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Mathematical modelling of the immune response to infectious diseases with the influence of environmental factors

Yaroslav Bihun and Oleh Ukrainets

Abstract. The mathematical model of the immune response to infectious diseases with the influences of environmental factors is investigated. The conditions for the existence and uniqueness of the solution to the mathematical model for $t > 0$ have been established. Stationary solutions have been identified, along with the conditions for their existence and asymptotic stability. The results are illustrated using a model example.
2020 Mathematics Subject Classification: 34K10; 34K33.
Keywords: immune response, infectious disease, mathematical model, stationary solution, stability of solutions, delay differential equations, Marchuk model, model of immune system.

Modelarea matematică a răspunsului imun la bolile infecţioase sub influenţa factorilor de mediu

Rezumat. Modelul matematic al răspunsului imun la bolile infecţioase sub influenţa factorilor de mediu este investigat. Au fost stabilite condiţiile de existenţă şi unicitate a soluţiei modelului matematic pentru $t > 0$. Soluţiile staţionare au fost identificate împreună cu condiţiile de existenţă şi stabilitate asimptotică. Rezultatele sunt prezentate folosind un exemplu model.
Cuvinte-cheie: răspuns imun, boală infecţioasă, model matematic, soluţie staţionară, stabilitatea soluţiilor, ecuaţii diferenţiale cu întârziere, modelul Marchuk, model al sistemului imunitar.

1. Introduction

Numerous works, including those [1]-[4], [8], [9] and others, are devoted to the mathematical modelling of the immune response. G.Bell proposed a predatory-prey model in an immune response to infections by antigens (viruses, bacteria or foreign cells) [1]. In 1980, G.I. Marchuk published a mathematical model that reflects the humoral immune response of the human body and is described by a system of delay differential equations [2]:

$$\begin{aligned}
\frac{dV}{dt} &= (\beta - \gamma F)V, \\
\frac{dC}{dt} &= \alpha\xi(m)V_\tau F_\tau - \mu_c(C - C^*), \\
\frac{dF}{dt} &= \rho C - \eta\gamma FV - \mu_f F, \\
\frac{dm}{dt} &= \sigma V - \mu_m m,
\end{aligned} \tag{1}$$

where variables represent the core factors of the infectious process. The immune response involves the production of specific objects (antibodies, $F(t)$), which are generated by a cascade of plasma cells $C(t)$. Antibodies are capable of neutralizing or destroying foreign materials (antigens), the amount $V(t)$ of which changes over time $t \geq t_0 = 0$. The models also include the relative mass of the affected target organ m(t), which serves as a generalized measure of organ damage caused by the virus, and $\xi(m) = 1$ for $m \in [0, m^*]$ and $\xi(m) = (m-1)/(m^*-1)$ for $m^* < m \leq 1$, having $m^* \in (0,1)$ and considering for $m \in [0, m^*]$ the immune system functions normally; $V_\tau(t) = V(t-\tau)$, $F_\tau(t) = F(t-\tau)$.

The delay factor $\tau > 0$ plays a crucial role in the model as it sets the time from the moment of infection to the activation of immune response mechanisms. More complex delay models have been developed for the immune response to hepatitis B and C, tuberculosis, and other diseases [2]-[6]. Various aspects of immune response dynamics have been studied in the works of U. Forys and M. Bodnar [4].

The course of infectious diseases, such as hepatitis and acute respiratory diseases, is influenced by factors such as air pollution, water contamination, industrial waste, noise pollution, chemical pollution and other environmental pollutants. The model represented in the current work and described subsequently takes into account an integral factor $E(t)$, which is the sum of m factors $E_i(t)$ and is represented as follows:

$$E(t) = a_1E_1(t) + \dots + a_mE_m(t),$$

where $a_i \geq 0$, $a_1 + \dots + a_m = 1$.

Let us assume that the change of $E(t)$ occurs according to the generalized Hutchinson equation [5], [7], which has the following form:

$$\frac{dE(t)}{dt} = r\left(1 - \left(\frac{E(t-\Delta)}{E^*}\right)^n\right)E(t), t > 0, \tag{2}$$

where $r > 0$ - coefficient of linear growth, $0 < \Delta$ - the average time for the restoration of ecological balance, amount of which is $E^* > 0$. Using the parameter $n > 0$, a more accurate shape of the curve can be selected for a better representation of the system dynamics. This flexibility allows the modelling of specific scenarios or data, ensuring a closer match to observed behavior in immune response or external factors dynamics (see Fig.1).

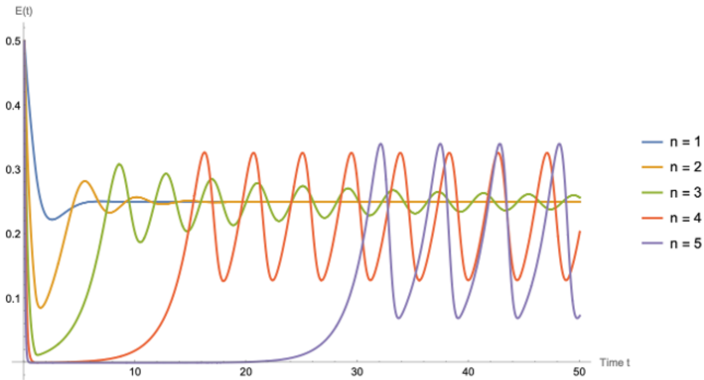


Figure 1. The dynamics of the generalized Hutchinson model for $n = 1, 2, 3, 4, 5$ and $r = 0.5, \Delta = 1, E^* = 0.25$

The change over time of the factors V, E, F, C and the measure m, $0 \le m(t) \le 1$ – the extent of organ damage against which the antigen V is directed – is proposed to be described by a system of equations:

$$\begin{aligned}
\frac{dV}{dt} &= (\beta - \gamma F)V, \\
\frac{dC}{dt} &= \alpha\xi(m)V_\tau F_\tau - \mu_c(C - C^*) - \varepsilon_c E, \\
\frac{dF}{dt} &= \rho C - (\mu_f + \eta\gamma V)F, \\
\frac{dm}{dt} &= \sigma V - \mu_m m + \varepsilon_m E,
\end{aligned} \tag{3}$$

The initial conditions for the system (3) solution have the following form:

$$\begin{aligned}
&V(t) = 0, t \in [-\tau, 0), V(0) = V_0 \ge 0; \\
&F(t) = F_0(t) \ge 0, t \in [-\tau, 0]; C(0) = C_0 \ge 0; m(0) = m_0 \in [0, 1).
\end{aligned} \tag{4}$$

The work explores issues of the existence and nonnegativity of solutions, identifies stationary solutions, establishes coefficient conditions for their stability, and conducts numerical modelling of the immune response for the model (3).

2. Nonnegativity and existence of a solution

It has been proven that the solution to the problem (3), (4) is nonnegative, which corresponds to the medical nature of the immune response process. It is known that the solution to equation (2) with initial condition $E_0(t) \geq 0$ for $t > 0$ exists for $t > 0$ and is bounded, that means $0 \leq E(t) \leq M$.

Theorem 2.1. *Let the coefficients of the system of equations (3) be nonnegative, and suppose there exists a solution for* $t > 0$ *and the condition*

$$\varepsilon_c M < \mu_c C^* \tag{5}$$

is satisfied. Then the solution of system (3) with initial conditions (4) is nonnegative for $t > 0$.

Proof. The solution of the equation (2) with initial function $E_0(t) \geq 0$ for the $t \in [-\Delta, 0]$ exists for $t > 0$ and limited [7] by

$$0 \leq E(t) \leq M, t \geq 0. \tag{6}$$

From the first equation of (3) after integration we obtain the following:

$$V(t) = V_0 exp(\int_0^t (\beta - \gamma F(s))\, ds) \geq 0.$$

From that follows that $V(t) \geq 0$ for $t > 0$, if $V_0 \geq 0$ and $V(t) > 0$ for $V_0 > 0$. From the equation for the $m(t)$, we obtain

$$m(t) = m_0 e^{-\mu_m t} + \int_0^t e^{-\mu_m(t-s)}(\sigma V(s) + \varepsilon_m E(s)\, ds) \geq 0. \tag{7}$$

Since $m(0) \geq 0$, $V(t) \geq 0$ and $E(t) \geq 0$, then $m(t) > 0$ for $t > 0$. The initial function $V(t) = 0$ for $t < 0$, then on the interval $[0, \tau]$

$$\frac{dC}{dt} = -\mu_c C + \mu_c C^* - \varepsilon_c E. \tag{8}$$

The solution of the equation (8) is the following:

$$C(t) = C^* + (C_0 - C^*)e^{-\mu_c t} - \varepsilon_c \int_0^t e^{-\mu_m(t-s)} E(s)\, ds,$$

Since $E(t) \leq M$ for $t > 0$, then

$$C(t) \geq C^* - \frac{\varepsilon_c M}{\mu_c}(1 - e^{-\mu_c t}) \geq C^* - \frac{\varepsilon_c M}{\mu_c} > 0.$$

From the condition $F(0) > 0$ we obtain $F(t) > 0$ on some interval $(0, t_1)$. Let us assume that $t_1 \leq \tau$ and $F(t_1) = 0$. Then $\frac{dF(t_1)}{dt} = 0$. At the same time,

$$\frac{dF(t_1)}{dt} = \rho C(t_1) - \eta\gamma F(t_1)V(t_1) - \mu_c F(t_1) = \rho C(t_1) > 0,$$

which contradicts the assumption. Thus, $F(t) > 0$ for $t \in [0, \tau]$. Since $\xi(m) \geq 0$ and, on the interval $[\tau, 2\tau]$, $F(t-\tau)V(t-\tau) \geq 0$, then

$$\frac{dC}{dt} = \xi(m(t))V(t-\tau)F(t-\tau) - \mu_c(C - C^*) - \varepsilon_c E \geq -\mu_c(C - C^*) - \varepsilon_c M.$$

From the estimate of the solution of the equation for F(t) on $[0, \tau]$, it follows that $C(t) > 0$ on $[\tau, 2\tau]$. Accordingly, $F(t) > 0$ on that interval. Using the step method, the positivity of $C(t)$ and $F(t)$ is similarly proven on $[2\tau, 3\tau]$, and so forth for subsequent intervals. ■

Theorem 2.2. *Let the coefficients and initial conditions at* $t = 0$ *for the solutions of equations (2) and (3) be positive numbers. Then there exists a unique solution to the problem (2), (3), defined on* $[0, \infty)$ *and differentiable on* $(0, \tau) \cup (\tau, \infty)$.

Proof. For equation (2), at each step $[k\Delta, (k+1)\Delta], k = 0, 1, \ldots$, a linear equation $\frac{dE}{dt} = qE(t)$ with a continuous function $q(t), t > 0$ is obtained. Therefore, there exists a unique solution to the equation (2) for $t > 0$, which is differentiable if the initial function $E_0 \in C[-\Delta, 0]$.

Let $V(0) > 0$. Then there exists a solution $V(t)$ on some interval $(0, a)$. Moreover, by Theorem 2.1, $V(t) > 0$. From this it follows that $F(t) > 0$ for $t \in (0, a)$. Thus on that interval

$$\frac{dV}{dt} = \beta V - \gamma F V \leq \beta V.$$

The solution to the linear equation $\frac{dV}{dt} = \beta V$ is defined for all $t > 0$. According to Wintner's theorem [6], the solution $V(t)$ of the first equation of (3) is defined for $t > 0$. Since the function $V_0(t)$ has a first-order discontinuity at $t = 0$, the function $V(t)$ is continuous for $(0, \infty)$ and differentiable over intervals $(0, \tau)$ and (τ, ∞).

From the form of the solution $m(t)$ according to formula (7), it follows that the solution $m(t)$ is defined for $t > 0$ and $m \in C^1(0, \infty)$.

The existence and uniqueness of solution $F \in C^1(0, \infty)$ is received from the differentiability of the right-hand side of the equation for F factor and an inequality

$$\frac{dF}{dt} = \rho C - (\eta\gamma F + \mu_f)F \leq \rho C,$$

using Winter's theorem.

