldova, 2001–2003), one European project FP7-PEOPLE-2012-IRSES (2012–2016) and two national projects (2011–2014, 2015–2019). Due to his prestige in the field of mathematics, he became member of the editorial boards of four accredited journals and co-president of the Seminar on Differential Equations and Algebras at Tiraspol State University/“Ion Creangă” State Pedagogical University. The Seminar works on regular basis since 2002, and it is designed for Bachelor, Master and PhD Degree students and scientific researches.

The special appreciation of his scientific work brings him several prizes, titles and medals, namely: prize “Academician Constantin Sibirschi” (2013); Medal “Nicolae Milescu Spătaru” (2019); Medal “Dimitrie Cantemir” (2023); Medal “Ion Creangă” (2023) given by „Ion Creangă” State Pedagogical University. Professor Alexandru Șubă is also Honorary citizen of the village of Dănceni from Ialoveni District (2023).

The present volume is dedicated to Professor Alexandru Șubă at the age of 70, very active in the academic community, full of vigor and optimism, who brought a significant contribution to the development of mathematics in the Republic of Moldova.

On the occasion of his 70th birthday we congratulate Professor Alexandru Șubă on his achievements and we wish him many returns of the day, good health, all the blessings of life, new scientific accomplishments and fruitful didactic activities.

Happy Anniversary, dear Professor Alexandru Șubă!

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Dedicated to Professor Alexandru Şubă on the occasion of his 70[th] birthday

Averaging in multifrequency systems with multi-point conditions and a delay

Yaroslav Bihun and Ihor Skutar

Abstract. For multifrequency system of differential equations with a discrete and integral delay we find conditions for the existence and uniqueness of the solution. Linear multipoint conditions are set for the solution. An estimate of the error of the averaging method is obtained, which clearly depends on the small parameter.

2010 Mathematics Subject Classification: 34K10, 34K33.

Keywords: multifrequency system, averaging method, resonance, integral delay, linearly transformed argument.

Medierea în sistemele multifrecvenţă cu condiţii multi-punct şi întârziere

Rezumat. Pentru sistemul multifrecvenţă de ecuaţii diferenţiale cu o întârziere integrală şi discretă găsim condiţii pentru existenţa şi unicitatea soluţiei. Condiţiile liniare multipunct sunt stabilite pentru soluţie. Se obţine o estimare a erorii metodei de mediere, care depinde în mod clar de parametrul mic.

Cuvinte-cheie: sistem multifrecvenţă, metoda medierii, rezonanţă, întârziere integrală, argument transformat liniar.

1. Introduction

In many cases, mathematical models of oscillating systems are described with differential equations of the form

$$\frac{da}{d\tau} = X(\tau, a, \varphi), \quad \frac{d\varphi}{d\tau} = \frac{\omega(\tau, a)}{\varepsilon} + Y(\tau, a, \varphi), \tag{1}$$

where $0 \le \varepsilon t = \tau$ – slow time, ε – positive small parameter, $a \in \mathbb{D} \subset \mathbb{R}^n$, $\varphi \in \mathbb{R}^m$. The system (1) is rigid, its research and construction of both analytical and numerical solutions is a complex and not always solvable task. Therefore, to simplify the system (1), the averaging procedure for fast variables $\varphi_1, \ldots, \varphi_m$ is used, which greatly simplifies it, reducing it to the form

$$\frac{d\overline{a}}{d\tau} = X_0(\tau, \overline{a}), \quad \frac{d\overline{\varphi}}{d\tau} = \frac{\omega(\tau, \overline{a})}{\varepsilon} + Y_0(\tau, \overline{a}). \tag{2}$$

In the general case, the deviation of solutions $\|a(\tau,\varepsilon) - \overline{a}(\tau)\|$ can become $O(1)$ on a finite segment $[0, L]$ or $\mathbb{R}_+ = (0, \infty)$ due to frequency resonance, the condition of which is

$$(k, \omega(\tau, a)) := k_1\omega_1(\tau, a) + \cdots + k_m\omega_m(\tau, a) \simeq 0, \quad k \neq 0. \tag{3}$$

Therefore, in order to justify the averaging method, additional conditions are imposed on the frequency vector $\omega(\tau, a)$ for the system to exit from a small circumference of resonance. The works of V. I. Arnold [1], E. O. Grebenikov [2], A. M. Samoilenko and R. I. Petryshyn [3] and many others are devoted to this issue.

The monograph [3] presents a new method of studying multifrequency systems (1), which is based on estimates of the corresponding oscillatory integrals, which made it possible to justify a wide class of multifrequency systems with initial and boundary conditions.

For adequate modeling of oscillating systems, it is also important to take into account informational, technological and other delays. Multifrequency systems with constant and variable delay were studied in the works [4, 5, 6]. In particular, systems in which the delay is specified with a linearly transformed argument of the form $\lambda\tau$, $\tau > 0$, $0 < \lambda \leq 1$ in [6, 7, 8]. A new resonance condition was obtained, including for systems with linearly transformed arguments and a frequency vector $\omega(\tau)$ in fast variables $\varphi(\theta_\nu\tau)$ of the form

$$\gamma_k(\tau) := \sum_{\nu=1}^{q} \theta_\nu\big(k_\nu, \omega(\theta_\nu\tau)\big) = 0. \tag{4}$$

The works [6, 7, 8, 9] are devoted to the substantiation of the averaging method for such systems with initial multipoint and integral conditions.

This article considers systems with both point and integral delay, which allows taking into account the background history of the process at some interval. Parabolic equations with such a delay were studied in [9] for functional differential equations in the monograph [11] and others.

2. Formulation of the problem

We investigate a system of differential equations with variable delay of the form

$$\frac{da(\tau)}{d\tau} = X\Big(\tau, a(\tau), a_\lambda(\tau), \int_{\Delta\tau}^{\tau} g(s)a(s)ds, \varphi_\Theta(\tau)\Big), \tag{5}$$

$$\frac{d\varphi(\tau)}{d\tau} = \frac{\omega(\tau)}{\varepsilon} + \varepsilon^\beta Y\Big(\tau, a(\tau), a_\lambda(\tau), \int_{\Delta\tau}^{\tau} g(s)a(s)ds, \varphi_\Theta(\tau)\Big), \tag{6}$$

where $\tau \in [0, L]$, $\varepsilon \in (0, \varepsilon_0]$, $a \in \mathbb{D} \subset \mathbb{R}^n$, $\varphi \in \mathbb{R}^m$; $0 < \lambda < 1$, $a_\lambda(\tau) = a(\lambda\tau)$; $\varphi_\Theta = \big(\varphi(\theta_1\tau), \ldots, \varphi(\theta_q\tau)\big)$, $0 < \theta_1 < \cdots < \theta_q \le 1$, $0 < \Delta < 1$, $\beta > 0$. Vector-functions X and Y are 2π-periodic by components of variables φ_{θ_ν}, $\nu = \overline{1, q}$.

For the solution of the system (5), (6), multipoint conditions are set

$$\sum_{\nu=1}^{r} A_\nu(\varepsilon) a|_{\tau=\tau_\nu} = d_1, \tag{7}$$

$$\sum_{\nu=1}^{r} B_\nu(\varepsilon) \varphi|_{\tau=\tau_\nu} = d_2, \tag{8}$$

where $0 \le \tau_1 < \tau_2 < \cdots < \tau_r \le L$, $A_\nu(\varepsilon)$ and $B_\nu(\varepsilon)$ are given matrices of order n and m, respectively, defined at $\varepsilon \in [0, \varepsilon_0]$ and vectors $d_1 \in \mathbb{R}^n$, $d_2 \in \mathbb{R}^m$.

The corresponding system (5), (6) averaged over fast variables on the mq-cube of periods takes the form

$$\frac{d\overline{a}(\tau)}{d\tau} = X_0\Big(\tau, \overline{a}(\tau), \overline{a}_\lambda(\tau), \int_{\Delta\tau}^{\tau} g(s)\overline{a}(s)ds\Big), \tag{9}$$

$$\frac{d\overline{\varphi}(\tau)}{d\tau} = \frac{\omega(\tau)}{\varepsilon} + \varepsilon^\beta Y_0\Big(\tau, \overline{a}(\tau), \overline{a}_\lambda(\tau), \int_{\Delta\tau}^{\tau} g(s)\overline{a}(s)ds\Big) \tag{10}$$

with multipoint conditions

$$\sum_{\nu=1}^{r} A_\nu(\varepsilon) \overline{a}|_{\tau=\tau_\nu} = d_1, \tag{11}$$

$$\sum_{\nu=1}^{r} B_\nu(\varepsilon) \overline{\varphi}|_{\tau=\tau_\nu} = d_2, \tag{12}$$

Now the problem (9), (11) can be solved separately and we can find the solution $\overline{a} = \overline{a}(\tau; \overline{y}, \varepsilon)$, $\overline{a}(0; \overline{y}, \varepsilon) = \overline{y}$. Solving the multipoint problem (11) is reduced to integration if the initial value for the solution component is known $\overline{\varphi} = \overline{\varphi}(\tau; \overline{y}, \overline{\psi}, \varepsilon)$, $\overline{\varphi}(0; \overline{y}, \overline{\psi}, \varepsilon) = \overline{\psi}$.

Suppose that the condition is satisfied:

Condition A. There is a unique solution of the averaged problem (9)–(12), whose component is $\overline{a}(\tau; \overline{y}, \varepsilon)$, $\overline{y} \in \mathbb{D}_1 \subset \mathbb{D}$, at $\tau \in [0, L]$ and $\varepsilon \in (0, \varepsilon_0]$ lies in the area $\mathbb{D}$ with some ρ-circumference.

In the work, sufficient conditions are established, under which there is a unique differentiable solution of the problem (5)–(8). The method of averaging is stipulated and the estimate of the deviation error of the solutions is constructed, which clearly depends on the small parameter ε and has the form

$$u(\tau;\varepsilon) := \|a(\tau;\overline{y}+\mu,\overline{\psi}+\xi,\varepsilon)-\overline{a}(\tau;\overline{y},\varepsilon)\|+ \\ +\|\varphi(\tau;\overline{y}+\mu,\overline{\psi}+\xi,\varepsilon)-\overline{\varphi}(\tau;\overline{y},\overline{\psi},\varepsilon)\| \le c_1\varepsilon^{\alpha}. \tag{13}$$

Here $\alpha = 1/(mq)$, $c_1 > 0$ and does not depend on ε, $a(0;\overline{y}+\mu,\overline{\psi}+\xi,\varepsilon) = \overline{y}+\mu(\varepsilon)$, $\varphi(0;\overline{y}+\mu,\overline{\psi}+\xi,\varepsilon) = \overline{\psi}+\xi(\varepsilon)$.

3. Auxiliary Statements

Lemma 3.1. *Let the matrix $B(\varepsilon) := \sum\limits_{\nu=1}^{r} B_\nu(\varepsilon)$ be nondegenerate for $\varepsilon \in [0,\varepsilon_0]$. Then there is a unique solution to the problem* (10), (12).

Proof. From the equation (10) we have

$$\varphi(\tau_\nu;\overline{y},0,\varepsilon) = \int\limits_0^{\tau_\nu}\Big(\frac{\omega(s)}{\varepsilon}+Y_0(s,\overline{a},\overline{a}_\lambda,\overline{v}_\Delta)\Big)ds,$$

where

$$\overline{v}_\Delta(\tau,\varepsilon) = \int\limits_{\Delta\tau}^{\tau} g(s)\overline{a}(s;\overline{y},\varepsilon)ds.$$

It follows from the condition (12) that

$$B(\varepsilon)\overline{\psi} = d_2 - \sum_{\nu=1}^{r}\varphi(\tau_\nu;\overline{y},0,\varepsilon),$$

wherefrom we find the initial value of $\overline{\psi}(\overline{y},\varepsilon)$. The solution to the problem (10), (12) takes the form

$$\varphi(\tau;\overline{y},\overline{\psi},\varepsilon) = \overline{\psi}(\overline{y},\varepsilon)+\varphi(\tau;\overline{y},0,\varepsilon).$$

□

Lemma 3.2. *Let*

1) number $d \ge 0$, $\lambda,\Delta \in (0,1)$;

2) f_1, f_2 and g – continuous functions on [0,L] with value in $\mathbb{R}_+ = [0,\infty)$;

$$0 \le u(\tau) \le d + \int\limits_0^{\tau} f_1(s)u(s)ds + \int\limits_0^{\lambda\tau} f_2(s)u(s)ds + \int\limits_0^{\tau}\Big(\int\limits_{\Delta s}^{s} g(z)u(z)dz\Big)ds. \tag{14}$$

Then

$$u(\tau) \le d\cdot exp\Big(\int\limits_0^{\tau}\big(f_1(s)+\lambda f_2(s)\big)ds + \int\limits_0^{\tau}\big(\int\limits_{\Delta s}^{s} g(z)dz\big)ds\Big), \quad 0 \le \tau \le L. \tag{15}$$

Proof. We denote by $w(\tau)$ the right-hand side of the inequality (14). Then $w(0) = d$, $u(\tau) \leq w(\tau)$ and $w^{'}(\tau) \geq 0$ for $\tau \in [0, L]$.

Then we have

$$v^{'}(\tau) = f_1(\tau)u(\tau) + \lambda f_2(\lambda\tau)u(\lambda\tau) + \int\limits_{\lambda\tau}^{\tau} g(s)u(s)ds \leq$$

$$\leq f_1(\tau)v(\tau) + \lambda f_2(\lambda\tau)v(\lambda\tau) + \int\limits_{\lambda\tau}^{\tau} g(s)v(s)ds \leq$$

$$\leq \Big(f_1(\tau) + \lambda f_2(\lambda\tau) + \int\limits_{\lambda\tau}^{\tau} g(s)ds\Big)v(\tau).$$

After integrating the inequality, we obtain the solution (15) of the integral inequality (14) $\square$

The article [10] substantiates the averaging method for a system of equations of a more general form than (5), (6) with initial conditions. The following condition is the condition for exiting the (4) resonance small circumference.

Condition B. Let $\omega \in \mathbb{C}^{mq}[0, L]$ and be constructed according to the mq system of functions $\{\omega(\theta_1\tau), \ldots, \omega(\theta_q\tau)\}$ Wronskian

$$W(\varphi_\Theta) \neq 0, \quad \tau \in [0, L].$$

Theorem 3.1. *Suppose that:*

1) *vector function $F(\tau, a, a_\lambda, w_\Delta, \varphi_\Theta) := (X, Y)$ is twice continuously differentiable over all arguments in the area $G = G_1 \times \mathbb{R}^m q$, $G_1 = [0, L] \times \mathbb{D} \times \mathbb{D} \times \mathbb{D}_\nu$, 2π-periodic in the components of the vectors $\varphi_\nu, \nu = \overline{1, q}$ and bounded together with the derivatives by the constant σ_1;*
2) *conditions A and B are satisfied;*
3) *for the Fourier coefficients F_k in the area G_1 the evaluation is performed:*

$$\sum_{k\neq 0}\Big(\sup_{G_1}\|F_k\| + \frac{1}{\|k\|_\Theta}\big(\sup_{G_1}\Big\|\frac{\partial F_k}{\partial \tau}\Big\| + \sup_{G_1}\Big\|\frac{\partial F_k}{\partial a}\Big\| + \sup_{G_1}\Big\|\frac{\partial F_k}{\partial a_\lambda}\Big\|$$

$$+(1-\Delta_\nu)\sup_{G_1}\Big\|\frac{\partial F_k}{\partial v_\lambda}\frac{\partial v_\lambda}{\partial \tau}\Big\|\big)\Big) \leq \sigma_2$$

where $\|k\|_\Theta = \sum\limits_{\nu=1}^{q} \theta_\nu \|k_\nu\|$.

Then for sufficiently small $\varepsilon_1 \in (0, \varepsilon_0]$ there exists a unique solution $\big(a(\tau; \overline{y}, \overline{\psi}, \varepsilon)$, $\varphi(\tau; \overline{y}, \overline{\psi}, \varepsilon)\big)$ with initial conditions $(\overline{y}, \overline{\psi})$ and the evaluation is performed

$$u(\tau; \varepsilon) := \|a - \overline{a}\| + \|v - \overline{v}\| \leq c_2 \varepsilon^{\alpha}, \tag{16}$$

for all $(\tau, \varepsilon) \in [0, L] \times (0, \varepsilon_1]$, $\alpha = (mq)^{-1}$, $c_2 > 0$ and does not depend on ε.

Remark 3.1. *If the vector functions X and Y are continuously differentiable mq once over the variable τ and $mq + 1$ the other time over the other variables, then condition 3) of Theorem 3.1 is satisfied and the estimate of the form* (16) *is correct for the derivatives of the deviation of the solutions for the initial variables y and ψ with the constant c_2.*

4. Justification of the Averaging Method

Let

$$A(\varepsilon) = \sum_{\nu=1}^{r} A_\nu(\varepsilon) \frac{\partial \overline{a}(\tau_\nu; \overline{y}, \varepsilon)}{\partial y}.$$

Theorem 4.1. *Suppose that:*

1) *condition 1) of Theorem 3.1 and conditions A and B are satisfied;*
2) *matrices $A_\nu(\varepsilon)$, $B_\nu(\varepsilon)$, $\nu = \overline{1, r}$ are continuous at $\varepsilon \in (0, \varepsilon_0]$, $A(\varepsilon)$, $B(\varepsilon)$ are non-degenerate and $\|A^{-1}(\varepsilon)\| \leq \sigma_2$, $\|B^{-1}(\varepsilon)\| \leq \sigma_3$;*
3) *$g \in \mathbb{C}[0, L]$.*

Then there exists such $\varepsilon^* \in (0, \varepsilon_0]$ that for each $\varepsilon \in [0, \varepsilon^*]$ there is a unique solution to the problem (5)–(8) *in the class $\mathbb{C}^1[0, L]$ and for all $(\tau, \varepsilon) \in [0, L] \times (0, \varepsilon^*]$ evaluation is performed* (13).

Besides

$$\|\mu\| \leq c_3 \varepsilon^{\alpha}, \quad \|\xi\| \leq c_4 \varepsilon^{\alpha}, \quad \alpha = 1/(mq). \tag{17}$$

Proof. Let

$$2 c_1 \varepsilon^{\alpha} \leq \rho, \quad \|\mu\| \leq c_3 \varepsilon^{\alpha} \leq \rho/2, \tag{18}$$

where the constant $c_3 > 0$ and will be determined further. Then based on the estimate (16) for all $\psi \in \mathbb{R}^m$, $(\tau, \varepsilon) \in (0, \varepsilon_2]$, $\varepsilon_2 = min\Big(\varepsilon_1, \big(\frac{\rho}{2c_2}\big)^{mp}, \big(\frac{\rho}{2c_3}\big)^{mp}\Big)$

$$\|a(\tau; \overline{y} + \mu, \psi) - \overline{a}(\tau; \overline{y} + \mu, \varepsilon)\| \leq c_2 \varepsilon^{\alpha}. \tag{19}$$

From equation (9) we have

$$v(\tau, \mu, \varepsilon) := \|\overline{a}(\tau; \overline{y} + \mu, \varepsilon) - \overline{a}(\tau; \overline{y}, \varepsilon)\| \leq$$

$$\leq \|\mu\| + \sigma_1 \int_0^{\tau} v(s, \mu, \varepsilon) ds + \sigma_1 \int_0^{\tau} v(\lambda s, \mu, \varepsilon) ds + \sigma_1 \int_0^{\tau} \int_{\Delta s}^{s} |g(z)| v(z, \mu, \varepsilon) dz ds.$$

Applying the estimate (15) gives

$$v(\tau,\mu,\varepsilon) \le \|\mu\| \exp\Big(2\sigma_1 + \int\limits_0^\tau \int\limits_{\Delta s}^{s} |g(z)|dzds\Big)\tau.$$

So for $\tau \in [0, L]$

$$v(\tau,\mu,\varepsilon) \le c_5\varepsilon^\alpha,$$

where $c_5 = c_3 \exp\Big(2\sigma_1 + \int\limits_0^L \int\limits_{\Delta s}^{s} |g(z)|dzds\Big)L.$

The solution $a(\tau;\overline{y}+\mu,\psi,\varepsilon)$ under the conditions (18) lies in the ρ circumference of the solution $\overline{a}(\tau;\overline{y},\varepsilon)$ and the evaluation is performed

$$\overline{w}(\tau;\mu,\psi,\varepsilon) := \|a(\tau;\overline{y}+\mu,\psi,\varepsilon) - \overline{a}(\tau;\overline{y},\varepsilon)\| \le$$

$$\le \|a(\tau;y+\mu,\psi,\varepsilon) - \overline{a}(\tau;\overline{y}+\mu,\varepsilon)\| + v(\tau,\mu,\varepsilon) \le c_6\varepsilon^\alpha,$$

where $c_6 = c_2 + c_5$.

We will show that there is μ that satisfies the condition (18) such that the solution a of the equation (5) satisfies the condition (7).

From the conditions (7) and (11) we have

$$\sum_{\nu=1}^{r} A_\nu(\varepsilon)\Big(\overline{a}(\tau_\nu;\overline{y}+\mu,\varepsilon) - \overline{a}(\tau_\nu;\overline{y},\varepsilon)\Big) =$$
$$-\sum_{\nu=1}^{r} A_\nu(\varepsilon)\Big(\big(a(\tau_\nu;\overline{y}+\mu,\overline{\psi}+\xi,\varepsilon) - \overline{a}(\tau_\nu;\overline{y}+\mu,\varepsilon)\big) + R_{1,\nu}(\mu,\varepsilon)\Big), \tag{20}$$

where

$$R_{1,\nu}(\mu,\varepsilon) = \overline{a}(\tau_\nu;\overline{y}+\mu,\varepsilon) - \overline{a}(\tau_\nu;\overline{y},\varepsilon) - \frac{\partial\overline{a}(\tau_\nu;y,\varepsilon)}{\partial\overline{y}}\mu.$$

From (19) we obtain the equation for μ:

$$\mu = \Phi_1(\mu,\xi,\varepsilon), \tag{21}$$

where $\Phi_1(\mu,\xi,\varepsilon) =$

$$= -A^{-1}(\varepsilon)\Big(\sum_{\nu=1}^{r} \Big(A_\nu(\varepsilon)\big(a(\tau_\nu;\overline{y}+\mu,\overline{\psi}+\xi,\varepsilon) - \overline{a}(\tau_\nu;\overline{y}+\mu,\varepsilon)\big) + R_{1,\nu}(\mu,\varepsilon)\Big)\Big)$$

It follows from the differentiability of the solution $\overline{a}(\tau;\mu,\varepsilon)$ over the variable $\overline{y}$ that

$$\|R_{1,\nu}(\mu,\varepsilon)\| \le c_{7,\nu}\|\mu\|^2, \quad \Big\|\frac{\partial R_{1,\nu}}{\partial\mu}\Big\| \le c_{8,\nu}\|\mu\|. \tag{22}$$

Considering the estimates (16) and (19), when $(\tau,\varepsilon) \in [0, L] \times (0, \varepsilon_0]$ we obtain

$$\big\|\Phi_1(\mu,\xi,\varepsilon)\big\| \le \sigma_2 \sum_{\nu=1}^{r} \max_{[0,\varepsilon_0]} \big\|A_\nu(\varepsilon)\big\|\Big(c_2\varepsilon^\alpha + c_{7,\nu}\|\mu\|^2\Big) = c_9\varepsilon^\alpha + c_{10}\|\mu\|^2.$$

Let in condition (18) be

$$c_3 = 2c_9, \quad c_3^2 c_{10} \varepsilon_3^\alpha \le c_9.$$

Then

$$\left\| \Phi_1(\mu, \xi, \varepsilon) \right\| \le 2c_9 \varepsilon^\alpha$$

for $\mu \le 2c_9\varepsilon^\alpha$, $\xi \in \mathbb{R}^m$ and $\varepsilon \in (0, \varepsilon_3]$.

So, $\Phi_1 : S_1 \to S_1$, $S_1 = \{\mu : \|\mu\| \le c_3\varepsilon^\alpha\}$.

Then we have

$$\frac{\partial \Phi_1}{\partial \mu} = -A^{-1}(\varepsilon) \sum_{\nu=1}^{r} A_\nu(\varepsilon) \frac{\partial}{\partial \mu} \big(a(\tau; \overline{y} + \mu, \overline{\psi} + \xi, \varepsilon) - \overline{a}(\tau; \overline{y} + \mu, \varepsilon) \big) -$$
$$-A^{-1}(\varepsilon) \sum_{\nu=1}^{r} A_\nu(\varepsilon) \frac{\partial R_{1,\nu}(\mu, \varepsilon)}{\partial \mu}.$$

From Theorem 3.1, condition 2) of Theorem 4.1 and estimates (22), we obtain

$$\left\| \frac{\partial \Phi_1}{\partial \mu} \right\| \le \sigma_2 c_2 \sum_{\nu=1}^{r} \max_{[0, \varepsilon_0]} \left\| A_\nu(\varepsilon) \right\| \varepsilon^\alpha + \sigma_2 c_3 \sum_{\nu=1}^{r} \max_{[0, \varepsilon_0]} \left\| A_\nu(\varepsilon) \right\| = c_{11} \varepsilon^\alpha < \frac{1}{4}, \tag{23}$$

if $\varepsilon \le \varepsilon_4 = (4c_{11})^{mq}$.

Similarly, we have

$$\left\| \frac{\partial \Phi_1}{\partial \psi} \right\| = \left\| A^{-1}(\varepsilon) \sum_{\nu=1}^{r} A_\nu(\varepsilon) \frac{\partial}{\partial \psi} \big(a(\tau; \overline{y} + \mu, \overline{\psi} + \xi, \varepsilon) - \overline{a}(\tau; \overline{y} + \mu, \varepsilon) \big) \right\| \le$$
$$\le \sigma_2 c_2 \sum_{\nu=1}^{r} \max_{[0, \varepsilon_0]} \left\| A_\nu(\varepsilon) \right\| \varepsilon^\alpha = c_{12} \varepsilon^\alpha < \frac{1}{4},$$

if $\varepsilon \le \varepsilon_5 = (4c_{12})^{mq}$.

Now from the conditions (8) and (12) we find

$$\xi = \Phi_2(\mu, \xi, \varepsilon),$$

where

$$\Phi_2(\mu, \xi, \varepsilon) = -B^{-1}(\varepsilon) \sum_{\nu=1}^{r} B_\nu(\varepsilon) \Big(\big(\varphi(\tau_\nu; \overline{y} + \mu, \overline{\psi} + \xi, \varepsilon) - \overline{\varphi}(\tau_\nu; \overline{y} + \mu, \varepsilon) \big) +$$
$$+ \big(\overline{\varphi}(\tau_\nu; \overline{y} + \mu, \varepsilon) - \overline{\varphi}(\tau_\nu; \overline{y}, \varepsilon) \big) \Big).$$

Based on the estimates (15) and (19) and condition 2) of Theorem 4.1, we obtain

$$\left\|\varPhi_2(\mu,\xi,\varepsilon)\right\| \leq \Big(\sigma_3 c_2 \sum_{\nu=1}^{r} \max_{[0,\varepsilon_0]} \left\|B_\nu(\varepsilon)\right\| + \sigma_3 c_5 \sum_{\nu=1}^{r} \max_{[0,\varepsilon_0]} \left\|B_\nu(\varepsilon)\right\|\Big)\varepsilon^\alpha \leq$$

$$\leq \sigma_3(c_2+c_5)\Big(\sum_{\nu=1}^{r} \max_{[0,\varepsilon_0]} \left\|B_\nu(\varepsilon)\right\|\Big)\varepsilon^\alpha = c_{13}\varepsilon^\alpha.$$

So $\varPhi_2 : S_2 \to S_2$, $S_2 = \{\varphi : \|\varphi - \overline{\varphi}\| \leq c_{13}\varepsilon^\alpha\}$, if

$$\|\xi\| \leq c_{13}\varepsilon^\alpha. \tag{24}$$

Then we have

$$\frac{\partial \varPhi_2}{\partial \mu} = -B^{-1}(\varepsilon)\sum_{\nu=1}^{r} B_\nu(\varepsilon)\Big(\frac{\partial}{\partial \mu}\big(\varphi(\tau_\nu;\overline{y}+\mu,\overline{\psi}+\xi,\varepsilon) - \overline{\varphi}(\tau_\nu;\overline{y}+\mu,\overline{\psi}+\xi,\varepsilon)\big)+$$

$$+\frac{\partial}{\partial \mu}\overline{\varphi}(\tau_\nu;\overline{y}+\mu,\overline{\psi}+\xi,\varepsilon)\Big),$$

$$\left\|\frac{\partial \varPhi_2}{\partial \mu}\right\| \leq \sigma_3 \sum_{\nu=1}^{r} \max_{[0,\varepsilon_0]} \left\|B_\nu(\varepsilon)\right\|\big(c_2\varepsilon^\alpha + c_{13}\varepsilon^\beta\big) \leq c_{14}\varepsilon^\gamma < \frac{1}{4},$$

if $\varepsilon \leq \varepsilon_6 = (4c_{14})^{-\gamma}$, $\gamma = \min(\alpha,\beta)$.

Let $\varPhi = col(\varPhi_1,\varPhi_2)$, $\eta = col(\mu,\psi)$. Then

$$\left\|\frac{\partial \varPhi}{\partial \eta}\right\| < 1,$$

from which, according to the fixed point theorem [13], it follows that there is a single fixed point (μ^*,ψ^*) if $\varepsilon < \varepsilon^* = \min_{\nu=1,6}\varepsilon_\nu$. Therefore, there exists a unique solution $\big(a(\tau;\overline{y}+\mu^*,\overline{\psi}+\xi^*,\varepsilon),\varphi(\tau;\overline{y}+\mu^*,\overline{\psi}+\xi^*,\varepsilon)\big)$ of the system (5), (6), which satisfies the conditions (7), (8).

From the equation (9), estimates (16), (24) we obtain

$$\overline{\overline{w}}(\tau;\mu,\xi,\varepsilon) = \|\overline{\varphi}(\tau;\overline{y}+\mu,\overline{\psi}+\xi,\varepsilon) - \overline{\varphi}(\tau;\overline{y},\overline{\psi},\varepsilon)\| \leq$$

$$\leq \|\xi\| + \sigma_1\int_0^\tau \Big(w(s,\mu,\xi,\varepsilon) + w(\lambda s,\mu,\xi,\varepsilon) + \int_{\Delta s}^{s} |g(z)|w(z,\mu,\xi,\varepsilon)dz\Big)ds.$$

So,

$$\overline{\overline{w}}(\tau;\mu,\xi,\varepsilon) = \Big(c_{12} + \sigma_1 c_6\int_0^L \big(2+(1-\Delta)|g(s)|\big)ds\Big)\varepsilon^\alpha = c_{14}\varepsilon^\alpha. \tag{25}$$

Based on evaluations (18) we get

$$u(\tau,\varepsilon) \le \overline{w}(\tau;\mu,\xi,\varepsilon) + \overline{\overline{w}}(\tau;\mu,\xi,\varepsilon)+$$
$$+\|a(\tau;\overline{y}+\mu,\overline{\psi}+\xi,\varepsilon) - a(\tau;\overline{y},\overline{\psi},\varepsilon)\| + \|\varphi(\tau;\overline{y}+\mu,\overline{\psi}+\xi,\varepsilon) - \overline{\varphi}(\tau;\overline{y},\overline{\psi},\varepsilon)\| \le$$
$$\le (c_5+c_{15})\varepsilon^\alpha + c_2\varepsilon^\alpha = c_1\varepsilon^\alpha, \quad (\tau,\varepsilon) \in [0,L]\times(0,\varepsilon^*),$$

where $c_1 = c_2 + c_5 + c_{15}$. □

Remark 4.1. *If $\beta = 0$ and no other conditions are imposed on the system* (5), (6) *or the conditions* (7), (8)*, then it is possible to prove only the existence of a solution based on the Brouwer's theorem* [13].

5. Model Example

Consider a single-frequency system

$$\frac{da(\tau)}{d\tau} = a(\lambda\tau) + \int\limits_{\lambda\tau}^{\tau} a(s)ds + \cos(k\varphi(\tau) + l\varphi(\theta\tau)),$$
$$\frac{d\varphi(\tau)}{d\tau} = \frac{1+2\tau}{\varepsilon}, \quad 0 \le \tau \le 1,$$

where $0 < \lambda < 1$, $0 < \theta < 1$; $k,l \in \mathbb{Z}\setminus\{0\}$, $k + l\theta = 0$.

If $\varphi(0) = 0$, then $\varphi(\tau) = \tau(1+\tau)/\varepsilon$, $k\varphi(\tau) + l\varphi(\theta\tau) = \kappa\tau^2/\varepsilon$, $\kappa = k + l\theta^2 \ne 0$.

At the point $\tau = 0$, the resonance condition is satisfied, since $\gamma_{kl} = 2\tau\kappa$.

Let us set the boundary condition

$$\alpha_0 a|_{\tau=0} + \alpha_1 a|_{\tau=1} = d, \quad |\alpha_0| + |\alpha_1| \ne 0. \tag{26}$$

The averaged equation for the slow variable

$$\frac{d\overline{a}(\tau)}{d\tau} = \overline{a}(\lambda\tau) + \int\limits_{\lambda\tau}^{\tau} \overline{a}(s)ds \tag{27}$$

with a boundary condition of the form (26) has a solution

$$\overline{a}(\tau;\overline{y}) = \overline{y}e^\tau, \quad \overline{y} = d/(\alpha_0 + \alpha_1 e).$$

Let $v(\tau;\mu,\varepsilon) = a(\tau;\overline{y}+\mu,\varepsilon) - \overline{a}(\tau;\overline{y}+\mu)$. Then

$$v(\tau;\mu,\varepsilon) = \int\limits_0^\tau v(\lambda\tau;\mu,\varepsilon)ds + \int\limits_0^\tau\int\limits_{\lambda s}^{s} v(z;\mu,\varepsilon)dzds + \int\limits_0^\tau \cos\frac{\kappa s^2}{\varepsilon}ds.$$

Applying the estimate of the Fresnel integral [12] we obtain

$$\int_0^\tau \cos\frac{\kappa s^2}{\varepsilon}ds = \frac{\sqrt{\varepsilon}}{\sqrt{\kappa}}\int_0^{\sqrt{\pi}\tau/\sqrt{\varepsilon}} \cos x^2 dx = \frac{\sqrt{\pi}}{2\sqrt{2\kappa}}\sqrt{\varepsilon} + O(\sqrt[4]{\varepsilon^3}) \le c_{16}\sqrt{\varepsilon},$$

where $c_{16} = \sqrt{\pi}/\sqrt{2\kappa}$, $\varepsilon \le 4\kappa/\pi^2$.

From the estimate for $v(\tau;\mu,\varepsilon)$ and $\tau \in [0,1]$ it follows

$$|v(\tau;\mu,\varepsilon)| \le \sqrt{\varepsilon}c_{16}\exp\big(1 + (1-\lambda)\tau/2\big)\tau \le c_{17}\sqrt{\varepsilon},$$

where $c_{17} = c_{16}\exp(3-\lambda)/2$.

From the boundary conditions for the solutions $a(\tau;\overline{y}+\mu,\varepsilon)$ and $\overline{a}(\tau;\overline{y})$ we find

$$\mu = -\big(\alpha_1/(\alpha_0+\alpha_1 e)\big)\big(a(1;\overline{y}+\mu,\varepsilon) - \overline{a}(1;\overline{y}+\mu,\varepsilon)\big),$$

hence it follows

$$|\mu| \le \big(\alpha_1 c_{17}/(\alpha_0+\alpha_1 e)\big)\sqrt{\varepsilon}.$$

Based on the estimates for $v(\tau;\mu,\varepsilon)$ and μ, we obtain

$$|a(\tau;\overline{y}+\mu,\varepsilon) - \overline{a}(\tau;\overline{y})| \le |v(\tau;\mu,\varepsilon)| + |\overline{a}(\tau;\overline{y}+\mu) - \overline{a}(\tau;\overline{y})| \le c_{18}\sqrt{\varepsilon},$$

where $c_{18} = c_{17}\big(1 + \alpha_1/(\alpha_0+\alpha_1 e)\big)$.

6. Conclusions

In the article the existence and uniqueness of the solution in $\mathbb{C}^1[0,L]$ is proved for the system of equations (5), (6) with linear multipoint conditions (7), (8) and an estimate of the error of the method of averaging and deviation of the initial conditions for slow and fast variables of order ε^α, $\alpha = 1/(mq)$ was obtained. The same result can be obtained by more complex technical transformations for an arbitrary finite number of arguments $a_{\lambda_1},\dots,a_{\lambda_p}$ and $v_{\Delta_1},\dots,v_{\Delta_r}$ in vector functions X and Y.

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Dedicated to Professor Alexandru Şubă on the occasion of his 70th birthday

Quartic differential systems with a non-degenerate monodromic critical point and multiple line at infinity

Alexandru Şubă and Olga Vacaraş

Abstract. The quartic differential systems with a non-degenerate monodromic critical point and non-degenerate infinity are considered. We show that in this family the maximal multiplicity of the line at infinity is seven. Modulo the affine transformation and time rescaling the classes of systems with the line of infinity of multiplicity two, three, . . . , seven are determined. In the cases when the quartic systems have the line at infinity of maximal multiplicity the problem of the center is solved.
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Keywords: quartic differential system, multiple invariant line, monodromic critical point.

Sistemele diferenţiale cuartice ce au punct critic monodromic nedegenerat şi linia de la infinit multiplă

Rezumat. În această lucrare sunt examinate sistemele diferenţiale cuartice cu un punct critic monodromic nedegenerat şi infinitul nedegenerat. Se arată că în această familie de sisteme multiplicitatea maximală a dreptei de la infinit este egală cu şapte. Cu exactitatea unei transformări afine de coordonate şi rescalarea timpului sunt determinate clasele de sisteme cu dreapta de la infinit de multiplicitatea doi, trei, . . . , şapte. În cazurile când sistemele cuartice au linia de la infinit de multiplicitate maximală problema centrului este rezolvată.
Cuvinte-cheie: sistem diferenţial cuartic, dreaptă invariantă multiplă, punct critic monodromic.

1. Introduction

We consider real polynomial differential systems

$$\dot{x} = p(x, y), \quad \dot{y} = q(x, y), \tag{1}$$

where $\dot{x} = dx/dt,\ \dot{y} = dy/dt$.

Let $n = max\{deg(p),\ deg(q)\}$. If $n = 2$ (respectively, $n = 3,\ n = 4$), then system (1) is called quadratic (respectively, cubic, quartic). Via an affine transformation of coordinates and time rescaling each non-degenerate quartic system with a non-degenerate infinity and a center-focus critical point, i.e. a critical point with pure imaginary eigenvalues, can be written in the form

$$\begin{cases} \dot{x} = y + p_2(x,y) + p_3(x,y) + p_4(x,y) \equiv p(x,y), \\ \dot{y} = -(x + q_2(x,y) + q_3(x,y) + q_4(x,y)) \equiv q(x,y), \\ \gcd(p,q) = 1, \ yp_4(x,y) + xq_4(x,y) \not\equiv 0, \end{cases} \tag{2}$$

where $p_i(x,y) = \sum_{j=0}^{i} a_{i-j,j}x^{i-j}y^j$, $q_i(x,y) = \sum_{j=0}^{i} b_{i-j,j}x^{i-j}y^j$, $i = 2,3,4$ are homogeneous polynomials in x and y of degree i with real coefficients.

The eigenvalues λ_1, λ_2 of a critical point $(0,0)$ of system (2) are complex, $\lambda_1\lambda_2 \neq 0$, $\lambda_2 = \overline{\lambda_1}$, and therefore $(0,0)$ is a non-degenerate monodromic critical point.

Remark 1.1. *Via a transformation of the form*

$$x \to \omega(x\cos\varphi - y\sin\varphi)), \ y \to \omega(x\sin\varphi + y\cos\varphi)), \omega \neq 0,$$

and time rescaling we can make in (2)

$$b_{40} = 1. \tag{3}$$

The homogeneous system associated to the quartic system (2) look as

$$\begin{cases} \dot{x} = yZ^3 + p_2(x,y)Z^2 + p_3(x,y)Z + p_4(x,y) \equiv P(x,y,Z), \\ \dot{y} = -(xZ^3 + q_2(x,y)Z^2 + q_3(x,y)Z + q_4(x,y)) \equiv Q(x,y,Z). \end{cases} \tag{4}$$

Denote $\mathbb{X}_\infty = P\,(x,y,Z)\,\frac{\partial}{\partial x} + Q\,(x,y,Z)\,\frac{\partial}{\partial y}$ and $\mathbb{E}_\infty = P \cdot \mathbb{X}_\infty(Q) - Q \cdot \mathbb{X}_\infty(P)$.