■

3. Stationary solutions and their stability

By substituting $E(t) = \overline{E}(t) + E^*, t = s\Delta$, equation (2) is transformed into the form

$$\frac{d\overline{E}(s)}{ds} = -rn\Delta\overline{E}(s-1) + f(\overline{E}(s-1)),$$

where $\lim_{x\to 0}\frac{f(x)}{x} = 0$. The roots of the characteristic equation $\lambda + rn\Delta e^{-\lambda} = 0$ have negative real parts if the following condition is satisfied [7]

$$0 < rn\Delta < \pi/2. \tag{9}$$

According to the theorem on stability by linear approximation, the solution $E = E^*$ of equation (2) is asymptotically stable under the fulfilment of the condition (9).

The stationary solutions of system (3) are derived by the system of equations

$$\begin{aligned}
&(\beta - \gamma F)V = 0,\\
&\alpha VF - \mu_c(C - C^*) - \varepsilon_c E = 0,\\
&\rho C - (\mu_f + \eta\gamma V)F = 0,\\
&\sigma V - \mu_m m + \varepsilon_m E = 0.
\end{aligned} \tag{10}$$

The medical justification of the solutions requires $\xi(m) = 1$, which is achieved when $m \le m^*$ means that the damage to the target organ does not exceed the critical level.

For the problem (2),(3), there always exists such a stationary solution

$$E_1 = E^*, V_1 = 0, C_1 = C^* - \frac{\varepsilon_c E^*}{\mu_c}, F_1 = \frac{\rho C_1}{\mu_f}, m_1 = \frac{\varepsilon_m E^*}{\mu_m} \tag{11}$$

that defines the state of a healthy organism under permissible environmental pollution levels. The stationary solution (11) has a medical justification, if it is nonnegative. This holds if the following conditions are met:

$$\varepsilon_c E^* < C^*\mu_c, \varepsilon_m E^* \le \mu_m m^* \tag{12}$$

Theorem 3.1. *If condition (9), (12) and condition*

$$\beta - \gamma F_1 < 0 \tag{13}$$

are satisfied, then solution (11) is locally asymptotically stable.

Proof. Let us perform a substitution in system (3): $E = \overline{E} + E^*, V = \overline{V}, C = \overline{C} + C_1, F = \overline{F} + F_1, m = \overline{m} + m_1$. Let $(\overline{V}, \overline{F}, \overline{C}, \overline{m})$ be a solution of (10), then the linearized system corresponding to (3) for this solution takes the form

$$\begin{aligned}
\frac{d\overline{V}}{dt} &= (\beta - \gamma\overline{F})V,\\
\frac{d\overline{C}}{dt} &= \alpha\overline{V}F_\tau + \alpha\overline{F}V_\tau - \mu_c C - \varepsilon_c E,\\
\frac{d\overline{F}}{dt} &= \rho\overline{C} - \mu_f\overline{F} - \eta\gamma(\overline{V}F + V\overline{F}),\\
\frac{d\overline{m}}{dt} &= \sigma\overline{V} - \mu_m\overline{m} + \varepsilon_m E.
\end{aligned} \tag{14}$$

If the conditions of the theorem are satisfied, the nonnegativity of the solution (11) is evident. The characteristic equation for the linearized system (14) for solution (11) takes the form

$$(\lambda + rne^{-\Delta})\begin{vmatrix} \beta - \gamma F_1 - \lambda & 0 & 0 & 0 \\ \alpha F_1 e^{-\lambda\tau} & -\mu_c - \lambda & 0 & 0 \\ -\eta\gamma F_1 & \rho & \mu_f - \lambda & 0 \\ \sigma & 0 & 0 & -\mu_m - \lambda \end{vmatrix} =$$

$$= (\beta - \gamma F_1 - \lambda)(\mu_c + \lambda)(\mu_f + \lambda)(\mu_m + \lambda) = 0,$$

If conditions (9) and (13) are satisfied, the roots are negative, and the stationary solution is locally asymptotically stable. It is worth noting that solution (11) can be interpreted as the state of a healthy organism under an acceptable level of environmental pollution. ■

Theorem 3.2. *Let condition (13) hold, and for the initial values C_0 and V_0 the inequalities*

$$C_0 > C^* + \frac{\varepsilon M}{\mu_c}, 0 < V_0 < V^* = \frac{\mu_f(\gamma F_1 - \beta)}{\eta\gamma\beta} + \frac{2\rho\varepsilon_C M}{\mu_c} \tag{15}$$

are satisfied. Then, the function $V(t)$ decreases for $t > 0$ and

$$\lim_{t\to\infty} V(t) = 0.$$

Proof. Let $c(t) = C(t) - C^*$, $c_0 = C_0 - C^*$. For $t \in [0, \tau]$ from second equation of model (3) and initial functions (4); the following equation is received:

$$\frac{dc}{dt} = -\mu_C c - \varepsilon_c E.$$

From the first inequality from (15) and the boundedness of the solution of equation (2) by the constant M is received:

$$c(t) = c_0 e^{-\mu_c t} - \varepsilon_c \int_0^t e^{-\mu_c(t-s)} E(s)\, ds \geq c_0 e^{-\mu_c t} - \frac{\varepsilon_c M}{\mu_c}(1 - e^{-\mu_c t)} \geq \frac{c_0}{2}. \tag{16}$$

Thus, for $t \in [0, \tau]$
$$C(t) \geq C^* + \frac{2\varepsilon_c M}{\mu_c}. \tag{17}$$

On the interval $[\tau, 2\tau]$, taking into account that $F(t) > 0$, we obtain
$$\frac{dc}{dt} = \alpha\xi(m)F(t-\tau)V(t-\tau) - \mu_c c(t) - \varepsilon_c E(t) \geq -\mu_c c(t) - \varepsilon_c E(t),$$
from which assessment (16) is received. So forth for subsequent intervals $[2\tau, 3\tau]$.

Since $F(t) > 0$ for $t > 0$ and $\beta - \gamma F^* < 0$, then function $V(t)$ decreases on the interval $(0, t_1)$, $t_1 > 0$ and $\frac{dV(t_1)}{dt} = 0$. Then $F(t_1) = \frac{\beta}{\gamma}$ and on the certain interval (t_1, t_2) the following conditions are satisfied: $\frac{dV(t)}{dt} \geq 0$,
$$\frac{dF(t)}{dt} \leq 0. \tag{18}$$

Let us consider the value of the derivative
$$\frac{dF(t_1)}{dt} = \rho C(t_1) - \eta\gamma F(t_1)V(t_1) - \mu_f F(t_1) > \rho(C^* + \frac{2\varepsilon_c M}{\mu_c}) - \eta\beta V_0 - \mu_f\frac{\beta}{\gamma}.$$
From the estimate (17) follows:
$$\frac{dF(t_1)}{dt} = \eta\beta\left(\frac{\gamma F_1 - \beta}{\beta\gamma\eta} + \frac{2\rho\varepsilon_c E}{\beta\eta\mu_c} - V_0\right) = V^* - V_0 \geq 0,$$
This contradicts estimate (18). Hence, the function $V(t)$ decreases for $t > 0$ and the limit for $t \to \infty$ is the stationary solution $V_1 = 0$. ■

Remark 3.1. *In the monograph* [2] *number V^* is called an immunological barrier. If, during antigen infection, its degree does not exceed V^*, then the disease will not develop.*

The problem (2), (3) may have another stationary solution that corresponds to the state of a chronic disease:
$$\begin{aligned} E_2 &= E^*, F_2 = \frac{\beta}{\gamma}, \\ V_2 &= \frac{\mu_c\mu_f\beta - \rho\gamma\mu_c C^* + \rho\gamma\varepsilon_c E^*}{\beta(\alpha\rho - \mu_c\eta\gamma)}, \\ C_2 &= \frac{\alpha\beta\mu_f - \eta\gamma^2\mu_c C^* + \eta\gamma^2\varepsilon_c E^*}{\gamma(\alpha\rho - \mu_c\eta\gamma)}, \\ m_2 &= \frac{\delta V_2 + E_2}{\mu_m}. \end{aligned} \tag{19}$$

A stationary solution (19) exists if either
$$\alpha\rho > \mu_c\eta\gamma, \rho\gamma\mu_c C^* < \mu_c\mu_f\beta + \rho\gamma\varepsilon_c E^*$$
or the inequality with the opposite sign is satisfied. If $V_2 > 0$, then $C_2 > 0$ accordingly.

The characteristic equation for system (19), corresponding to the stationary solution $X := (E_2, V_2, C_2, F_2, m_2)$ takes the form:

$$P_5(\lambda) := -(\mu_m + \lambda)(\lambda + rne^{-\lambda\Delta}) * \begin{vmatrix} -\lambda & 0 & -\gamma V_2 \\ 2F_2 e^{-\lambda\tau} & -\mu_c - \lambda & \alpha V_2 e^{-\lambda\tau} \\ -\eta\gamma F_2 & \rho & \eta\gamma V_2 - \mu_f - \lambda \end{vmatrix} =$$

$$= (\mu_m + \lambda)(\lambda + rne^{-\lambda\Delta})(\lambda^3 + c_1\lambda^2 + +c_2\lambda + c_3) = 0,$$

where $c_1 = \mu_c + \mu_f - \eta\gamma V_2$, $c_2(\lambda) = \mu_c\mu_f - (\eta\gamma + \alpha\rho e^{-\lambda\tau} + \eta\beta)V_2$, $c_3(\lambda) = (\alpha\rho e^{-\lambda\tau} - \eta\mu_c)\Delta V_2$.

If inequality (9) holds, the study of the asymptotic stability of the solution X reduces to finding the conditions under which $Re(\lambda) < 0$ for the roots of the quasi-polynomial $P_3 = 0$. Let us consider the case when $\tau = 0$, which is the case of an instantaneous immune system response to the infection of the human body. In this case, the problem reduces to studying the roots of a cubic equation

$$P_{3,0}(\lambda) = \lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0, \tag{20}$$

where $a_1 = c_1$, $a_2 = \mu_c\mu_f - ((1+\beta)\eta\gamma + \alpha\rho)V_2$, $a_3 = \beta(\alpha\rho - \eta\gamma\mu_c)V_2$. Let us consider the case of a strong immune response [2], when

$$\alpha\rho > \eta\gamma\mu_c, \tag{21}$$

in that case $a_3 > 0$.

From the Routh-Hurwitz criterion [6], it follows that the necessary and sufficient conditions for the asymptotic stability of solution X are the fulfilment of condition (21) and

$$a_1 > 0, a_1a_2 - a_3 > 0. \tag{22}$$

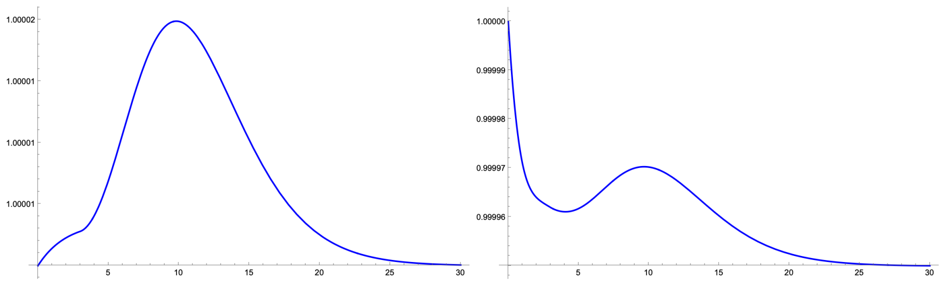
From the analysis of the roots of the characteristic equation of the linearized system, the conditions for the asymptotic stability and instability of solution (19) have been found. Therefore, sufficient conditions for either maintaining a chronic disease state or transitioning from a chronic condition to an acute form have been obtained.

4. Numerical modelling

Numerical simulations of the immune response were conducted using the Wolfram Mathematica computer algebra system, considering the influence of environmental factors. These simulations were based on the system (2), (3) with the following parameters: $\beta = 0.6, \gamma = 0.2, \alpha = 0.9, \mu_c = 0.5, C^* = 1, \rho = 0.9, \mu_f = 0.17, \eta = 0.8, \sigma = 0.35, \mu_m = 0.4, r = 0.5, \Delta = 1, E^* = 0.25, n = 1; V_0 = 0.000001, C_0 = F_0 = 1, E_0 =$

0.5, $m_0 = 0$. Simulations were performed under two distinct scenarios: Figures 2(a), 3(a) with $\varepsilon_c = \varepsilon_m = 0$, and Figures 2(b), 3(b) with $\varepsilon_c = \varepsilon_m = 0.0001$.

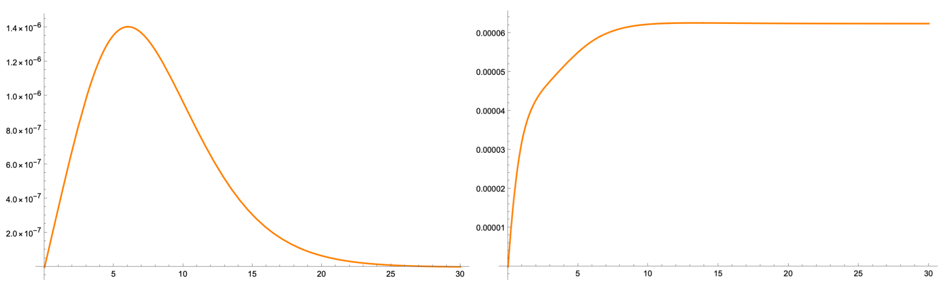
Figure 2(a) illustrates the change in the level of plasma cells $C(t)$ without the influence of the environmental factors $E(t)$. In Figure 2b, under the influence of $E(t)$, oscillations occur in the plasma cell population, and the weakened overall immune response is demonstrated.



a) b)

Figure 2. Dynamics in the immune response model factor $C(t)$ without (a) and with (b) the influence of environmental factors.

Figures 3(a) and 3(b) show the dynamics of the extent of damage $m(t)$ to the target organ. With pollution (Figure 3b), there remains relatively minor damage to the target organ according to the parameters set by this model example. The presence of the ecological factor leads to an overall destabilizing effect on the system's equilibrium. When $E(t) = 0$, then $m(t) \to 0$ for $t \to \infty$.



a) b)

Figure 3. Dynamics in the immune response model factor $m(t)$ without (a) and with (b) the influence of environmental factors.

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The problem of the center for cubic differential systems with two affine non-parallel invariant straight lines of total multiplicity three

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Abstract. In this paper, we show that a center-focus critical point of cubic differential systems with two affine non-parallel invariant straight lines of total multiplicity three is a center type if and only if the first five Lyapunov quantities vanish.

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Keywords: cubic differential system, multiple invariant line, the problem of the center.

Problema centrului pentru sistemele diferenţiale cubice cu două drepte afine invariante şi concurente de multiplicitate totală trei

Rezumat. În această lucrare se arată că punctul critic de tip centru-focar al sistemelor diferenţiale cubice cu două drepte afine invariante şi concurente de multiplicitate totală trei este centru, dacă şi numai dacă primele cinci mărimi Liapunov se anulează.

Cuvinte-cheie: sistem diferenţial cubic, dreaptă invariantă multiplă, problema centrului.

1. Introduction and statement of main results

We consider the real polynomial differential systems

$$\frac{dx}{dt} = P(x, y), \quad \frac{dy}{dt} = Q(x, y), \quad \gcd(P, Q) = 1 \tag{1}$$

and the vector fields $\mathbb{X} = P(x, y) \frac{\partial}{\partial x} + Q(x, y) \frac{\partial}{\partial y}$ associated to systems (1).

Denote $n = \max\{\deg(P), \deg(Q)\}$. If $n = 2$ (respectively, $n = 3$, $n = 4$), then the system (1) is called *quadratic* (respectively, *cubic, quartic*).

An algebraic curve $f(x, y) = 0$, $f \in \mathbb{C}[x, y]$ (a function $f = \exp[\frac{g}{h}]$, $g, h \in \mathbb{C}[x, y]$) is called *invariant algebraic curve* (*exponential factor*) of the system (1) if there exists a polynomial $K_f \in \mathbb{C}[x, y]$, $\deg(K_f) \leq n-1$ such that the identity $\mathbb{X}(f) \equiv f(x, y)K_f(x, y)$ holds. In particular, *a straight line* $\mathcal{L} \equiv \alpha x + \beta y + \gamma = 0$, $\alpha, \beta, \gamma \in \mathbb{C}$ is called *invariant* for the system (1) if there exists a polynomial $K_{\mathcal{L}} \in \mathbb{C}[x, y]$ such that the identity $\alpha P(x, y) + \beta Q(x, y) \equiv (\alpha x + \beta y + \gamma)K_{\mathcal{L}}(x, y)$, $(x, y) \in \mathbb{R}^2$, i.e. $\mathbb{X}(\mathcal{L}) \equiv \mathcal{L}(x, y)K_{\mathcal{L}}(x, y)$, $(x, y) \in \mathbb{R}^2$, holds. If a straight line $\mathcal{L}$ is described by the equation

$y = \alpha x + \beta$, $\beta \neq 0$ (respectively, $x = \alpha$), then $\mathcal{L}$ is invariant for (1) if the following identity in x:

$$(\alpha P(x, y) - Q(x, y))|_{y=\alpha x+\beta} \equiv 0 \qquad (2)$$

(respectively, in y: $P(\alpha, y) \equiv 0$) holds.

If $m_p(\mathcal{L})$ (respectively, $m_a(\mathcal{L})$) is the greatest natural number such that $\mathcal{L}^{m_p(\mathcal{L})}$ (respectively, $\mathcal{L}^{m_a(\mathcal{L})}$) divides $\mathbb{X}(\mathcal{L})$ (respectively, $E(\mathbb{X}) = P \cdot \mathbb{X}(Q) - Q \cdot \mathbb{X}(P)$), then we say that the invariant straight line $\mathcal{L}$ has *parallel multiplicity* (*algebraic multiplicity*, or in brief, *multiplicity*) $m_p(\mathcal{L})$ (respectively, $m_a(\mathcal{L})$).

Remark 1.1. $1 \leq m_p(\mathcal{L}) \leq n$ *and* $m_p(\mathcal{L}) \leq m_a(\mathcal{L})$.

The number $m_t(\mathcal{L}) = m_a(\mathcal{L}) - m_p(\mathcal{L}) + 1$ is called *transversally multiplicity* of the line $\mathcal{L}$.

Some notions on multiplicity (algebraic, integrable, infinitesimal, geometric) of an invariant algebraic line and its equivalence for polynomial differential systems are given in [1].

The cubic differential systems with multiple invariant straight lines (including the line at infinity) were studied in [5], [6], [10], [11], [12], [14].

Let $f_1, \ldots, f_r$ ($f_{r+1} = \exp(g_{r+1}/h_{r+1}), \ldots, f_s = \exp(g_s/h_s)$) be invariant algebraic curves (exponential factors) of (1) and let K_{f_j}, $j = \overline{1, s}$, be its cofactors [2]. The system (1) is called *Darboux integrable* if (1) has a first integral (an integrating factor) of the form $F(x, y) = f_1^{\alpha_1} \cdots f_s^{\alpha_s}$ ($\mu(x, y) = f_1^{\alpha_1} \cdots f_s^{\alpha_s}$), $\alpha_j \in \mathbb{C}$, $j = \overline{1, s}$. Note that the constants $\alpha_1, \ldots, \alpha_s$ are not all equal to zero.

It is easy to show that $F(x, y)$ ($\mu(x, y)$) is a Darboux first integral (a Darboux integrating factor) if and only if the following identity

$$\alpha_1 K_{f_1} + \alpha_2 K_{f_2} + \cdots + \alpha_s K_{f_s} \equiv 0$$
$$\left(\alpha_1 K_{f_1} + \alpha_2 K_{f_2} + \cdots + \alpha_s K_{f_s} + \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \equiv 0\right)$$

holds in x and y.

In this work we consider the cubic systems of the form

$$\begin{cases} \dot{x} = y + ax^2 + cxy + fy^2 + kx^3 + mx^2y + pxy^2 + ry^3 \equiv P(x, y), \\ \dot{y} = -(x + gx^2 + dxy + by^2 + sx^3 + qx^2y + nxy^2 + ly^3) \equiv Q(x, y), \\ \qquad \gcd(P, Q) = 1. \end{cases} \qquad (3)$$

The critical point $(0, 0)$ of the system (3) is of a center-focus type, i.e. is either a focus or a center. The problem of distinguishing between a center and a focus is called *the problem of the center* or *the center-focus problem*.

It is known that $(0,0)$ is a center for (3) if and only if the system has a nonconstant analytic first integral $F(x,y)$ (an analytic integrating factor $\mu(x,y)$) in a neighborhood of $(0,0)$. Also, it is known that there exists a formal power series $F(x,y) = x^2 + y^2 + \sum_{j\geq 3} F_j(x,y)$ such that the rate of change of $F(x,y)$ along trajectories of (3) is a linear combination of polynomials $\{(x^2+y^2)^j\}_{j=2}^{\infty}$, i.e. $\frac{dF}{dt} = \sum_{j=2}^{\infty} L_{j-1}(x^2+y^2)^j$. The quantities L_j, $j = \overline{1,\infty}$, are polynomials with respect to the coefficients of the system (3), called to be *the Lyapunov quantities*. For example, the first Lyapunov quantity looks as

$$L_1 = (bd - ac + 2bf - 2ag + dg - cf + 3k - 3l + p - q)/4.$$

The origin $(0,0)$ is a center for (3) if and only if $L_j = 0$, $j = \overline{1,\infty}$.

The problem of the center is completely solved for quadratic systems ($k = l = m = n = p = q = r = s = 0$) [4] and for symmetric cubic systems ($a = b = c = d = f = g = 0$) [8]. For other polynomial differential systems the necessary and sufficient conditions for the center-focus critical point to be a center were obtained in some particular cases (see, for example, [2], [7]).

The problem of coexistence in cubic systems of the distinct invariant straight lines and critical points of center type was studied in [2], [3], [9]. In [3] (see also [2]) it was proved that if the cubic system (3) has four distinct invariant straight lines of the form $1 + \alpha_j x + \beta_j y = 0$, $j = 1,2,3,4$ $(y \pm ix = 0, 1 + \alpha_j x + \beta_j y = 0,\ j = 1,2)$ and the Lyapunov quantity vanishes: $L_1 = 0$ ($L_1 = L_2 = 0$), then the origin is a center. In the cases when (3) has three distinct invariant straight lines then $(0,0)$ is a center if the first seven Lyapunov quantities vanish $L_j = 0$, $j = 1,\ldots 7$.

In this article we investigate the problem of the center for (3) with two invariant affine straight lines of total multiplicity three. Our main result is the following one:

Main Theorem. *Let the cubic system have two distinct affine non-parallel invariant straight lines $\mathcal{L}_1$, $\mathcal{L}_2$ and a critical point $M_0(x_0, y_0)$ with pure imaginary eigenvalues. If $m(\mathcal{L}_1) = 2$, then M_0 is a center if and only if the first five Lyapunov quantity vanishe* $(L_1 = L_2 = L_3 = L_4 = L_5 = 0)$.