The polynomial $\mathbb{E}_\infty$ has the form

$$\begin{aligned} \mathbb{E}_\infty = {} & A_2(x,y) + A_3(x,y)Z + A_4(x,y)Z^2 + A_5(x,y)Z^3 \\ & + A_6(x,y)Z^4 + A_7(x,y)Z^5 + A_8(x,y)Z^6 + A_9(x,y)Z^7 \\ & + A_{10}(x,y)Z^8 + A_{11}(x,y)Z^9, \end{aligned} \tag{5}$$

where $A_i(x,y)$, $i = 2, ..., 11$, are polynomials in x and y.

We say that for system (2) the line at infinity $Z = 0$ has *multiplicity* ν if $A_2(x,y) \equiv 0, ..., A_\nu(x,y) \equiv 0$, $A_{\nu+1}(x,y) \not\equiv 0$, i.e. $\nu - 1$ is the greatest positive integer such that $Z^{\nu-1}$ divides $\mathbb{E}_\infty$. If $A_2(x,y) \not\equiv 0$, then we say that $Z = 0$ has multiplicity one. Denote by $m(Z)$ the multiplicity of the line at infinity $Z = 0$.

About the notion of multiplicity of an invariant algebraic line and, in particular, of the line at infinity, we recommend the work [1] to the readers.

The quadratic (respectively, cubic; quartic) differential systems with multiple line at infinity was examined in [2] (respectively, [3] – [9]; [10]).

In this paper we will show that the maximal multiplicity of the line at infinity for quartic systems (2) is seven. Moreover, we determine the classes of systems $\{(2), (3)\}$ having the line at infinity of multiplicity two, three, ..., seven.

2. Quartic systems {(2), (3)} with the line at infinity $Z = 0$ of multiplicity $m(Z) = 2, 3, 4, 5, 6$

2.1. Systems {(2), (3)} with $m(Z) \geq 2$.

The multiplicity of the line at infinity is at least two if the identity $A_2(x, y) \equiv 0$ holds. The polynomial $A_2(x, y)$ looks as $A_2(x, y) = -A_{21}(x, y)A_{22}(x, y)$, where $A_{21}(x, y) = x^5 + (a_{40} + b_{31})x^4y + (a_{31} + b_{22})x^3y^2 + (a_{22} + b_{13})x^2y^3 + (a_{13} + b_{04})xy^4 + a_{04}y^5$, i.e. $A_{21}(x, y) = yp_4(x, y) + xq_4(x, y)$,

and

$A_{22}(x, y) = (a_{31} - a_{40}b_{31})x^6 + 2(a_{22} - a_{40}b_{22})x^5y + (3a_{13} - 3a_{40}b_{13} - a_{31}b_{22} + a_{22}b_{31})x^4y^2 + 2(2a_{04} - 2a_{40}b_{04} - a_{31}b_{13} + a_{13}b_{31})x^3y^3 - (3a_{31}b_{04} + a_{22}b_{13} - a_{13}b_{22} - 3a_{04}b_{31})x^2y^4 - 2(a_{22}b_{04} - a_{04}b_{22})xy^5 - (a_{13}b_{04} - a_{04}b_{13})y^6$.

As $A_{21} \not\equiv 0$, we require A_{22} to be identically equal to zero. Solving the identity $A_{22} \equiv 0$ we obtain the following result:

Lemma 2.1. *The line at infinity has for quartic system* {(2), (3)} *the multiplicity at least two if and only if the coefficients of* {(2), (3)} *verify the following conditions:*

$$a_{31} = a_{40}b_{31},\ a_{22} = a_{40}b_{22},\ a_{13} = a_{40}b_{13},\ a_{04} = a_{40}b_{04}. \tag{6}$$

2.2. Systems {(2), (3)} with $m(Z) \geq 3$.

The multiplicity $m(Z_\infty)$ of the line at infinity is at least three if $\{A_2(x, y) \equiv 0,\ A_3(x, y) \equiv 0.\}$ In the conditions of Lemma 2.1 the identity $A_3(x, y) \equiv 0$ leads us to the following two series of conditions:

$$a_{30} = a_{40}b_{30},\ a_{21} = a_{40}b_{21},\ a_{12} = a_{40}b_{12},\ a_{03} = a_{40}b_{03}; \tag{7}$$

$$\begin{aligned}&a_{03} = a_{40}b_{03} - a_{30}a_{40}^3 + a_{30}b_{13} - a_{30}a_{40}b_{22} + a_{40}^4b_{30} - a_{40}b_{13}b_{30}\\ &+a_{40}^2b_{22}b_{30} + a_{30}a_{40}^2b_{31} - a_{40}^3b_{30}b_{31},\ a_{12} = a_{30}a_{40}^2 + a_{40}b_{12} + a_{30}b_{22}\\ &-a_{40}^3b_{30} - a_{40}b_{22}b_{30} - a_{30}a_{40}b_{31} + a_{40}^2b_{30}b_{31},\ a_{21} = a_{40}b_{21} - a_{30}a_{40}\\ &+a_{40}^2b_{30} + a_{30}b_{31} - a_{40}b_{30}b_{31},\ b_{04} = a_{40}(b_{13} - a_{40}^3 - a_{40}b_{22} + a_{40}^2b_{31}).\end{aligned} \tag{8}$$

Lemma 2.2. *The line at infinity has for quartic system* {(2), (3)} *the multiplicity at least three if and only if the coefficients of* {(2), (3)} *verify one of the following two sets of conditions: 1)* {(6), (7)}*;* *2)* {(6), (8)}.

2.3. Systems {(2), (3)} with $m(Z) \geq 4$.

In each of the sets of equalities 1) and 2) of Lemma 2.2, the identity $A_4(x, y) \equiv 0$ yields the following series of conditions, respectively:

1) $\{(6), (7)\} \Rightarrow A_4(x,y) \equiv 0 \Rightarrow$

$$a_{20} = a_{40}b_{20},\ a_{11} = a_{40}b_{11},\ a_{02} = a_{40}b_{02}; \tag{9}$$

$$\begin{aligned} & a_{11} = -2a_{20}a_{40} + a_{40}b_{11} + 2a_{40}^2 b_{20} + a_{20}b_{31} - a_{40}b_{20}b_{31}, \\ & a_{02} = 3a_{20}a_{40}^2 + a_{40}b_{02} - 3a_{40}^3 b_{20} + a_{20}b_{22} - a_{40}b_{20}b_{22} \\ & -2a_{20}a_{40}b_{31} + 2a_{40}^2 b_{20}b_{31},\ b_{13} = a_{40}(4a_{40}^2 + 2b_{22} - 3a_{40}b_{31}), \\ & b_{04} = a_{40}^2(3a_{40}^2 + b_{22} - 2a_{40}b_{31}),\ \ a_{20} \neq a_{40}b_{20}; \end{aligned} \tag{10}$$

2) $\{(6), (8)\} \Rightarrow A_4(x,y) \equiv 0 \Rightarrow$

$$\begin{aligned} & a_{11} = -a_{30}^2 - 2a_{20}a_{40} + a_{40}b_{11} + 2a_{40}^2 b_{20} + a_{30}b_{21} + 3a_{30}a_{40}b_{30} \\ & -a_{40}b_{21}b_{30} - 2a_{40}^2 b_{30}^2 + a_{20}b_{31} - a_{40}b_{20}b_{31} - a_{30}b_{30}b_{31} + a_{40}b_{30}^2 b_{31}, \\ & a_{02} = 3a_{30}^2 a_{40} + 3a_{20}a_{40}^2 + a_{40}b_{02} + a_{30}b_{12} - 3a_{40}^3 b_{20} - a_{30}a_{40}b_{21} \\ & \quad +a_{20}b_{22} - a_{40}b_{20}b_{22} - 8a_{30}a_{40}^2 b_{30} - a_{40}b_{12}b_{30} + a_{40}^2 b_{21}b_{30} \\ & \quad -a_{30}b_{22}b_{30} + 5a_{40}^3 b_{30}^2 + a_{40}b_{22}b_{30}^2 - a_{30}^2 b_{31} - 2a_{20}a_{40}b_{31} \\ & \quad +2a_{40}^2 b_{20}b_{31} + 4a_{30}a_{40}b_{30}b_{31} - 3a_{40}^2 b_{30}^2 b_{31}, \\ & b_{13} = a_{40}(4a_{40}^2 + 2b_{22} - 3a_{40}b_{31}), \qquad b_{03} = 6a_{30}a_{40}^2 + a_{40}b_{12} \\ & -a_{40}^2 b_{21} + a_{30}b_{22} - 5a_{40}^3 b_{30} - a_{40}b_{22}b_{30} - 3a_{30}a_{40}b_{31} + 3a_{40}^2 b_{30}b_{31}; \end{aligned} \tag{11}$$

$$a_{20} = a_{40}b_{20},\ a_{11} = a_{40}b_{11},\ a_{02} = a_{40}b_{02},\ a_{30} = a_{40}b_{30}; \tag{12}$$

$$\begin{aligned} & a_{11} = -2a_{20}a_{40} + a_{40}b_{11} + 2a_{40}^2 b_{20} + a_{20}b_{31} - a_{40}b_{20}b_{31}, \\ & a_{02} = 3a_{20}a_{40}^2 + a_{40}b_{02} - 3a_{40}^3 b_{20} + a_{20}b_{22} - a_{40}b_{20}b_{22} - 2a_{20}a_{40}b_{31} \\ & +2a_{40}^2 b_{20}b_{31}, \quad a_{30} = a_{40}b_{30}, \quad b_{13} = a_{40}(4a_{40}^2 + 2b_{22} - 3a_{40}b_{31}). \end{aligned} \tag{13}$$

It is easy to see that the set of conditions $\{(6), (8), (12)\}$ is a particular case for the set of conditions $\{(6), (7), (9)\}$. The conditions $\{(6), (7), (10)\}$ and $\{(6), (8), (13)\}$ are the same.

Lemma 2.3. *The line at infinity has for quartic system* $\{(2), (3)\}$ *the multiplicity at least four if and only if the coefficients of* $\{(2), (3)\}$ *verify one of the following three sets of conditions: 1)* $\{(6), (7), (9)\}$*;* *2)* $\{(6), (7), (10)\}$*;* *3)* $\{(6), (8), (11)\}$.

2.4. Systems $\{(2), (3)\}$ with $m(Z) \geq 5$.

In the conditions of Lemma 2.3 we solve the identity $A_5(x,y) \equiv 0$. We have, respectively:

1) $\{(6), (7), (9)\} \Rightarrow A_5(x,y) \equiv 0 \Rightarrow$

$$\begin{aligned} & b_{31} = (3a_{40}^2 - 1)/a_{40}, \quad b_{22} = 3(a_{40}^2 - 1), \quad b_{13} = a_{40}(a_{40}^2 - 3), \\ & b_{04} = -a_{40}^2, \quad a_{40} \neq 0. \end{aligned} \tag{14}$$

2) $\{(6),(7),(10)\} \Rightarrow A_5(x,y) \equiv 0 \Rightarrow$

$$\begin{aligned}
& b_{21} = (1 - 3a_{40}^2 - a_{20}a_{40}b_{30} + a_{40}^2 b_{20}b_{30} + a_{40}b_{31} + a_{20}b_{30}b_{31} \\
& \quad -a_{40}b_{20}b_{30}b_{31})/(a_{20} - a_{40}b_{20}), \qquad b_{12} = (2a_{40} + a_{40}b_{22} \\
& \quad +a_{20}a_{40}^2 b_{30} - a_{40}^3 b_{20}b_{30} + a_{20}b_{22}b_{30} - a_{40}b_{20}b_{22}b_{30} - a_{40}^2 b_{31} \\
& \quad -a_{20}a_{40}b_{30}b_{31} + a_{40}^2 b_{20}b_{30}b_{31})/(a_{20} - a_{40}b_{20}), \\
& b_{03} = a_{40}(a_{40} + 3a_{40}^3 + a_{40}b_{22} + 3a_{20}a_{40}^2 b_{30} - 3a_{40}^3 b_{20}b_{30} \\
& \quad +a_{20}b_{22}b_{30} - a_{40}b_{20}b_{22}b_{30} - 2a_{40}^2 b_{31} - 2a_{20}a_{40}b_{30}b_{31} \\
& \quad +2a_{40}^2 b_{20}b_{30}b_{31})/(a_{20} - a_{40}b_{20}).
\end{aligned} \tag{15}$$

3) $\{(6),(8),(11)\} \Rightarrow A_5(x,y) \equiv 0 \Rightarrow$

$$\begin{aligned}
& b_{22} = 3a_{40}(b_{31} - 2a_{40}), \\
& b_{11} = (1 + 3a_{20}a_{30} - 3a_{40}^2 - 5a_{30}a_{40}b_{20} - a_{20}b_{21} + a_{40}b_{20}b_{21} \\
& \quad -2a_{30}^2 b_{30} - 4a_{20}a_{40}b_{30} + 6a_{40}^2 b_{20}b_{30} + a_{30}b_{21}b_{30} + 5a_{30}a_{40}b_{30}^2 \\
& \quad -a_{40}b_{21}b_{30}^2 - 3a_{40}^2 b_{30}^3 + a_{40}b_{31} + a_{30}b_{20}b_{31} + a_{20}b_{30}b_{31} \\
& -2a_{40}b_{20}b_{30}b_{31} - a_{30}b_{30}^2 b_{31} + a_{40}b_{30}^3 b_{31})/(a_{30} - a_{40}b_{30}), \\
& b_{12} = -8a_{30}a_{40} + 2a_{40}b_{21} + 5a_{40}^2 b_{30} + 2a_{30}b_{31} - 2a_{40}b_{30}b_{31}, \\
& b_{02} = (-2a_{30}^3 + a_{40} - a_{20}a_{30}a_{40} - 3a_{40}^3 - 2a_{30}a_{40}^2 b_{20} + a_{30}^2 b_{21} \\
& \quad -a_{20}a_{40}b_{21} + a_{40}^2 b_{20}b_{21} + 5a_{30}^2 a_{40}b_{30} + 3a_{40}^3 b_{20}b_{30} \\
& -a_{30}a_{40}b_{21}b_{30} - 3a_{30}a_{40}^2 b_{30}^2 + a_{20}a_{30}b_{31} + a_{40}^2 b_{31} - a_{30}^2 b_{30}b_{31} \\
& -a_{40}^2 b_{20}b_{30}b_{31} + a_{30}a_{40}b_{30}^2 b_{31})/(a_{30} - a_{40}b_{30}), \quad a_{30} \neq a_{40}b_{30}.
\end{aligned} \tag{16}$$

Lemma 2.4. *The line at infinity has for quartic system* $\{(2),(3)\}$ *the multiplicity at least five if and only if the coefficients of* $\{(2),(3)\}$ *verify one of the following three sets of conditions: 1)* $\{(6),(7),(9),(14)\}$; *2)* $\{(6),(7),(10),(15)\}$; *3)* $\{(6),(8),(11),(16)\}$.

2.5. Systems $\{(2),(3)\}$ with $m(Z) \geq 6$.

In each of the conditions 1) - 3) of Lemma 2.4 we solve the identity $A_6(x,y) \equiv 0$. We obtain the following results:

1) $\{(6),(7),(9),(14)\} \Rightarrow A_6(x,y) \equiv 0 \Rightarrow$

$$b_{03} = -a_{40}b_{30}, \; b_{12} = (a_{40}^2 - 2)b_{30}, \; b_{21} = (2a_{40}^2 - 1)b_{30}/a_{40}. \tag{17}$$

2) $\{(6),(7),(10),(15)\} \Rightarrow A_6(x,y) \equiv 0 \Rightarrow$

$$\begin{aligned}
& b_{02} = (a_{40}^2 - 3a_{40}^4 + a_{40}^3 b_{31} + a_{40}b_{30}\alpha - 3a_{40}^3 b_{30}\alpha + a_{40}^2 b_{30}b_{31}\alpha - \\
& \qquad -3a_{40}^2 b_{20}\alpha^2 + a_{40}b_{20}b_{31}\alpha^2 - 6a_{40}\alpha^3 + 2b_{31}\alpha^3)/\alpha^2, \\
& b_{11} = (a_{40} - 3a_{40}^3 + a_{40}^2 b_{31} + b_{30}\alpha - 3a_{40}^2 b_{30}\alpha + a_{40}b_{30}b_{31}\alpha - \\
& \qquad -2a_{40}b_{20}\alpha^2 + b_{20}b_{31}\alpha^2 + 2\alpha^3)/\alpha^2, \\
& b_{22} = 3a_{40}(b_{31} - 2a_{40}), \quad \alpha = a_{20} - a_{40}b_{20}, \quad \alpha \neq 0.
\end{aligned} \tag{18}$$

3) $\{(6), (8), (11), (16)\} \Rightarrow A_6(x, y) \equiv 0 \Rightarrow$

$$\begin{aligned} a_{20} &= (1 + a_{40}^2 + 2a_{30}a_{40}b_{20} + a_{30}^2 b_{30} - 2a_{40}^2 b_{20}b_{30} - 2a_{30}a_{40}b_{30}^2 + \\ &+ a_{40}^2 b_{30}^3)/(2(a_{30} - a_{40}b_{30})), \quad b_{21} = 3a_{30}, \ b_{31} = 4a_{40}. \end{aligned} \tag{19}$$

Lemma 2.5. *The line at infinity has for quartic system* $\{(2), (3)\}$ *the multiplicity at least six if and only if the coefficients of* $\{(2), (3)\}$ *verify one of the following sets of conditions: 1)* $\{(6), (7), (9), (14), (17)\}$*; 2)* $\{(6), (7), (10), (15), (18)\}$*; 3)* $\{(6), (8), (11), (16), (19)\}$.

3. Quartic systems $\{(2), (3)\}$ with the line at infinity $Z = 0$ of multiplicity seven

To obtain the quartic systems $\{(2), (3)\}$, which have the line at infinity of multiplicity seven, we solve the identity $A_7(x, y) \equiv 0$ in each of the series of conditions 1) – 3) of Lemma 2.5.

1) $\{(6), (7), (9), (14), (17)\} \Rightarrow A_7(x, y) \equiv 0 \Rightarrow$

$$b_{11} = (a_{40}^2 - 1)b_{20}/a_{40}, \qquad b_{02} = -b_{20}. \tag{20}$$

In this case the identity $A_8(x, y) = -(a_{40}x - y)(x + a_{40}y)^2((1 + 3a_{40}^2)x^2 - 4a_{40}xy + (3 + a_{40}^2)y^2)/a_{40} \not\equiv 0$, therefore the multiplicity of the line at infinity is exactly seven.

2) $\{(6), (7), (10), (15), (18)\} \Rightarrow A_7(x, y) \equiv 0 \Rightarrow$

$$\begin{aligned} b_{20} &= (2a_{40} + 2a_{40}^3 - 2\alpha^3 + 2a_{40}^2\beta + 2a_{40}^4\beta - 3a_{40}\alpha^3\beta + \\ &+ \alpha^3\beta^2)/(\alpha^2(a_{40} - \beta)(1 + a_{40}\beta)), \quad \beta = b_{31} - 3a_{40}, \quad \beta \neq 0, \\ b_{30} &= (a_{40}\beta - 3a_{40}^2 - 2)/(\alpha(a_{40} - \beta)), \quad (a_{40} - \beta)(1 + a_{40}\beta) \neq 0. \end{aligned} \tag{21}$$

Under these conditions the identity $A_8(x, y) \equiv 0$ does not give us real solutions, therefore the multiplicity of the line at infinity is exactly seven.

3) $\{(6), (8), (11), (16), (19)\}$. In this case the identity $A_7(x, y) \equiv 0$ does not give us real solutions, therefore the multiplicity of the line at infinity is exactly six.

Lemma 3.1. *The line at infinity has for quartic system* $\{(2), (3)\}$ *the multiplicity seven if and only if the coefficients of* $\{(2), (3)\}$ *verify one of the following sets of conditions:*
1) $\{(6), (7), (9), (14), (17), (20)\}$*; 2)* $\{(6), (7), (10), (15), (18), (21)\}$.

In this way we prove the statement of the following theorem.

Theorem 3.1. *In the class of quartic differential systems with a non-degenerate monodromic critical point and non-degenerate infinity the maximal multiplicity of the line at infinity is seven.*

4. Solution of the center problem for quartic systems $\{(2), (3)\}$ with the line at infinity of maximal multiplicity

Let $F(x,y) = x^2 + y^2 + F_3(x,y) + F_4(x,y) + \cdots + F_n(x,y) + \cdots$, where $F_k(x,y) = \sum_{i+j=k} f_{ij}x^i y^j$, $f_{0j} = 0$ if j is even, be a function such that

$$\frac{\partial F}{\partial x}p(x,y) + \frac{\partial F}{\partial y}q(x,y) \equiv \sum_{j=1}^{\infty} L_j(x^2+y^2)^{j+1}. \tag{22}$$

In (22) L_j are polynomials in coefficients of (2) and are called Lyapunov quantities. The critical point $(0,0)$ is a center for system (2) if and only if $L_j = 0,\ j = 1,2,3,\ldots$.

Using the identity (22) we calculate the first three Lyapunov quantities, i.e. we solve in f_{ij} and L_k the following systems of identities:

$$\begin{aligned}
&2xp_2 + yF'_{3x} - 2yq_2 - xF'_{3y} \equiv 0,\\
&2xp_3 + F'_{3x}p_2 + yF'_{4x} - 2yq_3 - F'_{3y}q_2 - xF'_{4y} \equiv L_1(x^2+y^2)^2,\\
&2xp_4 + F'_{3x}p_3 + F'_{4x}p_2 + yF'_{5x} - 2yq_4 - F'_{3y}q_3 - F'_{4y}q_2 - F'_{5y}x \equiv 0,\\
&F'_{3x}p_4 + F'_{4x}p_3 + F'_{5x}p_2 + F'_{6x}y - F'_{3y}q_4 - F'_{4y}q_3 - F'_{5y}q_2 - F'_{6y}x \equiv L_2(x^2+y^2)^3,\\
&F'_{4x}p_4 + F'_{5x}p_3 + F'_{6x}p_2 + F'_{7x}y - F'_{4y}q_4 - F'_{5y}q_3 - F'_{6y}q_2 - F'_{7y}x \equiv 0,\\
&F'_{5x}p_4 + F'_{6x}p_3 + F'_{7x}p_2 + F'_{8x}y - F'_{5y}q_4 - F'_{6y}q_3 - F'_{7y}q_2 - F'_{8y}x \equiv L_3(x^2+y^2)^4.
\end{aligned}$$

The first Lyapunov quantity looks as

$$L_1 = (a_{12} - a_{02}a_{11} - a_{11}a_{20} + 3a_{30} + 2a_{02}b_{02} - 3b_{03} + b_{02}b_{11} - 2a_{20}b_{20} + b_{11}b_{20} - b_{21})/4.$$

The quantities L_2 and L_3 are very cumbersome and cannot be brought here.

In the following we will solve the center problem for the system (2) under the conditions 1) and 2) of Lemma 3.1, i.e. when the line at infinity is of the maximal multiplicity.

The conditins 1) of Lemma 3.1 are

$$\begin{aligned}
&a_{20} = a_{40}b_{20}, a_{11} = a_{40}b_{11}, a_{02} = a_{40}b_{02}, b_{11} = (a_{40}^2 - 1)b_{20}/a_{40},\\
&b_{02} = -b_{20}, a_{30} = a_{40}b_{30}, a_{21} = a_{40}b_{21}, a_{12} = a_{40}b_{12}, a_{03} = a_{40}b_{03},\\
&b_{21} = (2a_{40}^2 - 1)b_{30}/a_{40}, b_{12} = (a_{40}^2 - 2)b_{30}, b_{03} = -a_{40}b_{30}, a_{31} = 3a_{40}^2 - 1,\\
&a_{22} = 3a_{40}(a_{40}^2 - 1), a_{13} = a_{40}^2(a_{40}^2 - 3), a_{04} = -a_{40}^3, b_{40} = 1,\\
&b_{31} = (3a_{40}^2 - 1)/a_{40}, b_{22} = 3(a_{40}^2 - 1), b_{13} = a_{40}(a_{40}^2 - 3), b_{04} = -a_{40}^2.
\end{aligned} \tag{23}$$

The first Lyapunov quantity calculated for system $\{(2), (23)\}$ is $L_1 = (1+a_{40}^2)^2 b_{30}/(4a_{40})$ and $L_1 = 0$ gives us $b_{30} = 0$. The transformation $X = a_{40}x - y$, $Y = x + a_{40}y$ reduces system $\{(2), (23), b_{30} = 0\}$ to the following system

$$\begin{aligned}
&\dot{X} = Y(a_{40} + b_{20}X + a_{40}^2 b_{20}X + XY^2 + a_{40}^2 XY^2))/a_{40},\\
&\dot{Y} = -X.
\end{aligned} \tag{24}$$

For system (24) the straight line $X = 0$ is an axis of symmetry. Therefore, the origin $(X, Y) = (0, 0)$ (respectively, $(x, y) = (0, 0)$) is for system (24) (respectively, $\{(2), (23), b_{30} = 0\}$) the critical point of a center type.

The exponential factors. Let $h, g \in \mathbb{C}[x, y]$ be relatively prime in the ring $\mathbb{C}[x, y]$. The function $\Phi = \exp(g/h)$ is called an *exponential factor* of system (2) if for some polynomial $K \in \mathbb{C}[x, y]$ of degree at most three it satisfies the identity

$$\frac{\partial \Phi}{\partial x} p(x, y) + \frac{\partial \Phi}{\partial y} q(x, y) \equiv \Phi \cdot K(x, y).$$

We remark that the system $\{(2), (23)\}$ has the following six exponential factors

$$\begin{aligned}
&\Phi_1 = \exp(x + a_{40}y),\\
&\Phi_2 = \exp((x + a_{40}y)^2),\\
&\Phi_3 = \exp((x + a_{40}y)^3),\\
&\Phi_4 = \exp(-4a_{40}y + (x + a_{40}y)^4),\\
&\Phi_5 = \exp(-5a_{40}y(x + a_{40}y) + (x + a_{40}y)^5),\\
&\Phi_6 = \exp(6a_{40}b_{20}y - 6a_{40}y(x + a_{40}y)^2 + (x + a_{40}y)^6).
\end{aligned}$$

The conditions 2) of Lemma 3.1 are

$$\begin{aligned}
a_{20} = {} & -(-2a_{40}^2 - 2a_{40}^4 + a_{40}\alpha^3 - 2a_{40}^3\beta - 2a_{40}^5\beta + \alpha^3\beta +\\
& 2a_{40}^2\alpha^3\beta)/(\alpha^2(a_{40} - \beta)(1 + a_{40}\beta)),\\
a_{11} = {} & (-2a_{40} + 2a_{40}^5 + a_{40}^2\alpha^3 - 2a_{40}^2\beta + 2a_{40}^6\beta - 4a_{40}\alpha^3\beta - \alpha^3\beta^2 -\\
& 4a_{40}^2\alpha^3\beta^2)/(\alpha^2(a_{40} - \beta)(1 + a_{40}\beta)),\\
a_{02} = {} & -(a_{40}(2a_{40} + 2a_{40}^3 + 2a_{40}^2\beta + 2a_{40}^4\beta - a_{40}\alpha^3\beta + 3\alpha^3\beta^2 +\\
& 2a_{40}\alpha^3\beta^3))/(\alpha^2(a_{40} - \beta)(1 + a_{40}\beta)),\\
b_{20} = {} & -(-2a_{40} - 2a_{40}^3 + 2\alpha^3 - 2a_{40}^2\beta - 2a_{40}^4\beta + 3a_{40}\alpha^3\beta -\\
& \alpha^3\beta^2)/(\alpha^2(a_{40} - \beta)(1 + a_{40}\beta)),\\
b_{11} = {} & -(2 - 2a_{40}^4 + 2a_{40}\beta - 2a_{40}^5\beta + 4\alpha^3\beta + a_{40}^2\alpha^3\beta + 4a_{40}\alpha^3\beta^2 -\\
& \alpha^3\beta^3)/(\alpha^2(a_{40} - \beta)(1 + a_{40}\beta)),\\
b_{02} = {} & -(2a_{40} + 2a_{40}^3 + 2a_{40}^2\beta + 2a_{40}^4\beta + 2\alpha^3\beta^2 + a_{40}^2\alpha^3\beta^2 +\\
& a_{40}\alpha^3\beta^3)/(\alpha^2(a_{40} - \beta)(1 + a_{40}\beta)),\\
a_{30} = {} & -(a_{40}(2 + 3a_{40}^2 - a_{40}\beta))/(\alpha(a_{40} - \beta)),\\
a_{21} = {} & -(3a_{40}(a_{40} + 2a_{40}^3 + \beta))/(\alpha(a_{40} - \beta)),\\
a_{12} = {} & -(3a_{40}^2(a_{40}^3 + 2\beta + a_{40}^2\beta))/(\alpha(a_{40} - \beta)),\\
a_{03} = {} & -(a_{40}^3(-a_{40} + 3\beta + 2a_{40}^2\beta))/(\alpha(a_{40} - \beta)),\\
b_{30} = {} & -(2 + 3a_{40}^2 - a_{40}\beta)/(\alpha(a_{40} - \beta)),\\
b_{21} = {} & -(3(a_{40} + 2a_{40}^3 + \beta))/(\alpha(a_{40} - \beta)),\\
b_{12} = {} & -(3a_{40}(a_{40}^3 + 2\beta + a_{40}^2\beta))/(\alpha(a_{40} - \beta)),
\end{aligned} \tag{25}$$

$$b_{03} = -(a_{40}^2(-a_{40} + 3\beta + 2a_{40}^2\beta))/(\alpha(a_{40} - \beta)),$$
$$a_{31} = a_{40}(3a_{40} + \beta),\ a_{22} = 3a_{40}^2(a_{40} + \beta),$$
$$a_{13} = a_{40}^3(a_{40} + 3\beta),\ a_{04} = a_{40}^4\beta,$$
$$b_{40} = 1,\ b_{31} = 3a_{40} + \beta,\ b_{04} = a_{40}^3\beta,$$
$$b_{22} = 3a_{40}(a_{40} + \beta),\ b_{13} = a_{40}^2(a_{40} + 3\beta).$$

Direct calculations show that in coefficients of differential system $\{(2), (25)\}$ the algebraic system $\{L_1 = 0, L_2 = 0, L_3 = 0\}$ is not compatible.

Example 4.1.

$$\begin{aligned} \dot{x} &= y + (19x(x - 2y))/(6^{1/3}95^{2/3}),\\ \dot{y} &= -(x + (6^{2/3}95^{1/3}(19x^2 - 150xy - 76y^2) - 306^{1/3}95^{2/3}x^2(x - 3y)+\\ &\quad 570x^3(x - 2y))/570). \end{aligned} \tag{26}$$

The coefficients of system (26) verify the set of conditions (25). The first two Lyapunov quantities vanish and the third one is not equal zero. This example shows that the ciclicity of the focus $(0, 0)$ in system $\{(2), (25)\}$ is at most three.

The system $\{(2), (25)\}$ has the following six exponential factors

$$\Phi_1 = \exp(x + a_{40}y),$$
$$\Phi_2 = \exp((x + a_{40}y)^2),$$
$$\Phi_3 = \exp((x + a_{40}y)^3 + 3\alpha y),$$
$$\Phi_4 = \exp((x + a_{40}y)^4 + 4y(\alpha(x + a_{40}y) + (2(1 + a_{40}^2))/(a_{40} - \beta))),$$
$$\begin{aligned}\Phi_5 = \exp(&(x + a_{40}y)^5 + 5y(\alpha(x + a_{40}y)^2 + (2(1 + a_{40}^2)(x + a_{40}y))/(a_{40} - \beta)+\\ &(4(1 + a_{40}^2)^2(1 + a_{40}\beta) + \alpha^3(a_{40} - \beta)(2 + 3a_{40}\beta - \beta^2))\\ &/(\alpha(a_{40} - \beta)^2(1 + a_{40}\beta)))),\end{aligned}$$
$$\Phi_6 = \exp((x + a40y)^6 + y(A(x + a_{40}y)^3 + B(x + a_{40}y)^2 + Cx + Dy + F)),$$

where

$$A = 6\alpha,\ \ B = (12(1 + a_{40}^2))/(a_{40} - \beta),$$
$$\begin{aligned}C = (&6(4\beta(1 + a_{40}^2)^2(1 + a_{40}) + \alpha^3(a_{40} - \beta)(2 + 3a_{40}\beta - \beta^2)))\\ &/(\alpha(a_{40} - \beta)^2(1 + a_{40}\beta)),\end{aligned}$$
$$\begin{aligned}D = (&3(8a_{40}(1 + a_{40}^2)^2(1 + a_{40}\beta) + (1 + a_{40}^2)\alpha^3(5 + 7a_{40}\beta - 10\beta^2)+\\ &\alpha^3(a_{40} - \beta)\beta(-13 + 3\beta^2) + \alpha^3(1 + \beta^2)(-5 + 3\beta^2)))\\ &/(\alpha(a_{40} - \beta)^2(1 + a_{40}\beta)),\end{aligned}$$
$$\begin{aligned}F = (&6(14(1 + a_{40}^2)^2\alpha^3\beta + 8(1 + a_{40}^2)^3(1 + a_{40}\beta) + 2\alpha^3\beta(1 + \beta^2)+\\ &\alpha^3(a40 - \beta)(-1 + 3\beta^2) + (1 + a_{40}^2)(\alpha^3(a_{40} - \beta)(9 - 20\beta^2)-\\ &2\alpha^3\beta(8 + 7\beta^2))))/(\alpha^2(a_{40} - \beta)^3(1 + a_{40}\beta)).\end{aligned}$$

In this way, we have proved the following two Theorems.

Theorem 4.1. *The origin* $(0, 0)$ *is a center for quartic differential systems* (2) *with the line at infinity of maximal multiplicity if and only if the first three Lyapunov quantities vanish* $L_1 = L_2 = L_3 = 0$.

Theorem 4.2. *System* (2) *with the line at infinity of maximal multiplicity has a center at the origin* $(0, 0)$ *if and only if its coefficients verify the set of conditions* $\{(23),\ b_{30} = 0\}$.

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Dedicated to Professor Alexandru Șubă on the occasion of his 70th birthday

Perturbation of singular integral operators with piecewise continuous coefficients

Vasile Neagu and Diana Bîclea

Abstract. In the paper it is shown that the property of singular integral operators with piecewise continuous coefficients to be Noetherian is stable with respect to their perturbation with certain non-compact operators. An example is constructed showing that the corner points of the integration contour significantly affect the Noetherian property of singular operators with translations. These results are obtained using the symbol of the singular operators on contours with corner points, symbol, which is also determined.
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Perturbarea operatorilor integrali singulari cu coeficienţi continui pe porţiuni

Rezumat. În lucrare se demonstrează că proprietatea operatorilor integrali singulari cu coeficienţi continui pe porţiuni de a fi noetherieni este stabilă în raport cu perturbarea lor cu anumiţi operatori necompacţi. Este construit un exemplu care demonstrează că punctele unghiulare ale conturului de integrare afectează în mod semnificativ proprietatea noetheriană a operatorilor singulari cu translaţii. Aceste rezultate sunt obţinute cu ajutorul simbolului operatorilor singulari pe contururi cu puncte unghiulare, simbol, care de asemenea este determinat.
Cuvinte cheie: operatori integrali singulari, operatori noetherieni, simbol, contur Lyapunov pe porţiuni.

1. Introduction

In the well-known monographs of N. Mushelyishvili and F. Gahov, operators of the form

$$(A\varphi)(t) = a(t)\varphi(t) + \int_\Gamma \frac{k(t,\tau)\varphi(\tau)}{\tau - t} d\tau, \ (t \in \Gamma), \tag{1}$$

are called *complete* singular integral operators. In relation (1) the functions $a(t)$ and $k(t,\tau)$ are functions that verify Holder's conditions on Γ, respectively on $\Gamma \times \Gamma$, and the integral is considered in the principal value sense. The operator A, defined by equality (1) can be represented as

$$A = aI + bS + T,$$

where $b(t) = \pi i k(t,t)$, S is the singular integral operator on the contour Γ,

$$(S\varphi)(t) = \frac{1}{\pi i}\int_{\Gamma} \frac{\varphi(\tau)}{\tau - t} d\tau,$$

and T is the integral operator with the kernel

$$k(t,\tau) = \frac{k(\tau,t) - k(t,t)}{\tau - t}. \tag{2}$$

When $k(t,\tau)$ satisfies Holder's conditions on $\Gamma\times\Gamma$, the kernel (2) has weak singularities and therefore the operator T is compact on the spaces $L_p(\Gamma,\rho)$, where

$$\rho(t) = \prod_{k=1}^{n} |t - t_k|^{\beta_k} \ (1 < p < +\infty, \ -1 < \beta_k < p-1, \ t_k \in \Gamma).$$

It hence follows that the operator $A = aI + bS + T$ is normally solvable and Noetherian if and only if this property is possessed by its characteristic part, $A_0 = aI + bS$. Moreover

$$dimker A = dimker A_0 \ \text{ and } \ dimker A^* = dimker A_0^*.$$

Based on this statement Noether's theory for singular integral operators developed into the foundation for characteristic singular integral operators. Remarkable results have been obtained in this direction: Noetherian criteria have been established for these operators with continuous piecewise coefficients, with coefficients, which have almost periodic discontinuities, with coefficients arbitrary (measurable and bounded). However, in various problems of mechanics, physics and other fields, which reduce to singular equations, it is not characteristic operators but complete operators that appear. In this contest it arises the necessity to study complete singular operators with discontinuous functions $a(t)$ și $k(t,\tau)$. The main difficulty in this direction consists in the fact that the operator T with kernel (2) may not be compact and (more importantly) may not represent a permissible perturbation for singular characteristic operators.

We will illustrate this with an example. Let Γ_0 be the unit circle, $\chi(t)$ be the characteristic function of the semicircle $\Gamma_0^+ = \{t \in \Gamma_0, Imt \geq 0\}$, $k(\tau,t) = \chi(t) - \chi(\tau)$, $\lambda \in C$ and

$$(A\varphi)(t) = \lambda\varphi(t) + \int_{\Gamma} \frac{k(t,\tau)}{\tau - t}\varphi(\tau)d\tau.$$

In this example $k(t,t) = 0$, therefore the characteristic part of the operator A is the scalar operator $(A_0\varphi)(t) = \lambda\varphi(t)$. In this example the operator can be represented as

$$A = \lambda I + \chi S - S\chi I, \tag{3}$$

and it follows that it is contained in the algebra $\sum_p(\Gamma_0)$ generated by singular operators with piecewise continuous coefficients. It is known that this algebra is a symbol algebra. The symbol of the operators is determined from the equalities [5], [9].

$$a(t,\xi) = \begin{Vmatrix} a(t+0)f(\xi) + (1-f(\xi))a(t) & h(\xi)(a(t+0)-a(t)) \\ h(\xi)(a(t+0)-a(t)) & (1-f(\xi))a(t+0) + a(t)f(\xi) \end{Vmatrix}, \qquad (4)$$

where

$$f(\xi) = \left[1 - exp\left(-2\pi\left(\xi + \frac{i}{p}\right)\right)\right]^{-1} \ (-\infty \le \xi \le +\infty),$$

$$S(t,\xi) = \begin{Vmatrix} 1 & 0 \\ 0 & -1 \end{Vmatrix}. \qquad (5)$$

In particular, in the case of the operator $A = \lambda I + \chi S - S\chi I$ and $p = 2$ we have:

$$detA(t,\xi) = \lambda^2 \ for \ t \neq \pm 1$$

and

$$detA(t,\xi) = \lambda^2 + 4\frac{e^{\xi}}{1+e^{\xi}} \ (-\infty \le \xi \le +\infty), \ \text{for } t = \pm 1.$$

The operator A is Noetherian in the space $L_2(\Gamma_0)$ if and only if $\lambda^2 + 4e^{\xi}/(1+e^{\xi}) \neq 0$ for any ξ, $-\infty \le \xi \le +\infty$. This is equivalent for $\lambda \neq \mu i$, where $\mu \in [-1,1]$.

Hence, for $\lambda = \mu i$, where $\mu \in [-1,1] \setminus \{0\}$, the operator A is not Noetherian and its characteristic part $A_0 = \lambda I$ is Noetherian. It follows that the operator $T = A - A_0$ is not a permissible perturbation to the characteristic part of operator A. It also follows that the operator $T = \chi S - S\chi I$ is not compact.

From this reasoning and the examined example, the following problem comes apparent. What (at least necessary) conditions should we impose on the operator kernel T,

$$k(t,\tau) = \frac{k(\tau,t) - k(t,t)}{\tau - t},$$

so that this operator does not influence the Noetherian conditions of the operator $A = aI + bS$, i.e. the operators $A_0 = aI + bS$ and $A = aI + bS + T$ are or are not Noetherian conditions and $IndA_0 = IndA$. If the function $k(t,\tau)$ is continuous or has weak singularities on the integration contour then the operator T is compact and it satisfies the conditions enumerated above. In this paper a class of operators T (non-compact) is described which also possess this property. In the construction of this class of operators an important role will be played by the symbol defined on algebra $\sum_p(\Gamma_0)$.

2. Perturbation of singular operators with operators from the set M

We denote by **M** the set of all operators in algebra $\sum_p(\Gamma)$ with the following properties. If $H \in \mathbf{M}$, then its symbol $H(t,\xi)$ has the form

$$H(t,\xi) = \begin{Vmatrix} 0 & m(t,\xi) \\ n(t,\xi) & 0 \end{Vmatrix},$$

where $m(t,\xi)\cdot n(t,\xi) \equiv 0$ and

$$m(t,\xi) = \frac{(\psi(t,\xi)-1)h(\xi)}{f(\xi)+(1-f(\xi))\psi(t,\xi)}$$

and the real function $\psi(t,\xi)$ satisfies the following conditions. If $\psi(t_0,\xi) \neq 1 \;\; (t_0 \in \Gamma)$, then the continuous function $\psi(t_0,\xi)$ is decreasing and $\psi(t_0,\xi) \to +\infty$ for $\xi \to -\infty$ and $\psi(t_0,\xi) \to 0$ for $\xi \to +\infty$. We mention that the set **M** includes all compact operators acting on the space $L_p(\Gamma)$. It will be shown below that some non-compact operators of the algebra $\sum_p(\Gamma)$ also belong to the set **M**. Thus, it will be shown that the conditions for singular integral operators to be Noetherian are stable with respect to their perturbations by non-compact operators. This will follow from the next theorem.

Theorem 2.1. *Let $H \in \mathbf{M}$. The operator*

$$A = aP + Q + H \quad \left(P = \frac{1}{2}(I+S),\; Q = \frac{1}{2}(I-S),\; a \in PC(\Gamma)\right)$$

is Noetherian on the space $L_p(\Gamma)$, if and only if this property is held by the operator $A_0 = aP + Q$.

Proof. The symbol $A(t,\xi)$ of the operator $A = aP + Q + H$ has the form

$$A(t,\xi) = \begin{Vmatrix} a(t+0)f(\xi)+(1-f(\xi))a(t) & m(t,\xi) \\ h(\xi)(a(t+0)-a(t))+n(t,\xi) & 1 \end{Vmatrix}$$

and

$$detA(t,\xi) = \frac{a(t+0)f(\xi)+(1-f(\xi))a(t)\psi(t,\xi)}{f(\xi)+(1-f(\xi))\psi(t,\xi)}. \tag{6}$$

Let the operator A be Noetherian, then $a(t\pm 0) \neq 0$ and

$$a(t+0)f(\xi)+(1-f(\xi))a(t)\psi(t,\xi) \neq 0 \tag{7}$$

for all $t \in \Gamma$ and $\xi \in \bar{R}$. We admit that the operator $A_0 = aP + Q$ is not Noetherian, then the determinant of its symbol cancels at a point (t_0,ξ_0):

$$a(t_0+0)f(\xi_0)+(1-f(\xi_0))a(t_0) = 0, \tag{8}$$

where $t_0 \in \Gamma$ and $\xi_0 \in \bar{R}$. Therefore $\psi(t_0,\xi) \neq 1$ and ratio $\frac{a(t_0)}{a(t_0+0)}$ can be written as

$$\frac{a(t_0)}{a(t_0+0)} = \exp\Big(2\pi(\xi_0 + i/p)\Big).$$

We will show that in this case $detA(t_0, \xi)$ vanishes at a point $\xi_1 \in R$. Indeed, from the relation (6) and condition (8) we get

$$detA(t_0, \xi) = \frac{(f(\xi)-1)a(t_0)}{f(\xi)+(1-f(\xi))\psi(t_0,\xi)}[e^{2\pi(\xi-\xi_0)} - \psi(t_0,\xi)].$$

From the properties of the function $\psi(t_0, \xi)$ follows that the equation $e^{2\pi(\xi-\xi_0)} - \psi(t_0, \xi)$ possesses a solution $\xi = \xi_1$. Thus, $detA(t_0, \xi_1) = 0$, which is in contradiction with conditions (7). Necessity is proved. Let A not be Noetherian, then or $a(t+0)\cdot a(t) = 0$ at a point $t_0 \in \Gamma$ or that

$$a(t+0)f(\xi) + (1-f(\xi))a(t)\psi(t,\xi) = 0$$

at the point (t_0, ξ_0) $(t_0 \in \Gamma,\ \xi_0 \in \bar{R})$. In the first case the operator A_0 also is not Noetherian, and in the second case we obtain that $\frac{a(t_0)}{a(t_0+0)} = 2\pi/p$. Then $a(t_0+0)f(\xi) + (1-f(\xi))a(t_0) = 0$ at the point $\xi_1 \in R$. Therefore, in this case also the operator A_0 is not Noetherian. □

Corollary 2.1. *The property of singular integral operators with piecewise continuous coefficients to be Noetherian is stable under the perturbation of these operators with the operators of the set* **M**.

3. Example of a non-compact operator from the set M

Let us consider operator H, defined by the equality

$$H = P - \sum_{j=1}^{m}\sum_{k=1}^{n}(t-t_j)^{\alpha_k}P(t-t_j)^{-\alpha_k}I, \tag{9}$$

where $t_1, t_2, ..., t_m$, are arbitrary, distinct points on Γ and $\alpha_k = \frac{k-1}{n} + \frac{1-n}{np}$.

In this section we prove that the operator H belongs to the set **M** and is not completely continous on the space $L_p(\Gamma)$.

Lemma 3.1. *Operator*

$$H_{j,k} = (t-t_j)^{\alpha_k}P(t-t_j)^{-\alpha_k}I$$

is bounded on the space $L_p(\Gamma)$.

Proof. First we are going to show that the operator $H_{j,k}$ is bounded on the space $L_p(\Gamma)$ if and only if the operator P is bounded on the space L_p with the weight $\rho(t) = |t-t_j|^{\alpha_k p}$. Since $-1 < \alpha_k p < p-1$, then, from Theorem of B. Khvedelidze [8], the operator P is bounded on the space $L_p(\Gamma, |t-t_j|^{\alpha_k p})$. □

Theorem 3.1. *The operator*

$$H_{j,k} = (t - t_j)^{\alpha_k} P (t - t_j)^{-\alpha_k} I$$

belongs to algebra $\sum_p(\Gamma)$ *and its symbol has the form*

$$H(t,\xi) = \begin{cases} \begin{Vmatrix} 1 & 0 \\ 0 & 0 \end{Vmatrix}, & \text{for } t \neq t_j, \\ \begin{Vmatrix} 1 & -h(\xi)\frac{1-\exp(2\pi i \alpha_k)}{(1-f(\xi))(1-\exp(2\pi i\alpha_k))} \\ 0 & 0 \end{Vmatrix}, & \text{for } t = t_j. \end{cases} \tag{10}$$

Proof. Let $\varphi_{i,k}(z)$ be a fixed branch of the function $z^{-\alpha_k}$, defined on the complex plane with the cut connecting the point 0 with ∞ and intersecting the contour Γ at a single point t_j. The function $\varphi_{j,k}(t)$ is continuous at every point $\Gamma \setminus \{t_j\}$ and

$$\frac{\varphi_{i,k}(t_j)}{\varphi_{i,k}(t_j+0)} = e^{-2\pi i \alpha_k}.$$

Since $1/p - 1 < \alpha_k < 1/p$, the function $\varphi_{j,k}(t)$ admits a factorization on the space $L_p(\Gamma)$ in the form

$$\varphi_{j,k}(t) = (t - t_j)^{-\alpha_k} \left(\frac{t - t_j}{t}\right)^{\alpha_k}.$$

Let us consider the operator $B_{j,k} = \varphi_{j,k}(t)P + Q$. This operator $B_{j,k}$ is invertible in space $L_p(\Gamma)$ and its inverse is defined by the equality

$$B^{-1}_{j,k} = (t - t_j)^{\alpha_k} P \left(\frac{t - t_j}{t}\right)^{-\alpha_k} I + \left(\frac{t - t_j}{t}\right)^{\alpha_k} Q \left(\frac{t - t_j}{t}\right)^{-\alpha_k} I.$$

It follows from this that

$$PB^{-1}_{j,k} = (t - t_j)^{\alpha_k} P \left(\frac{t - t_j}{t}\right)^{-\alpha_k} I.$$

Therefore,

$$(t - t_j)^{\alpha_k} P \left(\frac{t - t_j}{t}\right)^{-\alpha_k} I = PB^{-1}_{j,k} t^{-\alpha_k} I = P(P + t^{\alpha_k} Q)^{-1}. \tag{11}$$

Because the operator $P(P+t^{\alpha_k}Q)^{-1}$ belongs to algebra $\sum_p(\Gamma)$, then from (11) it results that the operator $H_{j,k} = (t-t_j)^{\alpha_k} P (t-t_j)^{-\alpha_k} I$ also belongs to algebra $\sum_p(\Gamma)$. By direct calculations, taking into account the equality (11), we make sure that the operator symbol $H_{j,k}$ coincides with the right-hand side of the equality (10). $\square$

Corollary 3.1. *The operator H, defined by equality (9), belongs to algebra $\sum_p(\Gamma)$ and its symbol has the form*

$$H_{j,k}(t,\xi)=\begin{cases}\begin{Vmatrix}0 & 0\\ 0 & 0\end{Vmatrix}, & \textit{for } t\neq t_j,\\ \begin{Vmatrix}1 & h(\xi)\frac{1-\exp(2\pi(n-1)\xi)}{(1-f(\xi))(1-\exp(2\pi(n\xi+i/p)))}\\ 0 & 0\end{Vmatrix}, & \textit{for } t=t_j.\end{cases} \tag{12}$$

From the equality (12) it results.

Corollary 3.2. *The operator H is not compact. Moreover*

$$\psi(t,\xi)=\begin{cases}1, & \textit{for } t\neq t_j,\\ \exp(-2\pi(n-1)\xi), & \textit{for } t=t_j.\end{cases}$$

From this and from Theorem 2.1 it results.

Theorem 3.2. *The operator*

$$A=(a+1)P+Q-\sum_{j=1}^{m}\sum_{k=1}^{n}(t-t_j)^{\alpha_k}P(t-t_j)^{-\alpha_k}I$$

is Noetherian on the space $L_p(\Gamma)$ if and only if the operator $A_0=aP+Q$ has the same property.

4. The symbol of singular operators on contours with angular points

Let the contour Γ consist of two semi-axes starting at the point $z=0$. We denote by α $(0<\alpha\leq\pi)$ the angle formed by these half lines. We will assume that one of these half lines corresponds to the half axis $R^+=[0,+\infty)$ and that the contour Γ is oriented in a such way that the orientation on $\Gamma\cap R^+$ coincide with the orientation on R^+.

Let $B=L_p(\Gamma,|t|^\beta)$ $(-1<\beta<p-1)$. We denote by $\lambda_0(\Gamma)$ the set of constant piecewise functions that receives on Γ two values: a value on R^+ and another value on $\Gamma\setminus R^+$. If $h\in\lambda_0$, then we denote

$$h(t)=\begin{cases}h_1, & \text{for } t\in R^+,\\ h_2, & \text{for } t\in\Gamma\setminus R^+,\end{cases}\quad h_j\in\mathbb{C}.$$

Thus, $h(0)=h_2,\ h(0+0)=h_1,\ h(\infty-0)=h_1,\ h(\infty+0)=h_2$.

The contour Γ will be considered compactified with a point at infinity, whose neighborhoods are complements of the neighborhoods of $z_0=0$. Evidently, the contour Γ is homeomorphic with an bounded contour $\tilde{\Gamma}$, having two angular points.

We denote by K_α the Banach algebra generated by the singular integration operator S_Γ and all multiplication operators to the functions $h \in \lambda_0(\Gamma)$. By K^+ we denote the subalgebra of algebra $L(L_p(R^+, |t|^\beta))$ generated by singular integral operators $aI + bS$ $(S = S_{R^+})$ with constant coefficients on R^+. As K^+ is commutative, then it possesses [5] a sufficient system of multiplicative functionals. The operator ν is linear and bounded

$$(\nu\varphi)(x) = (\varphi(x), \varphi(e^{i\alpha}x)) \ (x \in R^+),$$

acting from the space $L_p(\Gamma, |t|^\beta)$ on the space $L_p^2(R^+, t^\beta)$.

Let $\varphi \in L_p(\Gamma, |t|^\beta)$ and consider the equation

$$A\varphi = a\varphi + bS_\Gamma\varphi = \psi,$$

$$a(t) = \begin{cases} a_1, \text{ for } t \in R^+, \\ a_2, \text{ for } t \in \Gamma \setminus R^+, \end{cases} \quad b(t) = \begin{cases} b_1, \text{ for } t \in R^+, \\ b_2, \text{ for } t \in \Gamma \setminus R^+, \end{cases} \quad a_j,\ b_j \in \mathbb{C}.$$

This equation can be written as a system of two equations: in one equation $t \in \Gamma \setminus R^+$ and in the second equation $t \in R^+$. We get,

$$\begin{cases} a(t)\varphi(t) + \dfrac{b(t)}{\pi i}\displaystyle\int_{R^+} \dfrac{\varphi(\tau)}{\tau - t}d\tau + \dfrac{b(t)}{\pi i}\displaystyle\int_{\Gamma\setminus R^+} \dfrac{\varphi(\tau)}{\tau - t}d\tau = \psi(t),\ t \in R^+, \\ a(t)\varphi(t) + \dfrac{b(t)}{\pi i}\displaystyle\int_{R^+} \dfrac{\varphi(\tau)}{\tau - t}d\tau + \dfrac{b(t)}{\pi i}\displaystyle\int_{\Gamma\setminus R^+} \dfrac{\varphi(\tau)}{\tau - t}d\tau = \psi(t),\ t \in \Gamma \setminus R^+. \end{cases}$$

In the integral $\int_{\Gamma\setminus R^+} \frac{\varphi(\tau)}{\tau - t}d\tau$ we change the variable $\tau \longrightarrow e^{i\alpha}\tau$ and in the second equation of the obtained system, we change t by $e^{i\alpha}t$. We obtain:

$$\begin{cases} a_1\varphi_1(t) + \dfrac{b_1}{\pi i}\displaystyle\int_{R^+} \dfrac{\varphi_1(\tau)}{\tau - t}d\tau - \dfrac{b_1}{\pi i}\displaystyle\int_{R^+} \dfrac{\varphi_2(\tau)}{\tau - e^{-i\alpha}t}d\tau = \psi_1(t),\ t \in R^+, \\ a_2\varphi_2(t) + \dfrac{b_2}{\pi i}\displaystyle\int_{R^+} \dfrac{\varphi_2(\tau)}{\tau - e^{i\alpha}t}d\tau - \dfrac{b_2}{\pi i}\displaystyle\int_{R^+} \dfrac{\varphi_2(\tau)}{\tau - t}d\tau = \psi_2(t),\ t \in \Gamma \setminus R^+, \end{cases}$$

in which the notations can be used: $f_1(t) = f(t),\ f_2(t) = f(e^{i\alpha}t)$ $(t \in R^+)$.

Thus, the operator $\nu A \nu^{-1}$ has the form

$$\nu A \nu^{-1} = \begin{Vmatrix} a_1 I + b_1 S & -b_1 M \\ -b_2 N & a_2 I + b_2 S \end{Vmatrix}$$

where

$$(S\varphi)(t) = \frac{1}{\pi i}\int_{R^+} \frac{\varphi(\tau)}{\tau - t}d\tau,\ (M\varphi)(t) = \frac{1}{\pi i}\int_{R^+} \frac{\varphi(\tau)}{\tau - e^{-i\alpha}t}d\tau,$$

$$(N\varphi)(t) = \frac{1}{\pi i}\int_{R^+} \frac{\varphi(\tau)}{\tau - e^{i\alpha}t}d\tau,\ t \in \Gamma.$$

From the results of the works [6], [7], [10], it appears that operators M and N belong to the algebra K^+ generated by the operator S $(= S_{R^+})$ and the multiplication operators on constant functions. Therefore, $\nu K_\alpha \nu^{-1} \subset (K^+)^{2\times 2}$.

Let $\{\gamma_M\}$ be the system of homeomorphisms determining the symbol on the algebra K^+. For any operator $A \in K_\alpha$ we put

$$\tilde{\gamma}_M(A) = \|\gamma_M(A_{jk})\|^2_{j,k=1}, \text{ where } |A_{jk}\|^2_{j,k=1} = \nu A \nu^{-1}.$$

Theorem 4.1. *The operator $A \in K$ is Noetherian on the space $L_p(\Gamma, |t|^\beta)$ if and only if*

$$det\tilde{\gamma}_M(A) \neq 0.$$

Proof. The factor algebra $\hat{K}^+$ is commutative with respect to all compact operators in $L(L_p(R^+, t^\beta))$. Therefore, the elements of the matrix operator $||A_{jk}||^2_{j,k=1} = \nu A \nu^{-1}$ commute with the exactness of a compact. Then according to [1] the operator $||A_{jk}||^2_{j,k=1}$ is Noetherian in $L_p(R^+, t^\beta)$ if and only if the operator $\Delta = det||A_{jk}||$ is Noetherian in $L_p(R^+, t^\beta)$. But the operator $det||A_{jk}||$ is Noetherina if and only if $\gamma_M(det||(A_{jk})||) \neq 0$. As $\gamma_M(det||(A_{jk})||) = det||\gamma_M(A_{jk})||$, it follows that A is Noetherian, if and only if $det\tilde{\gamma}_M(A) \neq 0$. □

Conclusion 4.1. *Theorem 4.1 allows us to define the symbol on algebra K. Namely, the matrix $\tilde{\gamma}_M(A)$ is called the symbol of the operators $A \in K$. Then Theorem 4.1 can be formulated as follows.*

Theorem 4.2. *The operator $A \in K$ is Noetherian on the space $L_p(\Gamma, |t|^\beta)$ if and only if the determinant of its symbol is non-zero.*

Taking into account the results of the work [7], the symbol of the operators $H = hI$ $(h \in \lambda_0(\Gamma))$ and S_Γ has the form:

$$\tilde{\gamma}_M = \begin{Vmatrix} h_1 & 0 \\ 0 & h_2 \end{Vmatrix}, \ \tilde{\gamma}_M(S_\Gamma) = \begin{Vmatrix} z & (z-1)^{1-\frac{\alpha}{2\pi}}(z+1)^{\frac{\alpha}{2\pi}} \\ (z-1)^{\frac{\alpha}{2\pi}}(z+1)^{1-\frac{\alpha}{2\pi}} & -z \end{Vmatrix}.$$

The symbol of the operator S_Γ can be written in a more convenient form. For this we put

$$z = \frac{e^{2\pi(\xi+i\gamma)}+1}{e^{2\pi(\xi+i\gamma)}-1} = cth(\pi(\xi+i\gamma)) \ \left(-\infty \leq \xi \leq +\infty, \ \gamma = \frac{1+\beta}{p}\right).$$

Then

$$(z-1)^{1-\frac{\alpha}{2\pi}}(z+1)^{\frac{\alpha}{2\pi}} = 2\frac{e^{\alpha(\xi+i\gamma)}}{e^{\alpha(\xi+i\gamma)}-1} = \frac{e^{(\alpha-\pi)(\xi+i\gamma)}}{sh\pi(\xi+i\gamma)},$$

$$(z-1)^{\frac{\alpha}{2\pi}}(z+1)^{1-\frac{\alpha}{2\pi}} = 2\frac{e^{(2\pi-\alpha)(\xi+i\gamma)}}{e^{\alpha(\xi+i\gamma)}-1} = \frac{e^{(\pi-\alpha)(\xi+i\gamma)}}{sh\pi(\xi+i\gamma)}.$$

Therefore the symbol of the operator can be written in the form

$$\tilde{\gamma}_M(S_\Gamma) = \left\| \begin{array}{cc} \operatorname{cth}(\pi(\xi+i\gamma)) & \frac{e^{(\alpha-\pi)(\xi+i\gamma)}}{\operatorname{sh}\pi(\xi+i\gamma)} \\ \frac{e^{(\alpha-\pi)(\xi+i\gamma)}}{\operatorname{sh}\pi(\xi+i\gamma)} & -\operatorname{cth}\pi(\xi+i\gamma) \end{array} \right\|.$$

Remark 4.1. *Let $\alpha = \pi$, i.e. the contour Γ satisfies the Lyapunov conditions at the point $z_0 = 0$. Then the symbol of the operator $H = hI$ does not change and the symbol of the operator S_Γ has the form* [1], [11]

$$\tilde{\gamma}_M(S_\Gamma) = \left\| \begin{array}{cc} z & -\sqrt{z^2-1} \\ \sqrt{z^2-1} & -z \end{array} \right\| = \left\| \begin{array}{cc} \operatorname{cth}\pi(\xi+i\gamma) & -(\operatorname{sh}\pi(\xi+i\gamma))^{-1} \\ (\operatorname{sh}\pi(\xi+i\gamma))^{-1} & -\operatorname{cth}\pi(\xi+i\gamma) \end{array} \right\|.$$

Now we can define the symbol of singular integral operators with coefficients from $CP(\Gamma)$ in the case of a piecewise Lyapunov contour.

Let Γ be a piecewise closed Lyapunov contour. We denote by $t_1, ..., t_n$ all the corner points with angles $\alpha_k (0 < \alpha < \pi)(k = 1, ..., n)$ and

$$\rho(t) = \prod_{k=1}^{n} |t - t_k|^{\beta_k} \quad (1 < p < \infty, \ -1 < \beta_k < p - 1).$$

We denote by $\sum(\Gamma, \rho) (\subset L(L_p(\Gamma, \rho))$ the algebra generated by operators $(H\varphi)(t) = h(t)\varphi(t)$, $h(t) \in CP(\Gamma)$ and the operator S_Γ. We mention, that the ideal formed by compact operators acting on the space $L_p(\Gamma, \rho)$ is contained in the algebra $\sum(\Gamma, \rho)$. Let us define the symbol of the operator from $\sum(\Gamma, \rho)$. For this it is sufficient to define the symbol of the operator hI ($h \in CP(\Gamma)$) and the operator S_Γ. We will use the local principle of Simonenko [12]. The symbol $H(t, \xi)$ ($t \in \Gamma$, $\xi \in R$) of the operator hI will be defined as follows:

$$H(t,\xi) = \left\| \begin{array}{cc} h(t+0) & 0 \\ 0 & h(t-0) \end{array} \right\|. \tag{13}$$

We define the symbol $S_\Gamma(t, \xi)$ of the operator S_Γ as follows:

$$S(t,\xi) = \left\| \begin{array}{cc} \operatorname{cth}\pi(\xi+i\gamma) & -\frac{\exp((\alpha(t)-\pi)(\xi+i\gamma(t)))}{sh\pi(\xi+i\gamma(t))} \\ \frac{\exp((\pi-\alpha(t))(\xi+i\gamma(t)))}{sh\pi(\xi+i\gamma)} & -\operatorname{cth}\pi(\xi+i\gamma(t)) \end{array} \right\|, \tag{14}$$

where

$$\alpha(t) = \begin{cases} \alpha_k, \ for \ t = t_k \ (k = 1, 2, ..., n), \\ \pi, \ for \ t \in \Gamma \setminus \{t_1, t_2, ..., t_n\}, \end{cases} \quad \text{and}$$

$$\gamma(t) = \begin{cases} \frac{1+\beta_k}{p}, \ for \ t = t_k \ (k = 1, 2, ..., n), \\ \frac{1}{p}, \ for \ t \in \Gamma \setminus \{t_1, t_2, ..., t_n\}. \end{cases}$$

Theorem 4.3. *Let $A \in \sum(\Gamma, \rho)$ and $A(t, \xi)$ be its symbol. The operator A is Noetherian on the space $L_p(\Gamma, \rho)$ if and only if*

$$detA(t, \xi) \neq 0 \; (t \in \Gamma, \; -\infty \leq \xi \leq +\infty). \tag{15}$$

5. Singular operators with a shift on a piecewise Lyapunov contour

Let Γ be a closed piecewise Lyapunov contour, $\nu : \Gamma \to \Gamma$ and $(V\varphi)(t) = \varphi(\nu(t))$. On the space $L_p(\Gamma)$, we consider a linear singular integral operator with a shift $\nu(t)$ of the form

$$(A\varphi)(t) = a(t)\varphi(t) + \frac{b(t)}{\pi i} \int_\Gamma \frac{\varphi(\tau)}{\tau - t} d\tau + c(t)\varphi(\nu(t)) + \frac{d(t)}{\pi i} \int_\Gamma \frac{\varphi(\nu(t))}{\tau - t} d\tau, \tag{16}$$

where $a(t),\; b(t),\; c(t)$ and $d(t)$ are bounded measurable functions on Γ. Assume that the mapping ν satisfies the following conditions:

(1) Carleman conditions: $\nu(\nu(t)) = t$;

(2) the derivative $\nu'(t)$ has a finite number of discontinuity points of the first kind on Γ , and on the arcs l_k connecting the discontinuity points it satisfies Hölder's condition, $\nu'(t) \in H(l_k)$;

(3) $\nu'(t \mp 0) \neq 0 \; (\triangledown t \in \Gamma)$.

Along with the operator A of the form (16), we also consider the operator $\tilde{A}$ defined on the space $L_p^2(\Gamma) = L_p(\Gamma) \times L_p(\Gamma)$ by the equality

$$\tilde{A} = \left\| \begin{matrix} aI + bS & cI + dS \\ \tilde{c}I + \tilde{d}S & \tilde{a}I + \varepsilon\tilde{b}S \end{matrix} \right\| + \left\| \begin{matrix} 0 & 0 \\ \tilde{d}(VSV - \varepsilon S) & \tilde{b}(VSV - \varepsilon S) \end{matrix} \right\| = \tilde{A}_0 + R, \tag{17}$$

where $\tilde{f} = f(\nu(t))$ and $\varepsilon = 1$ $(\varepsilon = -1)$, if the mapping ν preserves (changes) its orientation on the contour Γ. As it is known (see [4], [2] and the bibliography given in these papers), if a, b, c and d are continuous functions and $\nu'(t) \in H(\Gamma)$, then the operator is R completely continuous in $L_p(\Gamma)$ and

Theorem 5.1. *The operator A defined by equality (16) is Noetherian on the space $L_p(\Gamma)$ if and only if the operator $\tilde{A}_0$ is Noetherian in the space $L_p^2(\Gamma)$. When these conditions are fulfilled, the index of the operator A is calculated by formula*

$$IndA = \frac{1}{2} Ind\tilde{A}_0.$$

In this section, we prove that this assertion ceases to be true if Γ has corner points. In this case, as a rule, the derivative $\nu'(t)$ has discontinuity points on Γ, and it turns out that if the operator A is Noetherian, then the operator $\tilde{A}_0$ is also Noetherian. However, the converse assertion does not hold.

Theorem 5.2. *If operator $A = a(t)I + b(t)S + (c(t)I + d(t)S)V$ $(a, b, c, d \in C(\Gamma))$ is Noetherian on the space $L_p(\Gamma)$, then operator $\tilde{A}_0$ is also Noetherian on the space $L_p^2(\Gamma)$.*

Proof. Indeed, operator $\tilde{A}_0$ is Noetherian if and only if the conditions

$$\Delta_1(t) = (a(t) - b(t))(\tilde{a}(t) - \varepsilon\tilde{b}(t)) - (c(t) - d(t))(\tilde{c}(t) - \varepsilon d(t)) \neq 0,$$

$$\Delta_2(t) = (a(t) + b(t))(\tilde{a}(t) + \varepsilon\tilde{b}(t)) - (c(t) + d(t))(\tilde{c}(t) + \varepsilon d(t)) \neq 0,$$

hold for all $t \in \Gamma$. Let the operator A be Noetherian. Then the determinant of its symbol [1] does not vanish: $detA(t, \xi)$ $(t \in \Gamma, -\infty \leq \xi \leq +\infty)$. It is directly verified that

$$detA(t, -\infty) \cdot detA(t, -\infty) = \Delta_1(t) \cdot \Delta_2(t).$$

Hence the operator $\tilde{A}_0$ is Noetherian in $L_p^2(\Gamma)$. Theorem is proved. □

The following example shows that Theorem 5.1 cannot be inverted. Let us change the orientation of the contour Γ and the corner point $t_0 \in \Gamma$ with the angle θ $(0 < \theta < \pi)$ be a fixed point of the mapping $\nu : \nu(t_0) = t_0$. In this case, it is easy to verify that the derivative $\nu'(t)$ is discontinuous at the point t_0, and $\nu'(t_0 - 0) = \exp(-i\theta - \sigma)$ and $\nu'(t_0 + 0) = \exp(i\theta + \sigma)$, where σ is some real number. Consider the operator

$$A = I + \delta SV,$$

where δ is a complex number. The corresponding operator $\tilde{A}$ has the form

$$\tilde{A} = \begin{Vmatrix} I & \delta S \\ -\delta S & I \end{Vmatrix} + \begin{Vmatrix} 0 & 0 \\ \delta(VSV - S) & 0 \end{Vmatrix} = \tilde{A}_0 + R.$$

If $\delta \neq \pm i$, then the operator $\tilde{A}_0$ is Noetherian. Let $A(t_0, \xi)$ $(-\infty \leq \xi \leq +\infty)$ be the symbol of the operator A at the point t_0. It is directly verified that

$$detA(t_0, \xi) = \delta^2 + 2(\gamma + \beta)\delta + 1,$$

where

$$\gamma = \frac{\exp[(2\pi - \theta - i\sigma)(\xi + i/p)]}{\exp(\xi + i/p) - 1} \text{ and } \beta = \frac{\exp[(\theta + i\sigma)(\xi + i/p)]}{\exp(\xi + i/p) - 1}.$$

Hence, by virtue of Theorem 1.1 from [5], it follows that for all $\delta = -(\gamma + \beta)^{-1} \pm \sqrt{(\gamma + \beta)^2 - 1}$ the operator A is not Noetherian on the space $L_p(\Gamma)$. Thus, the condition for the operator A to be Noetherian depends on the angle θ.

Theorems 5.1 and 5.2, and the above example imply the following assertions

Corollary 5.1. *Let $\nu'(t) \notin H(\Gamma)$. Then operator $VSV - \varepsilon S$ is not compact on the space $L_p(\Gamma)$.*

Corollary 5.2. *If the operator A, defined by equality (16), is Noetherian, then the operators $\tilde{A}$ and $\tilde{A}_0$ defined by equality (17) are also Noetherian.*

Corollary 5.3. *If the operator $\tilde{A}$ is Noetherian, then $\tilde{A}_0$ is also Noetherian. The converse is generally not true.*

In conclusion, we note that the corresponding example of a non-Noetherian operator A for which $\tilde{A}_0$ is Noetherian can also be given in the case when the mapping ν preserves the orientation of the contour Γ.

6. Examples

The symbol of the singular integral operators and Theorem 2.1 can be used in studying different classes of composite singular operators. The difficulties which can arise in this context are the following: to show that the operator under consideration belongs to an algebra of operators with symbol; to write in the explicit form the symbol of this operator; to show that the symbol can be expressed as a singular perturbed operator that satisfies the conditions of Theorem 2.1.

We will consider an example where these difficulties arise and are overcome. Studying singular integral operators with homographic translations on the real axis in the space

$$L_p^\gamma = \left\{\varphi : \int_{-\infty}^{+\infty} |\varphi(x)|^p |x-\delta|^\gamma dx < \infty\right\} \ (-1 < \gamma < p-1,\ \delta \in R),$$

operators of the following form are investigated (see [12])

$$H\varphi = a(x) + \frac{b(x)}{\pi i}\int_{-\infty}^{+\infty} \frac{\varphi(t)}{t-x} dt + c(x)\frac{(x-\delta)^\lambda}{\pi i}\int_{-\infty}^{+\infty} \frac{(t-\delta)^{-\lambda}\varphi(t)}{t-x} dt,$$
$$-\frac{1+\gamma}{p} < \lambda < 1 - \frac{1+\gamma}{p}. \tag{18}$$

To apply the conditions of Theorem 2.1, we will express the operator H as follows

$$H = aI + bS + cM, \tag{19}$$

where

$$S\varphi = \int_{-\infty}^{+\infty} \frac{\varphi(t)}{t-x} dt$$

and

$$M\varphi = \frac{(x-\delta)^\lambda}{\pi i}\int_{-\infty}^{+\infty} \frac{(t-\delta)^{-\lambda}\varphi(t)}{t-x} dt.$$

The expression (19) implies the operator H to be a singular integral operator perturbed with the operator cM.

Theorem 6.1. *Let $a, b, c \in C(\bar{R})$. The operator $H = aI + bS + cM$ is Noetherian in the space L_p^γ if and only if the operator $H_0 = aI + bS$ is Noetherian. Moreover, $Ind\, H = Ind\, H_0$.*

Thus, the operator cM is a permissible perturbation to the operator H_0 and as a result of this perturbation his index remains the same.

Proof. Denote by $H(x,\mu)$, respectively by $H_0(x,\mu)$ the symbols of operators H and H_0. The symbol of the operator M at the point $x = \delta$ has the form

$$M(\delta,\mu) = \left\| \begin{matrix} 0 & u(\mu) \\ 0 & 0 \end{matrix} \right\|, \tag{20}$$

where

$$u(\mu) = \frac{4ih(\mu)\sin\pi\lambda \cdot \exp(\pi i\lambda)}{2if(\mu)\sin\pi\lambda \cdot \exp(\pi i\lambda) + 1},$$

$$f(\mu) = \begin{cases} \frac{\sin\theta\mu\cdot\exp(i\theta\mu)}{\sin\theta\cdot\exp(i\theta)}, & \text{for } \theta \neq 0, \\ \mu, & \text{for } \theta = 0, \end{cases} \qquad \theta = \pi - \frac{2\pi(1+\gamma)}{p}$$

and

$$h(\mu) = \sqrt{f(\mu)(1 - f(\mu))},\ 0 \le \mu \le 1.$$

Obviously, the operator M has singularities only at the points $x = \delta$ and $x = \infty$, therefore it is equivalent to the null operator at the points $x \in \bar{R} \setminus \{\delta, \infty\}$. Thus

$$M(x,\mu) = \left\| \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \right\|,\ x \neq \delta,\ x \neq \infty.$$

To calculate the symbol of M at the point $x = \infty$, we proceed as follows. We consider the linear and bounded operator $A : L_p(R, |x-\delta|^\gamma) \to L_p(R, |x-\delta|^{p-2-\gamma})$ defined by the relation

$$(A\varphi)(t) = \frac{1}{t}\varphi\left(\frac{\delta t - 1}{t}\right)\left((A^{-1}\psi)(x) = \frac{1}{x-\delta}\varphi\left(-\frac{1}{x-\delta}\right)\right).$$

The symbol of the operator M at the point $x = \infty$ is defined as the symbol of the operator AMA^{-1} at the point $x = 0$. We calculate AMA^{-1}. Let $f(x) = (x-\delta)^\lambda$, then

$$AfA^{-1} = \left(-\frac{1}{t}\right)^\lambda = t^{-\lambda}e^{\pi i\lambda}I,$$

$$(AfA^{-1}\varphi)(t) = -\frac{1}{t}\cdot\frac{1}{\pi i}\int_{-\infty}^{+\infty} \frac{\frac{1}{x-\delta}\varphi\left(\frac{1}{x-\delta}\right)}{x - \frac{\delta t - 1}{t}}dx = -\frac{1}{t}\cdot\frac{1}{\pi i}\int_{-\infty}^{+\infty}\frac{\tau\varphi(\tau)}{\frac{\delta\tau-1}{\tau}\,\frac{\delta t-1}{\tau}}\frac{d\tau}{\tau^2}.$$

$$(AfA^{-1}\varphi)(t) = (S\varphi)(t).$$

Thus, $AMA^{-1} = t^{-\lambda}St^{\lambda}I$ and λ verifies the condition

$$\frac{1+p-\gamma-2}{p} < \lambda < 1 - \frac{1+p-\gamma-2}{p}.$$

Therefore, the symbol of the operator M at the point $x = \infty$ has the form (20), where λ must be replaced by $-\lambda$, and θ by $-\theta$. This is equivalent to the fact that the function $u(\mu)$ from (20) is replaced by $\overline{u(\mu)}$. Thus, the symbol of the operator (18) has the form

$$H(x,\mu) = \left\| \begin{array}{cc} a(x)+b(x) & c(x)\sigma(x,\mu) \\ 0 & a(x)-b(x) \end{array} \right\|,$$

where $\sigma(x,\mu) = 0$ for any $x \in \bar{R} \setminus \{\delta, \infty\}$, $\sigma(\delta,\mu) = u(x)$ and $\sigma(\infty,\mu) = \overline{u(\mu)}$. It follows from this that $det H(x,\mu) = det H_0(x,\mu)$ which means that both operators H and H_0 are or are not Noetherians and $Ind\, H = Ind\, H_0$. □

Remark 6.1. *The statements of Theorem 6.1 remain to be true even if the functions are replaced by matrices functions with elements from $C(\bar{R})$.*

Conclusion 6.1. *The results presented in this paper show us that the property of singular integral operators to be Noetherian is stable with respect to their perturbation with certain noncompact operators. This property was established due to the symbol of some operators with singularities was determined. It was shown that the determinant of the symbol of the original operator coincides with the determinant of the symbol of the perturbed operator. Moreover, the indices of these operators are also equal.*

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Dedicated to Professor Alexandru Șubă on the occasion of his 70th birthday

The method for solving the multi-criteria linear-fractional optimization problem in integers

Alexandra Tkacenko

Abstract. In the paper we propose a method for solving the linear-fractional multi-criteria optimization model with identical denominators in whole numbers. Such models are in increasing demand, especially from an application point of view. The solving procedure of these models initially involves assigning utilities (weights) to each criterion [15] and building the optimization model with a single criterion, which is a synthetic function of all criteria weighted. It was found that the optimal solution of the model does not depend on the values optimum of the criteria obtained in R^+ or in Z^+. So, the decision maker can combinatorially select the types of optimal values of criteria, a fact that represents the essential priority of the algorithm. By changing the utility values, at the decision maker's discretion, we will obtain a new optimal compromise solution of the model. Theoretical justification of the algorithm as well as a solved example are brought to work.

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Keywords: multi-criteria fractional model in integers, basic efficient solution, optimal compromise solution.

Metodă de soluţionare a problemei de optimizare multicriterială de tip liniar-fracţionar în numere întregi

Rezumat. În actuala lucrare propunem o metodă de rezolvare a modelului de optimizare multicriterial de tip liniar-fracţionar cu numitori identici în numere întregi. Acest tip de modele înregistrează o solicitare practică în creştere. Procedura de soluționare a modelului presupune atribuirea inițială a unor utilităţii (ponderi) fiecărui criteriu [15], apoi se construieşte modelul de optimizare de tip liniar-fracţionar în numere întregi cu un singur criteriu, care este o funcţie sinteză a criteriilor ponderate. S-a dovedit că soluţia de compromis optim a modelului nu depinde de tipul soluțiilor optime a fiecărui criteriu real sau întreg pentru funcția sinteză, astfel fiind posibilă selectarea combinatorială a acestora, iar modificând utilităţile, obținem o nouă soluţie a modelului. Justificarea teoretică a algoritmului, cât și un exemplu rezolvat se aduc în lucrare.

Cuvinte-cheie: model multicriterial în numere întregi, soluţie eficientă de bază, soluţie de compromis optimal

1. Introduction

There is currently a growing demand for solving integer optimization problems. This happens because many decision situations require solving only in whole numbers [16]. Of course, this condition requires increased efforts to solve the optimization problem. Among the practical domains, where are needed optimal integer solutions, there is the problem of bi and three-dimensional cutting of materials [8], [9], [17]. A number of studies of this type have been done to solve the problem of dynamic memory allocation for multiprocessor and positioning systems. Several researchers have proposed various studies on this topic (Dowsland and Dowsland 1992, Sweeny and Paternoster 1992, Dyckhoff 1990, Coffman 1984, Golden 1976, Gilmore 1966). All approaches of these researchers can be divided into 3 categories: precise, heuristic and metaheuristic. The exact methods were investigated by Gilmore and Gomory (1961) and are considered the first methods actually applied in the tailoring industry.The fundamental drawback of these approaches is their inability to effectively solve the problems of large-dimensions. However, this effort increases significantly when the problem is multicriteria in nature, even if it is of linear type [5], [6], [7]. The requirement that the choice variables be of integer type increases the problem's complexity and the length of the solving time [1], [2]. That is why, the interest in this fertile field of scientific research remains opening further[10], [11], [12]. From a practical point of view, there is an increased interest for the multicriterial optimization models of linear-fractional type in whole numbers, a fact that intensified my research on these types of issues. Next, I will propose a study specifically dedicated to this type of models.