2. Conditions of the existence of invariant straight lines

Let the system (3) have an invariant straight line $\mathcal{L}_1$. Using a transformation of the form $x \to \omega(x\cos\alpha - y\sin\alpha)$, $y \to \omega(x\sin\alpha + y\cos\alpha)$, we do $\mathcal{L}_1$ to be described by the equation $\mathcal{L}_1 \equiv x - 1 = 0$. The identity $P(1,y) \equiv 0$ gives us

$$k = -a,\ m = -c - 1,\ p = -f,\ r = 0. \tag{4}$$

For system $\{(3), (4)\}$ the identity (2) has the form

$$A_0 + A_1x + A_2x^2 + A_3x^3 \equiv 0,$$

where

$$\begin{aligned}
&A_0 = -\beta(\alpha + b\beta + f\alpha\beta + l\beta^2),\\
&A_1 = -1 - \alpha^2 - (d + 2b\alpha + c\alpha + 2f\alpha^2)\beta - (n - f\alpha + 3l\alpha)\beta^2,\\
&A_2 = -g - q\beta - \alpha(a + d - \beta - c\beta + 2n\beta) - \alpha^2(b + c - 2f\beta + 3l\beta) - f\alpha^3,\\
&A_3 = -s + (a - q)\alpha + (1 + c - n)\alpha^2 + (f - l)\alpha^3.
\end{aligned}$$

The system $\{A_0 \equiv 0, A_1 \equiv 0, A_2 \equiv 0, A_3 \equiv 0, \beta \neq 0\}$ has the solution

$$\begin{aligned}
&g = \alpha(d + c\alpha - a) - (q - \alpha - c\alpha)\beta + \alpha(2 - \alpha^2)/\beta,\\
&l = -(\alpha + b\beta + f\alpha\beta)/\beta^2,\\
&n = f\alpha + (2\alpha^2 - 1 - \beta(d - b\alpha + c\alpha - f\alpha^2))/\beta^2,\\
&s = \alpha(a - q + \alpha + c\alpha) + \alpha^2(d + c\alpha)/\beta + \alpha^2(1 - \alpha^2)/\beta^2.
\end{aligned} \tag{5}$$

Therefore, the system $\{(3),(4),(5)\}$ has the invariant straight lines $\mathcal{L}_1 \equiv x-1 = 0$, $\mathcal{L}_2 \equiv \alpha x - y + \beta = 0, \beta \neq 0$.

The invariant straight line $\mathcal{L}_1$ has parallel multiplicity two if $P(1, y)|_{\{(4),(5)\}} \equiv 0$, i.e. if

$$a = c + 2 = f = 0. \tag{6}$$

The equalities $\{(4), (5), (6)\}$ give us the first set of conditions

$$\begin{aligned}
&a = c + 2 = f = k = m - 1 = p = r = 0,\ l = -(\alpha + b\beta)/\beta^2,\\
&g = (2\alpha - \alpha(\alpha + \beta)^2 + \beta(d\alpha - q\beta))/\beta, n = (2\alpha^2 - d\beta + 2\alpha\beta + b\alpha\beta - 1)/\beta^2,\\
&s = \alpha(\alpha - \alpha(\alpha + \beta)^2 + \beta(d\alpha - q\beta))/\beta^2,
\end{aligned} \tag{7}$$

so that, the straight lines $\mathcal{L}_1$, $\mathcal{L}_2$ are invariant for (3) and $m_p(\mathcal{L}_1) = 2$.

If $m_a(\mathcal{L}_1) \geq 2 > m_p(\mathcal{L}_1) = 1$, then it is necessary that x^2 divide for $\{(3),(4),(5)\}$ the polynomial

$$\begin{aligned}
\kappa(x, y) = {}& x(\alpha x - y)(y^2(\alpha + b\beta + f\alpha\beta + f\beta^2) - xy(\alpha^2 - d\beta - c\alpha\beta - \beta^2 - c\beta^2 - 1)\\
&+x^2(\alpha - \alpha^3 + d\alpha\beta + c\alpha^2\beta + a\beta^2 - q\beta^2 + \alpha\beta^2 + c\alpha\beta^2))/\beta^2.
\end{aligned}$$

This implies that

$$b = -f(\alpha + \beta) - \alpha/\beta. \tag{8}$$

Taking into account (4) and (5), we obtain that

$$\beta^4(E(\mathbb{X})/((-1 + x)(-y + x\alpha + \beta)))|_{x=1} = f_1(y)f_2(y),$$

where

$$\begin{aligned}
f_1(y) &= \alpha^3 - \alpha - \beta(1 + d\alpha + c\alpha^2 - q\beta + \alpha\beta + c\alpha\beta) \\
&\quad + (\alpha^2 - 1 - \beta(d + \alpha + c\alpha))y + f\beta^2 y^2, \\
f_2(y) &= a\beta^2(d + q - a) + (c + 2)(\alpha + \beta)(\alpha^3 - \alpha - \beta - d\alpha\beta + 2\alpha^2\beta + q\beta^2 + \alpha\beta^2) \\
&\quad - \alpha\beta(c + 2)^2(\alpha + \beta)^2 + 2((\alpha^2 - 1)(a + f\alpha^2) - \beta(ad - a\alpha + 2f\alpha + df\alpha^2 \\
&\quad - 3f\alpha^3) - f\beta^2(1 + a\alpha + d\alpha - q\alpha - 3\alpha^2) - f\beta^3(a - q - \alpha) \\
&\quad - \beta(c + 2)(\alpha + \beta)(a + f\alpha^2 + f\alpha\beta))y(f\beta^2(a - d - q) \\
&\quad + (c + 2)(\alpha^2 - 1 - d\beta + \alpha\beta - f\alpha\beta^2 - f\beta^3) - \beta(\alpha + \beta)(c + 2)^2)y^2.
\end{aligned}$$

If $\{f_1(y) \equiv 0,\ \beta \neq 0\}$, then $\gcd(P, Q) = x - 1$, i.e. the system (3) is degenerate.

In the case $\{f_2(y) \equiv 0,\ \beta \neq 0\}$ we obtain the equalities (6) and the following three set of solutions

$$a = -(c + 2)(\alpha + \beta),\ d = ((\alpha + \beta)(\alpha - 2\beta - c\beta) - 1)/\beta, f = 0; \tag{9}$$

$$d = ((\alpha + \beta)(\alpha - 2\beta - c\beta) - 1)/\beta,\ q = (1 - \alpha^2 + a\beta - \alpha\beta)/\beta; \tag{10}$$

$$\begin{aligned}
a = -(\alpha + \beta)(2 + c + f\alpha + f\beta), q = -(df\beta^2 + 2\beta(d + \alpha + 2\beta) \\
+ (c + 2)(1 - \alpha^2) + c\beta(d + 3\alpha + 4\beta) \\
+ \beta(\alpha + \beta)(c^2 + 4f\beta + 2cf\beta) + f^2\beta^2(\alpha + \beta)^2)/(f\beta^2).
\end{aligned} \tag{11}$$

Equalities (6) lead us to a particular case of the set (7) and each of the equalities (9), (10), (11), together with (4), (5) and (8) give us, respectively, the following three series of conditions

$$\begin{aligned}
&a = -(2 + c)(\alpha + \beta), b = -\alpha/\beta, d = ((\alpha + \beta)(\alpha - 2\beta - c\beta) - 1)/\beta, f = 0, \\
&g = (\alpha + \alpha\beta(c + 1)(\alpha + \beta) - q\beta^2)/\beta, k = -a, l = 0, m = -c - 1, r = 0, \\
&n = (\alpha + 2\beta + c\beta)/\beta, p = -f, s = -(\alpha(q\beta + (\alpha + \beta)(\alpha + 2\beta + c\beta)))/\beta;
\end{aligned} \tag{12}$$

$$\begin{aligned}
&b = -(\alpha + f\alpha\beta + f\beta^2)/\beta, d = ((\alpha + \beta)(\alpha - 2\beta - c\beta) - 1)/\beta, m = -c - 1, \\
&g = (\alpha - \beta - a\alpha\beta - a\beta^2)/\beta, k = -a, l = f, n = (\alpha + 2\beta + c\beta)/\beta, \\
&p = -f, q = (1 - \alpha^2 + a\beta - \alpha\beta)/\beta, r = 0, s = -\alpha/\beta;
\end{aligned} \tag{13}$$

$$\begin{aligned}
&a = -(\alpha + \beta)(2 + c + f\alpha + f\beta), b = -(\alpha + f\alpha\beta + f\beta^2)/\beta, r = 0, \\
&g = (2f\alpha + (c + 2)(1 + d\beta) + (\alpha + \beta)(df\beta - (c + 2)(\alpha - 2\beta - c\beta)) \\
&- f(\alpha + \beta)^2(\alpha - 4\beta - 2c\beta) + f^2\beta(\alpha + \beta)^3)/(f\beta), k = -a, l = f, \\
&n = -(1 - \alpha^2 + d\beta + c\alpha\beta)/\beta^2, p = -f, q = -(df\beta^2 + 2\beta(d + \alpha + 2\beta) \\
&+ (c + 2)(1 - \alpha^2) + c\beta(d + 3\alpha + 4\beta) + \beta(\alpha + \beta)(c^2 + 4f\beta + 2cf\beta) \\
&+ f^2\beta^2(\alpha + \beta)^2)/(f\beta^2), s = \alpha(f\alpha + 2d\beta + 2\beta(\alpha + 2\beta) + (c + 2)(1 - \alpha^2) \\
&+ c\beta(d + 3\alpha + 4\beta) - (\alpha + \beta)(f(\alpha - 2\beta)(\alpha + \beta) - \beta(c^2 + df)) \\
&+ cf\beta(\alpha + \beta)^2)/(f\beta^2), m = -c - 1
\end{aligned} \tag{14}$$

such that the straight lines $\mathcal{L}_1$, $\mathcal{L}_2$ are invariant for (3) and $m_a(\mathcal{L}_1) \geq 2$.

3. Sufficient conditions of the center

Lemma 3.1. *The following set of conditions is sufficient for the origin* $(0, 0)$ *to be a center for the system* (3)

$$
\begin{aligned}
&a = \gamma(\gamma-\beta)/\beta, b = (\beta-\gamma)/\beta, c = -(\beta+\gamma)/\beta, d = (2\gamma^2 - 2\beta\gamma - 1)/\beta,\\
&f = 0, g = ((\beta-\gamma)(\gamma^2-1) - q\beta^2)/\beta, k = \gamma(\beta-\gamma)/\beta, l = 0, m = \gamma/\beta,\\
&n = 0, p = 0, r = 0, s = q(\beta-\gamma);
\end{aligned} \tag{15}
$$

Proof. In conditions (15) the system (3) has the integrating factor of the form

$$\mu(x, y) = \mathcal{L}_1^{\alpha_1}\mathcal{L}_2^{\alpha_2},$$

where $\mathcal{L}_1 = x - 1,\ \mathcal{L}_2 = (\beta-\gamma)x + y - \beta,\ \alpha_1 = -2, \alpha_2 = -1.$ □