2. Defining the problem with specific reasoning

The integer multicriteria linear optimization problem is typically represented by a collection of linear constraints, including on the variables restrictions of non-negativity and integrity, such as equations and/or inequalities. The mathematical model of this type of problem [16] is as follows:

$$\begin{cases} optimum\{F_k(x)\}, k = \overline{1, r} \\ x \in D \\ D = \textit{the field of the admissible solutions} \end{cases} \tag{1}$$

in which: $D = \{x = (x_1, x_2, ..., x_n)^T \, | Ax \leq b, x \in Z^+\}$.

The explicit form of model (1), in which the objective functions have the same denominator, being of linear-fractional type, is the following:

$$\begin{cases} \left\{ \begin{array}{c} \min \\ \max \end{array} \right\} F_k(x) = \frac{\sum\limits_{j=1}^{n} c_{kj}x_j}{\sum\limits_{j=1}^{n} d_j x_j} \\ Ax \le b \\ x \in Z^+ \end{cases} \tag{2}$$

in which: $D = \{x = (x_1, x_2, ..., x_n)^T \,|Ax \le b, x \in Z^+\}$, $A = ||a_{ij}||$ is an array of size $m \times n$ $(m < n)$, $C = ||c_{ij}||$ is an array of size $r \times n$ $(r < n)$, $d-$ is a n-dimensional line vector, x is a n-dimensional column vector and b is a m-dimensional column vector.

The parameters c_{kj}, as well as d_j, may be the most different, according of their practical meanings such as unit costs or benefits, unit of damages or others close in this meaning. The type of related objective function, minimum or maximum, is determined by their relevance. Similar to how the elements of the vector b indicate the resources available by types, the elements of the matrix A, $\{a_{ij}\}$, represent the specific consumption of the resource j for the creation of a product unit of type i.

In order to solve the model (2) obviously, the value of the denominator function, which is the same for all criteria, must be nonzero on the domain D, that is the following condition must be satisfied:

$$\sum_{j=1}^{n} d_j x_j \neq 0, (\forall) x \in D.$$

It should be noted, that if in model (2) there are criteria of both minimum and maximum type, it is not complicated to homogenize them, if necessary.

Unfortunately, it is well known that the multicriteria optimization model rarely admits an optimal solution. That's why, in order to solve the multicriteria model, the notion of a solution that achieves the best compromise, solution of the optimal compromise, non-dominant solution, efficient solution, optimal solution in the Pareto sense, etc. is used. In [13] different ways of defining the vector solution x^* of the best compromise for the real-type multicriteria optimization model are proposed. We will adapt some of them to solve the integer multicriteria linear-fractional optimization model (2) as follows.

1. The solution $x^* \in Z^+$ for the model (1) is the vector that optimizes a synthesis function of all r criteria, ie: $h(F) = h[F_1, F_2, ..., F_r]$. We mention that $h(\cdot)$ can be defined in various ways.

2. The solution $x^* \in Z^+$ is the vector which minimizes a single criterion such as: $\phi(x^*) = \min_{x \in D} h(\Psi_1(x - X_1), ..., \Psi_1(x - X_r))$, in which $X_j = (x_{1j}, x_{2j}, ..., x_{nj})^T, j = \overline{1, r}$

is the optimal solution of each criterion, F_j, and Ψ_k is a distance type function between the vector $x \in D$ and optimal solution X_k for each criterion F_k.

3. The solution $x^* \in Z^+$ is a vector which belongs to a set of efficient solutions of integer type. Because the model (1) is of multicriteria type, it is well known, that such of models in general rarely admit optimal solutions. Solving model (1) involves constructing a finite set of efficient integer solutions known as best compromise solutions [13], which we mentioned earlier. For the model (2) we will adapt the next definitions.

Definition 2.1. *The basic solution x^* of the model (2), where $x^* \in Z^+$, is called optimal overall if it is the optimal solution for each of criteria.*

Definition 2.2. *The basic solution $\overline{X}$, where $\overline{X} \in Z^+$ of the model (2) is a basic efficient one if and only if it doesn't exists any other basic solution $X \in Z^+$, where $X \neq \overline{X}$, which would improve the values of all criteria and at least one of criteria would be strictly improved.*

The more exact, mathematical version of the same definition is proposed below.

Definition 2.3. *The basic solution $\overline{X} \in Z^+$ of the model (2) is a basic efficient one if and only if for any other basic solution $X \in Z^+$, where $X \neq \overline{X}$, for which the relations $F_{j_1}(X) \geq F_{j_1}(\overline{X})$ are true, where $j_1 \in (1, ..., j_2)$, indices corresponding to the maximum type of criteria immediately exists at least one index $\exists j_l \in (j_2 + 1, ..., r)$, of the minimum type for which is true the relation: $F_{j_l}(X) > F_{j_1}(\overline{X})$ or, if the relation $F_{j_l}(X) \leq F_{j_1}(\overline{X})$ is true for all indices corresponding to the minimum type of criteria which are $j_l \in (j_2 + 1, ..., r)$, immediately exists at least one index from the set of indices of the maximum type of criteria $\exists j_1 \in (1, ..., j_2)$, for which the next relation $F_{j_2}(\overline{X}) < F_{j_2}(X)$ is true.*

3. Section plans method

In order to iteratively improve the integer solution of the optimization model, the section plans approach involves a sectioning procedure for the domain of admissible solutions. Sections are executed in accordance with predetermined rules. The section plans algorithm is often referred to as the "Cyclic Algorithm". The algorithm iteratively modifies one of the components of the admissible solution of the optimization problem, cutting each time the admissible domain, so that the new obtained solution remains admissible. Of course, at each iteration the value of the objective function changes in the direction opposite to the criterion type. Since the algorithm is convergent and finite, after a finite number of steps it determines the optimal integer solution of the model, if it exists. Despite the fact that the convergence of this algorithm has not been proven, no examples have been found that contradict it. This algorithm is also known as Gomory's algorithm [4], in honor of the American scientist, R. Gomory, who created it and published the method for the first time in 1958. We will describe further, applying mathematical formulas the sectioning process adapted for the linear-fractional optimization problem. Next we will consider the following couple of optimization problems:

$$(ILP)\begin{cases} (\max) f = \dfrac{cx + c_0}{dx + d_0} \\ Ax = b \\ x \geq 0 \\ x \in Z^+ \end{cases} \qquad (LP)\begin{cases} (\max) f = \dfrac{cx + c_0}{dx + d_0} \\ Ax = b \\ x \geq 0 \end{cases}$$

where the elements of the matrix A and the components of the vector b, c,d, and the constants c_0, d_0, all are of integer type.

We denote $D_0 = \{x|A \cdot x = b, x \in Z^+\}$ and $D = \{x|A \cdot x = b, x \geq 0\}$, where D_0 is the domain of admissible solutions of the problem (ILP) and D of the problem (LP), respectively. We will assume that the function at the denominator is different from zero in D, which appears like this: $dx + d_0 \neq 0, (\forall)x \in D$.

Algorithm stages with theoretical justifications

We'll start off assuming that x^* doesn't have all of the integer components. In this instance, a constraint of the fractional optimum x^* is constructed; nonetheless, it is satisfied by any admissible solution of whole type. It is added to the original problem noted with (LP_0), after which the optimal solution will be re-optimized. Let x^{**} be the optimal solution to the new constructed problem, denoted by (LP_1). Because of the way the additive constraint was defined, we will have the following true relationships between admissible domains: $D_{ILP} \subset D_{LP_1} \subset D_{LP_0} = D_{LP}$.

If x^{**} does not have all components of integer type, the described procedure is repeated: a new restriction is constructed, which is not satisfied by x^{**}, but is verified by the set of admissible solutions. This new restriction is added to (LP_1), resulting a new linear optimization problem (LP_2). The sectioning procedure is as follows: $D_{ILP} \subset D_{LP_2} \subset D_{LP_1} \subset D_{LP_0} = D_{LP}$.

After applying the reoptimization procedure of the new admissible solution, it is decided whether LP_2 admit or not optimal solution. The theory guarantees that, after a finite number of steps, we obtain a linear-fractional programming problem, (LP_{k-1}), whose optimal solution is $x^{k(*)}$, which has all integer components, hence it is the optimal solution of our proposed problem (ILP).

Geometrically, each new added constraint removes some part of the set of admissible solutions, thus cutting off an intrusive section of the entire admissible domain. Next, we will describe the admissible domain partitioning algorithm proposed in [16], adapted for the linear-fractional optimization problem by imposing additional partitioning restrictions. We shall take into consideration the vector $\overline{b}$ and matrix $\overline{A}$, which correspond to the optimal solution of model (LP). We'll assume that the vector $\overline{b}$ doesn't have all integer components. Let the vector component $\overline{b}$ with the largest fractional part be $\overline{b}_r$.

We can represent the constraint coefficients as follows:

$$\overline{b_r} = x_r + \sum_{j \in J} \overline{a_{rj}} x \tag{3}$$

which can be decomposed thus:

$$\left[\overline{b_r}\right] + \left\{\overline{b_r}\right\} = x_r + \sum_{j \in J} \left([\overline{a_{rj}}] + \{\overline{a_{rj}}\}\right) x_j. \tag{4}$$

Because we have true the relationship $0 < \left\{\overline{b_r}\right\} < 1$, the following equality is also true:

$$[\overline{b_r}] - \sum_{j \in J} [\overline{a_{rj}}] x_j - x_r = \sum_{j \in J} \left\{\overline{a_{rj}}\right\} x_j - \left\{\overline{b_r}\right\} \tag{5}$$

Let x be a whole admissible solution of the problem (ILP). Therefore, the left-hand member of the relation (5) is an integer, so we get the following relation true:

$$[\overline{b_r}] - \sum_{j \in J} [\overline{a_{rj}}] x_j - x_r \in Z \tag{6}$$

It follows that the right-hand side of equality (5), which is calculated in the same solution, is an integer, so we have true the next relationship:

$$\sum_{j \in J} \left\{\overline{a_{rj}}\right\} x_j - \left\{\overline{b_r}\right\} \in Z. \tag{7}$$

Obviously, the next real relation is true:

$$\sum_{j \in J} \left\{\overline{a_{rj}}\right\} x_j - \left\{\overline{b_r}\right\} \geq 0. \tag{8}$$

If, however, by absurdity, we assume that:

$$\sum_{j \in J} \left\{\overline{a_{rj}}\right\} x_j - \left\{\overline{b_r}\right\} < 0, \tag{9}$$

then from the equality (5) yields the following true inequality:

$$\left[\overline{b_r}\right] - \sum_{j \in J} \left[\overline{a_{rj}}\right] x_j - x_r < 0 \tag{10}$$

and from (6) results the true expression:

$$\left[\overline{b_r}\right] - \sum_{j \in J} \left[\overline{a_{rj}}\right] x_j - x_r \leq -1. \tag{11}$$

From (5), we will obtained the next relations: $\sum\limits_{j\in J}\left\{\overline{a_{rj}}\right\}x_j-\left\{\overline{b_r}\right\}\leq -1$, whence it follows:

$$\sum_{j\in J}\left\{\overline{a_{rj}}\right\}x_j\leq\left\{\overline{b_r}\right\}-1 \tag{12}$$

Since the relationship is true: $\left\{\overline{a_{rj}}\right\}\geq 0$, $(\forall)j\in J$, we get that the left member of the relationship (5) is also positive, while the right limb is negative, since $\left\{\overline{b_i}\right\}<1$. The obtained contradiction proves that the inequality (8) is true. Because x has been chosen arbitrarily, we conclude that the next restriction is also true:

$$\sum_{j\in J}\left\{\overline{a_{rj}}\right\}x_j\geq\left\{\overline{b_r}\right\} \tag{13}$$

and that this is verified by any admissible solution of integer type.

But in the optimal solution, which is not of integer type $x*$, we will get: $x_j^*=0, j\in J$. Inserting these values into (8), we obtain the inequality: $\left\{\overline{b_r}\right\}\leq 0$, which contradicts the hypothesis (4), according to which we had: $\left\{\overline{b_r}\right\}>0$. So, it turns out that the fractional optimum x^* does not verify the inequality (10). Adding this constraint to model (10), we obtain a new linear optimization problem with $(m+1)$ constraints, which we denote by (LP_1). By introducing a new deviation variable x_{n+1}, we will transform the added constraint into equality, after which we will apply the re-optimization procedure of the model solution. The new restriction introduced in (10): $-\sum\limits_{j\in J}\left\{\overline{a_{rj}}\right\}x_j+x_{n+1}=-\left\{\overline{b_r}\right\}$, is considered a *plane sectioning restriction*.

4. Method of maximizing global utility

The global utility maximization method is known as the method of Boldur Latescu, a Romanian researcher, who developed it, as mentioned in [13]. It is based on the idea of transforming of the objective functions of a multicriteria problem into utility functions in the sense of von Neumann—Morgenstern [3], which are to be summed to obtain a synthesis function. In the hypothesis of the existence of a multicriteria linear programming problem, this method can be used quite effectively even in the case of an infinite number of decision variants. We will extend the method for the case when the objective functions are of linear fractional type with the same denominator.

Definition 4.1. *Utility is a subjective amount of appreciation of the event by the decision maker on a certain scale of values depending on the specifics of the event [3].*

Definition 4.2. *Given n criteria $C_1, C_2, ..., C_n$, they are called mutually independent in the sense of the theory of utility, if and only if we have the true relation: $\omega_i \sim \omega_j$ for anything $(\omega_i , \omega_j) \in G$, where G is the events space.*

Since the additivity of utilities is obviously possible, we will have true the following relationship:

$$U(a_{i1}, a_{i2}, ..., a_{in}) = u_1(a_{i1}) + u_2(a_{i2}) + ... + u_n(a_{in}).$$

The independence of the criteria in the sense of the utility theory specifies that any consequence of the possible decision variant of a criterion always corresponds to the same a priori assigned utility.

Global utility maximization algorithm

We will present the algorithm of the global utility maximization method [13], considering the case of the linear multicriteria optimization problem (1).

Step 1. We will consider for each objective function its optimal value X_j, which is determined, where $F_j = optim_{x \in D} F_j(x)$ and Y_j is its pessimistic value, where $F_j^p = pessim_{x \in D} F_j(x)$. We note that in these cases we will solve fractional linear models, applying the adapted simplex algorithm [14].

Step 2. For all sets of optimal and worst values of the criteria, the corresponding values of utilities in the sense Neumann–Morgenstern [13] are associated as follows:

$$\left\{F_1, F_2, ..., F_r; F_1^p, F_2^p, ..., F_r^p\right\} \longrightarrow \{U_1, U_2, ..., U_r; U_{r+1}, U_{r+2}, ..., U_{2r}\}.$$

Step 3. The objective functions F_j are presented as utility functions FU_j, by solving r linear systems with $2r$ variables. The unknowns in these equations are the coefficients of the type: $\left\{(\alpha_j, \beta_j)\right\}_{j=\overline{1,r}}$.

Using the solutions of the r systems of linear equations of this type:

$$\begin{cases} \alpha_j F_j + \beta_j = u_j \\ \alpha_j F_j^p + \beta_j = u_{j+r} \end{cases}, \ j = \overline{1, r},$$

we will build the following r utility functions such as:

$$FU_j = \alpha_j F_j(X) + \beta_j, \ j = \overline{1, r}.$$

Step 4. At the final stage, we will solve a single problem of linear programming whose objective is to maximize the global utility function UG, which is as follows:

$$\max_{x \in D} UG = \max_{x \in D} \sum_{j=1}^{r} \pi_j FU_j,$$

where π_j is the weight coefficient of the criterion C_j, which, obviously, can be changed by the decision maker, thus obtaining another, a new linear optimization problem.

5. The generalized synthesis algorithm

A rather important problem, which we obviously face when solving the multicriteria optimization problem in integers, using the methods of synthesis functions, as mentioned in [16], is formulated as follow: what type of optimal solution of each criterion must be used to construct the synthesis function of all criteria, in R^+ or in Z^+, so that the final model efficiently solves the problem in Z^+ ?

In this paragraph we will answer and justify the answer to this question. We will adjust the global utility maximization method for the objective functions of the linear-fractional criteria, in order to use it in solving the proposed model (2). The algorithm will be performed in two stages.

Stage I:

1. At this stage it is necessary to solve $2r$ unicriteria linear fractional programming problem from model (2), of which r are of the type: $F_j = optim_{x \in D} F_j(x)$ and the other r of the type: $F_j^p = pessim_{x \in D} F_j(x)$ on the same admissible domain:
$$D = \{x \in R | Ax \leq b, x \geq 0\};$$

2. Next, we will analogically solve $2r$ linear fractional programming problems of integer type as follows, the first r of the type: $F_j = optim_{x \in D} F_j(x)$ and the others r of the type: $F_j^p = pessim_{x \in D} F_j(x)$ all on the domain: $D = \{x \in Z^+ | Ax \leq b, x \geq 0\}$;

3. We will build the vectors of records of the optimal values of the objective functions, using in each combinatorial vector both values of some criteria in Z^+ and of others in R^+. Analogously, we will build the vectors of the worst records of the objective functions in R^+ and Z^+. Since the size of the optimization problem is finite, it follows that the number of such combinations is also finite. These combinations can be described as follows:

$$\left\{ \begin{pmatrix} F_1(R^+) \\ F_2(R^+) \\ \dots \\ F_r(R^+) \end{pmatrix} \vee \begin{pmatrix} F_1(R^+) \\ F_2(Z^+) \\ \dots \\ F_r(Z^+) \end{pmatrix} \vee \begin{pmatrix} F_1(R^+) \\ F_2(R^+) \\ \dots \\ F_r(Z^+) \end{pmatrix} \vee \dots \vee \begin{pmatrix} F_1(Z^+) \\ F_2(Z^+) \\ \dots \\ F_r(Z^+) \end{pmatrix} \right\},$$

$$\left\{ \begin{pmatrix} F_1^p(R^+) \\ F_2^p(R^+) \\ \dots \\ F_r^p(R^+) \end{pmatrix} \vee \begin{pmatrix} F_1^p(R^+) \\ F_2^p(Z^+) \\ \dots \\ F_r^p(Z^+) \end{pmatrix} \vee \begin{pmatrix} F_1^p(R^+) \\ F_2^p(R^+) \\ \dots \\ F_r^p(Z^+) \end{pmatrix} \vee \dots \vee \begin{pmatrix} F_1^p(Z^+) \\ F_2^p(Z^+) \\ \dots \\ F_r^p(Z^+) \end{pmatrix} \right\}.$$

The total number of such vectors is: $N(V) = C_r^1 + C_r^2 + \dots + C_r^r$, analogously, the same number is for pessim type of criteria.

Stage II:

1. Randomly considering one of the vector records of the optimal values of the objective functions and correspondingly the vector of records of the worst values, we construct the synthesis function of the model, which expresses the summary utility of all criteria thus: $G = \sum_{j=1}^{r} \left(\alpha_j F_j + \beta_j\right)$, which is obviously to be maximized. The utility coefficients $\left\{\left(\alpha_j, \beta_j\right)\right\}_{j=\overline{1,r}}$ are determined by solving the systems of equations for each criterion, as previously described. Finally, we will solve the following model on linear programming:

$$\max_{x \in D} G = \sum_{j=1}^{r} \left(\alpha_j F_j\left(X\right) + \beta_j\right),$$

where $D = \{x|Ax = b, x \in Z^+\}$. The optimal solution of this problem is the optimal compromise solution for model (2). By calculating the values of each objective function of the model (2) in the obtained optimal solution we will construct the following vector of records of all objective functions:

$$\left\{\begin{array}{c} F_1(X^*) \\ F_2(X^*) \\ \ldots \\ F_r(X^*) \end{array}\right\}.$$

Theorem 5.1. *For any utility values assigned a priori to the objective functions in model (2), where the identical denominator is nonzero over the admissible domain, the optimal compromise solution corresponding to them remains the same for any combinatorial selection of the optimal values of the criteria and the corresponding pessimistic ones from in R^+ or in Z^+.*

Proof. Let X^1_{eff} be a solution of the optimal compromise for the model (2) of integer type, for a given a priori set of utilities, obtained by applying the global utility maximization method. Because the solution is of the optimal compromise, it turns out that it is the closest located to the optimal solutions of the whole type of each criterion. We will assume that the synthesis function of the model, which generated the given solution, was obtained using a certain combination of optimal and pessimistic values of the objective functions of the model (2), some being solved in R^+, others in Z^+.

Let $\begin{pmatrix} F_1(R^+) \\ F_2(R^+) \\ \ldots \\ F_r(Z^+) \end{pmatrix}$ be the vector of optimal and correspondingly pessimistic $\begin{pmatrix} F_1^p(R^+) \\ F_2^p(R^+) \\ \ldots \\ F_r^p(Z^+) \end{pmatrix}$ objective functions values. We will admit, analogously to the demonstration in [16], that

for another record values, different from the previous one, of the values of the objective functions of model (2), let it be
$\begin{pmatrix} F_1(Z^+) \\ F_2(R^+) \\ \dots \\ F_r(Z^+) \end{pmatrix}$ and corresponding vector of the pessimistic values $\begin{pmatrix} F_1^p(Z^+) \\ F_2^p(R^+) \\ \dots \\ F_r^p(Z^+) \end{pmatrix}$, the objective synthesis function admits another integer-optimal compromise solution that is different from the first one, either it is: X_{eff}^2. If $X_{eff}^1 . \neq X_{eff}^2$, there is at least one coordinate by which these solution vectors differ from each other. So, we will have at least one criterion of model (2), be it having indices i_1, for which the distance between its optimal solution in integers and the new solution is smaller than the previously received one. Therefore, we will have the following true relation: $\rho\left(X_{eff}^1 . X_{i_1}^*\right) > \rho\left(X_{eff}^2 . X_{i_1}^*\right)$, where $X_{i_1}^*$ is optimal solution in integer of criterion i_1, fact that contradicts the assumption that X_{eff}^1 is a optimal compromise solution of integer type for the model (2). Therefore, our assumption is wrong. So, in conclusion, we obtained, that model (2) admits a single optimal compromise solution in integers, regardless of the type of optimal values of the objective functions of the model solved in R^+ or Z^+, which is used to build the synthesis function of the proposed model.

Remark 5.1. *Obviously, for any values of the a priori utilities assigned to the criteria in the multicriteria optimization model (2), applying the global utility maximization method, we will obtain the optimal compromise solution of integer type corresponding to them.*

6. Conclusions

Multicriteria optimization models have always enjoyed increased interest. This trend is maintained even today, especially due to the fact that they more adequately describe the decision-making situations in the most diverse socio-economic fields, and the optimal compromise solution of such a model effectively solves the real situation described. In the current paper, an efficient algorithm is proposed for solving the multicriteria optimization model in integers, where the objective functions are of the linear-fractional type with identical denominators. Obviously, the complexity of such a problem is increased, but the practical necessity of its solution is certainly imposed. For this purpose, we focused on using the synthesis function methods, namely the global utility maximization method, adapted for solving the proposed multicriteria optimization model. This method leads to the determination of an optimal compromise integer solution for all criteria, which are of linear-fractional type and which is closest to the optimal integer solutions of each criterion.

As a result of the algorithm investigation, we obtained a significant result for determining the optimal compromise solution in whole numbers of the proposed model. Thus, for its determination, the decision-maker can use combinatorially both the optimal values of some criteria in whole numbers, as well as others calculated on the set of real numbers, all, of course, positive when constructing the synthesis function. Regardless of the configuration used to construct the synthetic function, its optimal integer solution does not change. Therefore, the decision-maker has the free choice to select more advantageous values of the criteria from his point of view for building the synthesis function of the model. This fact is quite important, making it possible to solve the model interactively, obviously increasing both the efficiency and the attractiveness of the algorithm.

Example 5.1. For the following linear-fractional multicriteria optimization model in integers and for the proposed values of the criteria's utilities, the optimal compromise solution is to be determined, using the global utility maximization method:

$$\min\left\{F_1(X)=\frac{x_1+2x_2+x_3}{x_1+x_2+x_3+1}\right\},\quad \max\left\{F_2(X)=\frac{2x_1+x_2+2x_3}{x_1+x_2+x_3+1}\right\},$$

$$\max\left\{F_3(X)=\frac{2x_1+3x_2+x_3}{x_1+x_2+x_3+1}\right\},$$

$$\begin{cases} 3x_1+5x_2+x_3 \le 18 \\ 5x_1+3x_2+2x_3 \le 20 \\ 2x_1+x_2+2x_3 \ge 5 \\ x_j \in Z^+. \end{cases}$$

F_1	F_2	F_3	F_1^p	F_2^p	F_3^p
$U_1=4$	$U_2=8$	$U_3=9$	$U_4=1$	$U_5=2$	$U_6=2$

Solving procedure: For solving the proposed model, we will apply the global utility maximization method, being one of synthesis type. Initially, we can observe, that in the model the value of the denominator will be non-zero in D. We will go through stage I of the algorithm. For this purpose we will solve six unicriteria linear- fractional programming problems in R^+, recording the optimal and worst values for each objective function. Next, in an analogous way, we will solve the same six unicriteria linear-fractional programming problems on the set Z^+, keeping the optimal and pessimistic values of each criterion. For the construction of the synthesis function, using the global utility maximization method, we will randomly select any combination of records of the corresponding optimal and worst criteria values, some criteria being solved in R^+ and others in Z^+.

These are as follows:

$$1)\left\{\begin{array}{c}F_1(R^+)\\F_2(R^+)\\F_3(R^+)\end{array}\right\},\left\{\begin{array}{c}F_1^p(R^+)\\F_2^p(R^+)\\F_3^p(R^+)\end{array}\right\};2)\left\{\begin{array}{c}F_1(Z^+)\\F_2(Z^+)\\F_3(Z^+)\end{array}\right\},\left\{\begin{array}{c}F_1^p(Z^+)\\F_2^p(Z^+)\\F_3^p(Z^+)\end{array}\right\};$$

$$3)\left\{\begin{array}{c}F_1(R^+)\\F_2(Z^+)\\F_3(Z^+)\end{array}\right\},\left\{\begin{array}{c}F_1^p(R^+)\\F_2^p(Z^+)\\F_3^p(Z^+)\end{array}\right\};4)\left\{\begin{array}{c}F_1(R^+)\\F_2(R^+)\\F_3(Z^+)\end{array}\right\},\left\{\begin{array}{c}F_1^p(R^+)\\F_2^p(R^+)\\F_3^p(Z^+)\end{array}\right\};$$

$$5)\left\{\begin{array}{c}F_1(R^+)\\F_2(Z^+)\\F_3(R^+)\end{array}\right\},\left\{\begin{array}{c}F_1^p(R^+)\\F_2^p(Z^+)\\F_3^p(R^+)\end{array}\right\};6)\left\{\begin{array}{c}F_1(Z^+)\\F_2(Z^+)\\F_3(R^+)\end{array}\right\},\left\{\begin{array}{c}F_1^p(Z^+)\\F_2^p(Z^+)\\F_3^p(R^+)\end{array}\right\};$$

$$7)\left\{\begin{array}{c}F_1(Z^+)\\F_2(R^+)\\F_3(Z^+)\end{array}\right\},\left\{\begin{array}{c}F_1^p(Z^+)\\F_2^p(R^+)\\F_3^p(Z^+)\end{array}\right\};8)\left\{\begin{array}{c}F_1(Z^+)\\F_2(R^+)\\F_3(R^+)\end{array}\right\},\left\{\begin{array}{c}F_1^p(Z^+)\\F_2^p(R^+)\\F_3^p(R^+)\end{array}\right\}.$$

The optimal solutions of the unicriteria models as well as the weight criteria, we placed them directly in the vectors of value combinations of the objective functions proposed above. Next, we solved 24 systems of linear equations in order to determine the weight coefficients of each criterion in the synthesis function: $\left\{\left(\alpha_j,\beta_j\right)\right\}_{j=\overline{1,r}}$. For each of the selected combinations of objective function values we have built the corresponding synthesis functions using the same criterion utility table for the proposed model. We obtained the following utility functions:

$$F_1\left(U\right)=\frac{1,73x_1+1,63x_2+1,09x_3}{x_1+x_2+x_3+1}\longrightarrow\max$$

$$F_2\left(U\right)=\frac{1,83x_1+1,75x_2+1,13x_3}{x_1+x_2+x_3+1}\longrightarrow\max$$

$$F_3\left(U\right)=\frac{1,85x_1+1,8x_2+1,15x_3}{x_1+x_2+x_3+1}\longrightarrow\max$$

$$F_4\left(U\right)=\frac{1,85x_1+1,8x_2+1,15x_3}{x_1+x_2+x_3+1}\longrightarrow\max$$

$$F_5\left(U\right)=\frac{1,73x_1+1,63x_2+1,09x_3}{x_1+x_2+x_3+1}\longrightarrow\max$$

$$F_6\left(U\right)=\frac{1,7x_1+1,57x_2+1,07x_3}{x_1+x_2+x_3+1}\longrightarrow\max$$

$$F_7\left(U\right)=\frac{1,83x_1+1,75x_2+1,13x_3}{x_1+x_2+x_3+1}\longrightarrow\max$$

$$F_8\left(U\right)=\frac{1,7x_1+1,57x_2+1,07x_3}{x_1+x_2+x_3+1}\longrightarrow\max$$

The above expressions express the summary utility of all criteria for the corresponding weights and are to be maximized over the admissible domain of the model, given by the same restrictions:

$$\begin{cases} 3x_1 + 5x_2 + x_3 \le 18 \\ 5x_1 + 3x_2 + 2x_3 \le 20 \\ 2x_1 + x_2 + 2x_3 \ge 5 \\ x_j \in Z^+ \end{cases}$$

By solving these 8 constructed problems, which are of integer linear programming type, we obtained the same optimal compromise solution:

$$X^1_{eff} = X^2_{eff} = X^3_{eff} = X^4_{eff} = X^5_{eff} = X^6_{eff} = X^7_{eff} = X^8_{eff} = \{x_1 = 1, x_2 = 3, x_3 = 0\}.$$

Further we calculated the values of utility functions, which are as follows:

$F_1(U) \approx 2,38; F_2(U) \approx 2,41; F_3(U) \approx 2,4; F_4(U) \approx 2,4;$

$F_5(U) \approx 2,38; F_6(U) \approx 2,39; F_7(U) \approx 2,41; F_8(U) \approx 2,39;$

$$F\left(X^1_{eff}\right) = \left\{\begin{array}{c} F_1\left(X^1_{eff}\right) \\ F_2\left(X^1_{eff}\right) \\ F_3\left(X^1_{eff}\right) \end{array}\right\} = F\left(X^2_{eff}\right) = \left\{\begin{array}{c} F_1\left(X^2_{eff}\right) \\ F_2\left(X^2_{eff}\right) \\ F_3\left(X^2_{eff}\right) \end{array}\right\} =$$

$$= F\left(X^3_{eff}\right) = \left\{\begin{array}{c} F_1\left(X^3_{eff}\right) \\ F_2\left(X^3_{eff}\right) \\ F_3\left(X^3_{eff}\right) \end{array}\right\} = F\left(X^4_{eff}\right) = \left\{\begin{array}{c} F_1\left(X^4_{eff}\right) \\ F_2\left(X^4_{eff}\right) \\ F_3\left(X^4_{eff}\right) \end{array}\right\} =$$

$$= F\left(X^5_{eff}\right) = \left\{\begin{array}{c} F_1\left(X^5_{eff}\right) \\ F_2\left(X^5_{eff}\right) \\ F_3\left(X^5_{eff}\right) \end{array}\right\} = F\left(X^6_{eff}\right) = \left\{\begin{array}{c} F_1\left(X^6_{eff}\right) \\ F_2\left(X^6_{eff}\right) \\ F_3\left(X^6_{eff}\right) \end{array}\right\} =$$

$$= F(X^7_{eff}) = \left\{\begin{array}{c} F_1\left(X^7_{eff}\right) \\ F_2\left(X^7_{eff}\right) \\ F_3\left(X^7_{eff}\right) \end{array}\right\} = F(X^8_{eff}) \left\{\begin{array}{c} F_1\left(X^8_{eff}\right) \\ F_2\left(X^8_{eff}\right) \\ F_3\left(X^8_{eff}\right) \end{array}\right\} = \left\{\begin{array}{c} 7 \\ 5 \\ 11 \end{array}\right\}.$$

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Dedicated to Professor Alexandru Șubă on the occasion of his 70th birthday

On stability of some examples of ternary differential critical systems with quadratic nonlinearities

Natalia Neagu and Mihail Popa

Abstract. Starting with Example 1 of A.M. Lyaponov's thesis [1] (§32), which represents a ternary differential system with quadratic nonlinearities, examples of differential systems of the generalized Darboux type were constructed in the critical case.The stability conditions of the unperturbed motion described by these systems were determined.
2010 Mathematics Subject Classification: 34C05, 34C14, 34C40.
Keywords: differential system, stability of unperturbed motion, differential system of generalized Darboux type, the intrinsic transmission dynamics of tuberculosis.

Despre stabilitatea unor exemple de sisteme critice diferenţiale ternare cu neliniarităţi pătratice

Rezumat. Pornind de la Exemplul 1, din teza lui A. M. Lyaponov [1] (§32), ce constă dintr-un sistem diferenţial ternar cu nelinearităţi pătratice, în cazul critic, au fost construite mai multe exemple de sisteme diferenţiale de tip generalizat Darboux. Pentru aceste sisteme au fost determinate condiţiile de stabilitate a mişcării neperturbate.
Cuvinte-cheie: sistem diferenţial, stabilitatea mişcării neperturbate, sistem diferenţial de tip generalizat Darboux, sistemul dinamicii răspândirii tuberculozei.

1. Introduction

Systems of autonomous differential equations of the first order are mathematical models of many processes in everyday life, for example the system of intrinsic transmission dynamics of tuberculosis (TB).

This mathematical model is described by ternary differential systems with quadratic nonlinearities, which are contained in ternary differential systems with quadratic nonlinearities generalized Darboux type. In the world, there are countless dedicated works to TB problems, both in medicine and in mathematics. For example, among the works devoted to the problem of dynamics of TB, in medicine and mathematics, we can mention the paper [2].

In the Institute of Mathematics and Computer Science of ASM, there were carried researches in the field of TB within a project [3], without examining the stability of the unperturbed motion governed by the system mentioned above.

Also, starting with Example 1 from A. M. Lyaponov's thesis [1] (§32), which consisted of a ternary differential system with quadratic nonlinearities, in the critical case, there were constructed some examples of differential systems of the generalized Darboux type.

In this work, there were determined the stability conditions of unperturbed motion in the critical case for the differential system aimed at the intrinsic transmission dynamics of tuberculosis TB in society and examples of differential systems of generalized Darboux type.

2. Intrinsic transmission dynamics of tuberculosis (TB)

The intrinsic transmission dynamics of tuberculosis [2, 4], represents a mathematical model, in which the entire population is divided into:

- the sensitive population (S);
- the population carrying latent infection (L);
- the population with active tuberculosis (T).

This dynamics is defined by the following system of differential equations:

$$\frac{dS}{dt} = \tau - \mu S - \beta ST \equiv P, \quad \frac{dL}{dt} = -\delta L - \mu L + (1-p)\beta ST \equiv Q,$$
$$\frac{dT}{dt} = \delta L - (\mu + \nu)T + p\beta ST \equiv R. \tag{1}$$

The variables and parameters of system (1) are described in the Table 1:

Table 1. The variables and parameters of the system (1)

Value	**Description**
$S(t)$	number of sensible persons in the moment t
$L(t)$	number of infected persons in the moment t
$T(t)$	number of infectious persons in the moment t
$\lambda(t)$	force of infection per capita in the moment t
τ	influx of young people
μ	average mortality from causes not related to TB
p	probability of rapid progression of the disease
δ	speed constant of reactivation of TB infection
ν	additional mortality caused by active TB
β	transfer coefficient of TB infection

The intrinsic transmission dynamics of tuberculosis [3] (1), through the affine transformation

$$x = \tau - \mu S; \quad y = L; \quad z = T, \tag{2}$$

and $\mu \neq 0$, according to the medical meaning of this variable, can be brought to the form

$$\begin{aligned} \frac{dx}{dt} &= ax + bz + 2gxz; \\ \frac{dy}{dt} &= cy + dz + 2hxz; \\ \frac{dz}{dt} &= ey + fz + 2kxz. \end{aligned} \tag{3}$$

where

$$\begin{gathered} a = -\mu \neq 0,\ b = \beta\tau,\ c = -\delta - \mu, d = \frac{(1-p)\beta\tau}{\mu},\ e = \delta,\ f = \frac{p\beta\tau}{\mu} - \mu - \mu_T, \\ g = -\frac{\beta}{2},\ h = -\frac{(1-p)\beta}{2\mu},\ k = -\frac{p\beta}{2\mu}. \end{gathered} \tag{4}$$

The characteristic equation of system (3) is

$$\rho^3 + (-a - c - f)\rho^2 + (ac + af + cf - de)\rho - a(cf - de) = 0. \tag{5}$$

Taking into account the medical meaning of the variables ($a \neq 0$), for equation (5) to have one zero root, we obtain the relation $cf - de = 0$.