Lemma 3.2. *The following six sets of conditions are sufficient for the origin* $(0, 0)$ *to be a center for the system* (3)

$$
\begin{aligned}
&b = f = g = l = p = q = r = s = 0,\ d = (\beta^2-1)/\beta,\\
&a - \beta = c + 3 = k + \beta = m - 2 = n + 1 = 0;
\end{aligned} \tag{16}
$$

$$
\begin{aligned}
&b = 1, c = 0, d = -1/\beta, f = 0, g = -2, k = -a, l = 0,\\
&m = -1, n = 1, p = 0, q = (1 + a\beta)/\beta, r = 0, s = 1;
\end{aligned} \tag{17}
$$

$$
\begin{aligned}
&a = -1/(\beta(1+\beta^2)), b = 1, c = -2\beta^2/(1+\beta^2), d = -1/\beta, f = -\beta/(1+\beta^2),\\
&g = -2, k = 1/(\beta(1+\beta^2)), l = -\beta/(1+\beta^2), m = (\beta^2-1)/(1+\beta^2),\\
&n = (1-\beta^2)/(1+\beta^2), p = \beta/(1+\beta^2), q = \beta/(1+\beta^2), r = 0, s = 1;
\end{aligned} \tag{18}
$$

$$
\begin{aligned}
&a = \gamma^2(1 + \beta\gamma - \gamma^2)/(2\beta(1-\gamma^2)), c = \gamma(\beta\gamma + \gamma^2 - 1)/(\beta(1-\gamma^2)),\\
&b = ((\beta-\gamma)(\gamma^2-1)(\gamma^2-2) - \beta\gamma^2)/(2\beta(1-\gamma^2)),\\
&d = (\gamma(\beta-\gamma)(2\gamma^2-3) - 1)/(\beta(1-\gamma^2)),\\
&f = \gamma((\beta-\gamma)(\gamma^2-1) - \beta)/(2\beta(\gamma^2-1)),\\
&g = (\gamma^2-2)((\beta-\gamma)(\gamma^2-1) - \beta)/(2\beta(\gamma^2-1)),\\
&k = \gamma^2(1 + \beta\gamma - \gamma^2)/(2\beta(\gamma^2-1)), r = 0, s = (\beta-\gamma)/\beta,\\
&l = \gamma((\beta-\gamma)(\gamma^2-1) - \beta)/(2\beta(\gamma^2-1)),\\
&m = (\beta - \gamma + \gamma^3)/(\beta(\gamma^2-1)), n = 1/(1-\gamma^2),\\
&p = \gamma((\beta-\gamma)(\gamma^2-1) - \beta)/(2\beta(1-\gamma^2)),\\
&q = (1 + \beta\gamma - \gamma^2)(\gamma^2-2)/(2\beta(\gamma^2-1));
\end{aligned} \tag{19}
$$

$$\begin{aligned}
&b = (\beta - \gamma - f\beta\gamma)/\beta,\\
&c = (2\beta(f-\gamma) - \gamma(a+f)(2a\beta - 2f\beta + 3\beta\gamma - \gamma^2 - 1))/(\beta(\gamma\\
&\qquad +(a+f)(\gamma^2-1))),\ d = ((3a-\gamma)(\beta-\gamma)\gamma - \gamma(1 - f\beta + 3f\gamma)\\
&\qquad +(a+f)(1 + 2a\beta\gamma^2 - 2f\beta\gamma^2))/(\beta(\gamma + (a+f)(\gamma^2-1))),\\
&g = (\gamma - 2\beta - a\beta\gamma)/\beta,\ k = -a,\ l = f,\ m = (\beta(a - f + \gamma)+\\
&\qquad \gamma(a+f)(-1 + 2a\beta - 2f\beta + 2\beta\gamma - \gamma^2))/(\beta(\gamma + (a+f)(\gamma^2-1))),\\
&n = ((1 + 2\gamma(a+f))(\gamma^2 + f\beta - a\beta - \beta\gamma))/(\beta(\gamma + (a+f)(\gamma^2-1))),\\
&p = -f,\ q = (1 + a\beta + \beta\gamma - \gamma^2)/\beta,\ r = 0,\ s = (\beta - \gamma)/\beta,\\
&((\beta-\gamma)\gamma^3 - 2\beta(f - \gamma - a\gamma^2) - \gamma^2)(a\beta\gamma + (a+f)(a\beta - \gamma^2 - f\beta\gamma^2)) = 0;
\end{aligned} \tag{20}$$

$$\begin{aligned}
&a = -\gamma(c + 2 + f\gamma),\ b = (\beta - \gamma - f\beta\gamma)/\beta,\ k = \gamma(c + 2 + f\gamma),\ l = f,\\
&g = (\beta\gamma(c+2)^2 + (c+2)(1 + d\beta + \beta\gamma - \gamma^2 + 2f\beta\gamma^2) + f((\beta-\gamma)(\gamma^2-2)\\
&\qquad +d\beta\gamma + f\beta\gamma^3))/(f\beta),\ n = ((\beta-\gamma)(\beta + c\beta - \gamma) - d\beta - 1)/\beta^2,\\
&m = -c - 1,\ p = -f,\ q = -((c+2)^2\beta\gamma + (c+2)(d\beta + \beta\gamma + 2f\beta^2\gamma - \gamma^2\\
&\qquad +1) + f\beta^2(d + f\gamma^2))/(f\beta^2),\ r = 0,\ s = (\gamma - \beta)((c+2)^2\beta\gamma\\
&\qquad +(c+2)(1 + d\beta + \beta\gamma - \gamma^2 + f\beta\gamma^2) + df\beta\gamma f(\beta-\gamma)(\gamma^2-1))/(f\beta^2),\\
&\qquad 2(c+2)^2\beta\gamma + (c+2)(1 + d\beta - f\beta + 4f\beta\gamma^2) + f\beta\gamma(d - 2f + 2f\gamma^2).
\end{aligned} \tag{21}$$

Proof. In each of the sets of conditions (16)−(21), the system (3) has the integrating factor of the form

$$\mu(x,y) = \mathcal{L}_1^{\alpha_1}\mathcal{L}_2^{\alpha_2}\mathcal{L}_3^{\alpha_3} \tag{22}$$

and therefore, in all cases the origin $(0,0)$ is a center for (3). Indeed, in *Case* (16):

$$\mathcal{L}_1 = x - 1,\ \mathcal{L}_2 = y - \beta,\ \mathcal{L}_3 = \beta y + 1,\ \alpha_1 = -3, \alpha_2 = -1, \alpha_3 = 1;$$

in *Case* (17):

$$\begin{aligned}
&\mathcal{L}_1 = x - 1,\ \mathcal{L}_2 = \beta x + y - \beta,\ \mathcal{L}_3 = \exp[y/(x-1)],\\
&\alpha_1 = 2a\beta - 2\beta^2 - 1,\ \alpha_2 = 2\beta^2 - 2a\beta - 1,\ \alpha_3 = -2\beta;
\end{aligned}$$

in *Case* (18):

$$\begin{aligned}
&\mathcal{L}_1 = x - 1,\ \mathcal{L}_2 = \beta x + y - \beta,\ \mathcal{L}_3 = \beta^2 x + \beta y - \beta^2 - 1,\\
&\alpha_1 = -3,\ \alpha_2 = 1,\ \alpha_3 = -2;
\end{aligned}$$

in *Case* (19):

$$\begin{aligned}
&\mathcal{L}_1 = x - 1,\ \mathcal{L}_3 = \exp[\beta\gamma(\gamma - y)(2\beta - \gamma - \beta\gamma^2 + \gamma^3)/(2(\gamma^2-1)(x-1))],\\
&\mathcal{L}_2 = (\beta - \gamma)x + y - \beta,\ \alpha_1 = (1 - 2\beta(2\gamma - \beta) - \gamma^2(\gamma - \beta)^2)/(\gamma^2 - 1),\\
&\alpha_2 = (1 + (\beta-\gamma)^2(\gamma^2 - 2))/(\gamma^2 - 1),\ \alpha_3 = -2/(\beta\gamma);
\end{aligned}$$

in *Case* (20):

$$
\begin{aligned}
&\mathcal{L}_2 = (\beta-\gamma)x + y - \beta,\ \ \mathcal{L}_3 = \exp[(f\beta^2(y-\gamma))/(x-1)],\\
&\alpha_1 = (2\beta^2(\gamma-\beta)\gamma^2 f^2 + \beta(2a\beta^2 + 3\gamma - 2a\beta\gamma - 2\beta\gamma^2 + \gamma^3 + 2a\beta\gamma^3\\
&\qquad +\beta^2\gamma^3 - \beta\gamma^4)f + \gamma(a\beta - 2a^2\beta^2 - 3\beta\gamma - a\beta^2\gamma + \gamma^2 - a\beta\gamma^2\\
&\qquad +\beta^2\gamma^2 - 2\beta\gamma^3 + \gamma^4))/(\beta\gamma(\gamma + (a+f)(\gamma^2-1))),\\
&\alpha_2 = (2\beta(\beta-\gamma)\gamma^2 f^2 + (\gamma - 2a\beta^2 + 2a\beta\gamma + 2\beta\gamma^2 - 3\gamma^3 - 2a\beta\gamma^3\\
&\qquad -\beta^2\gamma^3 + \beta\gamma^4)f + \gamma(a + 2a^2\beta - \gamma + a\beta\gamma - 3a\gamma^2 - \beta\gamma^2\\
&\qquad +\gamma^3))/(\gamma(\gamma + (a+f)(\gamma^2-1))),\\
&\alpha_3 = (2a\beta - \gamma^2 - 2f\beta\gamma^2 + \beta\gamma^3 - \gamma^4)/(\beta^2\gamma(\gamma + (a+f)(\gamma^2-1)));
\end{aligned}
$$

in *Case* (21):

$$
\begin{aligned}
&\mathcal{L}_1 = x-1,\ \mathcal{L}_2 = (\beta-\gamma)x + y - \beta,\\
&\mathcal{L}_3 = \exp[(f\beta^2 y + \gamma^2 - d\beta - 3\beta\gamma - c\beta\gamma - f\beta^2\gamma - 1)/(x-1)],\\
&\alpha_1 = -((c+2)^2\beta^2\gamma(1+2\gamma^2) + \beta(c+2)(2\beta^2\gamma^2 - 2\gamma^4 + (1+d\beta)(1+3\gamma^2)\\
&\qquad +f\beta(\beta^2 + 2\gamma^2(1+\beta^2+\gamma^2))) + \beta^2\gamma d^2 + \beta\gamma d(3 + 2f\beta + \beta^2 + f\beta^3 - \gamma^2\\
&\qquad +2f\beta\gamma^2) - 2\gamma(-(1+f\beta)(1+f\beta^3+f\beta^3\gamma^2) - 2\beta^2 + \gamma^2 + f\beta\gamma^4))/(\beta^2\gamma),\\
&\alpha_2 = \beta((c+2)(f\beta + 2\gamma^2 + 2f\beta\gamma^2) + \gamma(1+f\beta)(d + 2f + 2f\gamma^2))/\gamma,\\
&\alpha_3 = -\beta(c+2)(1+2\gamma^2) + 2\gamma + d\beta\gamma + 2f\beta\gamma(1+\gamma^2)/(\beta^2\gamma).
\end{aligned}
$$

□

Lemma 3.3. *The following three sets of conditions are sufficient for the origin* $(0,0)$ *to be a center for the system* (3)

$$
\begin{aligned}
&a = 0, b = 1, d = -1/\beta, f = 0, g = -1 - q\beta, k = 0,\\
&l = 0, m = -1 - c, n = 1 + c, p = 0, r = 0, s = q\beta;
\end{aligned}
\tag{23}
$$

$$
\begin{aligned}
&a = 0, b = 1, c = -2, d = -1/\beta, g = -2, k = 0, l = f,\\
&m = 1, n = -1, p = -f, q = 1/\beta, r = 0, s = 1;
\end{aligned}
\tag{24}
$$

$$
\begin{aligned}
&a = \gamma^2(1+\beta\gamma-\gamma^2)/(2\beta(1-\gamma^2)),\\
&b = (2\beta+\beta\gamma^2(\gamma^2-4)-\gamma(\gamma^2-1)(\gamma^2-2))/(2\beta(1-\gamma^2)),\\
&c = (\beta\gamma^2+2\gamma^3+\beta\gamma^4-\gamma^5-4\beta-\gamma)/(2\beta(1-\gamma^2)),\\
&d = (\gamma^4(\gamma^2-\beta\gamma-4)+(1+\beta\gamma)(5\gamma^2-2))/(2\beta(1-\gamma^2)),\\
&f = \gamma(\beta\gamma^2-\gamma^3-2\beta+\gamma)/(2\beta(\gamma^2-1)),\\
&g = (\gamma^2-2)(\beta\gamma^2-\gamma^3-2\beta+\gamma)/(2\beta(\gamma^2-1)),\\
&k = \gamma^2(1+\beta\gamma-\gamma^2)/(2\beta(\gamma^2-1)),\\
&l = \gamma(\beta\gamma^2-\gamma^3-2\beta+\gamma)/(2\beta(\gamma^2-1)),\\
&m = (2\beta+\gamma(\gamma^2-1)(\gamma^2-\beta\gamma-1))/(2\beta(1-\gamma^2)),\\
&n = (1+\gamma^2)((\beta-\gamma)(\gamma^2-1)-\beta)/(2\beta(1-\gamma^2)),\\
&p = \gamma((\beta-\gamma)(\gamma^2-1)-\beta)/(2\beta(1-\gamma^2)),\\
&q = (1+\beta\gamma-\gamma^2)(\gamma^2-2)/(2\beta(\gamma^2-1)), r=0, s=(\beta-\gamma)/\beta.
\end{aligned}
\tag{25}
$$