By a center-affine transformation

$$\bar{x} = -ey + cz; \quad \bar{y} = y; \quad \bar{z} = x + z \ (\Delta \equiv c \neq 0) \tag{6}$$

and $cf - de = 0$ or $f = \frac{de}{c}$, according to Hurwitz's theorem [5], the system (3) can be brought to the critical Lyapunov form

$$\begin{aligned} \frac{dx}{dt} &= \frac{2(ck - eh)}{c^2}(-x^2 - 2exy + cxz - e^2y^2 + ceyz); \\ \frac{dy}{dt} &= \frac{d}{c}x + (c + \frac{de}{c})y + \frac{2h}{c^2}(-x^2 - 2exy + cxz - e^2y^2 + ceyz); \\ \frac{dz}{dt} &= \frac{-a + f + b}{c}x + \frac{(-a + c + f + b)e}{c}y + az + \\ &+ \frac{2(g + k)}{c^2}(-x^2 - 2exy + cxz - e^2y^2 + ceyz), \end{aligned} \tag{7}$$

where

$$-a - c - f > 0, \quad a(c + f) > 0. \tag{8}$$

According to Lemma 4.2 [6], we have

$$
\begin{aligned}
&C_1 = 0, \quad C_2 = \frac{2}{c^2}(-1 + cB_1 - eA_1)(1 + eA_1)(-eh + ck), \\
&C_3 = \frac{2}{c^2}[cB_2 - 2eA_2 + ce(A_2B_1 + A_1B_2) - 2e^2A_1A_2](-eh + ck), \\
&C_4 = \frac{2}{c^2}[cB_3 - 2eA_3 + ce(A_3B_1 + A_2B_2 + A_1B_3) - e^2(A_2^2 + 2A_1A_3)](-eh + ck), \\
&C_5 = \frac{2}{c^2}[cB_4 - 2eA_4 + ce(A_4B_1 + A_3B_2 + A_2B_3 + A_1B_4)- \\
&\quad -2e^2(A_2A_3 + A_1A_4)](-eh + ck), \\
&C_6 = \frac{2}{c^2}[cB_5 - 2eA_5 + ce(A_5B_1 + A_4B_2 + A_3B_3 + A_2B_4 + A_1B_5)- \\
&\quad -e^2(A_3^2 + 2A_2A_4 + 2A_1A_5)](-eh + ck), \\
&C_7 = \frac{2}{c^2}[cB_6 - 2eA_6 + ce(A_6B_1 + A_5B_2 + A_4B_3 + A_3B_4 + A_2B_5 + A_1B_6)- \\
&\quad -2e^2(A_3A_4 + A_2A_5 + A_1A_6)](-eh + ck), \\
&C_8 = \frac{2}{c^2}[cB_7 - 2eA_7 + ce(A_7B_1 + A_6B_2 + A_5B_3 + A_4B_4 + A_3B_5+ \\
&\quad +A_2B_6 + A_1B_7) - e^2(A_4^2 + 2A_3A_5 + 2A_2A_6 + 2A_1A_7)](-eh + ck), \\
&C_9 = \frac{2}{c^2}[cB_8 - 2eA_8 + ce(A_8B_1 + A_7B_2 + A_6B_3 + A_5B_4 + A_4B_5+ \\
&\quad +A_3B_6 + A_2B_7 + A_1B_8) - 2e^2(A_4A_5 + A_3A_6 + A_2A_7 + A_1A_8)](-eh + ck), \\
&C_{10} = \frac{2}{c^2}[cB_9 - 2eA_9 + ce(A_9B_1 + A_8B_2 + A_7B_3 + A_6B_4 + A_5B_5 + A_4B_6+ \\
&\quad +A_3B_7 + A_2B_8 + A_1B_9) - e^2(A_5^2 + 2A_4A_6 + 2A_3A_7+ \\
&\quad +2A_2A_8 + 2A_1A_9)](-eh + ck), \ldots,
\end{aligned}
\tag{9}
$$

where

$$
\begin{aligned}
&A_1 = -\frac{d}{c^2 + de}, \quad B_1 = \frac{(a - b)c}{a(c^2 + de)}; \\
&A_2 = -\frac{2}{c(c^2 + de)}[(-1 + cB_1 - eA_1)(1 + eA_1)h], \\
&B_2 = -\frac{2}{ac^3(c^2 + de)}(-1 + cB_1 - eA_1)(1 + eA_1)(c^3g + cdeg + aceh- \\
&\quad -bceh - c^2eh - de^2h + c^3k + cdek),
\end{aligned}
$$

$$A_3 = -\frac{2}{c(c^2+de)}(cB_2 - 2eA_2 + ce(A_2B_1 + A_1B_2) - 2e^2A_1A_2)h,$$

$$B_3 = -\frac{2}{ac^3(c^2+de)}(cB_2 - 2eA_2 + ce(A_2B_1 + A_1B_2) -$$
$$-2e^2A_1A_2)(c^3g + cdeg + aceh - bceh - c^2eh - de^2h + c^3k + cdek),$$

$$A_4 = -\frac{2}{c(c^2+de)}[cB_3 - 2eA_3 + ce(A_3B_1 + A_2B_2 + A_1B_3) - e^2(A_2^2 + 2A_1A_3)]h,$$

$$B_4 = -\frac{2}{ac^3(c^2+de)}[cB_3 - 2eA_3 + ce(A_3B_1 + A_2B_2 + A_1B_3) -$$
$$-e^2(A_2^2 + 2A_1A_3)](c^3g + cdeg + aceh - bceh - c^2eh - de^2h + c^3k + cdek),$$

$$A_5 = -\frac{2}{c(c^2+de)}[cB_4 - 2eA_4 + ce(A_4B_1 + A_3B_2 + A_2B_3 + A_1B_4) -$$
$$-2e^2(A_2A_3 + A_1A_4)]h,$$

$$B_5 = -\frac{2}{ac^3(c^2+de)}[cB_4 - 2eA_4 + ce(A_4B_1 + A_3B_2 + A_2B_3 + A_1B_4) -$$
$$-2e^2(A_2A_3 + A_1A_4)](c^3g + cdeg + aceh - bceh - c^2eh - de^2h + c^3k + cdek),$$

$$A_6 = -\frac{2}{c(c^2+de)}[cB_5 - 2eA_5 + ce(A_5B_1 + A_4B_2 + A_3B_3 + A_2B_4 + A_1B_5) -$$
$$-e^2(A_3^2 + 2A_2A_4 + 2A_1A_5)]h,$$

$$B_6 = -\frac{2}{ac^3(c^2+de)}[cB_5 - 2eA_5 + ce(A_5B_1 + A_4B_2 + A_3B_3 + A_2B_4 + A_1B_5) -$$
$$-e^2(A_3^2 + 2A_2A_4 + 2A_1A_5)](c^3g + cdeg + aceh - bceh - c^2eh - de^2h + c^3k + cdek),$$

$$A_7 = -\frac{2}{c(c^2+de)}[cB_6 - 2eA_6 + ce(A_6B_1 + A_5B_2 + A_4B_3 + A_3B_4 +$$
$$+A_2B_5 + A_1B_6) - 2e^2(A_3A_4 + A_2A_5 + A_1A_6)]h,$$

$$B_7 = -\frac{2}{ac^3(c^2+de)}[cB_6 - 2eA_6 + ce(A_6B_1 + A_5B_2 + A_4B_3 + A_3B_4 +$$
$$+A_2B_5 + A_1B_6) - 2e^2(A_3A_4 + A_2A_5 + A_1A_6)](c^3g +$$
$$+cdeg + aceh - bceh - c^2eh - de^2h + c^3k + cdek),$$

$$A_8 = -\frac{2}{c(c^2+de)}[cB_7 - 2eA_7 + ce(A_7B_1 + A_6B_2 + A_5B_3 + A_4B_4 +$$
$$+A_3B_5 + A_2B_6 + A_1B_7) - e^2(A_4^2 + 2A_3A_5 + 2A_2A_6 + 2A_1A_7)]h,$$

$$B_8 = -\frac{2}{ac^3(c^2+de)}[cB_7 - 2eA_7 + ce(A_7B_1 + A_6B_2 + A_5B_3 + A_4B_4 + A_3B_5+$$
$$+A_2B_6 + A_1B_7) - e^2(A_4^2 + 2A_3A_5 + 2A_2A_6 + 2A_1A_7)](c^3g+$$
$$+cdeg + aceh - bceh - c^2eh - de^2h + c^3k + cdek),$$
$$A_9 = -\frac{2}{c(c^2+de)}[cB_8 - 2eA_8 + ce(A_8B_1 + A_7B_2 + A_6B_3 + A_5B_4 + A_4B_5+$$
$$+A_3B_6 + A_2B_7 + A_1B_8) - 2e^2(A_4A_5 + A_3A_6 + A_2A_7 + A_1A_8)]h,$$
$$B_9 = -\frac{2}{ac^3(c^2+de)}[cB_8 - 2eA_8 + ce(A_8B_1 + A_7B_2 + A_6B_3 + A_5B_4 + A_4B_5+$$
$$+A_3B_6 + A_2B_7 + A_1B_8) - 2e^2(A_4A_5 + A_3A_6 + A_2A_7 + A_1A_8)](c^3g+ \tag{10}$$
$$+cdeg + aceh - bceh - c^2eh - de^2h + c^3k + cdek),$$
$$A_{10} = -\frac{2}{c(c^2+de)}[cB_9 - 2eA_9 + ce(A_9B_1 + A_8B_2 + A_7B_3 + A_6B_4 + A_5B_5+$$
$$+A_4B_6 + A_3B_7 + A_2B_8 + A_1B_9) - e^2(A_5^2 + 2A_4A_6 + 2A_3A_7 + 2A_2A_8 + 2A_1A_9)]h,$$
$$B_{10} = -\frac{2}{ac^3(c^2+de)}[cB_9 - 2eA_9 + ce(A_9B_1 + A_8B_2 + A_7B_3 + A_6B_4 + A_5B_5+$$
$$+A_4B_6 + A_3B_7 + A_2B_8 + A_1B_9) - e^2(A_5^2 + 2A_4A_6 + 2A_3A_7 + 2A_2A_8+$$
$$+2A_1A_9)](c^3g + cdeg + aceh - bceh - c^2eh - de^2h + c^3k + cdek), \ldots$$

Taking into account the medical meaning of the parameters from system (3) – (4), we mention that the denominators in (9) and (10) are different from zero. We get

Lemma 2.1. *Let $a+c+f < 0$ and $a(c+f) > 0$. Then the stability of unperturbed motion governed by system* (7) *includes all possible cases in the following two:*

I. *When $(-1+cB_1-eA_1)(1+eA_1)(ck-eh) \neq 0$ the unperturbed motion is* ***unstable****;*

II. *When $(-1+cB_1-eA_1)(1+eA_1)(ck-eh) = 0$ the unperturbed motion is* ***stable****.*

In the last case, the unperturbed motion belongs to some continuous series of stabilized motion. For sufficiently small perturbations, any perturbed motion will asymptotically approach to one of the stabilized motions of the mentioned series. Moreover, this motion is also asymptotic stable [5].

Proof. To prove Lemma we use the Lyapunov's theorem [1] (§32) in the ternary case. Next we analyze the coefficients of the series C_i from (9).

If $C_2 \neq 0$, then we obtain the Case I of Lemma 2.1.

If $-eh+ck = 0$, then $C_i = 0$ $(\forall i)$ and if $(-1+cB_1-eA_1)(1+eA_1) = 0$, then $A_i = B_i = 0$ $(i \geq 2)$ from (10). This implies $C_i = 0$ $(i \geq 3)$. Lemma 2.1 is proved. □

3. Stability conditions of unperturbed motion for some differential systems of generalized Darboux type with quadratic nonlinearities

Following Example 1 from [1] (§32), which in the critical equation has 2 parameters, we will examine a few cases of ternary systems of Lyapunov critical canonical form:

Example 3.1. *We will examine the ternary differential system with three parameters in the critical equation of the form*

$$\begin{aligned}\frac{dx}{dt} &= (ax+by+cz)(-x+y+z),\\ \frac{dy}{dt} &= x-y+(x-y+2z)(-x+y+z),\\ \frac{dz}{dt} &= x-z+(x+2y-z)(-x+y+z),\end{aligned} \tag{11}$$

where a, b, c are real arbitrary coefficients.

The characteristic equation of the linear part is

$$\rho^3+2\rho^2+\rho=0, \tag{12}$$

where

$$\rho_1=0, \quad \rho_2=\rho_3=-1. \tag{13}$$

According to Lemma 4.2 [6], we have

$$\begin{gathered}C_1=0, \quad C_2=a+b+c,\\ C_3=4a+6b+6c=2[2a+3(b+c)],\\ C_4=20a+38b+38c=2[10a+19(b+c)],\\ C_5=116a+254b+254c=2[58a+127(b+c)],\\ C_6=740a+1774b+1774c=2[370a+887(b+c)],\\ C_7=5028a+12822b+12822c=2[2514a+6411(b+c)],\\ C_8=35700a+95190b+95190c=2[17850a+47595(b+c)],\\ C_9=261780a+721870b+721870c=2[130890a+360935(b+c)],\\ C_{10}=1967300a+5569118b+5569118c=2[982650a+2784559(b+c)],\dots,\end{gathered} \tag{14}$$

where

$$A_1 = B_1 = 1; \quad A_2 = B_2 = 2, \quad A_3 = B_3 = 10, \quad A_4 = B_4 = 58, \quad A_5 = B_5 = 370,$$
$$A_6 = B_6 = 2514, \quad A_7 = B_7 = 17850, \quad A_8 = B_8 = 130890, \quad A_9 = B_9 = 983650, \tag{15}$$
$$A_{10} = B_{10} = 7536418, \ldots$$

As the characteristic equation (12) of system (11) has the roots (13), then according to Lyapunov's Theorem [1] (§32), in the ternary case, we obtain

Lemma 3.1. *The stability of the unperturbed motion, governed by system* (11)*, includes all possible cases in the following four:*

I. $a + b + c \neq 0$*, then the unperturbed motion is* ***unstable****;*

II. $a > 0$*, then the unperturbed motion is* ***stable****;*

III. $a < 0$*, then the unperturbed motion is* ***unstable****;*

IV. $b + c = -a = 0$*, then the unperturbed motion is* ***stable****.*

In the last case, the unperturbed motion belongs to some continuous series of stabilized motion. For sufficiently small perturbations, any perturbed motion will asymptotically approach to one of the stabilized motions of the mentioned series. Moreover, this motion is also asymptotic stable [5].

Proof. According to Lyapunov's Theorem [1] (§32), we analyze the coefficients of the series C_i, from (14). If $C_2 \neq 0$, then we get the case I of Lemma 3.1.

If $C_2 = 0$, then $b + c = -a$. Substituting in C_3 we obtain $C_3 = -2a$. Depending on the sign of this expression, we get the cases II and III of Lemma 3.1.

If $C_3 = 0$, then $b + c = -a = 0$. In this case, $C_i = 0$ $(i \geq 4)$. Lemma 3.1 is proved. □

Example 3.2. *We examine the ternary differential system with six parameters in the critical equation of the form*

$$\begin{aligned}
\frac{dx}{dt} &= a_1x^2 + a_2y^2 + a_3z^2 + 2a_4xy + 2a_5xz + 2a_6yz,\\
\frac{dy}{dt} &= x - y + (x - y + 2z)(-x + y + z),\\
\frac{dz}{dt} &= x - z + (x + 2y - z)(-x + y + z),
\end{aligned} \tag{16}$$

where a_i $(i = \overline{1, 6})$ *are real arbitrary coefficients.*

According to Lemma 4.2 [6], we have

$$C_1 = 0, \quad C_2 = a_1 + (a_2 + a_3 + 2a_6) + 2(a_4 + a_5),$$
$$C_3 = 4[(a_2 + a_3 + 2a_6) + (a_4 + a_5))],$$
$$C_4 = 4[6(a_2 + a_3 + 2a_6) + 5(a_4 + a_5)],$$

$$\begin{gathered}
C_5 = 4[39(a_2 + a_3 + 2a_6) + 29(a_4 + a_5)], \\
C_6 = 4[268(a_2 + a_3 + 2a_6) + 185(a_4 + a_5)], \\
C_7 = 12[639(a_2 + a_3 + 2a_6) + 419(a_4 + a_5)], \\
C_8 = 60[942(a_2 + a_3 + 2a_6) + 595(a_4 + a_5)], \\
C_9 = 20[21319(a_2 + a_3 + 2a_6) + 13089(a_4 + a_5)], \\
C_{10} = 4[819096(a_2 + a_3 + 2a_6) + 491825(a_4 + a_5)], \ldots,
\end{gathered} \tag{17}$$

where

$$\begin{gathered}
A_1 = B_1 = 1; \;\; A_2 = B_2 = 2, \;\; A_3 = B_3 = 10, \;\; A_4 = B_4 = 58, \\
A_5 = B_5 = 370, \;\; A_6 = B_6 = 2514, \;\; A_7 = B_7 = 17850, \\
A_8 = B_8 = 130890, \;\; A_9 = B_9 = 983650, \\
A_{10} = B_{10} = 7536418, \ldots
\end{gathered} \tag{18}$$

We introduce the notation

$$\begin{gathered}
L_1 = a_2 + a_3 + 2a_6; \;\; L_2 = -a_1 - 2(a_4 + a_5); \\
L_3 = -a_1 - (a_4 + a_5); \;\; L_4 = -(a_4 + a_5).
\end{gathered} \tag{19}$$

As the characteristic equation (12) of the system (16) has the roots (13), then according to Lyapunov's Theorem [1] (§32), in the ternary case, we obtain

Lemma 3.2. *The stability of the unperturbed motion, governed by system* (16)*, includes all possible cases in the following five:*

I. $L_1 \neq L_2$*, then the unperturbed motion is* ***unstable****;*

II. $L_1 = L_2,\ L_3 < 0$*, then the unperturbed motion is* ***stable****;*

III. $L_1 = L_2,\ L_3 > 0$*, then the unperturbed motion is* ***unstable****;*

IV. $L_1 = L_2,\ L_3 = 0,\ L_4 \neq 0$*, then the unperturbed motion is* ***unstable****;*

V. $a_2 + a_3 + 2a_6 = a_1 = 0,\ a_4 = -a_5$*, then the unperturbed motion is* ***stable****.*

In the last case, the unperturbed motion belongs to some continuous series of stabilized motion. For sufficiently small perturbations, any perturbed motion will asymptotically approach to one of the stabilized motions of the mentioned series. Moreover, this motion is also asymptotic stable [5]*. The expressions* L_i $(i = \overline{1,4})$ *are given in* (19).

Proof. According to Lyapunov's Theorem [1] (§32), we analyze the coefficients of the series C_i from (17). If $C_2 \neq 0$, then $L_1 - L_2 \neq 0$. We obtain the Case I of Lemma 3.2.

If $C_2 = 0$, then $L_1 = L_2$, and

$$C_3 = 4[L_1 + (a_4 + a_5)] = 4[L_2 + (a_4 + a_5)] = 4[-a_1 - (a_4 + a_5)] = 4L_3.$$

Depending on the sign of this expression, we get the Cases II and III of Lemma 3.2.

If $C_2 = C_3 = 0$, then $L_1 = L_2$ and $L_3 = 0$ implies $a_1 = -(a_4 + a_5)$ and

$$C_4 = 4[6L_1 + 5(a_4 + a_5)] = 4[6L_2 + 5(a_4 + a_5)] = 4[-6a_1 - 7(a_4 + a_5)] = 4L_4.$$

Assume that $L_4 \neq 0$. In this case we get the Case IV of Lemma 3.2.

If $C_2 = C_3 = C_4 = 0$, then $L_1 = L_2, L_3 = L_4 = 0$ or $a_2 + a_3 + 2a_6 = a_1 = 0$, $a_4 = -a_5$. In this case, all $C_i = 0$ $(i \geq 5)$. Lemma 3.2 is proved. □

Example 3.3. *We examine the ternary differential system with 6 parameters in the critical equation, of which three form the common factor of the quadratic part, of the form*

$$\begin{aligned} \frac{dx}{dt} &= (a_1x + b_1y + c_1z)(ax + by + cz), \\ \frac{dy}{dt} &= x - y + (x - y + 2z)(ax + by + cz), \\ \frac{dz}{dt} &= x - z + (x + 2y - z)(ax + by + cz), \end{aligned} \tag{20}$$

where a, b, c, a_1, b_1, c_1 are real arbitrary coefficients.

We introduce the notation

$$\begin{aligned} M_1 = a + b + c;\quad M_2 = a_1 + b_1 + c_1; \\ M_3 = -aa_1 + (b + c)(b_1 + c_1);\quad M_4 = (b + c)(b_1 + c_1). \end{aligned} \tag{21}$$

According to Lemma 4.2 [6], we have

$$\begin{gathered} C_1 = 0,\quad C_2 = M_1M_2,\quad C_3 = (M_1M_2 + M_3)A_2, \\ C_4 = M_4A_2^2 + (M_1M_2 + M_3)A_3, \\ C_5 = 2M_4A_2A_3 + (M_1M_2 + M_3)A_4, \\ C_6 = M_4(2A_2A_4 + A_3^2) + (M_1M_2 + M_3)A_5, \\ C_7 = 2M_4(A_2A_5 + A_3A_4) + (M_1M_2 + M_3)A_6, \\ C_8 = M_4(2A_2A_6 + 2A_3A_5 + A_4^2) + (M_1M_2 + M_3)A_7, \\ C_9 = 2M_4(A_2A_7 + A_3A_6 + A_4A_5) + (M_1M_2 + M_3)A_8, \\ C_{10} = M_4(2A_2A_8 + 2A_3A_7 + 2A_4A_6 + A_5^2) + (M_1M_2 + M_3)A_9, \ldots, \end{gathered} \tag{22}$$

where

$$A_1 = B_1 = 1;\quad A_2 = B_2 = 2(a + b + c),$$

$$A_3 = B_3 = (a + 3b + 3c)A_2,$$

$$
\begin{aligned}
A_4 = B_4 &= (b+c)A_2^2 + (a+3b+3c)A_3,\\
A_5 = B_5 &= 2(b+c)A_2A_3 + (a+3b+3c)A_4,\\
A_6 = B_6 &= (b+c)(2A_2A_4 + A_3^2) + (a+3b+3c)A_5,\\
A_7 = B_7 &= 2(b+c)(A_2A_5 + A_3A_4) + (a+3b+3c)A_6,\\
A_8 = B_8 &= (b+c)(2A_2A_6 + 2A_3A_5 + A_4^2) + (a+3b+3c)A_7,\\
A_9 = B_9 &= 2(b+c)(A_2A_7 + A_3A_6 + A_4A_5) + (a+3b+3c)A_8,\\
A_{10} = B_{10} &= (b+c)(2A_2A_8 + 2A_3A_7 + 2A_4A_6 + A_5^2) + (a+3b+3c)A_9, \ldots
\end{aligned}
\tag{23}
$$

As the characteristic equation (12) of system (20) has the roots (13), then according to Lyapunov's Theorem [1] (§32), in the ternary case, we obtain

Lemma 3.3. *The stability of the unperturbed motion governed by the system* (20) *includes all possible cases in the following six:*

I. $M_1M_2 \neq 0$*, then the unperturbed motion is* ***unstable****;*

II. $M_2 = 0,\ M_1M_3 < 0$*, then the unperturbed motion is* ***stable****;*

III. $M_2 = 0,\ M_1M_3 > 0$*, then the unperturbed motion is* ***unstable****;*

IV. $M_1M_4 \neq 0$*, then the unperturbed motion is* ***unstable****;*

V. $M_4 = 0$*, then the unperturbed motion is* ***stable****;*

VI. $M_1 = 0$*, then the unperturbed motion is* ***stable****;.*

In the last case, the unperturbed motion belongs to some continuous series of stabilized motion. For sufficiently small perturbations, any perturbed motion will asymptotically approach to one of the stabilized motions of the mentioned series. Moreover, this motion is also asymptotic stable [5]. *The expressions* M_i $(i = \overline{1,4})$ *are given in* (21).

Proof. According to Lyapunov's Theorem [1] (§32), we analyze the coefficients of the series C_i from (22). Suppose that $M_1 \neq 0$.

If $C_2 \neq 0$, then $M_1M_2 \neq 0$ and we get the Case I of Lemma 3.3.

If $C_2 = 0$, then $M_2 = 0$, and $C_3 = (M_1M_2 + M_3)A_2 = 2M_1M_3$. Depending on the sign of this expression, we obtain the Cases II and III of Lemma 3.3.

If $C_2 = C_3 = 0$, then $M_2 = M_3 = 0$ and $C_4 = M_4A_2^2 + (M_1M_2 + M_3)A_3 = 4M_1^2M_4$. If $M_1M_4 \neq 0$, we get the Case IV of Lemma 3.3.

If $C_2 = C_3 = C_4 = 0$, then $M_2 = M_3 = M_4 = 0$, and all $C_i = 0$ $(i \geq 5)$. We have the Case V of Lemma 3.3.

If $M_1 = 0$, then all $C_i = 0$ $(\forall i)$ and we obtain the Case VI. Lemma 3.3 is proved. □

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Dedicated to Professor Alexandru Șubă on the occasion of his 70th birthday

Tuning method of automatic controllers to object models with second order advance-delay and dead time

Dumitru Moraru

Abstract. This article explores the application of mathematical models in the design and analysis of automatic control system. By integrating mathematical concepts such as linear algebra, mathematical analysis, the performance and reliability of automatic control systems can be optimized. In the paper, an efficient procedure has been developed for tuning the standardized P, PI, PD, and PID control algorithms to mathematical models of second-order advance-delay with dead time control objects with known parameters, using the maximal stability degree method with iterations.The advantages of the maximum stability degree method with reduced calculations and minimal time are highlighted.

2010 Mathematics Subject Classification: 93A30, 93C85, 44A10.

Keywords: mathematical model, advance-delay control object, transfer function, automatic system, tuning methods, differential equation, system performances.

Metodă de acordare a regulatoarelor automate la modele de obiecte cu anticipaţie-întârziere de ordinul doi și timp mort

Rezumat. Acest articol explorează aplicarea modelelor matematice în proiectarea și analiza sistemelor automate. Prin integrarea conceptelor matematice, cum ar fi algebra liniară, analiza matematică, se pot optimiza performanțele și fiabilitatea sistemelor de conducere automată. În lucrare s-a elaborat o procedură eficientă de acordare a algoritmilor tipizați P, PI, PD şi PID la modele matematice ale obiectelor de reglare cu anticipație-întârziere de ordinul doi cu timp mort cu parametrii cunoscuţi după metoda gradului maximal de stabilitate cu iterații. Se evidențiază avantajele metodei gradul maximal de stabilitate cu iterații cu calcule reduse și timp minim.

Cuvinte-cheie: model matematic, obiect de reglare cu anicipație-întârziere, funcţie de transfer, sistem automat, metode de acordare, ecuație diferențială, performanţele sistemului.

1. Introduction

Automatic control systems are complex entities that can adapt their behavior based on external conditions or inputs. These can be mathematically modeled using differential equations, Laplace transforms, and transfer functions. Differential equations are used to describe the relationships between the input and output variables of a system as a function of time. In the context of automatic systems, these equations model the dynamic behavior of the system. The Laplace transform is used to convert differential equations into transfer functions, which represent algebraic equations of complex variables. This facilitates the analysis and solving of dynamic system problems. Control theory deals with the design and analysis of controllers that influence the behavior of a system. There are two main types of control: open-loop and closed-loop. In open-loop control, the input is set without considering the output, whereas in closed-loop control, the input is adjusted based on the output magnitude to achieve a desired behavior.

According to the concept of automatic control theory, the technological process presents the control object with the variables that interact in the process: the input flow is called the control variable, denoted by the vector $x(t)$, the characteristic variables $y_1, ..., y_n$, which represents the output flow known as the controlled variable, denoted by the vector $y(t)$ and disturbances denoted by the vector $p(t)$ (Figure 1), where FP is fixed part of control object [5].

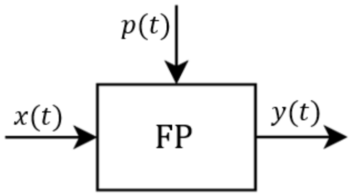

Figure 1. The block diagram of the control object.

The control object represents a technical, industrial, biological, economic, social, etc. process that requires control for optimal operation.

In paper is discussed the mathematical model of the control object, characterized as a advance-delay object with second-order inertia and dead time, described by the transfer function $H_{FP}(s)$ in form [2], [3]:

$$H_{FP}(s) = e^{-\tau s} \frac{k(T_1 s + 1)}{(T_2 s + 1)(T_3 s + 1)} = e^{-\tau s} \frac{b_0 s + b_1}{a_0 s^2 + a_1 s + a_2}, \tag{1}$$

where k is the transfer coefficient, T_1, T_2, and T_3 are the time constants of the process, τ is the dead time and the generic coefficients are $b_0 = kT_1$, $b_1 = k$, $a_0 = T_2 T_3$, $a_1 = T_2 + T_3$, $a_2 = 1$.

For the control object model (1), it is necessary to synthesize the control algorithm. In the practice of automation various industrial processes, controllers with a fixed PID structure have a wide range of applications [1], [6], [7].

There are several methods for tuning the standard PID control algorithm to the model object (1): the frequency-domain method, the pole-zero allocation method, the polynomial method, the Ziegler-Nichols method, etc [2], [3], [4], [9].

The application of the frequency-domain method involves calculations in the frequency domain and graphical constructions, which can lead to difficulties in synthesizing control algorithms.

The pole-zero allocation method (or model-based method) is an analytical approach. Based on the model of the control object (1) and the performance requirements imposed on the designed system, PI and PID control algorithms are synthesized. This is done by solving a system of matrix equations to determine the control algorithm parameters that meet the stability, performance, and robustness requirements of the system. As a result, the control algorithm synthesis procedure involves iterations and can become challenging [3], [4].

The polynomial method is also an analytical approach that leads to solving the control algorithm synthesis problem. However, it can be challenging to determine the characteristic equation of the designed system [8].

The basic experimental method includes the Ziegler-Nichols (ZN) method, which is widely used in practice for tuning standard PID algorithms for the model (1), but it may lead to reduced system performance [4].

In the paper, a procedure for tuning the PID controller for the control object model (1) has been developed based on the maximum stability degree method with Iterations (MSDI) [1], [6], [7].

To verify and compare the obtained results, both the Ziegler-Nichols and parametric optimization (PO) methods are applied.

2. Tuning the controller using the Maximum Stability Degree Method with Iterations

The structural block diagram of the automatic control system, consisting of the object model with transfer function $H_{FP}(s)$ and the controller with transfer function $H_R(s)$, is shown in Figure 2. Here, $r(t) = 1(t)$ represents the unit reference, $e(t)$ is the system error, $u(t)$ is the command generated by the controller, and $y(t) = h(t)$ is the step response of the system.

The standardized control algorithms P, PI, PD and PID are represented by the transfer function:

$$H_P(s) = k_p, \tag{2}$$

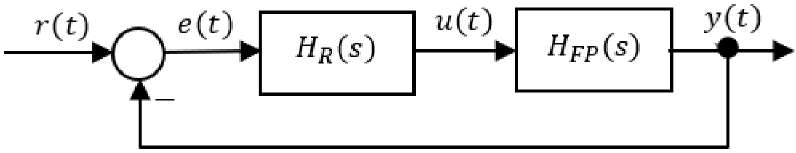

Figure 2. Structural block diagram of the automatic system.

$$H_{PI}(s) = k_p + \frac{k_i}{s} = \frac{k_p s + k_i}{s}, \tag{3}$$

$$H_{PD}(s) = k_p + k_d s, \tag{4}$$

$$H_{PID}(s) = k_p + \frac{k_i}{s} + k_d s = \frac{k_d s^2 + k_p s + k_i}{s}, \tag{5}$$

where k_p, k_i, and k_d are the tuning parameters of the proportional, integral and derivative components of the P, PI, PD and PID algorithms [1], [4], [8].

Tuning the P, PI, PD and PID control algorithms to the model (1) based on the maximum stability degree method of the designed system in the classical version becomes challenging when determining the algebraic equation for finding the maximum stability degree J.

The procedure for tuning the PID control algorithm according to the proposed method involves obtaining the characteristic equation of the closed-loop system. The notion of stability degree is introduced into the characteristic equation as a new unknown variable. Through operations of differentiation on this variable, relationships are derived that express the PID tuning parameters as nonlinear functions of the stability degree J and the known parameters of the object model parameters.

The transfer function of the closed-loop system with a P controller is given by:

$$H_0(s) = \frac{H_d(s)}{1 + H_d(s)} = \frac{k_p e^{-\tau s}(b_0 s + b_1)}{a_0 s^2 + a_1 s + a_2 + k_p e^{-\tau s}(b_0 s + b_1)} = \frac{C(s)}{D(s)}, \tag{6}$$

where $H_0(s)$is the transfer function of closed-loop system, $H_d(s)$ - transfer function of open-loop system, k_p - the parameter of the P controller, $C(s)$ and $D(s)$ - the system polynomials.

The characteristic equation of the automatic control system is the polynomial $D(s)$:

$$D(s) = a_0 s^2 + a_1 s + a_2 + k_p e^{-\tau s}(b_0 s + b_1) = 0. \tag{7}$$

According to the maximum stability degree method algorithm, it is substituted $s = -J$, and after some transformations, it is obtained the expression:

$$D(-J) = a_0J^2 - a_1J + a_2 + k_pe^{\tau J}(b_1 - b_0J) =$$

$$= \frac{e^{-\tau J}(a_0J^2 - a_1J + a_2)}{b_1 - b_0J} + k_p = 0. \tag{8}$$

In the case of a system with a P controller, expression (8) is differentiated once with respect to J and the resulting expression is:

$$\dot{D}(-J) = \frac{e^{-\tau J}(d_0J^3 - d_1J^2 + d_2J - d_3)}{b_0^2J^2 - 2b_0b_1J + b_1^2} = 0. \tag{9}$$

where $d_0 = a_0b_0\tau$, $d_1 = a_0b_0 + a_0b_1\tau + a_1b_0\tau$, $d_2 = 2a_0b_1 + a_1b_1\tau + a_2b_0\tau$, $d_3 = a_1b_1 - a_2b_0 + a_2b_1\tau$.

The optimal degree value J_{opt} is the smallest positive root of the expression:

$$e^{-\tau j}[a_0b_0J^3\tau - a_2b_1\tau + J^2(-a_0b_0 - a_0b_1\tau - a_1b_0\tau) +$$

$$+J(2a_0b_1 + a_1b_1\tau + a_2b_0\tau) - a_1b_1 + a_2b_0] = 0. \tag{10}$$

To determine the tuning parameter for the P controller from (8), the following relationship is used:

$$k_p = \frac{e^{-\tau J}(-a_0J^2 + a_1J - a_2)}{b_1 - b_0J} = f_p(J). \tag{11}$$

Further, the calculation mathematical expression for the tuning parameters k_p, k_i, k_d of the PI, PD, and PID control algorithms are presented using the MSDI to the object model (1) in a simplified form.

Mathematical expressions for determine of tuning parameters of PI controller are:

$$k_p = \frac{e^{-\tau J}(-d_0J^4 + d_1J^3 - d_2J^2 + d_3J - d_4)}{b_0^2J^2 - 2b_0b_1J + b_1^2} = f_p(J), \tag{12}$$

$$k_i = \frac{e^{-\tau J}(a_0J^3 - a_1J^2 + a_2J)}{b_1 - b_0J} + k_pJ = f_i(J), \tag{13}$$

where $d_0 = a_0b_0\tau$, $d_1 = 2a_0b_0 + a_0b_1\tau + a_1b_0\tau$, $d_2 = 3a_0b_1 + a_1b_0 + a_1b_1\tau + a_2b_0\tau$, $d_3 = 2a_1b_1 + a_2b_1\tau$, $d_4 = a_2b_1$.

Mathematical expressions for determine of tuning parameters of PD controller are:

$$k_p = \frac{e^{-\tau J}\left(a_0J^3 - a_1J^2 + a_2J\right)}{b_1 - b_0J} + k_dJ = f_p(J), \tag{14}$$

$$k_d = \frac{e^{-\tau J}(-d_0J^4 + d_1J^3 - d_2J^2 + d_3J - d_4)}{b_0^2J^2 - 2b_0b_1J + b_1^2} = f_d(J), \tag{15}$$

where $d_0 = a_0b_0\tau$, $d_1 = 2a_0b_0 + a_0b_1\tau + a_1b_0\tau$, $d_2 = 3a_0b_1 + a_1b_0 + a_1b_1\tau + a_2b_0\tau$, $d_3 = 2a_1b_1 + a_2b_1\tau$.

Mathematical expressions for determine of tuning parameters of PID controller are:

$$k_p = \frac{e^{-\tau J}\left(-d_0J^4 + d_1J^3 - d_2J^2 + d_3J - d_4\right)}{b_0^2J^2 - 2b_0b_1J + b_1^2} + 2k_dJ = f_p(J), \tag{16}$$

$$k_i = \frac{e^{-\tau J}(a_0J^3 - a_1J^2 + a_2J)}{b_1 - b_0J} - k_dJ^2 + k_pJ = f_i(J), \tag{17}$$

$$k_d = \frac{e^{-\tau J}(-d_5J^6 + d_6J^5 - d_7J^4 + d_8J^3 - d_9J^2 + d_{10}J - d_{11})}{2(b_0^4J^4 - 4b_0^3b_1J^3 + 6b_0^2b_1^2J^2 - 4b_0b_1^3J + b_1^4)} = f_d(J), \tag{18}$$

where $d_0 = a_0b_0\tau$, $d_1 = 2a_0b_0 + a_0b_1\tau + a_1b_0\tau$, $d_2 = 3a_0b_1 + a_1b_0 + a_1b_1\tau + a_2b_0\tau$, $d_3 = 2a_1b_1 +$
$+ a_2b_1\tau$, $d_4 = a_2b_1$, $d_5 = a_0b_0^3\tau^2$, $d_6 = 4a_0b_0^3\tau + 3a_0b_0^2b_1\tau^2 + a_1b_0^3\tau^2$, $d_7 = 2a_0b_0^3 + 14a_0b_0^2b_1\tau +$
$+ 3a_0b_0b_1^2\tau^2 + 2a_1b_0^3\tau + 3a_1b_0^2b_1\tau^2 + a_2b_0^3\tau^2$, $d_8 = 8a_0b_0^2b_1 + 16a_0b_0b_1^2\tau + a_0b_1^3\tau^2 + 8a_1b_0^2b_1\tau +$
$+ 3a_1b_0b_1^2\tau^2 + 3a_2b_0^2b_1\tau^2$, $d_9 = 12a_0b_0b_1^2 + 6a_0b_1^3\tau + 10a_1b_0b_1^2\tau + a_1b_1^3\tau^2 + 2a_2b_0^2b_1\tau + 3a_2b_0b_1^2\tau^2$, $d_{10} = 6a_0b_1^3 + 2a_1b_0b_1^2 + 4a_1b_1^3\tau + 2a_2b_0^2b_1 + 4a_2b_0b_1^2\tau + a_2b_1^3\tau^2$, $d_{11} = 2a_1b_1^3 - 2a_2b_0b_1^2 + 2a_2b_1^3\tau$.

3. Applications and computer simulation

The mathematical model of object described by the transfer function (1) is considered with the following numerical values: $\tau = 2$, $b_0 = 0.35$, $b_1 = 0.2313$, $a_0 = 1$, $a_1 = 0.3872$, and $a_2 = 0.04851$.

$$H_{PF}(s) = e^{-\tau s}\frac{b_0s + b_1}{a_0s^3 + a_1s^2 + a_2s} = e^{-2s}\frac{0,35s + 0,2313}{s^3 + 0,3872s^2 + 0,04851s}. \tag{19}$$

It is required: to tune the P, PI, PD and PID controllers.

Solution. The parameters of the control algorithms P, PI, PD and PID of the automatic system with the model of object in (1) with the given parameters and the respective controller according to relations (12)-(18) are calculated.

Substitute the numerical data in (11) and it is obtained the mathematical calculation expression for the P controller:

$$k_p = \frac{e^{-2J}(-J^2 + 0.3872J - 0.04851)}{0.2313 - 0.35J}. \tag{20}$$

The value of stability degree J is varied from 0.01 to 4.8, and the dependence $k_p = f(J)$ is plotted (Figure 3).

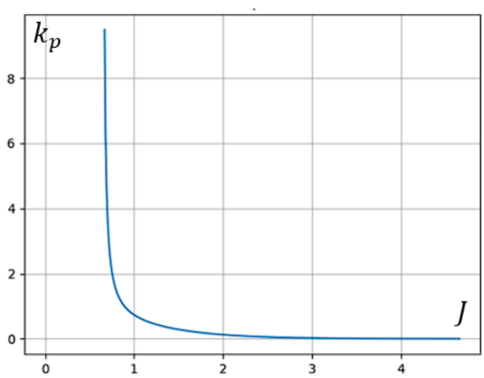

Figure 3. Dependence of $k_p = f(J)$.

Substitute the numerical data in (12), (13) and obtain the mathematical calculation expressions for the PI controller:

$$k_p = \frac{e^{-2J}(-0.7J^4 + 1.433J^3 - 1.042J^2 + 0.201J - 0.011)}{0.122J^2 - 0.162J + 0.053}, \tag{21}$$

$$k_i = \frac{e^{-2J}(J^3 - 0.3872J^2 + 0.04851J)}{0.2313 - 0.35J} + k_pJ. \tag{22}$$

The value of stability degree J is varied from 0.76 to 1.9, and the dependencies $k_p = f(J)$, $k_i = f(J)$ are plotted (Figure 4).

Substitute the numerical data in (14), (15) and obtain the mathematical calculation expressions for the PD controller:

$$k_p = \frac{e^{-2J}(-J^2 + 0.3872J - 0.04851)}{0.2313 - 0.35J} + k_dJ, \tag{23}$$

$$k_d = \frac{e^{-2J}(0.7J^3 - 1.083J^2 + 0.6756J - 0.095)}{0.122J^2 - 0.162J + 0.053}. \tag{24}$$

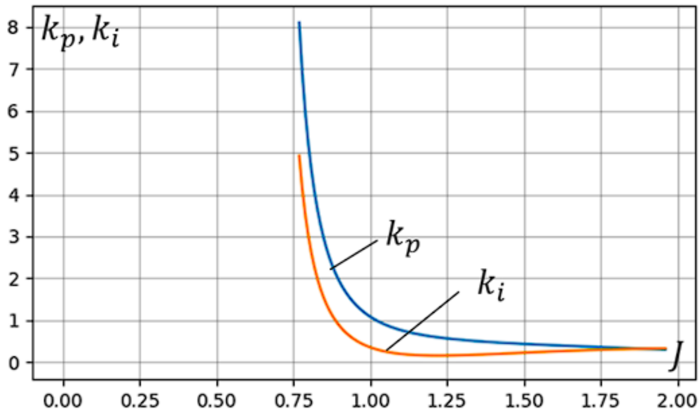

Figure 4. Dependencies of $k_p = f(J)$, $k_i = f(J)$.

The value of stability degree J is varied from 0.76 to 3, and the dependencies $k_p = f(J)$, $k_d = f(J)$ are plotted (Figure 5).

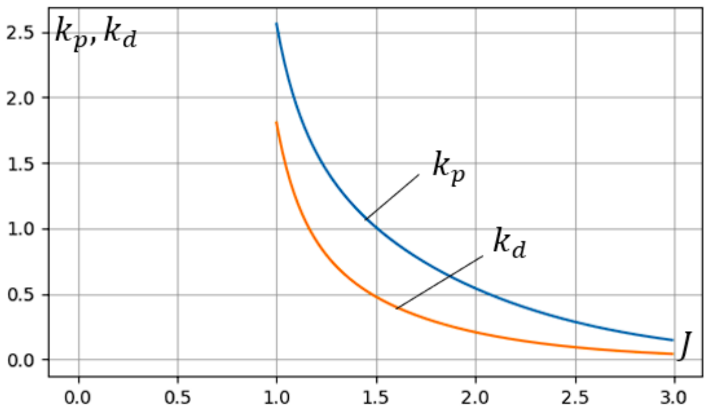

Figure 5. Dependencies of $k_p = f(J)$, $k_d = f(J)$.

Substitute the numerical data in (16), (17), (18) and obtain the mathematical calculation expressions for the PID controller:

$$k_p = \frac{e^{-2J}(-0.7J^4 + 1.433J^3 - 1.042J^2 + 0.201J - 0.011)}{0.122J^2 - 0.162J + 0.053} + 2k_dJ, \tag{25}$$

$$k_i = \frac{e^{-2J}(J^3 - 0.3872J^2 + 0.04851J)}{0.2313 - 0.35J} - k_dJ^2 + k_pJ, \tag{26}$$

$$k_d = \frac{e^{-2J}(-0.1715J^6 + 0.343J^5 - 1.31J^4 + 1.182J^3 - 0.553J^2 + 0.139J - 0.01)}{2(0.015J^4 - 0.0396J^3 + 0.0393J^2 - 0.0173J + 0.0028)}. \tag{27}$$

The value of stability degree J is varied from 0.01 to 0.57, and the dependencies $k_p = f(J)$, $k_i = f(J)$, $k_d = f(J)$ is plotted (Figure 6).