Proof. If one of the conditions (23) - (25) holds, then the cubic system (3) has a first integral $F(x,y)$ of the form

$$\mathcal{L}_1^{\alpha_1}\mathcal{L}_2^{\alpha_2}\mathcal{L}_3^{\alpha_3}\mathcal{L}_4^{\alpha_4}. \tag{26}$$

In *Case* (23):

$$
\begin{aligned}
&\mathcal{L}_1 = x-1,\ \mathcal{L}_2 = \beta x+y-\beta,\ \mathcal{L}_3 = \exp[(\beta(c+2)y+q\beta-1)/(x-1)],\\
&\mathcal{L}_4 = (c+1)x+1,\ \alpha_1 = (1+c)(1-3q\beta-cq\beta+(2+c)^2\beta^2),\\
&\alpha_2 = -(1+c)(2+c)^2\beta^2,\ \alpha_3 = (1+c)(2+c),\ \alpha_4 = -1-c-q\beta;
\end{aligned}
$$

in *Case* (24):

$$
\begin{aligned}
&\mathcal{L}_1 = x-1,\ \mathcal{L}_2 = \beta x+y-\beta,\ \mathcal{L}_3 = \exp[y/(x-1)],\\
&\mathcal{L}_4 = \exp[(x^2+f\beta y^2-1)/(x-1)^2],\ \alpha_1 = 2(1+\beta^2+f\beta^3),\\
&\alpha_2 = -2\beta^2(1+f\beta),\ \alpha_3 = 2\beta(1+f\beta),\ \alpha_4 = -1;
\end{aligned}
$$

in *Case* (25):

$$
\begin{aligned}
&\mathcal{L}_1 = x-1,\ \mathcal{L}_2 = (\beta-\gamma)x+y-\beta,\\
&\mathcal{L}_4 = ((\beta-\gamma)(\gamma^2-1)-\beta)(x+\gamma y)+2\beta,\\
&\mathcal{L}_3 = \exp[\beta\gamma(\gamma-y)(\gamma+\beta\gamma^2-\gamma^3-2\beta)/(2(x-1)(1-\gamma^2))],\\
&\alpha_1 = \beta(2+\beta\gamma)(1-\beta\gamma+\gamma^2)((\beta-\gamma)(\gamma^2-1)-\beta),\\
&\alpha_2 = \beta^2\gamma(1+\beta\gamma-\gamma^2)((\beta-\gamma)(\gamma^2-1)-\beta),\\
&\alpha_3 = 2(\gamma^2-1)(1-\beta\gamma+\gamma^2),\ \alpha_4 = 4\beta^2.
\end{aligned}
$$

□

Lemma 3.4. *The following sets of conditions are sufficient for the origin* $(0,0)$ *to be a center for the system* (3)

$$\begin{aligned}&a=0, b=1, c=(3\gamma^2-2)/(1-\gamma^2), d=\gamma^3/(\gamma^2-1), f=\gamma/(1-\gamma^2),\\&g=(\gamma^2-2)/(1-\gamma^2), k=0, l=\gamma/(1-\gamma^2), m=(2\gamma^2-1)/(\gamma^2-1),\\&n=-1, p=\gamma/(\gamma^2-1), q=0, r=0, s=1/(1-\gamma^2);\end{aligned}\tag{27}$$

$$\begin{aligned}&a=(4\gamma^3)/((\gamma^2-3)(9+\gamma^2)), b=((3+\gamma^2)(5\gamma^2-9))/((\gamma^2-3)(9+\gamma^2)),\\&c=-18/(9+\gamma^2), d=(\gamma(9+16\gamma^2-\gamma^4))/((\gamma^2-3)(9+\gamma^2)),\\&f=(6\gamma(3+\gamma^2))/((3-\gamma^2)(9+\gamma^2)), l=(6\gamma(3+\gamma^2))/((3-\gamma^2)(9+\gamma^2)),\\&k=(4\gamma^3)/((3-\gamma^2)(9+\gamma^2)), g=(2(3+\gamma^2)(2\gamma^2-9))/((3-\gamma^2)(9+\gamma^2)),\\&m=(9-\gamma^2)/(9+\gamma^2), n=(3((3+\gamma^2)^2-4\gamma^2))/((\gamma^2-3)(9+\gamma^2)),\\&p=(6\gamma(3+\gamma^2))/((\gamma^2-3)(9+\gamma^2)), q=(\gamma(3+\gamma^2)^2)/((3-\gamma^2)(9+\gamma^2)),\\&r=0, s=(3+\gamma^2)/(3-\gamma^2);\end{aligned}\tag{28}$$

$$\begin{aligned}&a=\delta(1+\beta\delta)/\nu,\ b=(\beta^2\delta^2-\delta^2-\beta\delta-1)/(\beta\delta\nu),\ c=-(1+3\beta\delta+3\delta^2)/\nu,\\&d=((1+\beta\delta)(1+2\beta\delta+\beta\delta^3)-\delta^4)/(\beta\delta^2\nu),\ f=-\delta/\nu,\ k=-(\delta(1+\beta\delta))/\nu,\\&g=(\nu-\beta\delta-2\beta^2\delta^2)/(\beta\delta\nu),\ l=-\delta/\nu,\ m=(\delta(\beta+2\delta))/\nu,\ n=(1-\delta^2)/\nu,\\&p=\delta/\nu,\ q=-(1+\beta\delta)/(\delta\nu),\ r=0,\ s=(\beta\delta)/\nu,\ \textit{where}\ \nu=1+2\beta\delta+\delta^2;\end{aligned}\tag{29}$$

$$\begin{aligned}&a=(\delta-\beta)/\sigma,\ b=\delta(\beta^2+\beta\delta-\delta^2-1)/(\beta\sigma),\ c=(3-3\beta\delta+\delta^2)/\sigma,\\&d=(1-(\beta-\delta)(\beta+2\beta\delta^2-\delta^3))/(\beta\sigma),\ f=-\delta/\sigma,\ k=(\beta-\delta)/\sigma,\\&g=-\delta(2\beta^2+\beta\delta-\delta^2-1)/(\beta\sigma),\ l=-\delta/\sigma,\ m=(\beta\delta-2)/\sigma,\\&n=(1-\delta^2)/\sigma,\ p=\delta/\sigma,\ q=(\beta-\delta)\delta^2/\sigma,\ r=0,\ s=\beta\delta/\sigma,\\&\textit{where}\ \ \sigma=2\beta\delta-\delta^2-1;\end{aligned}\tag{30}$$

$$\begin{aligned}&b=2(1+a^2),\ c=2a^2-3,\ d=(1+a^2)/a,\ f=-2a,\ g=-2,\ k=-a,\\&l=-2a,\ m=2(1-a^2),\ n=4a^2-1,\ p=2a,\ q=-4a^3,\ r=0,\ s=-2a^2;\end{aligned}\tag{31}$$

$$\begin{aligned}&a=(u^2-4f^2-1)/(8f),\\&b=(32f^2+16f^4\mp32f^2u+2u^2-u^4-1)/(4(1\mp u)(-1+4f^2+u^2)),\\&c=(4f^2\pm12u-3u^2-9)/(4(1\mp u)),\ d=((4f^2-1)^2+u^2(u^2-56f^2-2)\\&\quad\mp u((u^2-1)^2-8f^2(7+6f^2+u^2)))/(8f(1\mp u)(4f^2+u^2-1)),\\&g=(1-20f^2-u^2\mp u(4f^2+u^2-1))/(2(4f^2+u^2-1)),\\&k=(1+4f^2-u^2)/(8f),\ l=f,\ m=(3u^2\mp8u-4f^2+5)/(4(1\mp u)),\\&n=(4f^2-u^2-1\pm2u)/(2(1\mp u)),\ s=(4f^2+u^2-1)/(4(-1\pm u)),\\&p=-f,\ q=f(1+4f^2-u^2)/(2(u\mp1)^2),\ r=0.\end{aligned}\tag{32}$$

Proof. In each of the sets of conditions (27)–(32), the system (3) has an integrating factor of the form (26) $\mu(x,y)=\mathcal{L}_1^{\alpha_1}\mathcal{L}_2^{\alpha_2}\mathcal{L}_3^{\alpha_3}\mathcal{L}_4^{\alpha_4}$.

In *Case* (27):

$$\begin{aligned}&\mathcal{L}_1 = x-1;\ \mathcal{L}_2 = x-\gamma y+\gamma^2-1,\ \mathcal{L}_3 = \exp[(1-\gamma^2)(y-\gamma)/(\gamma(x-1))],\\&\mathcal{L}_4 = x-\gamma y-1,\ \alpha_1=-3,\ \alpha_2=1,\\&\alpha_3 = -\gamma^2/(\gamma^2-1)^2,\ \alpha_4=(\gamma^2-2)/(1-\gamma^2);\end{aligned}$$

in *Case* (28):

$$\begin{aligned}&\mathcal{L}_1 = x-1,\ \mathcal{L}_2=(\gamma^2+3)x-2y\gamma+\gamma^2-3,\\&\mathcal{L}_3=\exp[3(\gamma-y)(\gamma^4-9)/(2\gamma(x-1)(9+\gamma^2))],\\&\mathcal{L}_4 = 3(3+\gamma^2)x-6y\gamma-\gamma^2-9,\ \alpha_1=-3,\ \alpha_2=1,\\&\alpha_3=-2\gamma^2(9+\gamma^2)/(3(\gamma^2-3)^2),\ \alpha_4=(18+3\gamma^2+\gamma^4)/(3(\gamma^2-3));\end{aligned}$$

in *Case* (29):

$$\begin{aligned}&\mathcal{L}_1 = x-1,\ \mathcal{L}_2=x-y\delta+\beta\delta,\ \mathcal{L}_3=\exp[1+\beta\delta+\delta^2+y\beta\delta^2/(x-1)],\\&\mathcal{L}_4=\beta\delta(x-y\delta)-\delta^2-2\beta\delta-1,\ \alpha_1=-3,\ \alpha_4=1,\\&\alpha_2=(1+2\beta\delta+\beta^2\delta^2-2\beta\delta^3-\delta^4)/(\delta^2\nu),\ \alpha_3=(1+\beta\delta+\delta^2)/(\beta\delta^3\nu);\end{aligned}$$

in *Case* (30):

$$\begin{aligned}&\mathcal{L}_1 = x-1,\ \mathcal{L}_2=y-\beta+x\delta,\ \mathcal{L}_3=\exp[(1+y\beta-\beta\delta+\delta^2)/(x-1)],\\&\mathcal{L}_4=\beta(\delta x+y)-\sigma,\ \ \alpha_1=-3,\ \alpha_2=-(-1+2\beta\delta+\beta^2\delta^2-2\beta]\delta^3+\delta^4)/\sigma,\\&\alpha_3=-(\delta(-1+\beta\delta-\delta^2))/(\beta\sigma),\ \alpha_4=1;\end{aligned}$$

in *Case* (31):

$$\begin{aligned}&\mathcal{L}_1 = x-1,\ \mathcal{L}_2=2a^2x-ay+1,\ \mathcal{L}_3=\exp[2(1+2a^2-ay)/(a^2(x-1))],\\&\mathcal{L}_4=ax-y-a,\ \alpha_1=-3,\ \alpha_2=1,\ \alpha_3=-a^2(1+2a^2),\ \alpha_4=4a^4+4a^2-1;\end{aligned}$$

in *Case* (32):

$$\begin{aligned}&\mathcal{L}_1 = x-1,\ \mathcal{L}_2=(4f^2+u^2-1)(2fx+(1\mp u)y)+8f(1\mp u),\\&\mathcal{L}_3=\exp[\tfrac{(64f^3(2f(3+4f^2+u^2\mp 4u)+(1\mp u)(4f^2+u^2-1)y))}{((1\mp u)(1+4f^2+u^2-1)^3(x-1))}],\\&\mathcal{L}_4=8f^2x+4f(1\mp u)y+1-4f^2-u^2,\ \alpha_1=-3,\ \alpha_2=1,\\&\alpha_3=-(((-1+4f^2+u^2)^2(3+4f^2\mp 4u+u^2))/(128f^2(u\mp 1)^2)),\\&\alpha_4=((4f^2-1)(7+4f^2)\pm 8u(3-4f^2+u^2)+u^2(8f^2+u^2-26))/(8(1\mp u)^3).\end{aligned}$$