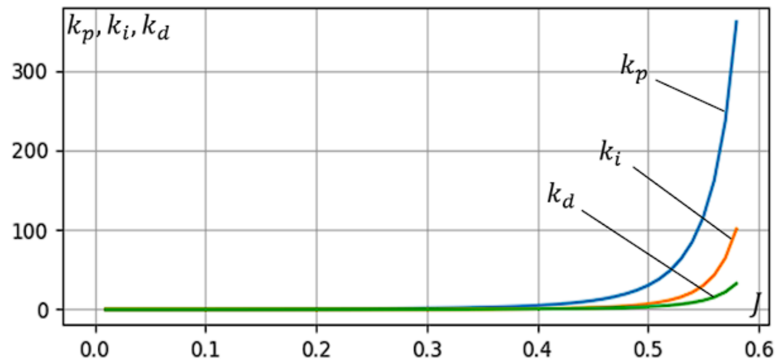

Figure 6. Dependencies of $k_p = f(J)$, $k_i = f(J)$ and $k_d = f(J)$.

To verify the tuning results of the controller, the system is simulated in the MATLAB software package, and the step responses (set point = 80) of the system with the respective P, PI, PD and PID controllers are illustrated in Figure 7.

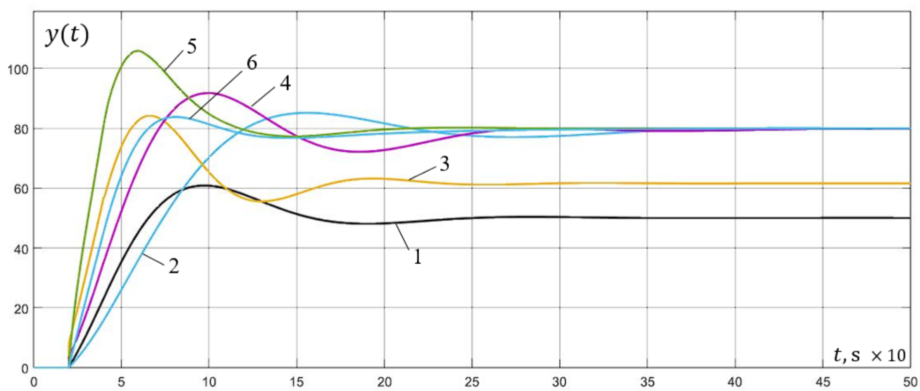

Figure 7. The step responses of the system with different controller types: P, PI, PD and PID: curve 1 is with P controller, 2 - PI, 3 - PD, 4 - PID tuned with MSDI, 5 - PID with Ziegler-Nichlos method, 6 - PID with parametric optimization method.

In Table 1, the performance of the simulated automatic control system in the MATLAB software package is presented with different P, PI, PD, and PID controllers tuned using the MSDI, Ziegler-Nichols, and parametric optimization methods.

Table 1. Controller parameters and simulated automated system performances

Iter. Nr.	**Tune method**	**Contr. type**	**Controller parameters**					**System performances**			
			J	k_p	k_i	T_i, s	k_d	t_c, s	σ, %	t_r, s	n
1	MSDI	P	1.38	0.35	-	-	-	29.3	31.97	187.2	2
2	MSDI	PI	1.35	0.209	0.032	31.25	-	57.5	12.02	263.4	2
3	MSDI	PD	1.80	0.701	-	-	0.29	18.7	57.49	245.3	4
4	MSDI	PID	0.22	0.424	0.046	21.73	0.119	28.1	8.75	82.8	1
5	ZN	PID	-	0.8922	0.1965	5.08	1.029	18.12	32.12	102.2	1
6	PO	PID	-	0.404	0.0501	19.96	0.572	36.88	-	60.94	-

4. Conclusions

Based on the conducted study, the following conclusions are formulated:

1. Good performances of the automatic control system were obtained for the version with PID controller tuned by the MSDI (Figure 7, curve 4, iteration 4, Table 1), having the settling time $t_r = 82.8$ s, the overshoot $\sigma = 8.75$ % and a deviation $n = 1$.
2. The best performance of the automatic control system was obtained for the system with PID controller tuned according to the parametric optimization method (Figure 7, curve 6, iteration 6, Table 1), having the lowest settling time $t_r = 60.94$ s, no overshoot $\sigma = 0$ and no oscillation $n = 0$.
3. The MSDI tuning method is the least computationally intensive and performs satisfactorily compared to the ZN and PO.
4. It is not recommended to use the P and PD controllers for the system with the given mathematical model of object (1) because they have a high stationary error (Figure 7, curve 1 and 3, Table 7, iteration 1 and 3).

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Dedicated to Professor Alexandru Şubă on the occasion of his 70th birthday

Generalized Hausdorff compactifications

Laurenţiu Calmuţchi

Abstract. This article investigates some properties of generalized Hausdorff compactifications of topological T_0-spaces. In particular, it is show that the totality of these compactifications forms a lattice of g-extensions in which there is the maximum element.
2010 Mathematics Subject Classification: 54D30, 54D40.
Keywords: continuous application, extension, g-extension, compactification, lattice, space.

Compactificări generalizate Hausdorff

Rezumat. În acest articol se studiază unele proprietăţi ale compactificărilor generalizate Hausdorff ale T_0-spaţiilor topologice. În particular, se demonstrează că totalitatea compactificărilor formează o latice de g-extensii în care există elementul maximal.
Cuvinte-cheie: aplicaţie continuă, extensie, g-extensie, compactificare, latice, spaţiu.

1. Extensions

Let us mention, that in case there are no concrete indications, then the topological space is considered T_0–space.

Definition 1.1. *A pair (Y, f) it is called a generalized extension or g–extension of space X, where Y is a space, $f : X \to Y$ is a continuous mapping and the set $f(X)$ is dense in Y. If f is an embadding of space X in Y, i.e. an omeomorphism of space X on subspace $f(X)$ of Y, then the pair (Y, f) is called an extension of space X.*

If (Y, f) is an extension of space X, then, as a rule, the point $x \in X$ is identified with $f(x) \in Y$ and it is considered to be $X \subseteq Y$. In this case $f(x) = x$ for any $x \in X$.

Let $GE(X)$ be the set of all g–extensions of the space X and $E(X)$ be the set of all extensions of X. Obviously, $E(X) \subseteq GE(X)$.

In class $GE(X)$ the binary increased relationship is introduced. If (Y, f) and (Z, g) are two g–extensions of X space, then it is considered $(Z, g) \leq (Y, f)$. If there is a continuous mapping $\varphi : Y \to Z$, for which $g(x) = \varphi(f(x))$ for any $x \in X$, i.e. $g = \varphi \circ f$ and Diagram 1 is commutative.

$$\begin{tikzcd} & Y \arrow[dr, "\varphi"] & \\ X \arrow[ur, "f"] \arrow[rr, "g"'] & & Z \end{tikzcd}$$

Diagram 1

$$\begin{tikzcd} & f(X) \arrow[dr, "h"] & \\ X \arrow[ur, "f"] \arrow[rr, "g"'] & & f(Y) \end{tikzcd}$$

Diagram 2

Figure 1. Diagrams 1 and 2.

If $(Y, f) \leq (Z, g)$ and $(Z, g) \leq (Y, f)$, then these g–extensions (Y, f) and (Z, g) are called *equivalent* and we denote this by $(Y, f) \sim (Z, g)$.

Proposition 1.1. *If (Y, f) and (Z, g) are two g–extensions of space X, $(Z, g) \leq (Y, f)$ and $(Z, g) \in E(X)$, then $(Y, f) \in E(X)$.*

Proof. Let $\varphi : Y \rightarrow Z$ be a continuous mapping and $g = \varphi \circ f$. According to the definition of relationship $\leq$, g is a dive. Let us denote $h = \varphi \mid f(X) \; : f(X) \rightarrow g(X)$. Then we get Diagram 2. As g is a bijection and f, h are surjections, it turns out that f and h are bijections. We have $f(A) = h^{-1}(g(A))$. Therefore, for any open set U of X the set $f(U)$ is open in $f(X)$, and the mapping $f^{-1} : f(X) \rightarrow X$ is continuous. So, f is a dive. Obviously, h is a homomorphism. Proposition 1.1 is proved. □

Corollary 1.1. *If (Y, f) and (Z, g) are two g–extensions equivalent of space X and one of them is extension, then the other one is extension.*

The pair (X, f), where $f(x) = x$ for any $x \in X$ is an extension of space X. This is the trivial extension or maximum extension. Let us denote this extension by (X, e_X).

Let S be a space consisting of a single point and let $s_X(x) = S$ for any $x \in X$. Then $(S,\ s_X)$ is called g–extension minimal or g–zero extension of space X.

Let P be a property of topological spaces. The property P is called *multiplicative* if the product of a set of spaces with the property P is a space with the property P.

The property P is called *hereditary closed* if any closed subspace of a space with the property P is a space with the property P.

Property P is called *additive* if the reunion space of a finite number of subspaces with the property P is a space with the property P.

Example 1.1. *The property of being compact space is multiplicative, hereditary closed and additive.*

Example 1.2. *The property of being countable compact space is hereditary, additive but not multiplicative. The product of two countable compact spaces can not be a countable compact space ([4], Example 3.10.19).*

Example 1.3. *The property of being pseudocompact is additive, but it is neither multiplicative and not hereditary closed [4].*

Example 1.4. *The property of being space is multiplicative, hereditary and additive. This property is called trivial property.*

2. Lattice of extensions

Let us fix a property P of topological spaces. We denote by $PGE(X)$ the totality of g–extensions (Y, f) with the property P, i.e. Y possesses the property P and denote $Y \in P$.

Let $PE(X) = E(X) \cap PGE(X)$. If P is a trivial property, then $PE(X) = E(X)$ and $PGE(X) = GE(X)$.

Definition 2.1. *If L is a nonempty set of $PGE(X)$ and $(Y, f) \in PGE(X)$, then:*

(1) *the extension (Y, f) is called the upper bound of a set L in $PGE(X)$ and denote $(Y, f) \in \vee L$, if $(Z, g) \leq (Y, f)$ for any $(Z, g) \in L$. If $(Y_1, f_1) \in PGE(X)$ and $(Z, g) \leq (Y, f)$ for any $(Z, g) \in L$, then $(Y, f) \leq (Y_1, f_1)$;*

(2) *the extension (Y, f) is called the lower bound of a set L in $PGE(X)$ and denote $(Y, f) \in \wedge L$, if $(Y, f) \leq (Z, g)$ for anything $(Z, g) \in L$. If $(Y_1, f_1) \in PGE(X)$ and $(Y, f) \leq (Z, g)$ for any $(Z, g) \in L$, then $(Y, f) \leq (Y_1, f_1)$.*

Proposition 2.1. *Let P be a multiplicative and hereditary closed property. Then for any nonempty set $L \subseteq PGE(X)$ there are extensions $(Y, f) \in \vee L$.*

Proof. Let $L = \{(Y_\mu, f_\mu) : \mu \in M\}$, $f(x) = (f_\mu(x) : \mu \in M) \in \prod\{Y_\mu : \mu \in M\}$ for any $x \in X$ and let Y be the adherence of a set $f(x)$ in $\prod\{Y_\mu : \mu \in M\}$. Then $(Y, f) \in \vee L$. Proposition 2.1 is proved. □

Definition 2.2. *The set $L \subseteq PGE(X)$ is called:*

(1) *the upper semilattice of extensions, if L is nonempty and for any nonempty subset $M \subseteq L$ there exists $(Y, f) \in \vee M$.*

(2) *the lower semilattice of extensions, if L is nonempty and for any nonempty subset $M \subseteq L$ there exists $(Y, f) \in \wedge M$;*

(3) *the lattice of extensions, if it is an upper semilattice and a lower semilattice of extensions.*

Proposition 2.2. *Let P be a multiplicative and closed hereditary property. Then for any nonempty set $H \subseteq PGE(X)$ there exists an upper semilattice of extensions $L^*(H)$ with properties:*

1. *$H \subseteq L^*(H)$;*
2. *if L is an upper semilattice of extensions and if $H \subseteq L \subseteq L^*(H)$, then $L = L^*(H)$.*

Proof. Let us fix $(Y_M, f_M) \in \vee M$ for any nonempty subset $M \subseteq H$. If $M = \{(Y, f)\}$, then $Y_M = Y$ and $f_M = f$. They can be obtained by constructing (Y_M, f_M) as in the proof of Proposition 2.1.

Let us denote $L^*(H) = \{(Y_M, f_M) : M \subseteq H, M \neq \emptyset\}$. Obviously, $H \subseteq L^*(H)$. If $M \subseteq K \subseteq H$, then $(Y_M, f_M) \leq (Y_K, f_K)$. According to construction $L^*(H)$ is an upper semilattice. If $K = \left\{(Y_{M_\alpha}, f_{M_\alpha}) : \alpha \in A\right\}$ and $M = \cup\{M_\alpha : \alpha \in A\}$, then $(Y_M, f_M) \in \vee K$. The proof is complete. □

Definition 2.3. *The upper semilattice $L^*(H)$ built in the proof of Proposition 2.2 is called the upper semilattice generated by set H.*

Corollary 2.1. *Let P be a multiplicative and closed hereditary property. Suppose that the continuous image of a space with property P is a space with property P. Then any nonempty set $H \subseteq PGE(X)$ is contained in a lattice of extensions of $PGE(X)$.*

Proof. Let (Z_0, g_0) be the extension, where Z_0 is a space consisting of a single point, and let $g_0 : X \to Z_0$ be the only possible application. It is clear that $(Z_0, g_0) \leq (Y, f)$ for any $(Y, f) \in GE(X)$. Let us denote $L(H) = L^*(H \cup \{(Z_0, g_0)\})$. Obviously, $L(H)$ is an upper semilattice. As the upper lattice $L(H)$ contains an element of $\wedge L(H)$, it is a lattice. But $(Z_0, g_0) \in \wedge L(H)$. The proof is complete. □

Definition 2.4. *A g–extension (Y, f) of the space X is called correct, if the family $\{cl_Y f(A) : A \subseteq X\}$ forms a closed base of the space Y.*

Let us denote by $KGE(X)$ the totality of correct g–extensions of the space X and let $KE(X) = E(X) \cap KGE(X)$.

Proposition 2.3. *If (Y, f), (Z, g) are two correct and equivalent g–compactifications of the space X, then $(Y, f) = (Z, g)$, i.e. the continuous application $\varphi : Y \to Z$ for any $g = \varphi \circ f$ is a homeomorphism of the space Y onto the space Z.*

Proof. Let $\varphi : Y \to Z$ and $\psi : Z \to Y$ be two continuous applications, for which $g = \varphi \circ f$ and $f = \psi \circ g$. If $A \subseteq X$, then $\varphi(cl_Y f(A)) \subseteq cl_Z g(A)$ and $\psi(cl_Z g(A)) \subseteq (cl_Y f(A))$. Hence, $\varphi(cl_Y f(A)) = cl_Z g(A)$ and $\psi(cl_z g(A)) = cl_Y f(A)$. From these two equalities we conclude that φ, ψ are reciprocal bijective applications and $\varphi^{-1} = \psi$. Proposition 2.3 is proved. □

3. Compacts

For topological spaces the notion of compact space was introduced by P.S. Alexandroff and P.S. Urysohn (see [1]).

Definition 3.1. *The class P of topological spaces is called strict compactness if it satisfies the conditions:*

$(C1)$ *class P is not empty;*

$(C2)$ *in P there is a space X containing at least two different points;*

$(C3)$ *class P is multiplicative;*

$(C4)$ *class P is closed hereditary;*

$(C5)$ *if Y is a dense subspace of the space $X \in P$, then $\{cl_X A : A \subseteq Y\}$ is a closed basis of the space X.*

Definition 3.2. *The class P of spaces with properties $(C1)$–$(C4)$ is called quasi-compactness.*

Definition 3.3. *A quasi-compactness P of Hausdorff spaces is called compactness.*

Proposition 3.1. *If P is a strict compactness, then:*

(1) *$PGE(X) = KPGE(X)$ for any space X;*

(2) *$PGE(X)$ is a set for any space X;*

(3) *$PGE(X)$ is a lattice of extensions for any space X.*

Proof. Equality (1) is a consequence of condition (C5) in Definition 3.1. It follows from Proposition 2.3 that $PGE(X)$ is a set. Since $L^*PGE(X)) = L(PGE(X)) = PGE(X)$, from Corollary 2.1 we obtain that $PGE(X)$ is a lattice of extensions. The proof is complete. □

Proposition 3.2. *If P is a compactness, then $PGE(X)$ is a lattice of extensions for any space X.*

Proof. If (Y, f) is a g–extension of the space X, Y is a Hausdorff space, and τ is the power of the set X, then the weight $\omega(Y) \leq 2^\tau$. Therefore, $PGE(X)$ is a set. Then, based on Corollary 2.1, we obtain that $L^*(PGE(X)) = L(PGE(X)) = PGE(X)$ is a lattice of extensions. The proof is complete. □

Corollary 3.1. *Let P be a compactness or a strict compactness, X be a space and suppose that $PE(X) \neq \emptyset$. Then $PE(X)$ is a upper semilattice of extensions.*

Corollary 3.2. *Let P be a compactness or a strict compactness. Then:*

(1) *for any space X a unique maximal g–extension $(\beta_P X,\ \beta_X) \in PGE\,(X)$ is determined;*

(2) *for any continuous mapping $\varphi : X \to Y$ there is a unique continuous mapping $\beta_{P\varphi} : \beta_P X \to \beta_P Y$, for which $\beta_Y \circ \varphi = \beta_{P\varphi} \circ \beta_X$, i.e. Diagram 3 is commutative.*

Proof. The statement (1) follows from Propositions 3.1 and 3.2. If $\varphi : X \to Y$ is continuous mapping and $(Z,\ g) \in PGE\,(X)$, then $(Z,\ g \circ \varphi) \in PGE(X)$. This fact proves the presence of $\beta_{P\varphi}$. The proof is complete. □

4. Generalized Hausdorff Compactifications

Let us denote by $HGC(X)$ the totality of g–compactifications $(bX,\ b_X)$ of the space X for which bX is a Hausdorff space.

Theorem 4.1. *The totality of $HGC\,(X)$ is a complete lattice of g–extensions.*

Proof. We will prove this theorem after the following steps:

(1) Let us note that the totality of $HGC(X)$ is not empty, since it contains the minimal extension $(mX,\ m_X)$ of a point.

(2) If Y is a Hausdorff space, then for the power (cardinality) of the set Y we have $|Y| \leq \exp\left(\exp d\,(Y)\,\right)$, where $d(Y)$ is the density of the space Y (see [4], Theorem 1.5.3). If $(Y,\ f) \in GE(X)$, then $d\,(Y) \leq |X|$. Hence, $|Y| \leq \exp\left(\exp|X|\,\right)$ for any Hausdorff g–compactification $(Y,\ f) \in HGC(X)$. But all topological spaces of power $\leq \exp\left(\exp|X|\,\right)$ form a set, which contains the entirety of $HGC(X)$. So the totality of $HGC(X)$ is a set.

(3) If (Y, f) and (Z, g) are two equivalent Hausdorff g–compactifications, then they coincide. Let $\varphi : Y \to Z$ and $\psi : Z \to Y$ be two continuous maps for which $g = \varphi \circ f$ and $f = \psi \circ g$. Let us prove that $\psi = \varphi^{-1}$. We examine the application $h = \psi \circ \varphi : Y \to Y$. This mapping is continuous and $h\,(y) = y$ for any $y \in f(X)$. Indeed, let $y = f(x)$ and $x \in X$. Then $\varphi(y) = \varphi\,(f\,(x)) = g(x)$ and $\psi\,(\varphi\,(y)) = \psi\,(g\,(x)) = f(x) = y$. Therefore, $h(y) = y$. The space Y is Hausdorff and $Y_1 = \{y \in Y : h(y) = y\}$ contains the set $f(X)$. So the set Y_1 is dense in Y. Now let us prove that $Y_1 = Y$. Assume that $y_0 \in Y \backslash Y_1$. Then $y_1 = h\,(y_0) \neq y_0$ and there are two open sets U, V in Y for which $y_1 \in U$, $y_0 \in V$ and $U \cap V$=$\emptyset$. The set $W = U \cap h^{-1}\,(V)$ is open in Y and $y_1 \in W$. If $Y \in W$, then $h\,(y) \in V$ and $h\,(y) \neq y$. Hence, $W \cap Y_1 = \emptyset$. Therefore, the set Y_1 is closed in Y. But Y_1 is dense in Y, and a dense and closed set in Y coincides with Y. So, $Y_1 = Y$. We proved that $h(y) = y$ for any $y \in Y$. Therefore, $\psi = \varphi^{-1}$.

(4) The property H to be compact and Hausdorff space is multiplicative and hereditary over closed subspaces. Applying Proposition 3.2 we obtain that $HGC(X)$ is a complete lattice. Theorem 4.1 is proved.

□

Definition 4.1. *The maximal element of the lattice $HGC(X)$ is denoted by $(\beta X, \beta_X)$ and is called the Stone-Čech g–compactification.*

Corollary 4.1. *If $f\colon X \to Y$ is a continuous mapping, then there is a unique continuous mapping $\beta f : \beta X \to \beta Y$ for which $\beta f \circ \beta_X = \beta_Y \circ f$, i.e. Diagram 4 is commutative.*

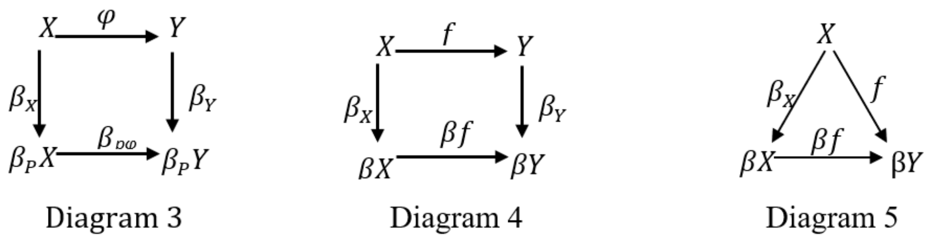

Figure 2. Diagrams 3, 4 and 5.

Corollary 4.2. *If $f\colon X \to Y$ is a continuous application of X space in the Hausdorff and compact space Y, then there is a unique continuous mapping $\beta f : \beta X \to Y$ for which $f = \beta f \circ \beta_X$, i.e. Diagram 5 is commutative.*

Corollary 4.3. *(See [4], Chapter 3). Let $HC(X) = E(X) \cap HGC(X)$ be the set of Hausdorff compactifications of the space X. Then:*

(1) *if $HC(X) \neq \emptyset$, then $HC(X)$ is a complete upper semilattice with maximal element βX;*

(2) *the following statements are equivalent:*

(2.1) *$HC(X) \neq \emptyset$;*

(2.2) *X is a T_0–completely regular space;*

(2.3) *βX is an extension of the space X.*

Theorem 4.2. *(see [4], for T_1 spaces). For any continuous application $f : X \to Y$ in a compact Hausdorff space Y there is a unique continuous application $\omega f : \omega X \to Y$ for which $f = \omega X \mid X$. The mapping ωf is always perfect.*

Proof. Denote $\varphi(x) = f(x)$ for any $x \in X$ and let $\varphi(\xi) = \cap \{cl_Y f(H) : H \in \xi\}$ for any ultrafilter $\xi \in \omega X \backslash X$. Let $y, z \in cl_Y f(X)$ be two different points. There are two open sets

U and V in Y for which $x \in U$, $y \in V$, and the sets $F = cl_Y U$, $\Phi = cl_Y V$ do not intersect. Then $cl_{\omega X} f^{-1}(F) \cap cl_{\omega X} f^{-1}(\Phi) = \emptyset$. If $X \backslash f^{-1}(U) \in \xi$, then $y \notin \varphi(\xi)$. If $X \backslash V \in \xi$, then $z \notin \varphi(\xi)$. But $\xi \cap \left\{X \backslash f^{-1}(U), X \backslash f^{-1}(V)\right\} = \emptyset$. So the mapping $\varphi \colon \omega X \to Y$ is unique and $f = \varphi \mid X$. The set $Z = \{cl_Y A : A \subseteq f(x)\}$ forms a closed basis of the space $Z = cl_Y f(X)$. If A is closed in $f(X)$ and $y \in cl_Y A$, then there exists an ultrafilter η of closed sets in $f(X)$ for which $\{y\} = \cap \{cl_Y H : H \in \eta\}$. There exists at least one ultrafilter $\xi \in \omega X$ for which $f^{-1}(\eta) \subseteq \xi$. Then $\varphi(\xi) = y$. Therefore, $\varphi^{-1}(A) = cl_{\omega X} f^{-1}(A)$ is a closed set in ωX. So, φ is a continuous mapping. From the construction and continuity of the mapping φ we obtain its uniqueness. If the set F is closed in ωX, then $\varphi(F)$ is a compact set. The compact set in a Hausdorff space is closed. So, φ is a closed mapping. Theorem 4.2 is proved. □

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Dedicated to Professor Alexandru Şubă on the occasion of his 70th birthday

First integrals in a cubic differential system with one invariant straight line and one invariant cubic

Dumitru Cozma

Abstract. In this paper we find conditions for a singular point $O(0,0)$ of a center or a focus type to be a center, in a cubic differential system with one invariant straight line and one invariant cubic. The presence of a center at $O(0,0)$ is proved by constructing Darboux first integrals.

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Keywords: cubic differential system, invariant algebraic curve, Darboux integrability, the problem of the center.

Integrale prime pentru un sistem diferenţial cubic cu o dreaptă invariantă şi o cubică invariantă

Rezumat. În lucrare se examinează sistemul diferenţial cubic cu punctul singular $O(0,0)$ de tip centru sau focar, care are o dreaptă invariantă şi o cubică invariantă. Pentru acest sistem sunt determinate condiţiile de existenţă a centrului în $O(0,0)$ prin construirea integralelor prime de forma Darboux.

Cuvinte-cheie: sistem diferenţial cubic, curbă algebrică invariantă, integrabilitatea Darboux, problema centrului şi focarului.

1. Introduction

We consider the cubic differential system

$$\begin{cases} \dot{x} = y + ax^2 + cxy + fy^2 + kx^3 + mx^2y + pxy^2 + ry^3 \equiv P(x,y), \\ \dot{y} = -(x + gx^2 + dxy + by^2 + sx^3 + qx^2y + nxy^2 + ly^3) \equiv Q(x,y), \end{cases} \tag{1}$$

where $P(x,y)$ and $Q(x,y)$ are real and coprime polynomials in the variables x and y, $\dot{x} = dx/dt, \dot{y} = dy/dt$. The origin $O(0,0)$ is a singular point of a center or a focus type for (1), i.e. a week focus. It arises the problem of distinguishing between a center and a focus (called *the problem of the center*), i.e. the problem of finding the coefficient conditions under which $O(0,0)$ is a center.

The problem of the center is equivalent to the problem of local integrability of a differential system in the neighboarhood of a singular point with pure imaginary eigenvalues.

It is known [1] that a singular point $O(0,0)$ is a center for (1) if and only if the system has a holomorphic first integral of the form $F(x,y) = C$ in some neighborhood of $O(0,0)$.

Although the problem of the center dates from the end of the 19th century, it is completely solved only for: quadratic systems $\dot{x} = y + p_2(x,y),\ \dot{y} = -x + q_2(x,y)$; cubic symmetric systems $\dot{x} = y + p_3(x,y),\ \dot{y} = -x + q_3(x,y)$; Kukles system $\dot{x} = y,\ \dot{y} = -x + q_2(x,y) + q_3(x,y)$ and a few particular cases in families of polynomial systems of higher degree, where $p_j(x,y)$ and $q_j(x,y)$ are homogeneous polynomials of degree j in the variables x and y.

If the cubic system (1) contains both quadratic and cubic nonlinearities, then the problem of the center is still open. For such systems the necessary and sufficient conditions for the origin to be a center were obtained in some particular cases (see, for example, [9], [20], [21], [22]), [25].

The problem of the center was solved for some families of cubic differential systems with algebraic solutions: four invariant straight lines [3], [4], [5], [9], [18]; three invariant straight lines [9], [24]; two parallel invariant straight lines [14], [23]; two invariant straight lines and invariant conic [6], [7], [9]; two invariant straight lines and invariant cubic [10], [11]. It was proved that every center in the cubic differential system (1) with two invariant straight lines and one invariant conic comes from a Darboux integrability.

The integrability conditions for some families of cubic differential systems having invariant algebraic curves were found in [2], [8], [9], [12]–[17], [19], [25].

The goal of this paper is to obtain the center conditions for cubic differential system (1) with one invariant straight line and one irreducible invariant cubic by using the method of Darboux integrability. Our main result is the following one.

Theorem 1.1. *The origin is a center for cubic differential system* (1)*, with one invariant straight line and one irreducible invariant cubic, if one of the following conditions holds:*

(i) $a = d = k = r = 0,\ g = c + 1 - b,\ l = bf,\ m = -(c+1),\ n = [b(2c+3-b)]/2,$ $p = -f,\ q = -f(c+1+b),\ s = -b(c+1),\ b^2 - 2f^2 - b = 0;$

(ii) $d = 2a,\ k = -a,\ l = [f(2b-c-1)]/3,\ m = -(c+1),\ n = [(2b-c-2)(c+1)]/2,$ $p = -f,\ q = a(2b-c-3),\ r = 0.$

The paper is organized as follows. In Section 2 we present the known results concerning the relation between algebraic solutions and Darboux integrability. In Sections 3 and 4 we determine the integrability conditions for cubic differential system (1) with one invariant straight line and one invariant cubic by constructing Darboux first integrals. Finally in Section 5 we prove the Theorem 1.1.

2. Algebraic solutions and Darboux first integrals

An important problem for differential system (1) is whether the trajectories to (1) can be described by an algebraic formula, for example, $\Phi(x, y) = 0$, where Φ is a polynomial.

Definition 2.1. *An algebraic invariant curve of* (1) *is the solution set in* $\mathbb{C}^2$ *of an equation* $\Phi(x, y) = 0$, *where* Φ *is a polynomial in* x, y *with complex coefficients such that*

$$\frac{\partial \Phi}{\partial x} P(x, y) + \frac{\partial \Phi}{\partial y} Q(x, y) = K(x, y)\Phi(x, y)$$

for some polynomial in x, y, $K = K(x, y)$ *with complex coefficients, called the cofactor of the invariant algebraic curve* $\Phi = 0$.

We say that the invariant algebraic curve $\Phi(x, y) = 0$ is an *algebraic solution* of (1) if and only if $\Phi(x, y)$ is an irreducible polynomial in $\mathbb{C}[x, y]$. We shall study the problem of the center for cubic differential system (1) assuming that (1) has algebraic solutions: one invariant straight line and one invariant cubic.

By Definition 2.1 a straight line

$$1 + Ax + By = 0, \ (A, B) \neq 0, \ A, B \in \mathbb{C} \tag{2}$$

is said to be invariant for (1), if there exists a polynomial with complex coefficients $K(x, y)$ such that the following identity holds

$$AP(x, y) + BQ(x, y) \equiv (1 + Ax + By)K(x, y).$$

Let the cubic system (1) have a real invariant straight line of the form (2). Then by rotating the system of coordinates ($x \to x\cos\varphi - y\sin\varphi, y \to x\sin\varphi + y\cos\varphi$) and rescaling the axes of coordinates ($x \to \alpha x,\ y \to \alpha y$), we can make the line to be $1 - x = 0$.

In [9] it was proved the following Lemma

Lemma 2.1. *The cubic system (1) has the invariant straight line* $1 - x = 0$ *if and only if the following set of conditions holds*

$$k = -a,\ m = -c - 1,\ p = -f,\ r = 0. \tag{3}$$

Suppose the set of conditions (3) is realized, then the cubic system (1) can be written as follows

$$\begin{cases} \dot{x} = (1 - x)(y + xy + ax^2 + cxy + fy^2) \equiv P(x, y), \\ \dot{y} = -(x + gx^2 + dxy + by^2 + sx^3 + qx^2y + nxy^2 + ly^3) \equiv Q(x, y), \end{cases} \tag{4}$$

We are interested in finding the conditions under which the cubic system (4) has one real irreducible invariant cubic curve. According to [9], a real irreducible invariant cubic curve of (1) can have one of the following two forms

$$a_{30}x^3 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3 + a_{20}x^2 + a_{11}xy + a_{02}y^2 + a_{10}x + a_{01}y + 1 = 0 \quad (5)$$

or

$$a_{30}x^3 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3 + x^2 + y^2 = 0, \quad (6)$$

where $(a_{30}, a_{21}, a_{12}, a_{03}) \neq 0,\ \ a_{ij} \in \mathbb{R}$.

By Definition 2.1, the cubic curve (5) ((6)) is said to be an invariant cubic for (1), if there exists a polynomial with real coefficients $K(x, y) = c_{10}x+c_{01}y+c_{20}x^2+c_{11}xy+c_{02}y^2$ such that the following identity holds

$$\frac{\partial \Phi}{\partial x}P(x, y) + \frac{\partial \Phi}{\partial y}Q(x, y) \equiv \Phi(x, y)K(x, y).$$

Definition 2.2. *System (1) is integrable on an open set D of R^2 if there exists a nonconstant analytic function $F : D \to R$ which is constant on all solution curves $(x(t), y(t))$ in D, i.e. $F(x(t), y(t)) = C$ for all values of t where the solution is defined. Such an F is called a first integral of the system on D.*

When F exists in D all the solutions of the differential system in D are known [1], since every solution is given by $F(x, y) = C$, for some $C \in \mathbb{R}$. Clearly F is a first integral of (1) on D if and only if

$$P\frac{\partial F}{\partial x} + Q\frac{\partial F}{\partial y} \equiv 0. \quad (7)$$

A first integral constructed from invariant algebraic curves $f_j(x, y) = 0,\ \ j = \overline{1, q}$

$$F(x, y) \equiv f_1^{\alpha_1} f_2^{\alpha_2} \cdots f_q^{\alpha_q} = C \quad (8)$$

with $\alpha_j \in \mathbb{C}$ not all zero is called *a Darboux first integral* [21], [25].

By constracting Darboux first integrals (8), the center conditions where obtained for cubic system (1) with two invariant straight lines and one invariant conic in [9], with two invariant straight lines and one invariant cubic of the form (6) in [10] and [11].

The qualitative investigation in 2-dimensional parameter space of cubic systems (1) with a center and having the Darboux first integral of the form

$$(1 + Ax + By)^2\Phi = 0,$$

where $\Phi = 0$ is an irreducible invariant cubic curve of the form (6), was done in [26].

In [13] it was found the center conditions for cubic differential system (1) by constructing Darboux integrating factors of the form

$$\mu(x, y) = (1 - x)^{\alpha}\Phi^{\beta},$$

where $\Phi = 0$ is an irreducible invariant cubic of the form (6) and $\alpha, \beta \in \mathbb{R}$.

In this paper, using the equation (7), we find the conditions under which the cubic differential system (1) has Darboux first integrals of the form

$$F(x, y) \equiv (1 - x)^{\alpha}\Phi^{\beta} = C \tag{9}$$

composed of one invariant straight line $1 - x = 0$ and one irreducible invariant cubic $\Phi = 0$ of the form (5) ((6)), where $\alpha, \beta \in \mathbb{R}$.

3. One invariant straight line and one invariant cubic of the form (5)

In this section we find Darboux integrability conditions for cubic differential system (1) with one invariant straight line and one invariant cubic of the form (5).

Lemma 3.1. *The cubic differential system (1) with one invariant straight line* $1 - x = 0$ *and one invariant cubic (5) has a Darboux first integral of the form (9) if and only if one of the following two sets of conditions is satisfied:*

(i) $a = d = k = r = 0$, $g = c + 1 - b$, $l = bf$, $m = -(c + 1)$, $n = [b(2c + 3 - b)]/2$, $p = -f$, $q = -f(c + 1 + b)$, $s = -b(c + 1)$, $b^2 - 2f^2 - b = 0$*;*

(ii) $d = 2a$, $k = -a$, $l = [f(2b-c-1)]/3$, $m = -(c+1)$, $n = [(2b-c-2)(c+1)]/2$, $p = -f$, $q = a(2b - c - 3)$, $r = 0$.

Proof. Let the cubic system (1) have the invariant straight line $1 - x = 0$ and an invariant cubic $\Phi = 0$ of the form (5). In this case the system (1) will have a Darboux first integral of the form (9) if and only if the identity (7) holds. Identifying the coefficients of the monomials $x^i y^j$ in (7), we obtain a system of twenty equations

$$\{U_{ij} = 0, \quad i + j = 1, \dots, 5\} \tag{10}$$

for the unknowns a_{30}, a_{21}, a_{12}, a_{03}, a_{20}, a_{11}, a_{02}, a_{10}, a_{01}, α, β and the coefficients of system (1).

When $i + j = 1$, the equations $U_{10} = 0$ and U_{01}=0 of (10) yield $a_{01} = 0$ and $\alpha = a_{10}\beta$. If $i + j = 2$, the equations $U_{20} = 0$, $U_{11} = 0$ and $U_{02} = 0$ of (10) imply $a_{11} = 0$ and $a_{20} = (2a_{02} + a_{10}^2 + a_{10})/2$.

When $i + j = 3$, the equations $U_{ij} = 0$ of (10) give $a_{21} = 2aa_{02}$, $a_{12} = a_{02}(a_{10} + 2b)$, $a_{30} = (6a_{02}a_{10}+8ba_{02}-4ca_{02}+4ga_{02}+a_{10}^3+3a_{10}^2+2a_{10})/6$ and $a_{03} = [2a_{02}(2a-d+f)]/3$. We express l, n, q, s from the equations $U_{04} = 0$, $U_{13} = 0$, $U_{22} = 0$, $U_{31} = 0$ of (10). Then $U_{40} \equiv 2a\beta a_{02}(a_{10} + 2b - c - 1) = 0$, where $\beta a_{02} \neq 0$. We divide the investigation into two cases: $\{a = 0\}$; $\{a_{10} = c - 2b + 1, a \neq 0\}$.

1. Let $a = 0$. Then $U_{05} \equiv d(d - f)(a_{10} + 3b) = 0$.

1.1. Assume that $d = f$. In this case $U_{14} \equiv f(a_{10} + 2b + 1)(a_{10} + 2b) = 0$.

1.1.1. Suppose $a_{10} = -2b$. Then $U_{32} \equiv f_1 f_2 f_3 = 0$, where

$$f_1 = f,\ f_2 = 2b - 3,\ f_3 = (2b - 1)(b - 1)b + (b + c - g)a_{02}.$$

If $f_2 = 0$ or $f_3 = 0$, then the right-hand sides of (1) have a common factor $cx+x+fy+1$. The case $f_1 = 0$ is contained in (ii) $(a = f = 0, c = -1)$.

1.1.2. Suppose $a_{10} = -2b - 1$. Then $U_{32} \equiv g_1 g_2 g_3 = 0$, where

$$g_1 = f,\ g_2 = 2b + 1,\ g_3 = (2b - 1)(b - 1)b + (b + c - g)a_{02}.$$

If $g_2 = 0$ or $g_3 = 0$, then the right-hand sides of (1) have a common factor $cx+x+fy+1$. The case $g_1 = 0$ is contained in (ii) $(a = f = 0, c = -2)$.

1.1.3. Suppose $(a_{10} + 2b + 1)(a_{10} + 2b) \neq 0$ and let $f = 0$. Then $U_{32} \equiv 0$ and $U_{23} = 0$ yields $a_{10} = c - 2b + 1$. This case is contained in (ii) $(a = f = 0)$.

1.2. Assume that $d \neq f$ and let $d = 0$. Then $U_{14} \equiv (a_{10} + 2b - c - 1)(a_{10} + 3b) = 0$.

1.2.1. The case $a_{10} = c + 1 - 2b$ is contained in (ii) $(a = 0)$.

1.2.2. If $a_{10} \neq c + 1 - 2b$ and $a_{10} = -3b$, then from equations $\{U_{23} = 0, U_{32} = 0, U_{41} = 0\}$ of (10) we get $2f^2 - b^2 + b = 0$, $g = c - b + 1$ and $a_{02} = (3b - 3b^2)/2$. In this case we obtain the set of conditions (i) for the existence of the first integral (9) with $\alpha = -3b,\ \beta = 1$ and

$$\Phi \equiv 2(1 - bx)^3 + b(b - 1)(3bx - 2fy - 3)y^2 = 0.$$

1.3. Assume that $d(d - f) \neq 0$ and let $a_{10} = -3b$. In this case $U_{05} \equiv 0$ and

$$U_{32} \equiv (3b - 1)(3b - 2)(3b(b - 1) + 2a_{02}) = 0.$$

1.3.1. If $b = 1/3$ or $b = 2/3$, then $U_{14} \equiv d(9(d - f)^2 + 1) \neq 0$.

1.3.2. If $a_{02} = (3b(1 - b))/2$ and $(3b - 1)(3b - 2) \neq 0$, then $U_{41} \equiv U_{32} \equiv 0$. The equations $U_{23} = 0, U_{14} = 0$ yield $b^2 - b - 2(d - f)^2 = 0$. In this case, the right-hand sides of (1) have a common factor $cx + x + fy + 1$.