□

Lemma 3.5. *The following three sets of conditions are sufficient for the origin* $(0,0)$ *to be a center for the system* (3)

$$\begin{aligned}&a=0, b=1, c=(3\gamma^2-2)/(1-\gamma^2), d=\gamma^3/(\gamma^2-1), f=\gamma/(1-\gamma^2),\\&g=(\gamma^2-2)/(1-\gamma^2), k=0, l=\gamma/(1-\gamma^2), m=(2\gamma^2-1)/(\gamma^2-1),\\&n=-1, p=\gamma/(\gamma^2-1), q=0, r=0, s=1/(1-\gamma^2);\end{aligned}\tag{33}$$

$$
\begin{aligned}
&a = -k = \gamma\delta(4 + 2c + 3\gamma^2 + 2c\gamma^2),\ b = -\delta(10 + 4c + 7\gamma^2 + 4c\gamma^2),\\
&d = \delta((1 + \gamma^2)(9\gamma^2 + 6c\gamma^2 - 4c) - 8)/\gamma,\ f = l = -p = \gamma\delta,\\
&g = \delta(16 + 6c + 11\gamma^2 + 6c\gamma^2),\ m = -(c + 1),\ n = 2\delta,\\
&q = \delta(24 + 23\gamma^2 + 6\gamma^4 + 2c(1 + \gamma^2)(10 + 2c + 5\gamma^2 + 2c\gamma^2))/\gamma,\ r = 0,\\
&s = -\delta(5 + 2c + 3\gamma^2 + 2c\gamma^2)(6 + 2c + 3\gamma^2 + 2c\gamma^2),\ \delta = -1/(2(1 + \gamma^2));
\end{aligned} \tag{34}
$$

$$
\begin{aligned}
&a = -1/(2\gamma),\ b = 1/2,\ c = -3/2,\ d = -(1 + 2\gamma^2)/(2\gamma),\\
&f = l = n = p = r = 0,\ g = -1,\ k = 1/(2\gamma),\ m = 1/2,\ q = \gamma/2,\ s = 1/2.
\end{aligned} \tag{35}
$$

Proof. In this Lemma, the existence of the center is guaranteed by the presence of the integrating factor of the form

$$
\mu(x, y) = \mathcal{L}_1^{\alpha_1}\mathcal{L}_2^{\alpha_2}\Phi^{\alpha_3},
$$

where $\mathcal{L}_1$, $\mathcal{L}_2$ are invariant straight lines and Φ is an invariant conic. Indeed,
in Case (33):

$$
\begin{aligned}
&\mathcal{L}_1 = x - 1,\ \mathcal{L}_2 = y - \beta,\ \Phi = 4(1 + \beta^2)(x - 1) - (x + y\beta)^2,\\
&\alpha_1 = -2,\ \alpha_2 = -1,\ \alpha_3 = -1/2;
\end{aligned}
$$

in Case (34):

$$
\begin{aligned}
&\mathcal{L}_1 = x - 1,\ \mathcal{L}_2 = \gamma(1 + \gamma^2) + 2y(2 + c + \gamma^2 + c\gamma^2) - x\gamma(5 + 2c + 3\gamma^2 + 2c\gamma]2),\\
&\Phi = 4(1 + \gamma^2) - 8x(3 + c + 2\gamma^2 + c\gamma^2) - 4y\gamma(5 + 2c + 3\gamma^2 + 2c\gamma^2)\\
&\qquad +(6x + 2cx + y\gamma + 3x\gamma^2 + 2cx\gamma^2)^2,\ \alpha_1 = -2,\ \alpha_2 = -1,\ \alpha_3 = -1/2;
\end{aligned}
$$

in Case (35):

$$
\begin{aligned}
&\mathcal{L}_1 = x - 1,\ \mathcal{L}_2 = \gamma x + y - 2\gamma,\ \Phi = 2(1 + \gamma^2) - (x + \gamma y)(2(1 + \gamma^2) - \gamma(\gamma x + y)),\\
&\alpha_1 = -2,\ \alpha_2 = 1,\ \alpha_3 = -1.
\end{aligned}
$$

□

Lemma 3.6. *The following set of conditions is sufficient for the origin* $(0,0)$ *to be a center for the system* (3)

$$\begin{aligned}
&a = -(f(1-6u^2+u^4))/(2(1-u^2)^2),\\
&b = (f(1+u^2)^2(1-6u^2+u^4)+32u^3(1-u^2))/(4(1-u^2)u(1+u^2)^2),\\
&c = (f(1-14u^2+u^4)+8u(u^2-1))/(4(1-u^2)u),\\
&d = (3-u^2)(3u^2-1)(f(1+u^2)^2+4u(u^2-1))/(2(u^2-1)^2(1+u^2)^2),\\
&g = (2fu(1+u^2)^2+(u^2-1)(1+10u^2+u^4))/((1-u^2)(1+u^2)^2),\\
&k = (f(1-6u^2+u^4))/(2(u^2-1)^2),\ l = f,\\
&m = (f(1-14u^2+u^4)+4u(u^2-1))/(4u(u^2-1)),\\
&n = (4u(f(1+u^2)^2-2u(u^2-1)))/(u^2-1)(1+u^2)^2,\ p = -f,\\
&q = (f(1+u^2)^2(1+10u^2+u^4)+4(u^2-1)u(u^2-3)(3u^2-1))\\
&\qquad /(2(u^2-1)^2(1+u^2)^2),\ r = 0,\\
&s = (u(f(1+u^2)^4+8u(u^2-1)^3))/((u^2-1)^3(1+u^2)^2);
\end{aligned} \tag{36}$$

Proof. Under the conditions (36), there is an invertible transformation of the form

$$x = \frac{a_1X+b_1Y}{a_3X+B_3Y-1}, \quad y = \frac{a_2X+b_2Y}{a_3X+B_3Y-1}, \tag{37}$$

in a neighborhood of $O(0,0)$, where $a_j, b_j \in \mathbb{R}$, $j = 1,2,3$ and $a_1b_2 - b_1a_2 \neq 0$. The transformation (37) brings the system (3) to the polynomial system

$$\dot{X} = Y + M(X^2, Y), \quad \dot{Y} = -X(1+N(X^2,Y)). \tag{38}$$

This system has an axis of symmetry $X = 0$ and therefore $O(0,0)$ is a center for (38) and for the initial system (3) (see, [2], pp.29-31).

In *Case* (36) the transformation (37) looks as

$$x = \frac{2uX+(1-u^2)Y}{2uX-u^2-1}, \quad y = \frac{(u^2-1)X+2uY}{2uX-u^2-1}$$

and the system (38) has the form

$$\begin{aligned}
\dot{X} = {}& Y + (4fu^2(1+u^2)^4X^4 - X^2(f(1+u^2)^6 + 16u^3(u^2-1)^2(1+u^2)Y-\\
& 16u^3(u^2-1)^3Y^2) - 4u(u^2-1)^3(1+u^2)^2Y^2)/(4u(u^2-1)^2(1+u^2)^3),\\
\dot{Y} = {}& X(1+(4X^2fu^2(1+u^2)^4(1+u^2+(1-u^2)Y)+(1-u^2)(1+u^2)^2(f-8u\\
& +4fu^2+8u^3+6fu^4-8u^5+4fu^6+8u^7+fu^8)Y+(u^2-1)^2(1+u^2)(f-4u\\
& +4fu^2-20u^3+6fu^4+20u^5+4fu^6+4u^7+fu^8)Y^2\\
& -16u^3(u^2-1)^4Y^3)/(4u(-1+u^4)^3).
\end{aligned}$$

□

4. Solution of the problem of the center

4.1. Centers in the conditions (7).

Lemma 4.1. *Under the conditions* (7) *the system* (3) *has a center at the origin* $(0,0)$ *if and only if the first four Lyapunov quantities vanish* $L_1 = L_2 = L_3 = L_4 = 0$.

Proof. The Lemma 4.1 is proved in {[2], pp. 111–116}. □

4.2. Centers in the conditions (12).

Let $\alpha = \gamma - \beta$.

Lemma 4.2. *When conditions* (12) *hold, the system* (3) *has a center at the origin* $(0,0)$ *if and only if the first three Lyapunov quantities vanish* $L_1 = L_2 = L_3 = 0$.

Proof. In conditions (12) the first Lyapunov quantity is $L_1 = f_0 f_1 f_2$, where $f_0 = \gamma$, $f_1 = c\beta + \beta + \gamma$, $f_2 = q\beta + (1+c)(\beta-\gamma)\gamma - c - 3$. If $f_0 = 0$, then Lemma {3.3, (23)} and if $f_1 = 0$, then Lemma {3.1, (15)}. Assume that $f_0 f_1 \neq 0$ and let $f_2 = 0$. Then $q = (c+3+\gamma(1+c)(\gamma-\beta))/\beta$ and $L_2 = (c+2)(2c\beta+5\beta+\gamma)$. If $c+2=0$, then $\gcd(P,Q) = x-1$. Let $2c\beta + 5\beta + \gamma = 0 \Rightarrow \gamma = -\beta(5+2c) \Rightarrow L_3 = c+3 = 0 \Rightarrow$ Lemma {3.2, (16)}. □

4.3. Centers in the conditions (13).

Lemma 4.3. *Under the conditions* {(13), $\alpha = -\beta$}, *the system* (3) *has a center at the origin* $(0,0)$ *if and only if the first four Lyapunov quantities vanish* $L_1 = L_2 = L_3 = L_4 = 0$.

Proof. The system {(3), (13), $\alpha = -\beta$ } has the invariant straight lines $\mathcal{L}_1 = x-1$, $\mathcal{L}_2 = \beta x + y - \beta$ and the exponential factor $\mathcal{L}_3 = \exp[y/(x-1)]$. For {(3), (13), $\alpha = -\beta$ } we calculate at $(0,0)$ the first four Lyapunov quantities L_1, L_2, L_3 and L_4. In the sequel, in expressions of Lyapunov quantities we always neglect the non-zero factors. The first one look as

$$L_1 = ac + (c+2)f.$$

If $c=0$, then {Lemma 3.2, (17)}. Let $c \neq 0$. Then $L_1 = 0 \Rightarrow a = -f(c+2)/c \Rightarrow L_2 = f(c+2)l_2$, where

$$l_2 = c^3\beta + 2c^2\beta + 6cf - 12f^2\beta + 4cf^2\beta.$$

If $f=0$, then {Lemma 3.3, (23), $q = 1/\beta$}, and if $c=-2$, then {Lemma 3.3, (24)}. Suppose $cf(c+2) \neq 0$. Reducing L_3 by l_2 with respect to the variable c we obtain

$L_3 = (c - 2f\beta)(12f\beta + cf\beta - c)$. If $c - 2f\beta = 0$, then $l_2 = f(1+\beta^2) + \beta = 0 \Rightarrow f = -\beta/(1+\beta^2) \Rightarrow \{$Lemma 3.2, (18)$\}$. If $12f\beta + c(f\beta - 1) = 0 \Rightarrow c = 12f\beta/(1 - f\beta) \Rightarrow$

$$\begin{aligned} &l_2 = 24\beta^2(5f\beta + 1) + 5(f\beta - 1)^2(f\beta + 1),\\ &L_4 = 15(f\beta + 1)(17f\beta - 11) + 50\beta^4(24 - 43f^2 + 120f\beta) - 2\beta^2(271 + 2290f\beta). \end{aligned}$$

The system $\{l_2 = 0, L_4 = 0\}$ has no real solutions with respect to the unknowns f and β. □

Lemma 4.4. *Let the conditions* $\{(13)\}$ *hold for the system* (3). *Then the origin* $(0,0)$ *is a center if and only if the first four Lyapunov quantities vanish* $L_1 = L_2 = L_3 = L_4 = L_5 = 0$.