2. Assume that $a_{10} = c - 2b + 1$ and let $a \neq 0$. Then the equations of (10) imply $d = 2a$. In this case we obtain the set of conditions (ii) for the existence of the first integral (9) with $\alpha = c - 2b + 1,\ \beta = 1$ and

$$\Phi \equiv a_{30}x^3 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3 + a_{20}x^2 + a_{11}xy + a_{02}y^2 + a_{10}x + a_{01}y + 1 = 0,$$

where $a_{01} = a_{11} = 0$, $a_{10} = c+1-2b$, $a_{20} = (2a_{02}+a_{10}^2+a_{10})/2$, $a_{02} = [(c+1-2b)(2b-c-2)(2b-c-3)(2b-c-4)]/[2(2b-c-2g-3)(2b-c-4)+12s]$, $a_{21} = 2aa_{02}$, $a_{12} = a_{02}(a_{10} + 2b)$, $a_{03} = [2a_{02}(2a - d + f)]/3$, $a_{30} = (6a_{02}a_{10} + 8ba_{02} - 4ca_{02} + 4ga_{02} + a_{10}^3 + 3a_{10}^2 + 2a_{10})/6$.

In each of the cases (i) and (ii), the system (1) has a Darboux first integral of the form (9) and therefore the origin is a center for (1). □

4. One invariant straight line and one invariant cubic of the form (6)

In this section we find Darboux integrability conditions for cubic differential system (1) with one invariant straight line and one invariant cubic of the form (6).

Lemma 4.1. *The cubic differential system (1) with one invariant straight line* $1 - x = 0$ *and one invariant cubic (6) has a Darboux first integral of the form (9) if and only if the following set of conditions is satisfied:*

(iii) $d = 2a$, $k = -a$, $l = [f(2b-c-1)]/3$, $m = -(c+1)$, $n = [(2b-c-2)(c+1)]/2$, $p = -f$, $q = a(2b-c-3)$, $r = 0$, $s = [(c-2b+2g+3)(2b-c-4)]/6$.

Proof. Let the cubic system (1) have the invariant straight line $1 - x = 0$ and an invariant cubic $\Phi = 0$ of the form (6). In this case the system (1) will have a Darboux first integral of the form (9) if and only if the identity (7) holds. Identifying the coefficients of the monomials $x^i y^j$ in (7), we obtain a system of fifteen equations

$$\{V_{ij} = 0, \quad i + j = 3, 4, 5\} \tag{11}$$

for the unknowns a_{30}, a_{21}, a_{12}, a_{03}, α, β and the coefficients of system (1).

When $i + j = 3$, the equations of (11) yield $a_{21} = 2a$, $\alpha = \beta(a_{12} - 2b)$, $g = (3a_{30} - 3a_{12} + 2b + 2c)/2$, $d = (4a + 2f - 3a_{03})/2$, where $\beta \neq 0$.

We express l, n, q, s from the equations $V_{04} = 0$, $V_{13} = 0$, $V_{22} = 0$, $V_{31} = 0$ of (11). Then $V_{05} \equiv (3a_{03} - 2f)(a_{12} + b)a_{03} = 0$. We divide the investigation into three cases:

$$\{a_{03} = (2f)/3\};\ \{a_{03} = 0, f \neq 0\};\ \{a_{12} = -b, a_{03}(3a_{03} - 2f) \neq 0\}.$$

1. Let $a_{03} = (2f)/3$. Then $V_{40} \equiv a(a_{12} - c - 1) = 0$.

1.1. Assume that $a_{12} = c+1$. Then we obtain the set of conditions (iii) for the existence of the first integral (9) with $\alpha = c - 2b + 1$, $\beta = 1$ and

$$\Phi \equiv 3(x^2 + y^2) + (c - 2b + 2g + 3)x^3 + 6ax^2y + 3(c + 1)xy^2 + 2fy^3 = 0.$$

1.2. Assume that $a_{12} \neq c + 1$ and let $a = 0$. Then $V_{14} \equiv f(a_{12} + b) = 0$.

If $f = 0$, then $V_{23} \equiv a_{12}(a_{12} + 1) = 0$. When $a_{12} = 0$ or $a_{12} = -1$ the right-hand sides of (1) have a common factor $cx + x + 1$.

If $f \neq 0$ and $a_{12} = -b$, then $a_{30} = (2-7b)/3$. In this case $V_{41} \equiv (3b-1)(3b-2) = 0$. When $b = 1/3$ or $b = 2/3$, we have that $V_{23} \neq 0$.

2. Let $a_{03} = 0$ and $f \neq 0$. Then $V_{40} \equiv a(a_{12} - c - 1) = 0$.

2.1. Assume that $a = 0$. Then $V_{14} \equiv a_{12}(a_{12} + 1) = 0$. If $a_{12} = 0$, then $b = 3/2$ and the right-hand sides of (1) have a common factor $cx + x + fy + 1$.

If $a_{12} = -1$, then $V_{32} \equiv (a_{30} + 1)(2b + 1) = 0$. When $a_{30} = -1$, the cubic curve (6) is reducible and when $b = (-1)/2$, the right-hand sides of (1) have a common factor $cx + x + fy + 1$.

2.2. Assume that $a_{12} = c + 1$ and let $a \neq 0$. Then $V_{50} \neq 0$.

3. Let $a_{12} = -b$ and $(3a_{03} - 2f)a_{03} \neq 0$. Then $V_{40} \equiv a(b + c + 1) = 0$.

3.1. Assume that $a = 0$. Then $V_{41} \equiv f_1 f_2 = 0$, where

$$f_1 = b + c + 1,\ f_2 = (6b - 3)a_{30} + 5b^2 - 2b.$$

If $f_1 = 0$, then $c = -b - 1$ and $V_{23} = 0$ yields $a_{30} = (2 - 7b)/3$. In this case $V_{32} \equiv (3b - 1)(3b - 2) = 0$. If $b = 1/3$ or $b = 2/3$, then $V_{14} \neq 0$.

Suppose that $f_2 = 0$ and $f_1 \neq 0$. Then $a_{30} = (b(5b - 2))/(3(1 - 2b))$ and $V_{32} \equiv (3b - 1)(3b - 2) = 0$. If $b = 1/3$ or $b = 2/3$, then $V_{14} \neq 0$.

3.2. Assume that $c = -b - 1$ and $a \neq 0$. Then $V_{50} \neq 0$.

In the case (iii), the system (1) has a Darboux first integral of the form (9) and therefore the origin is a center for (1). □

5. Proof of the Main Theorem

The proof of the main result, Theorem 1.1, follows directly from Lemmas 3.1 and 4.1. The existence of a center for system (1), in Cases (i), (ii) and (iii), is equivalent to the existence of the Darboux first integrals of the form (9) defined in a neighborhood of the origin [25]. It is easy to verify that the Case (iii) is contained in the Case (ii) ($s = [(c - 2b + 2g + 3)(2b - c - 4)]/6$).

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Dedicated to Professor Alexandru Șubă on the occasion of his 70th birthday

T-quasigroups with Stein 2-nd and 3-rd identity

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Abstract. In this paper we prolong research of T-quasigroups with Stein 2-rd ($xy \cdot x = y \cdot xy$) and Stein 3-rd ($xy \cdot yx = y$) identities [9].

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Keywords: quasigroup, loop, groupoid, Schröder quasigroups, Stein identity.

T-cvasigrupuri cu a doua şi a treia identitate Stein

Rezumat. În această lucrare sunt prelungite cercetările T-cvasigrupurilor cu a 2-a identitate Stein ($xy \cdot x = y \cdot xy$) şi a 3-a identitate Stein ($xy \cdot yx = y$) [9].

Cuvinte-cheie: cvasigrup, buclă, grupoid, cvasigrupuri Schröder, identitate Stein.

1. Introduction

Necessary definitions can be found in [2, 3, 4, 5, 7, 10, 14].

Definition 1.1. *Binary groupoid $(Q, \circ)$ is called a left quasigroup if for any ordered pair $(a, b) \in Q^2$ there exist the unique solution $x \in Q$ to the equation $a \circ x = b$* [2].

Definition 1.2. *Binary groupoid $(Q, \circ)$ is called a right quasigroup if for any ordered pair $(a, b) \in Q^2$ there exist the unique solution $y \in Q$ to the equation $y \circ a = b$* [2].

Definition 1.3. *Binary groupoid $(Q, \cdot)$ is called medial if this groupoid satisfies the following medial identity:*

$$xy \cdot uv = xu \cdot yv \tag{1}$$

for all $x, y, u, v \in Q$ [2].

We recall

Definition 1.4. *Quasigroup $(Q, \cdot)$ is a T-quasigroup if and only if there exists an abelian group $(Q, +)$, its automorphisms φ and ψ and a fixed element $a \in Q$ such that $x \cdot y = \varphi x + \psi y + a$ for all $x, y \in Q$* [6].

We mantion that a T-quasigroup with the additional condition $\varphi\psi = \psi\varphi$ is medial.

2. T-quasigroups with Stein 2-rd ($xy \cdot x = y \cdot xy$) identity

Theorem 2.1. *In T-quasigroup $(Q, \cdot)$ of the form $x \cdot y = \varphi x + \psi y$ Stein 2-nd identity ($xy \cdot x = y \cdot xy$) is true if $\varphi y + \psi^2 y = \varphi\psi y$, $\varphi^2 x + \psi x = \psi\varphi x$.*

Proof. From identity

$$xy \cdot x = y \cdot xy \tag{2}$$

we obtain

$$\varphi(\varphi x + \psi y) + \psi x = \varphi y + \psi(\varphi x + \psi y), \tag{3}$$

$$\varphi^2 x + \varphi\psi y + \psi x = \varphi y + \psi\varphi x + \psi^2 y, \tag{4}$$

If in (4) $x = 0$, then

$$\varphi y + \psi^2 y = \varphi\psi y. \tag{5}$$

If in (4) $y = 0$, then

$$\varphi^2 x + \psi x = \psi\varphi x. \tag{6}$$

□

Corollary 2.1. *In medial quasigroup $(Q, \cdot)$ of the form $x \cdot y = \varphi x + \psi y$, Stein 2-nd identity ($xy \cdot x = y \cdot xy$) is true if $(\varphi + \psi - 1)(\varphi - \psi) = 0$.*

Proof. From the mediality of quasigroup $(Q, \cdot)$, equalities (5), (6) we have the following

$$(\varphi + \psi - 1)(\varphi - \psi) = 0. \tag{7}$$

Indeed, $\varphi^2 x + \psi x = \varphi x + \psi^2 x$, $\varphi^2 x - \psi^2 x = \varphi x - \psi x$, $(\varphi x - \psi x)(\varphi x + \psi x) = \varphi x - \psi x$, $(\varphi x - \psi x)(\varphi x + \psi x - 1) = 0$ □

Example 2.1. *Suppose we have the group Z_n of residues modulo n. We define quasigroup $(Q, \circ)$ in the following way: $x \circ y = 4 \cdot x + 2 \cdot y \pmod 5$.*

Check: $(x \circ y) \circ x = y \circ (x \circ y)$, $16x + 8y + 2x = 4y + 8x + 4y \pmod 5$, $18x + 8y = 8x + 8y \pmod 5$, $10x = 0 \pmod 5$.

Example 2.2. *Suppose we have the group Z_n of residues modulo n. We define quasigroup $(Q, \circ)$ in the following way: $x \circ y = 2 \cdot x + 4 \cdot y \pmod{10}$.*

Verify: $(x \circ y) \circ x = y \circ (x \circ y)$, $4x + 8y + 4x = 2y + 8x + 16y \pmod{10}$, $8x + 8y = 8x + 18y \pmod{10}$, $0 = 10y \pmod{10}$.

Example 2.3. *Let us consider the group Z_n of residues modulo n. We define quasigroup $(Q, \circ)$ in the following way: $x \circ y = 11 \cdot x + 3 \cdot y \pmod{13}$.*

Check: $(x \circ y) \circ x = y \circ (x \circ y)$, $121x + 33y + 3x = 11y + 33x + 9y \pmod{13}$, $0 = 91x + 13y \pmod{13}$, $0 = 0 \pmod{13}$.

Example 2.4. *Suppose we have the group Z_n of residues modulo n. We define quasigroup $(Q, \circ)$ in the following way:* $x \circ y = 7 \cdot x + 11 \cdot y \pmod{17}$.

Verify: $(x \circ y) \circ x = y \circ (x \circ y)$, $49x + 77y + 11x = 7y + 77x + 121y \pmod{17}$, $60x + 77y = 128y + 77x \pmod{17}$, $0 = 0 \pmod{17}$.

Example 2.5. *Suppose we have the group Z_n of residues modulo n. We define quasigroup $(Q, \circ)$ in the following way:* $x \circ y = 21 \cdot x + 9 \cdot y \pmod{29}$.

Check: $(x \circ y) \circ x = y \circ (x \circ y)$, $441x + 189y + 9x = 21y + 189x + 81y \pmod{29}$, $450x + 189y = 189x + 102y \pmod{29}$, $261x + 87y = 0 \pmod{29}$, $0 = 0 \pmod{29}$.

3. T-quasigroups with Stein 3-rd identity $xy \cdot yx = y$

T-quasigroups with Stein 3-rd identity are researched in [13]. Sufficiently big number of simple medial qusigroups with 3-rd Stein identity is constructed in [12].

Theorem 3.1. *In T-quasigroup $(Q, \cdot)$ of the form $x \cdot y = \varphi x + \psi y$ Stein 3-rd identity is true if and only if $\varphi^2 + \psi^2 = 0$, $\varphi\psi y + \psi\varphi y = \varepsilon$* [13].

Corollary 3.1. *In medial quasigroup $(Q, \cdot)$ of the form $x \cdot y = \varphi x + \psi y$ Stein 3-rd identity is true if and only if $\varphi^2 + \psi^2 = 0$, $2\varphi\psi = \varepsilon$* [13].

Example 3.1. *Let Z_n be the group of residues modulo n. Assume that $\varphi = 8$, $\psi = 20$. Then* $\varphi^2 + \psi^2 = 64 + 400 = 464 = 0 \pmod{29}, n = 29$. *Next,* $2\varphi\psi = 2 \cdot 8 \cdot 20 = 320 = \varepsilon = 1 \pmod{29}$, $x \cdot y = 8x + 20y \pmod{29}$.

Verify: $8(8x + 20y) + 20(8y + 20x) = y \pmod{29}$, $64x + 160y + 160y + 400x = y \pmod{29}$, $y = y \pmod{29}$.

Example 3.2. *Let Z_n be the group of residues modulo n. We consider $\varphi = 9$, $\psi = 21$. Then* $\varphi^2 + \psi^2 = 81 + 441 = 522 = 0 \pmod{29}, n = 29$. *Next,* $2\varphi\psi = 2 \cdot 9 \cdot 21 = 378 = \varepsilon = 1 \pmod{29}$, $x \cdot y = 9x + 21y \pmod{29}$.

Verify: $9(9x + 21y) + 21(9y + 21x) = y \pmod{29}$, $81x + 189y + 189xy + 441x = y \pmod{29}$, $y = y \pmod{29}$.

Example 3.3. *Let Z_n be the group of residues modulo n. Let $\varphi = 3$, $\psi = 11$. In this case* $\varphi^2 + \psi^2 = 9 + 121 = 130 = 0 \pmod{65}, n = 65$. *Next,* $2\varphi\psi = 2 \cdot 3 \cdot 11 = 66 = \varepsilon = 1 \pmod{65}$, $x \cdot y = 3x + 11y \pmod{65}$.

Verify: $3(3x+11y)+11(3y+11x) = y \pmod{65}$, $9x+33y+33y+121x = y \pmod{65}$, $y = y \pmod{65}$.

Example 3.4. *Let Z_n be the group of residues modulo n. Let $\varphi = 11$, $\psi = 41$. Then we get* $\varphi^2+\psi^2 = 121+1681 = 1802 = 0 \pmod{53}, n = 53$. *Next,* $2\varphi\psi = 2\cdot 41\cdot 11 = 902 = \varepsilon = 1 \pmod{53}$, $x \cdot y = 11x + 41y \pmod{53}$.

Check: $11(11x+41y)+41(11y+41x) = y \pmod{53}$, $121x+451y+451y+1681x = y \pmod{53}$, $y = y \pmod{53}$.

Example 3.5. *Let Z_n be the group of residues modulo n. We consider $\varphi = 12$, $\psi = 42$. Then* $\varphi^2 + \psi^2 = 144 + 1764 = 1908 = 0 \pmod{53}, n = 53$. *Next,* $2\varphi\psi = 2 \cdot 42 \cdot 12 = 1008 = \varepsilon = 1 \pmod{53}$, $x \cdot y = 12x + 42y \pmod{53}$.

Verify: $12(12x+42y)+42(12y+42x) = y \pmod{53}$, $144x+504y+504y+1764x = y \pmod{53}$, $y = y \pmod{53}$.

Example 3.6. *Let Z_n be the group of residues modulo n. Let $\varphi = 55$, $\psi = 5$. Then* $\varphi^2 + \psi^2 = 3050 \pmod{61}, n = 61$. *Next,* $2\varphi\psi = 2 \cdot 5 \cdot 55 = 550 = \varepsilon = 1 \pmod{61}$, $x \cdot y = 55x + 5y \pmod{61}$.

Check. $55(55x + 5y) + 5(55y + 5x) = y \pmod{61}$, $3025x + 275y + 275y + 25x = y \pmod{61}$, $y = y \pmod{61}$.

Similar example gives us numbers $x\cdot y = 56x+6y \pmod{61}$. *Indeed, we have to check.* $56(56x + 6y) + 6(56y + 6x) = y \pmod{61}$, $3136x + 336y + 336y + 36x = y \pmod{61}$, $0 = 0 \pmod{61}$. *Notice, the last quasigroup is idempotent.*

Example 3.7. *Let Z_n be the group of residues modulo n. Assume that $\varphi = 59$, $\psi = 13$. Then we have* $\varphi^2+\psi^2 = 59^2+13^2 = 3650 = 0 \pmod{73}, n = 73$. *Next,* $2\varphi\psi = 2\cdot 59\cdot 13 = 1534 = \varepsilon = 1 \pmod{73}$, $x \cdot y = 59x + 13y \pmod{73}$.

Verify: $59(59x+13y)+13(59x+13y) = y \pmod{73}$, $3481x+767y+767y+169x = y \pmod{73}$, $y = y \pmod{73}$.

Idempotent example gives us numbers φ =60, ψ=14. Check: $60(60x+14y)+14(60x+14y) = y \pmod{73}$, $3600x + 840y + 840y + 196x = y \pmod{73}$, $y = y \pmod{73}$.

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Dedicated to Professor Alexandru Șubă on the occasion of his 70th birthday

Qualitative analysis of polynomial differential systems with the line at infinity of maximal multiplicity: exploring linear, quadratic, cubic, quartic, and quintic cases

Vadim Repeşco

Abstract. This article investigates the phase portraits of polynomial differential systems with maximal multiplicity of the line at infinity. The study explores theoretical foundations, including algebraic multiplicity definitions, to establish the groundwork for qualitative analyses of dynamical systems. Spanning polynomial degrees from linear to quintic, the article systematically presents transformations and conditions to achieve maximal multiplicity of the invariant lines at infinity. Noteworthy inclusions of systematic transformations, such as Poincaré transformations, simplify analysis and enhance the accessibility of phase portraits.

2010 Mathematics Subject Classification: 34C05.
Keywords: polynomial differential system, phase portrait, infinity, multiplicity of an invariant algebraic curve, Poincaré transformation.

Studiul calitativ al sistemelor diferenţiale polinomiale cu linia de la infinit de multiplicitate maximală: studierea cazurilor liniare, pătratice, cubice, cuartice şi cuintice

Rezumat. Acest articol investighează portretele de fază ale sistemelor diferențiale polinomiale cu multiplicitatea maximă a liniei de la infinit. Studiul explorează fundamentele teoretice, inclusiv definițiile multiplicității algebrice, pentru a stabili baza pentru analize calitative ale sistemelor dinamice. Acoperind grade polinomiale de la liniar la quintic, articolul prezintă în mod sistematic transformări și condiții pentru a obține multiplicitatea maximală a dreptei invariante de la infinit. Incluziile notabile ale transformărilor sistematice, cum ar fi transformările Poincaré, simplifică analiza și îmbunătățesc accesibilitatea portretelor de fază.

Cuvinte-cheie: sistem diferenţial polinomial, portret fazic, infinit, multiplicitatea curbei algebrice invariante, transformarea Poincaré.

1. Introduction

Phase portraits serve as visual representations illustrating the temporal evolution of a differential equation system, offering insights into the long-term dynamics of the system. The complexity of the phase portrait for a polynomial differential system, characterized by the maximal multiplicity of the line at infinity, can be intricate, showcasing diverse behaviors.

The exploration of invariant algebraic curves holds significant importance in the qualitative analysis of dynamical systems [1, 2, 3, 4]. The inquiry into the maximal number of invariant straight lines within a polynomial differential system is addressed in [5]. Moreover, the incorporation of invariant straight lines in the derivation of Darboux first integrals is a notable area of investigation, as detailed in [6]. The study demonstrates that a polynomial differential system can yield a Darboux first integral when a sufficient number of invariant straight lines, considering their multiplicities, is present.

This article concentrates on the phase portraits of polynomial differential systems expressed as equations of the form:

$$\frac{dx}{dt} = P(x, y), \ \frac{dy}{dt} = Q(x, y), \tag{1}$$

where x and y are dependent variables, and t is the independent variable. The functions $P(x, y)$ and $Q(x, y)$ are polynomials in x and y. The focus is on obtaining phase portraits for polynomial differential systems with degrees $n \in \{1, 2, 3, 4, 5\}$ that possess an invariant straight line at infinity of maximal multiplicity.

Definition 1.1. *An algebraic curve $f(x, y) = 0$, $f \in \mathbb{C}[x, y]$ is said to be an invariant algebraic curve for system* (1)*, if there exists a polynomial $K(x, y)$ such that the identity*

$$\mathbb{X}(f) = f(x, y)K(x, y)$$

holds.

To rigorously address the characterization of the invariant algebraic curve in a differential system, it becomes imperative to introduce the concept of multiplicity. Multiplicity, within this context, encompasses various facets such as algebraic, geometric, and infinitesimal type. For the specific analytical framework employed herein, we shall adopt the algebraic multiplicity as defined in reference [7].

Definition 1.2. *Let $\mathbb{C}_m[x]$ be the $\mathbb{C}$-vector space of polynomials in $\mathbb{C}[x]$ of degree at most m. Then it has dimension $R = \binom{n+m}{n}$. Let $v_1, v_2, \ldots, v_R$ be a base of $\mathbb{C}_m[x]$. If k is the greatest positive integer such that the k-th power of f divides $\det M_R$, where*

$$M_R = \begin{pmatrix} v_1 & v_2 & \ldots & v_R \\ \mathbb{X}(v_1) & \mathbb{X}(v_2) & \ldots & \mathbb{X}(v_R) \\ \ldots & \ldots & \ldots & \ldots \\ \mathbb{X}^{R-1}(v_1) & \mathbb{X}^{R-1}(v_2) & \ldots & \mathbb{X}^{R-1}(v_R) \end{pmatrix},$$

then the invariant algebraic curve f of degree m of the vector field $\mathbb{X}$ has algebraic multiplicity k.

Various methodologies exist for examining the behaviour at infinity within a polynomial differential system. In this context, I will employ a straightforward approach, leveraging one of the Poincaré transformations. Through this transformation, the infinity locus is effectively mapped onto one of the axes in the newly defined coordinates, thereby assuming the role of an invariant straight line within the finite part of the phase plane. Subsequently, the analytical tools elucidated earlier will be applied.

2. Linear and quadratic differential systems

A general linear differential system has the form:

$$\begin{cases} \dot{x} = a_{00} + a_{10}x + a_{01}y, \\ \dot{y} = b_{00} + b_{10}x + b_{01}y, \end{cases} \tag{2}$$

Utilizing the Poincaré transformation, denoted as $x \to \frac{1}{z}, y \to \frac{y}{z}$, followed by a subsequent adjustment for enhanced visual clarity $y \to x$ and $z \to y$, the transformed system is expressed as follows:

$$\begin{cases} \dot{x} = -b_{10} + (a_{10} - b_{01})\,x - b_{00}y + a_{01}x^2 + a_{00}xy, \\ \dot{y} = y\,(b_{00} + b_{10}x + b_{01}y)\,. \end{cases} \tag{3}$$

In this representation, the variable y corresponds to the invariant straight line characterizing the infinity of the system specified by equation (2).

Our objective is to achieve the maximum multiplicity for the system (2), which is equivalent to ensuring that the system (3) also attains its maximum multiplicity. Notably, the degree of the polynomial $\det M_r$ is equal to 3

$$\det M_R = A_1(x)y + A_2(x)y^2 + A_3(x)y^3,$$

implying that the system (2) can theoretically exhibit a multiplicity of up to 3, where

$$A_1\,(x) = b_{10}\,(a_{10}b_{01} - a_{01}b_{10}) + (b_{01} - a_{10})\,(a_{10}b_{01} - a_{01}b_{10})\,x + a_{01}\,(a_{01}b_{10} - a_{10}b_{01})\,x^2,$$

$$A_2\,(x) = -a_{10}^2b_{00} + a_{10}b_{00}b_{01} + a_{00}a_{10}b_{10} - 2a_{01}b_{00}b_{10} + a_{00}b_{01}b_{10} + \Big(-a_{01}a_{10}b_{00} -$$
$$-a_{00}a_{10}b_{01} - a_{01}b_{00}b_{01} + a_{00}b_{01}^2 + 2a_{00}a_{01}b_{10}\Big)\,x,$$
$$A_3\,(x) = -a_{00}a_{10}b_{00} - a_{01}b_{00}^2 + a_{00}b_{00}b_{01} + a_{00}^2b_{10}.$$

Requiring both $A_1(x)$ and $A_2(x)$ to be zero, while simultaneously ensuring $A_3(x) \neq 0$, entails solving a straightforward system of algebraic equations. Upon resolution, it is shown that the infinity of the system (2) possesses a multiplicity of 3. The system can be expressed in the following form:

$$\begin{cases} \dot{x} = 1, \\ \dot{y} = x. \end{cases} \tag{4}$$

The phase portrait on the Poincaré disk is presented in Figure 1.a) for the given system. Employing a parallel methodology, we confirm that the infinity of the general quadratic differential system (5)

$$\begin{cases} \dot{x} = a_{00} + a_{10}x + a_{01}y + a_{20}x^2 + a_{11}xy + a_{02}y^2, \\ \dot{y} = b_{00} + b_{10}x + b_{01}y + b_{20}x^2 + b_{11}xy + b_{02}y^2, \end{cases} \tag{5}$$

achieves a multiplicity of 5 and can be expressed as

$$\begin{cases} \dot{x} = 1, \\ \dot{y} = x^2. \end{cases} \tag{6}$$

The corresponding phase portrait is depicted in Figure 1.b).

As stated in [8], the maximal multiplicity of the line at infinity for cubic systems is identified as seven, and these systems can be reformulated into the following two configurations:

$$\begin{cases} \dot{x} = 1, \\ \dot{y} = x^3 + ax, \end{cases} \qquad a \in \mathbb{R}; \tag{7}$$

and

$$\begin{cases} \dot{x} = -x, \\ \dot{y} = x^3 + 2y. \end{cases} \tag{8}$$

The phase portraits depicted in Figure 1.c) and 1.d) for these two systems were demonstrated in [9].

3. Quartic and quintic differential systems

As outlined in [10], a quartic polynomial differential system with maximal multiplicity can be transformed into the following canonical form:

$$\begin{cases} \dot{x} = -3x + ay^4, \\ \dot{y} = y, \quad a > 0. \end{cases} \tag{9}$$

This system features an invariant line at infinity with a multiplicity of 10. Referring to [11], the phase portrait of this system can be constructed. However, to facilitate this analysis, it is necessary to relocate the singular points at infinity to the ends of the Oy axis by implementing the transformation $x \to y$, $y \to x$ (Figure 1.e)).

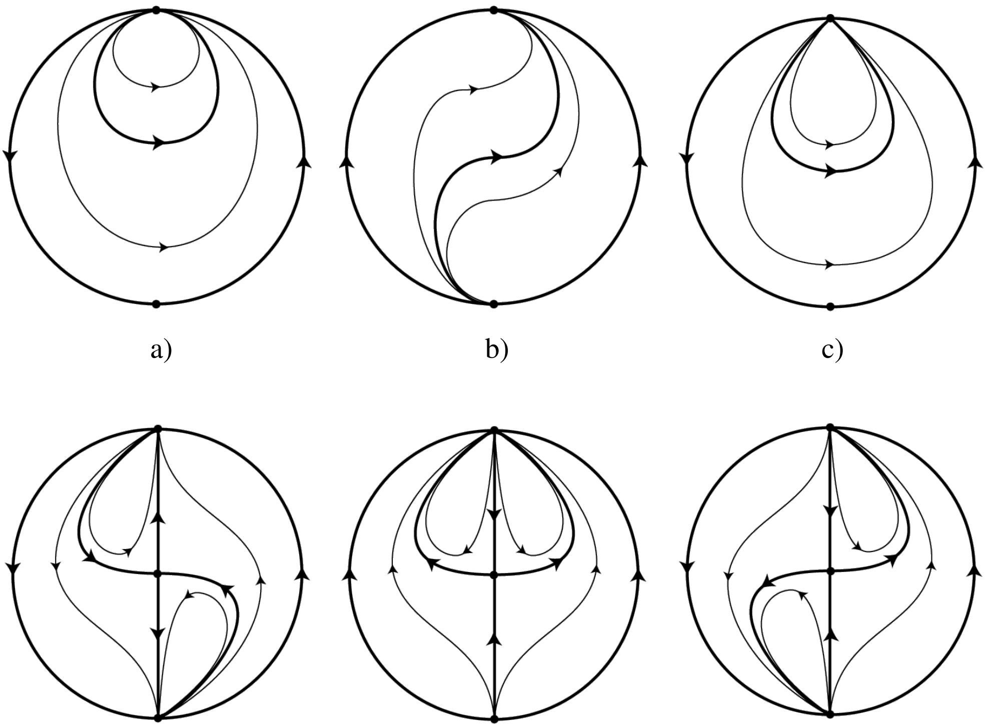
As indicated in [12], a quintic polynomial differential system with the line at infinity exhibiting maximal multiplicity can be written into the following form:

$$\begin{cases} \dot{x} = x, \quad a \neq 0, \\ \dot{y} = -4y + ax^5. \end{cases} \tag{10}$$

The system's structure remains unaltered under the transformations $x \to x$, $y \to -y$, $a \to -a$, with the additional condition $a > 0$ imposed for generality. To align its phase portrait with others, we apply the transformation $x \to y$, $y \to x$ to system (10), resulting in the following transformed system:

$$\begin{cases} \dot{x} = -4x + ay^5, \\ \dot{y} = y, \quad a > 0. \end{cases} \tag{11}$$

The phase portrait depicted in Figure 1.f) is obtained from the analysis presented in [9].



a) b) c)

d) e) f)

Figure 1. Phase portraits of all polynomial ($n \leq 5$) differential systems with infinity of maximal multiplicity.

4. Conclusions

This article explores phase portraits for polynomial differential systems, emphasizing the significance of invariant algebraic curves, particularly those associated with maximal multiplicity at the line at infinity. Theoretical foundations, including algebraic multiplicity definitions, lay the groundwork for qualitative dynamical system analysis.

Examining polynomial degrees from linear to quintic, the article systematically presents transformations and conditions to achieve desired invariant structures at infinity, offering nuanced insights into the dynamics of polynomial differential equations. Systematic transformations, like Poincaré transformations, simplify analysis and enhance accessibility to phase portraits.

Encompassing various polynomial degrees, including linear, quadratic, cubic, quartic, and quintic systems, the article contributes to a comprehensive understanding of the interplay between invariant algebraic curves, multiplicity, and resulting phase portraits in polynomial differential dynamics.

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Dedicated to Professor Alexandru Șubă on the occasion of his 70th birthday

Discovering the mysteries of Pi number using AR technologies

Inga Titchiev, Olesea Caftanatov, and Dan Talambuta

Abstract. The integration of Augmented Reality (AR) in education requires a strategic approach in order to ensure effectiveness in the learning process. AR technologies are constantly evolving, offering new possibilities for educational content, making it an evolving and innovative tool for educators. Exploring the mathematical world of Pi through AR can be an engaging and interactive experience for learners. This article presents the approach used in the development of an Augmented Reality application intended for Pi learning.

2010 Mathematics Subject Classification: 97C70, 68T05.
Keywords: Augmented Reality, challenges, education, Pi number.

Descoperirea misterelor numărului Pi utilizând tehnologiile RA

Rezumat. Integrarea Realității Augmentate (RA) în educație necesită o abordare strategică pentru a asigura eficacitatea procesului de învățare. Tehnologiile RA evoluează constant, oferind noi posibilități pentru conținutul educațional, făcându-l un instrument evolutiv și inovator pentru profesori. Explorarea numărului Pi prin RA poate fi o experiență captivantă și interactivă pentru instruiți. În acest articol este prezentată abordarea utilizată în dezvoltarea unei aplicaţii de Realitate Augmentată destinată învățării numărului Pi.

Cuvinte-cheie: Realitate Augmentata, provocări, educație, numărul Pi.

1. Introduction

Augmented Reality (AR) technology in education offers transformative opportunities to enhance learning experiences by overlaying digital content onto the physical world. AR creates immersive and interactive learning environments that engage students by bringing abstract concepts to life, making learning more engaging and memorable. It allows students to visualize complex subjects, explore 3D models, and interact with digital content, making abstract concepts more tangible and easier to understand.

AR applications [4] can be tailored to accommodate diverse learning styles, allowing students to learn at their own pace and providing personalized learning experiences.

By merging the physical and digital worlds, AR captures students' attention, fostering curiosity, and promoting active participation in the learning process.

Moreover, Augmented Reality facilitates interdisciplinary learning by connecting various subjects, enabling students to explore the connections between different fields of study in a more interactive manner. It helps develop critical thinking, problem-solving, and spatial reasoning skills as students engage in interactive AR experiences that require analysis and decision-making. AR bridges the gap between theory and practice by simulating real-world scenarios, allowing students to apply their knowledge in practical contexts.

In this article we will present how AR technology can be used to explore the world of Pi by offering a dynamic and immersive approach to learning mathematics. By integrating AR experiences into educational settings, students can engage with Pi's concepts in creative and interactive ways, fostering a deeper appreciation for its significance in mathematics and beyond.

From ancient civilizations, where approximations of Pi were etched into clay tablets, to the modern era of supercomputers and advanced mathematical theories, the quest to understand Pi has been an enduring journey. The intrigue surrounding Pi has led to profound discoveries about the nature of numbers, the limits of calculation, and the beauty inherent in mathematical patterns. Thus, we decided to develop an augmented mobile application that would be used as a tool in learning this transcendental number.

The Pi Journey App that we developed is an immersive exploration of the mysteries and intricacies of the mathematical constant π (Pi) using cutting-edge AR technology. This unique experience takes users on a visual and interactive journey, unraveling the significance of Pi in a three-dimensional augmented space.

2. Challenges associated with using AR technology in education

However, integrating AR into education faces challenges like infrastructure limitations, content creation complexities, teacher training needs, and ensuring equitable access for all students. Overcoming these hurdles requires investment in resources, professional development, and a commitment to adapting pedagogical approaches to harness the full potential of AR in education.

In order to develop a useful, interactive and efficient application, the following aspects are taken into consideration:

- Defining learning goals and objectives that align with the curriculum or educational outcomes where AR can enhance understanding or engagement.
- Identifying subjects or topics where visualizing, interacting with 3D models, or experiencing immersive content can significantly benefit student.

- Elaborating user-friendly AR application for both educators and students that align with educational goals.
- Developing AR content that supports the learning objectives. This might involve creating 3D models, animations, or utilizing existing AR resources.
- Integrating AR activities into the curriculum to complement and enhance traditional teaching methods, but not as standalone activities.
- Testing the effectiveness of AR in enhancing learning outcomes. In order to refine the approach feedback from both teachers and students are gathered.
- Developing assessment methods to measure the impact of AR on learning outcomes. To continuously improve AR content and teaching strategies feedback can be used.
- Sharing best practices and success stories in integrating AR, encouraging collaboration and innovation.
- Improving and updating AR content and methodologies continuously.

By following these steps and integrating AR strategically, educators can leverage this technology to create immersive, engaging, and effective learning experiences for students across various subjects and educational levels.

3. The mysteries and importance of the mathematical constant Pi

In the realm of mathematics, few constants have captured the imagination and curiosity of scholars and enthusiasts alike as much as the mysterious and revered number π (Pi) [6]. Defined as the ratio of a circle's circumference to its diameter, pi is an irrational and transcendental number with a decimal representation that stretches into infinity without repeating. As a fundamental constant, Pi plays a pivotal role in a myriad of mathematical equations, geometry, and scientific principles, transcending its utilitarian purpose to become a symbol of mathematical beauty and intrigue.

The enigma of Pi lies not only in its seemingly infinite and non-repeating decimal expansion but also in its ubiquitous presence across diverse mathematical landscapes. Its significance extends far beyond the simple geometry of circles, permeating areas such as calculus, trigonometry, and even physics. Pi has become a symbol of mathematical elegance and complexity, challenging mathematicians throughout history [7] to explore its mysteries and pushing the boundaries of mathematical knowledge.

In this exploration, we will delve into the mysteries of Pi, unravelling its infinite decimals, exploring its irrationality and transcendence, and discovering its unexpected appearances in various mathematical and scientific realms. Beyond its numerical significance, we will also delve into the cultural, artistic, and philosophical dimensions of Pi, examining how this mathematical constant has left an indelible mark on human thought and creativity. For this purpose, we designed 10 cards.

The mysterious nature of Pi lies in several intriguing aspects:

- **Irrationality**: Pi is an irrational number, meaning it cannot be expressed as a simple fraction. Its decimal representation goes on forever without repeating, and it cannot be precisely represented by any finite ratio of integers. This property was proven by Johann Lambert in 1768.
- **Transcendence**: Pi is not only irrational but also transcendental. This means that Pi is not the root of any non-zero polynomial equation with rational coefficients. Ferdinand von Lindemann established the transcendence of Pi in 1882. The combination of irrationality and transcendence makes Pi particularly mysterious in the realm of mathematics.
- **No Discernible Pattern**: Despite extensive computation and exploration, mathematicians have not discovered a discernible pattern or sequence within the digits of Pi. The randomness and lack of repetition in its decimal expansion contribute to the mystery surrounding this mathematical constant.
- **Computational Challenges**: The quest to calculate Pi to as many digits as possible has been ongoing throughout history. From manual calculations to modern supercomputers, mathematicians and computer scientists continually strive to push the boundaries of Pi's decimal expansion. Calculating Pi to a high degree of precision poses computational challenges. While modern computers have calculated Pi to trillions of digits, the process remains resource-intensive, emphasizing the vastness and complexity of Pi's decimal expansion.
- **Cultural and Philosophical Significance**: Pi has cultural and philosophical significance beyond its mathematical properties. Its mysterious and infinite nature has inspired contemplations about the limits of human knowledge and the nature of mathematical reality.

The exploration of Pi's digits continues to be a captivating pursuit in the field of mathematics and not only. Through, Pi Journey application we intend to express a part of people's passion to this mysterious number.

4. Some consideration in Pi application development

AR fosters collaboration and teamwork as students engage in shared AR experiences, encouraging peer-to-peer learning and cooperation. It has the potential to make learning

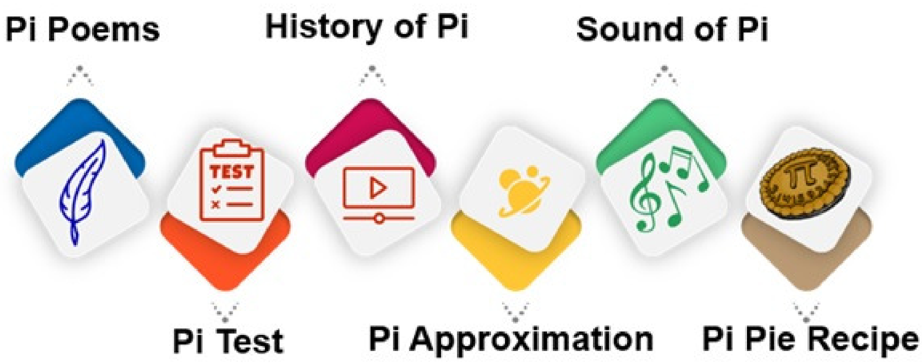

Figure 1. Mysteries of Pi App

more accessible for students with diverse learning needs [5] by providing alternative ways to access and interact with educational content.

In order to facilitate learning, increase study efficiency and successfully adapt to the multitude of learning situations, it is necessary to determine the specific preferences of the personal learning style so that they can be applied in a targeted manner.

Stable individual differences in the way of learning affect the rhythms and quality of learning, and especially determine the option for one or another learning strategy as one's own and personal way of approaching a learning situation.

In the application development the VAK (Visual, Auditory, Kinesthetic) learning style theory was applied.

For each category of learners the following discovery Pi activities were proposed:

- **Visual Learners**: Graphical Representations like diagrams, charts, and visual aids to illustrate Pi's relationship with circles, showing geometric models and patterns visually. Video displaying Pi's digits, sequences, or relationships with shapes, allowing exploration through interactive visual elements.
- **Auditory Learners**: Explanations or discussions about Pi's significance, history, and applications in an auditory format, such as podcasts or recorded lectures. Pi Chants or Songs, in particular Pi Symphony by Lars Ericksone.
- **Kinesthetic Learners**: Engage learners in measuring circles, calculating circumferences, and experimenting with circular objects to explore Pi's mathematical properties practically.

4.1. Designing augmented Pi artifacts and Pi markers

An augmented artifact and marker has quite a few tasks to accomplish [1]. Besides the fact that it has to capture student's attention, entice them to pick up their mobile and scan

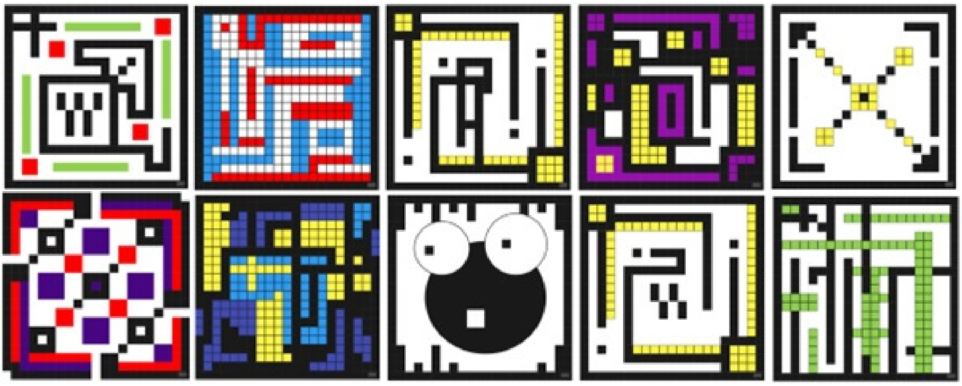

Figure 2. Augmented Pi artifacts and Pi markers

the image - it should have a high quality to let the AR experience come to life. Therefore, in this section, we will describe the best practice of designing augmented artifacts and markers that we observed as a result of the working process and testing.

Moreover, we will describe our experiences with low and high star rating image targets. We consider that "markers" are the digital form of image targets that Vuforia Engine can detect and track by comparing extracted natural features from the camera image against a known image target resources database.

Markers come in various forms: simple, flat image targets, curled targets in the form of cylindrical shapes, or multi-targets in the composition of a box. We define "artifacts" as the physical form of markers. They can also come in various forms: cards, papers, newspapers, posters, objects, etc. In our cases, it is a matte laminated image with size 10 x 10 cm, see Figure 1. The main purpose of the artifact is to trigger the augmentation content when it is scanned by camera.

In the next section we will give some examples of these Augmented Pi Artifacts.

4.2. Pi Artifact that explains the relationship between Pi and the circumference of a circle

The relationship between Pi (π) and the circumference of a circle is defined by a fundamental geometric formula. The circumference (C) of a circle is calculated using Pi and the circle's diameter (D) or radius (r). The formula for the circumference of a circle is as follows:

$$C = 2\pi r \tag{1}$$

or equivalently

$$C = \pi D, \tag{2}$$

where C is the circumference of the circle, π (Pi) is the mathematical constant approximately equal to 3.14159, r is the radius of the circle, D is the diameter of the circle.

This relationship is derived from the definition of Pi, which represents the ratio of a circle's circumference to its diameter. Essentially, Pi is the constant that relates the size of a circle to its "wrap-around" distance. The formula (1) expresses that the circumference is equal to twice the product of Pi and the radius. Alternatively, the formula (2) emphasizes that the circumference is equal to Pi multiplied by the diameter. Since the diameter is twice the radius, these two formulas are equivalent.

The augmented reality Pi artifacts utilize graphical representations to elucidate the correlation between Pi and the circumference of a circle, catering to diverse learning styles through visual and interactive elements.

4.3. Pi Artifacts that represents digits as melodic elements

Pi's influence extends into the realm of music, where musicians and composers have explored creative ways to incorporate the mathematical constant into their works. Pi-themed compositions showcase a unique fusion of mathematics and art, offering a creative interpretation of this transcendental number.

Musicians have experimented with using the digits of Pi as melodic elements in their compositions. Assigning musical notes or intervals to the numerical digits allows for the creation of melodies that directly reflect the numerical sequence of Pi. Composers have employed Pi to structure the rhythmic and harmonic elements of their music. For example, using the digits of Pi to determine the length of musical phrases, the arrangement of sections, or the timing of specific musical events can result in compositions with a distinct mathematical foundation.

We designed an artifact that contains a playlist with the following musics [2]:

- **Pi Symphony by Lars Ericksone**: Lars Erickson, captivated by the enigmatic nature of Pi (π), composed "Pi Symphony" in the early 1990s. Crafting a melody from the seemingly random digits of Pi, Erickson's magnum opus demonstrated that, contrary to expectations, a composition based on Pi could be as majestic as a symphony.
- **Pi Symphony by Jim Zamerski**: Jim Zamerski crafted a melody using 226 digits of π, by utilizing the 12 tones in music as the foundation. While sharing the numerical essence of π with Lars Erickson's piece, Zamerski's composition takes on a lighter ambiance. It initiates with a touch of melodrama, swiftly transitioning into a danceable tune with a dynamic tempo that fluctuates throughout its entirety.

- **Pi Symphony by David Macdonald**: Another example is David Macdonald's "Pi Symphony", which transforms the first 100 digits of Pi into a musical composition. The sequence of digits dictates the pitch, rhythm, and dynamics of the piece, offering an auditory experience that mirrors the mathematical constant. David Macdonald added a diverse angle to Pi-inspired music. Incorporating harmonic elements played by the left hand, Macdonald's composition concealed the randomness of Pi. The poignant piece seemed to transport listeners to a mythical realm in their minds.

This artifact employs auditory elements to cater to a learning style based on sound and hearing.

4.4. Pi poems Artifacts

While Pi (π) is primarily a mathematical constant, its intriguing nature has found its way into literature, where writers and authors have incorporated it as a symbol, metaphor, or even as a theme. Pi poems may use the actual numerical sequence of Pi (3.14159...) to determine elements of the poem, such as the number of syllables in each line or the length of stanzas. For example, the number of syllables in each line might correspond to the digits of Pi (3 syllables for the first line, 1 for the second, 4 for the third, and so on).We dedicated a few artifacts for learning Pi poems in the Romanian Language, i.e. "Iarna lui Pi" by Iuliana Ciubuc, see Table [3]. Pi poem artifacts use the characteristic of visual learning style.

Table 1. Romanian Pi poem with the sequence of Pi 3,141592653589793

Iar	e	rece	E	iarna	adevarata	Cu	zapada	multa	nor	Ceata	cumplita	tematoare	Troiene	visoclite	are
3	1	4	1	5	9	2	6	5	3	5	8	9	7	9	3

4.5. Pi Artifacts for visualizing and learning the first 100 decimal digits

Creating Pi artifacts for visualizing and learning the first 100 decimal expansions of Pi can be an engaging way to explore this mathematical constant. In Figure 2, two instances are depicted: in the left image, the user correctly inputs the first 8 decimal digits; while in the right image, the user inaccurately inputs the 9th digit, resulting in a red highlighted input section.

4.6. Pi Artifacts for assessing knowledge of the number Pi

For the evaluation of knowledge about the number Pi, a scenario was developed that represents a five-item test. After answering to them, feedback is given about which items were answered correctly and which were incorrect.

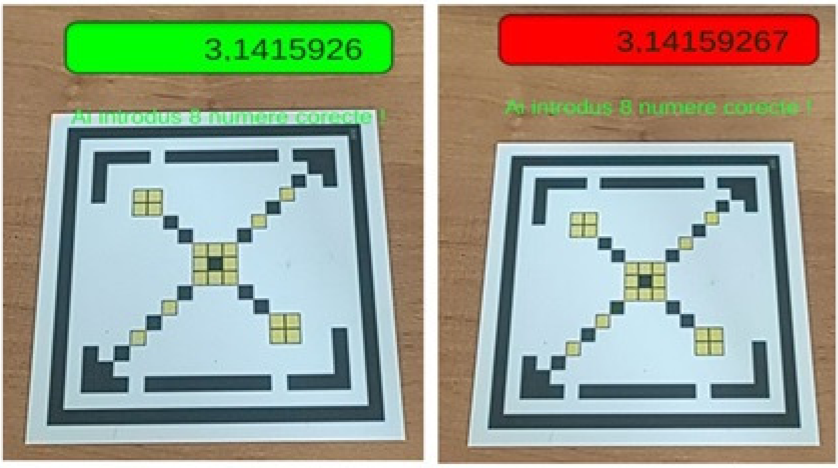

Figure 3. Pi artifacts for visualizing and learning the first 100 decimal digits

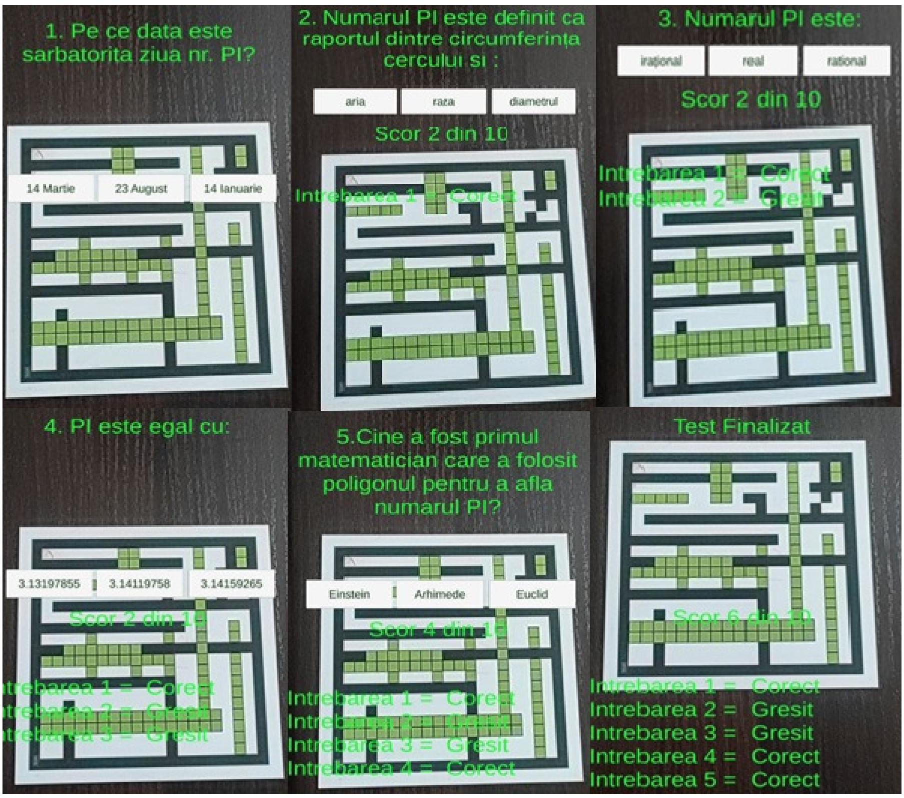

Figure 4. Test for assessing knowledge of the number PI

5. Conclusions

In this article we will present how AR technology can be used to explore the world of Pi by offering a dynamic and immersive approach to learning mathematics. By integrating AR experiences into educational settings, students can engage with Pi's concepts in creative and interactive ways, fostering a deeper appreciation for its significance in mathematics and beyond.

The educational Application Pi Journey is delivered via mobile device that engages pupils with a wide range of multi-sensory learning experiences, provide rich, contextualized learning for understanding the concepts related to transcendental number Pi.

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Dedicated to Professor Alexandru Şubă on the occasion of his 70th birthday

Application of genetic algorithm to solving the optimization problem of locations graph vertices in the line

Liubomir Chiriac, Natalia Lupashco, and Maria Pavel

Abstract. This article examines genetic algorithms that are built on the ”survival of the fittest” principle enunciated by Charles Darwin. By applying genetic algorithms to solving optimization problems, it is not always possible to guarantee the determination of the global optimum in polynomial time. This fact does not occur because only brute force search methods allow us to find the global optimum. Instead the genetic algorithm allows selecting good decisions, in a reasonable time, compared to other well-known deterministic or heuristic search engine optimization algorithms. The authors of this article develop an algorithm of solving the optimization problem of locations graph vertices in the line.

2010 Mathematics Subject Classification: 05C85, 05C90.
Keywords: genetic algorithm, optimization problem, location problem, graphs algorithms.

Aplicarea algoritmului genetic la rezolvarea problemei de optimizare privind amplasarea vârfurilor grafului în linie

Rezumat. Acest articol examinează algoritmii genetici care se bazează pe principiul ”supravieţuirii celui mai adaptat” enunţat de Charles Darwin. Prin aplicarea algoritmilor genetici la rezolvarea problemelor de optimizare, nu este întotdeauna posibil să se garanteze determinarea optimului global în timp polinomial. Acest fapt nu se întâmplă deoarece numai metodele de căutare prin forţă brută ne permit să găsim optimul global. În schimb, algoritmul genetic permite selectarea unor decizii bune, într-un timp rezonabil, în comparaţie cu alţi algoritmi de optimizare a motoarelor de căutare deterministe sau euristice bine cunoscute. Autorii acestui articol dezvoltă un algoritm de rezolvare a problemei de optimizare a amplasării vârfurilor grafurilor în linie.

Cuvinte-cheie: algoritm genetic, problema de optimizare, problema de localizare, algoritmi de grafuri.

1. History of the development of the evolutionary calculus

The approach regarding the application of evolutionary principles (evolutionary computation) in the automated solving of problems dates back long before the emergence and development of modern computers.

As early as 1948, Alan Turing introduced a new approach applied to problem-solving called evolutionary or genetic approach. Subsequently, in the 1960s, Dr. Lawrence Jerome Fogel (March 2, 1928 - February 18, 2007), a pioneer in evolutionary computation, along with Wlash (later David B. Fogel, born on February 2, 1964), introduced and developed the concept of evolutionary programming. During the same period, Holland focused on genetic algorithms. Hans-Paul Schwefel (born on December 4, 1940), a German computer scientist and emeritus professor at the University of Dortmund, and Ingo Rechenberg (November 20, 1934 - September 25, 2021), a German researcher and professor in the field of bionics, launched and developed evolutionary strategies as alternative methods for automated problem-solving. Later, in the 1990s, J. R. Koza developed genetic programming, a new technique for searching solutions.

Therefore, ***evolutionary computation*** is a field of modern computer science with a strong emphasis on mathematics, inspired by the natural evolutionary process. The fundamental concept underlying evolutionary computation is the interconnection between natural evolution and the trial-and-error problem-solving technique [1].

In the context of the above, evolutionary computation is currently an important research field in computer science. As known, this field derives from the natural evolutionary process. The algorithms that emerge and develop in this field are called evolutionary algorithms, and they include significant and promising subdomains such as:

- Evolutionary programming;
- Evolutionary strategies;
- Genetic programming;
- Genetic algorithms.

2. Genetic Algorithms

The fact that mathematics and computer science are widely applied in various sciences, including biology, is a well-known and appreciated phenomenon. However, the reciprocity of this relationship does not always occur. For instance, in modern science, there are not many instances where mechanisms, concepts and basic notions from biology that are widely and efficiently used in mathematics and computer science.

In this context, the genetic algorithm is an eloquent and convincing example. Genetic algorithms represent adaptive heuristic search techniques that are implemented based on the principles of natural selection and genetics.

The mechanisms of the Genetic Algorithm are similar to natural evolution and rely on the principle stated by Charles Darwin, ”survival of the fittest”, meaning that the most well-adapted individuals, not necessarily the strongest or most intelligent, survive.

Thus, the Genetic Algorithm represents a computer-mathematical model that mimics the evolutionary biological model to solve search and optimization problems. The Genetic Algorithm is determined by a set of elements representing a population, consisting of chromosomes (binary strings), and a set of genetic operators (selection, crossover, and mutation) that influence the population's structure.

Genetic algorithms are commonly used for problems where finding the optimal solution is not simple or at least inefficient due to the characteristics of probabilistic search. Genetic algorithms encode a possible solution to a specific problem in a unique data structure called a "chromosome" and apply genetic operators to these structures to maintain critical information. The implementation process of genetic algorithms starts with an "initial set of possible solutions" to the examined problem (usually chosen randomly) referred to in the literature as "population" [2], [3].

Each individual in the "examined population" represents a potential solution to the problem and is called a "chromosome", which is a string of symbols, typically expressed as a string of bits. The examined chromosomes evolve over successive iterations, symbolically called generations. In each generation, these chromosomes are evaluated, using fitness measures.

To generate the next population (generations), the most "efficient" or "best" chromosomes from the current generation are selected. New chromosomes are formed, using one of the three (or even all three) central genetic operators: selection, crossover and mutation.

Selection ensures the process from the following perspective: certain chromosomes from the examined (current) generation are copied, depending on their fitness value, in accordance with the problem requirements into the new generation. This indicates that chromosomes with high significance have a high probability of contributing to the formation of the new generation.

The genetic operator crossover represents the process by which, based on two individuals (chromosomes) from the current population, two individuals (chromosomes), called descendants, are formed for the next population. Mutation is the genetic operator that represents the process through which a chromosome from the current population is modified and saved in the new population.

Genetic algorithms have been successfully applied to a variety of NP-complete problems that require global optimization of the solution and, in this regard, there is no iterative method for resolution [4], [5].

In genetic algorithms, the individuals in a population are represented by chromosomes with encoded sets, task parameters, for example, solutions otherwise called points in the search space or search points. In some works, individuals are called organisms. In this sense, we will clarify the meaning of the following biological concepts from the perspective of computer science.

Darwin's concept of evolution is adapted to the functioning of the genetic algorithm to find solutions to a problem expressed through the fitness function (objective function or adaptation function).

The fitness function represents a measure of the adaptability of a given individual within each generation. This characteristic allows the evaluation of the adaptation degree of individuals in the population and selects the most adapted ones, i.e., those with the highest values of fitness function, following the evolution principle of the survival of the fittest [6], [7].

Thus, selection represents the choice of individuals with the best aptitude for reproduction (sorting by the value of the objective function). The better an individual's fitness is, the greater the chances of crossing and passing on its genes to the next generation are. The crossover operator is analogous to biological reproduction and crossover and usually it is applied to individuals in the intermediate population. Two individuals are selected from the intermediate population, and certain portions of their two chromosomes are exchanged.

In simple terms, mutation can be defined as a small random modification of the chromosome to obtain a new solution. Mutation is used to maintain and introduce diversity into the genetic population. Mutation is the part of the Genetic Algorithm related to "exploring" the search space [4].

3. Introductory Concepts

Genetic Algorithms are algorithms of evolutionary computation, inspired by Darwin's Theory of Evolution. In 1960, Ingo Rechenberg (November 20, 1934 - September 25, 2021) introduced the idea of evolutionary computation in a work titled "Evolution strategies". Rechenberg, a German researcher and professor in the field of bionics, was a pioneer in the fields of evolutionary computation and artificial evolution. In the 1960s and 1970s, he invented several optimization methods known as evolution strategies (in German, Evolutionsstrategie). His research team successfully applied these algorithms to optimization problems, including the aerodynamic design of wings. These were the first serious technical applications of artificial evolution, an important component of bionics and artificial intelligence [8].

In 1975, John Henry Holland (February 2, 1929 - August 9, 2015), an American scientist and professor of psychology and electrical engineering and computer science at the University of Michigan, introduced and analyzed a mathematical model that, through adaptive procedures, relied on a mechanism of natural selection and genetic evolution called genetic algorithm. He was a pioneer in what became known as genetic algorithms [1], [9].

Genetic algorithms are used often in cases where the optimal solution involves searches among all combinations, permutations or probabilistic arrangements, a very complex and sometimes inefficient process. They implement specific data structures called ”chromosomes” to encode, through genetic operators, the possible solution to a particular problem while retaining important information.

Usually, to solve a problem using a genetic algorithm, the so-called ”population” is identified, constructed randomly based on the ”initial set of possible solutions”. Each individual or ”chromosome” (a string of characters expressed as a sequence of bits) in the examined ”population” represents a possible solution to the problem. Through consecutive iterations, the evolution of the ”chromosomes” occurs at the ”generation” level, each of which is validated by an evaluation function called fitness. Using one of the three main genetic operators (*selection*, *crossover*, and *mutation*) new ”chromosomes” identified from the current generation as the most ”efficient” are generated for the future population. Thus, just like in biology, the most ”powerful chromosomes”, with a higher probability, *are selected* from the given generation to transmit their characteristics (values of the ”evaluation function”, according to the requirements of the problem), to the next generation, ensuring the perpetuation of the entire process. Using the *crossover* genetic operator combines information from two individuals (”parents”) from the current population to generate one or more descendants. *Mutation*, on the other hand, allows the random modification of a gene or a small section of the ”chromosome” to ensure diversity in the future population.

The success of genetic algorithms is ensured by their implementation in solving a series of NP-complete problems, whose solutions cannot be identified through iterative methods, but rather by obtaining the optimal solution globally.

In genetic algorithms, individuals in a population are represented by chromosomes with encoded sets of task parameters, e.g., solutions, otherwise called points in the search space (search points). In some works, individuals are called organisms.

In this sense, the following biological concepts, borrowed by computer scientists from the perspective of genetic algorithms, will be clarified:

Chromosomes: Ordered sequences of genes.

Gene: Also called a property, sign, or detector, is the atomic element of the genotype, especially of chromosomes.

Genotype: The set of chromosomes of a given individual. Consequently, individuals in a population can be genotypes or unique chromosomes (in a common case when the genotype consists of a single chromosome).

Phenotype: A set of values that correspond to a specific genotype or set of task parameters (solution, search space point).

Allele: The value of a specific gene, also defined as the property value or property variant.

Locus: The position indicating the location of a specific gene in a chromosome (chain). The set of gene positions represents loci.

Genome: The totality of the genetic material of an organism or species, determining the development, functioning and transmission of hereditary traits from one generation to another.

Individual: A unique entity that has a specific set of chromosomes inherited from parents. In genetic algorithms, an individual represents a possible solution or a combination of parameters that can be optimized over time by selection and recombination processes and is evaluated within a specific problem.

Population: A group of individuals sharing a common set of genetic characteristics that occupy a certain type of environment. Genetic variation within populations is important for adaptation to environmental changes.

Mapping: An essential evaluation function that assigns a numerical value to each individual in the given population, reflecting the quality or appropriateness of that solution within the optimization problem. The mapping process is also called morphogenesis.

A crucial concept in genetic algorithms is the function that measures the degree of adaptability known as the fitness function.

The fitness function is a measure of the adaptability of a given individual within each generation. This characteristic allows the evaluation of the adaptation degree of individuals in the population and the most adapted individuals, those with the highest values of the fitness function, are selected in accordance with the evolutionary principle of the survival of the fittest.

The fitness function got its name directly from genetics. It has a strong impact on the functioning of genetic algorithms and must have precision and correct definition. In optimization problems, the fitness function is usually optimized (maximized or minimized) and is called the objective function.

At each iteration of the genetic algorithm, the fitness (adaptation degree) of each individual in a particular population is estimated using the fitness function. Based on this, the next generation (population of individuals) is generated, constituting the possible set of solutions to the examined problem [1], [4].

4. Implementation of the Genetic Algorithm

In the specialized literature, the Genetic Algorithm involves a series of steps (originally proposed by John Henry Holland) intended to conclude with an optimal solution to the examined problem. Below, we will outline the steps regarding the implementation of the Genetic Algorithm.

***Step 1*:** Creating/generating the initial population.

***Step 2*:** Evaluating the fitness function value of each individual (a mechanism used to measure and evaluate the state of a chromosome).

***Step 3*:** Selection - considering the characteristics of each individual, during this stage, some individuals may reproduce more frequently than others.

***Step 4*:** Crossover.

***Step 5*:** Mutation.

***Step 6*:** Replacing the old population of chromosomes with the new population of chromosomes.

Step 7*:** Finding the best solutions (but if the optimization criteria are not met, then the method requires returning to ***Step 2 and ultimately selecting the best individual as the final solution).

Genetic algorithms generate a new population composed of individuals with better and more adapted characteristics to the environment than those of the previous population. The logical scheme related to the implementation of the Genetic Algorithm is presented in Figure 1.

The process begins by initializing a random genetic pool through the creation of a set of chromosomes according to a predefined template, where the values of all genes are randomly selected for each chromosome. These initial chromosomes correspond to the individuals in the initial population. Typically, the number of individuals (and implicitly chromosomes) in the population remains constant at different generations, although this is not always the case. The Genetic Algorithm starts with a set of permissible solutions called the "population" (created arbitrarily, as mentioned earlier), each of which represents a potential solution to the problem, called a "chromosome" [10].

Once the population is established, it evolves towards better solutions through various genetic processes (*selection*, *crossover*, *mutation*) that lead to a better fitness function value, used to evaluate the state of each chromosome.

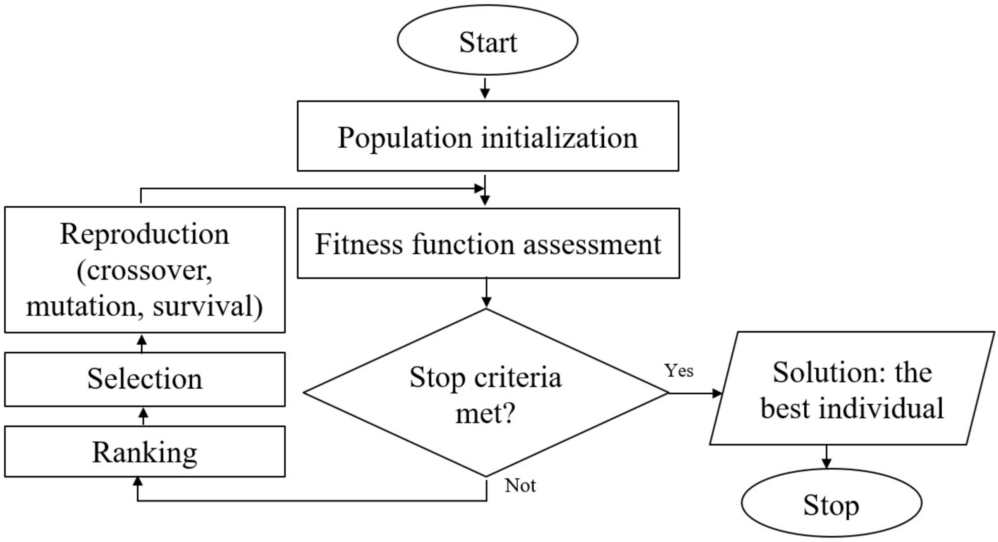


Figure 1. Logical Scheme of the Genetic Algorithm

5. Genetic algorithm for solving the optimization problem of locations graph vertices in the line

The problem of optimal placement of vertices of an undirected graph on a linear grid is a classic problem that requires knowledge in mathematics, computer science, and, evidently, genetic algorithms. In this section, the authors propose an algorithm that solves this problem under certain conditions.

Problem Statement: Given a graph G, where $n = |G|$ is the number of vertices of graph G. The goal is to find the best placement of the vertices of graph G on a linear grid after performing a genetic algorithm for k iterations (k generations), where $k < n$. It is assumed that the distances between the vertices of the graph are equal.

Solution: The total length of the edges of graph G is calculated according to the following formula:

$$L(G) = \sum_{i=1}^{n} \sum_{j=1}^{n} d_{i,j} a_{i,j} \tag{1}$$

where $n = |G|$ is the number of vertices, $d_{i,j}$ represents the distance between vertices of graph G, v_i and v_j, on the examined line. The distance in this case is measured in the number of edges of the graph, $a_{i,j}$ is the corresponding element of the adjacency matrix (0 or 1). In other words, the task is to find $\min L(G)$ after changing k generations (after performing k iterations).

The main goal of placement algorithms is to minimize the total length of the edges of a graph or hypergraph. Let us formulate this placement problem as an optimization problem. Formula (1) is selected as the objective function, which needs to be minimized.

Notations. The index i for chromosomes $C_1^i, C_2^i, \ldots, C_k^i$ represents the number of the generation to which chromosomes numbered $1, 2, \ldots, k$ belong.

The output will be the positions of each vertex. The genetic algorithm applied consists of the following steps:

***Step 1*:** Create the initial population consisting of k chromosomes, each composed of n elements (vertices of the graph):

$$C_1^i, C_2^i, \ldots, C_k^i, \text{ where } i = 0, 1, 2, \ldots, n-1.$$

***Step 2*:** Place the vertices of the graph on the linear grid according to the values of the examined chromosomes: $C_1^i, C_2^i, \ldots, C_k^i$.

Step 3*:** Calculate the length of each chromosome $C_1^i, C_2^i, \ldots, C_k^i$. This involves calculating the number of horizontal segments that connect the vertices of the graph, according to the placement made in ***Step 2, for each individual chromosome. Obtain lengths:

$$L_1(C_1^i), L_2(C_2^i), \ldots, L_k(C_k^i).$$

***Step 4*:** Calculate the total sum of the edges of the graph according to the placement of the vertices on the grid determined by that particular population of chromosomes:

$$S(i) = L_1(C_1^i) + L_2(C_2^i) + \ldots + L_k(C_k^i), \; i = 0, 1, 2, \ldots, n-1 \tag{2}$$

where i is the number of the iteration or generation of the population.

***Step 5*:** Choose the ”fittest” chromosome (with the smallest length) from the population $C_1^i, C_2^i, \ldots, C_k^i$. Let this chromosome be C_r^i, where $1 \leq r \leq k$.

Step 6*:** Apply the ***inverse mutation genetic operator on chromosome C_r^i after the first element. In other words, in the first iteration, the first element of the chromosome remains in place and the other elements are written in reverse order, starting with the last one, which will already be in the second position in the chromosome representation. In the second iteration, the first and second elements remain intact and the other elements are written in reverse order, starting with the last one, which will already be in the third position. And so on for each iteration. Denote the newly obtained chromosome after performing the inverse mutation by C_{rm}^i.

***Step 7*:** Calculate the length of the newly obtained chromosome C_{rm}^i.

Step 8: Identify and eliminate the weakest chromosome (with the maximum length) from the first generation of chromosomes, which is subsequently replaced by the chromosome C^i_{rm}.

Step 9: Build the next generation of chromosomes (which already includes the new chromosome C_{rm} and excludes the weakest chromosome) and then proceed to ***Step 2***.

Step 10: The process of the genetic algorithm stops after performing k iterations or, in other words, after constructing k generations of chromosomes. At each iteration, calculate $S(0), S(1), \ldots, S(k)$, where for each sum, the condition

$$S(m) \geq S(m+1),\ m = 0, 1, \ldots, k \tag{3}$$

is satisfied. The best placement of the vertices of the graph is obtained after completing the last iteration, in which the last genetically modified chromosome C^i_{rm} represents the solution $\min L(G)$.

6. Example of genetic algorithm application for solving the optimization problem of locations graph vertices in the line

Problem: Find the best placement of the vertices of graph G in Figure 2 on a line after performing three iterations of the genetic algorithm. The graph G and the initial population consisting of 3 chromosomes are given below.

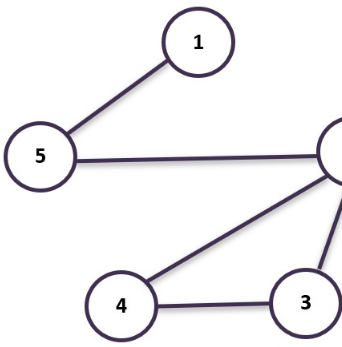


Figure 2. The examined graph, $n = 5$, and $m = 5$.

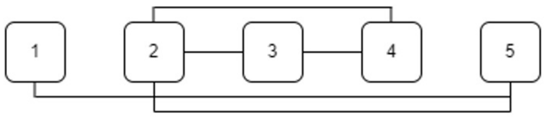
Solution: Place the vertices of the graph on the line according to the initial population of chromosomes to calculate the length of each chromosome.

Therefore, we get the graphical representation of the chromosomes.

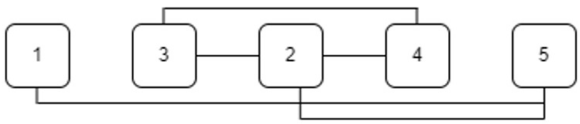
Table 1. Initial Population of Chromosomes (Generation 0)

Chromosome C_1^0	1	2	3	4	5
Chromosome C_2^0	1	3	2	4	5
Chromosome C_3^0	5	2	3	4	1

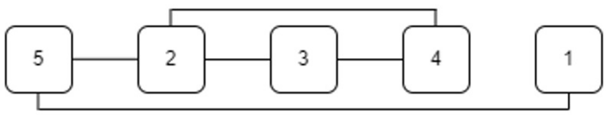
Chromosome C_1^0



Chromosome C_2^0



Chromosome C_3^0



We calculate the number of horizontal segments between the vertices of the graph (between chromosome elements). We obtain:

$$L_1(C_1^0) = 1 + 4 + 4 + 2 = 11;$$
$$L_2(C_2^0) = 1 + 3 + 4 + 2 = 10;$$
$$L_3(C_3^0) = 2 + 3 + 3 + 1 = 9.$$

Calculate the total sum of the edges of the graph according to the placement of the vertices on the grid determined by that particular population of chromosomes:

$$S(0) = L_1(C_1^0) + L_2(C_2^0) + L_3(C_3^0) = 11 + 10 + 9 = 30.$$

Among the examined chromosomes, chromosome C_3^0 has the minimum length, which is 9, so it is the fittest. Consider chromosome C_3^0 selected.

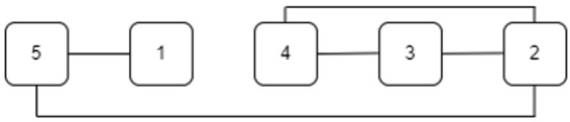
Chromosome C_3^0	5	2	3	4	1

Apply the inverse mutation genetic operator on chromosome C_3^0 to obtain a new chromosome, which we will denote as C_1^1.

Chromosome C_1^1	5	1	4	3	2

In other words, in the first iteration, the first element of the chromosome (5) remains in place, and the other elements are written in reverse order, starting with the last one, which will already be in the second position in the chromosome representation.

Chromosome C_1^1



Calculate the length of chromosome C_1^1:

$$L_1(C_1^1) = 2 + 1 + 3 + 3 = 9.$$

Thus, from the initial generation of the population, we replace the less fit chromosome C_1^0 with the more fit chromosome C_1^1 with a length of 9. Chromosome C_2^0 will be denoted as C_2^1, and chromosome C_3^0 will be denoted as C_3^1. Thus, we obtain Generation 1 of the population of chromosomes.

Table 2. Population of Chromosomes (Generation 1)

Chromosome C_1^1	5	1	4	3	2
Chromosome C_2^1	1	3	2	4	5
Chromosome C_3^1	5	2	3	4	1

The length of each chromosome is:

$$L_1(C_1^1) = 2 + 1 + 3 + 3 = 9;$$

$$L_2(C_2^1) = 1 + 3 + 4 + 2 = 10;$$

$$L_3(C_3^1) = 2 + 3 + 3 + 1 = 9.$$

The total sum of the edges of the graph according to the placement of the vertices on the grid determined by that particular population of chromosomes in the first iteration is:

$$S(1) = L_1(C_1^1) + L_2(C_2^1) + L_3(C_3^1) = 9 + 10 + 9 = 28.$$

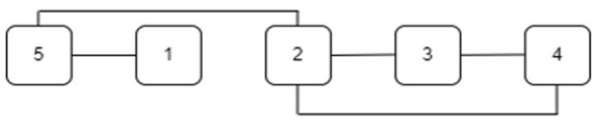
Among the two chromosomes with a length of 9, we select chromosome C_1^1 as the fittest from Generation 1.

Chromosome C_1^1	5	1	4	3	2

Apply the inverse mutation genetic operator on chromosome C_1^1 to obtain a new chromosome C_2^2. In other words, in iteration 2, the first and second elements of the chromosome (5, 1) remain intact, and the other elements are written in reverse order, starting with the last one, which will already be in the third position in the chromosome representation. Thus, we obtain:

Chromosome C_2^2	5	1	2	3	4

Chromosome C_2^2



Calculate the length of chromosome C_2^2. We have

$$L_2(C_2^2) = 2 + 1 + 2 + 2 = 7.$$

Further, in Generation 1 of the population, we have two chromosomes with a maximum length of 9, considered the least fit: C_3^1 and C_2^1. We replace less fit chromosome C_2^1 with fitter chromosome C_2^2 with a length of 7. Thus, we obtain Generation 2 of the population of chromosomes.

Table 3. Population of Chromosomes (Generation 2)

Chromosome C_1^2	5	1	4	3	2
Chromosome C_2^2	5	1	2	3	4
Chromosome C_3^2	5	2	3	4	1

The length of each chromosome is:

$$L_1(C_1^2) = 2 + 1 + 3 + 3 = 9;$$
$$L_2(C_2^2) = 2 + 1 + 2 + 2 = 7;$$
$$L_3(C_3^2) = 2 + 3 + 3 + 1 = 9.$$

The total sum of the edges of the graph according to the placement of the vertices on the grid determined by that particular population of chromosomes in the second iteration is:

$$S(2) = L_1(C_1^2) + L_2(C_2^2) + L_3(C_3^2) = 9 + 7 + 9 = 25.$$

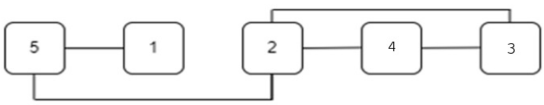
Choose the best chromosome from Generation 2. Clearly, we need to choose the chromosome of minimum length, which is C_2^2:

Chromosome C_2^2	5	1	2	3	4

Apply the inverse mutation genetic operator on chromosome C_2^2 to obtain a new chromosome C_3^3. In other words, in iteration 3, the first, second and third elements (5, 1, 2) of the chromosome remain in place, and the other elements are written in reverse order, starting with the last one, which will already be in the fourth position in the chromosome representation. Thus, we obtain:

Chromosome C_3^3	5	1	2	4	3

Chromosome C_3^3



Calculate the length of chromosome C_3^3:

$$L_3(C_3^3) = 2 + 1 + 2 + 2 = 7.$$

Further, in Generation 2 of the population, we select the chromosome with the maximum length equal to 9, considered the least fit, either C_1^2 or C_3^2. We replace the less fit chromosome C_3^2 with fitter chromosome C_3^3 with a length of 7. Thus, we obtain Generation 3 of the population of chromosomes.

Table 4. Population of Chromosomes (Generation 3)

Chromosome C_1^3	5	1	4	3	2
Chromosome C_2^3	5	1	2	3	4
Chromosome C_3^3	5	1	2	4	3

The length of each chromosome is:

$$L_1(C_1^3) = 1 + 1 + 3 + 2 = 7;$$
$$L_2(C_2^3) = 1 + 1 + 2 + 2 = 7;$$
$$L_3(C_3^3) = 1 + 1 + 3 + 2 = 7.$$

The total sum of the edges of the graph according to the placement of the vertices on the grid determined by that particular population of chromosomes in the third iteration is:

$$S(3) = L_1(C_1^3) + L_2(C_2^3) + L_3(C_3^3) = 7 + 7 + 7 = 21.$$

The evolution of lengths in the case of generations 0, 1, 2, 3 is as follows:

$$S(0) = 30 > S(1) = 28 > S(2) = 25 > S(3) = 21.$$

The best placement of the vertices of the graph on the line is obtained in the last iteration, in which the last genetically modified chromosomes C_2^3 and C_3^3 represent the solution with $\min L(G) = 7$ and $S(3) = 21$.

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