Proof. Denote $\alpha + \beta = \gamma \neq 0$. Then under the conditions $\{(13), \alpha = \gamma - \beta\}$ we calculate the first five Lyapunov quantities. The first one look as $L_1 = A_c \cdot c + B_c$, where

$$\begin{aligned} &A_c = \beta((a+f)(1-\gamma^2) - \gamma),\\ &B_c = 2\beta(f - \gamma) - (a+f)\gamma(-1 + 2a\beta - 2f\beta + 3\beta\gamma - \gamma^2). \end{aligned}$$

Remark that in conditions $\{(13), \alpha = \gamma - \beta\}$ the system (3) has the invariant straight lines $\mathcal{L}_\infty = x - 1$, $\mathcal{L}_2 = (\beta - \gamma)x + y - \beta]$ and the exponential factor $\mathcal{L}_3 = \exp[(fy\beta^2 + \gamma^2 - \beta\gamma(3 + c + f\beta) - d\beta - 1)/(x-1)]$.

Let $A_c = 0$. Then $\gamma^2 - 1 \neq 0$ and the system $\{A_c = 0,\ B_c = 0\}$ gives us

$$a = \gamma^2(1 + \beta\gamma - \gamma^2)/(2\beta(1-\gamma^2)),\ \ f = \gamma(\gamma - 2\beta + \beta\gamma^2 - \gamma^3)/(2\beta(\gamma^2 - 1)).$$

Substituting the expression of a and f into L_2, L_3, L_4 and L_5 we obtain that $L_2 = \gamma f_0 f_1 f_2$, $L_3 = \gamma f_0 f_1 f_3$, $L_4 = \gamma f_0 f_1 f_4$, $L_5 = \gamma f_0 f_1 f_5$, where

$$\begin{aligned} &f_0 = c\beta(\gamma^2 - 1) + \gamma(\gamma^2 + \beta\gamma - 1),\\ &f_1 = 4\beta - 2c\beta(\gamma^2 - 1) + \gamma(\gamma^2 - 1)^2 - \beta\gamma^2(\gamma^2 + 1),\\ &f_2 = (3\gamma - c\beta)(\gamma^2 - 1) - 2\beta(3\gamma^2 - 1) \end{aligned}$$

and f_3, f_4, f_5 are polynomials in variables c, β, γ. If $f_0 = 0$, then $\{$Lemma 3.2, (19)$\}$, and if $f_1 = 0$, then $\{$Lemma 3.3, (25)$\}$.

Assume that $(\gamma^2 - 1)f_0 f_1 \neq 0$ and let $f_2 = 0$. From $f_2 = 0$ we calculate $c = (6\beta\gamma^2 - 3\gamma^3 - 2\beta + 3\gamma)/(\beta(1-\gamma^2))$ and substitute it in L_3, L_4, L_5: $L_3 = \varphi_0\varphi_3, L_4 = \varphi_0\varphi_4, L_5 = \varphi_0\varphi_5$, where

$$\varphi_0 = 1 + \beta\gamma - \gamma^2,\ \varphi_3 = \gamma(14 + 17\gamma^2 - \gamma^4)\beta + (1 - \gamma^2)(2 + 9\gamma^2 - \gamma^4),$$

and φ_4, φ_5 are polynomials in β, γ. If $\varphi_0 = 0$, then {Lemma 3.4, (27)}, and if $\varphi_3 = 0$, then $\beta = (\gamma^2 - 1)(2 + 9\gamma^2 - \gamma^4)/(\gamma(14 + 17\gamma^2 - \gamma^4)) \Rightarrow$

$$\begin{aligned}
\varphi_4 &= 212 - 976\gamma^2 + 149\gamma^4 + 39\gamma^6 - 21\gamma^8 - 3\gamma^{10},\\
\varphi_5 &= 341744 + 8972240\gamma^2 + 20117464\gamma^4 - 330716296\gamma^6 + 161032455\gamma^8\\
&\quad -51877847\gamma^{10} - 26524163\gamma^{12} + 4576265\gamma^{14} + 470531\gamma^{16} - 577165\gamma^{18}\\
&\quad -126177\gamma^{20} - 8877\gamma^{22} - 174\gamma^{24}.
\end{aligned}$$

The polynomials φ_4 and φ_5 have no common solutions.

Let now $\beta\gamma A_c \neq 0$ and express c from $L_1 = 0$. Substituting

$$c = (2\beta(f - \gamma) - \gamma(a + f)(2a\beta - 2f\beta + 3\beta\gamma - \gamma^2 - 1))/(\beta(\gamma + (a + f)(\gamma^2 - 1)))$$

in L_2, L_3, L_4 and L_5 we obtain that $L_2 = \psi_0\psi_2$, $L_3 = \psi_0\psi_3$, $L_4 = \psi_0\psi_4$, $L_5 = \psi_0\psi_5$, where

$$\begin{aligned}
\psi_0 &= ((\beta - \gamma)\gamma^3 - 2\beta(f - \gamma - a\gamma^2) - \gamma^2)(a\beta\gamma + (a + f)(a\beta - \gamma^2 - f\beta\gamma^2)),\\
\psi_2 &= A_\beta \cdot \beta + B_\beta,\\
A_\beta &= 2f + 2(a + f)(2f\gamma - 1) + 2(a + f)^2(f - 4\gamma + f\gamma^2) + (a + f)^3(3 - 5\gamma^2),\\
B_\beta &= (a + f)((a + f)(3 - \gamma^2) - 2\gamma)
\end{aligned}$$

and ψ_3, ψ_4, ψ_5 are polynomials in a, f, β, γ. If $\psi_0 = 0$, then {Lemma 3.2, (20)}. Let $\psi_0 \neq 0$. If $a + f = 0$, then $\psi_2 = 0 \Rightarrow a = f = 0 \Rightarrow \gcd(P, Q) = x - 1 \neq 1$. The system $\{A_\beta = 0,\ B_\beta = 0,\ \gamma(a + f) \neq 0\} \Rightarrow$ gives us

$$a = 4\gamma^3/((\gamma^2 - 3)(9 + \gamma^2)),\ f = 6\gamma(3 + \gamma^2)/((3 - \gamma^2)(9 + \gamma^2))$$

$\Rightarrow \psi_3 = \eta_0\eta_3$, $\psi_4 = \eta_0\eta_4$, $\psi_5 = \eta_0\eta_5$, where

$$\eta_0 = 3 + 2\beta\gamma - \gamma^2,\ \eta_3 = 16\gamma(9 + 9\gamma^2 + \gamma^4)\beta - (\gamma^2 + 9)(\gamma^2 - 3)(1 + 3\gamma^2).$$

If $\eta_0 = 0$, then {Lemma 3.4,(28)}. If $\eta_3 = 0$, then $\beta = (\gamma^2-3)(9+\gamma^2)(1+3\gamma^2)/(16\gamma(9+9\gamma^2 + \gamma^4))$ and

$$\begin{aligned}
\eta_4 &= 1359 - 3582\gamma^2 - 1524\gamma^4 - 178\gamma^6 - 11\gamma^8,\\
\eta_5 &= 399256533 + 7147083924\gamma^2 + 16160765949\gamma^4 - 88245537822\gamma^6\\
&\quad -98340968934\gamma^8 - 42412220400\gamma^{10} - 10500825742\gamma^{12} - 1982042948\gamma^{14}\\
&\quad -323807311\gamma^{16} - 38802452\gamma^{18} - 2858767\gamma^{20} - 107822\gamma^{22}.
\end{aligned}$$

The polynomials η_4 and η_5 have not common roots.

Suppose now that $\beta\gamma(f + a)A_cA_\beta \neq 0$. From $\psi_2 = 0$ we express β : $\beta = -B_\beta/A_\beta$ and substitute it in ψ_3, ψ_4, ψ_5. We obtain: $\psi_3 = \delta_0\delta_3$, $\psi_4 = \delta_0\delta_4$, $\psi_5 = \delta_0\delta_5$, where

$$\delta_0 = a - f(a + f)^2 + (a + f)(3a + f)\gamma + (a + f)^2(2a + f)\gamma^2.$$

First we will examine the equality $\delta_0 = 0$. Each of the following three sets 1) $\{f = 0, a = -(1/\gamma)\}$, 2) $\{f = 0, a = -1/(2\gamma)\}$, 3) $\{f = -2a \neq 0, \gamma = (1+2a^2)/a\}$ vanish δ_0. In the case 1) (respectively, 2), 3)) we have Lemma $\{3.2, (16), \beta = -1/\gamma\}$ (respectively, Lemma 3.5, (35), Lemma 3.4, (31)). Suppose $f(f+2a) \neq 0$ and $a = (u^2-4f^2-1)/(8f)$. Then $\delta_0 = 0 \Rightarrow \gamma = 2f(3 + 4f^2 \mp 4u + u^2)/((-1 \pm u)(4f^2 + u^2 - 1)) \Rightarrow$ Lemma 3.4, (32).

Let $\delta_0 \neq 0$ and $\delta_3 = \delta_4 = \delta_5 = 0$. Suppose that $\mathcal{R}_0 = 0$, where $\mathcal{R}_0 = (\gamma^2 - 3)(9f + 6\gamma + f\gamma^2) + 36\gamma$. Taking into account that $\gamma \neq 0$, the equality $\mathcal{R}_0 = 0$ gives us $f = 6\gamma(3+\gamma^2)/((3-\gamma^2)(9+\gamma^2))$. Substituting the expression of f in $\delta_3, \delta_4, \delta_5$, we obtain: $\delta_3 = \Delta_0\Delta_3$, $\delta_4 = \Delta_0\Delta_4$, $\delta_5 = \Delta_0\Delta_5$, where

$$\begin{aligned}
&\Delta_0 = (3-\gamma^2)(9+\gamma^2)a + 4\gamma^3,\\
&\Delta_3 = 5a^2(\gamma^2-3)^2(1+\gamma^2)(3+\gamma^2)(9+\gamma^2)^2 -\\
&8a\gamma(\gamma^2-3)(9+\gamma^2)(117+129\gamma^2+49\gamma^4+5\gamma^6) +\\
&6(3+\gamma^2)(81+702\gamma^2+540\gamma^4+186\gamma^6+11\gamma^8)
\end{aligned}$$

and Δ_4, Δ_5 are polynomial in a, γ (here and in the future we neglect the nonzero factors). If $\Delta_0 = 0$, then $\delta_0 = 0$. We calculate the following two resultants with respect to the variable a:

$$\begin{aligned}
Ra43 \equiv Resultant[\Delta_4, \Delta_3, a] = {}& -5314410 - 10333575\gamma^2 - 10482291\gamma^4\\
& -7030476\gamma^6 - 3361176\gamma^8 - 757026\gamma^{10} + 22734\gamma^{12} + 52740\gamma^{14} + 11850\gamma^{16}\\
& -5775\gamma^{18} + 605\gamma^{20},\\
Ra53 \equiv Resultant[\Delta_5, \Delta_3, a] = {}& -204368836893016410 - 7812485204504756674 5\gamma^2\\
& -16375606609634273122 2\gamma^4 - 96691944562452884637\gamma^6\\
& -65295919868671875114\gamma^8 - 231514833275418305043\gamma^{10}\\
& -349818971483621819394\gamma^{12} - 225165183477638890419\gamma^{14}\\
& +80965486019817254 16\gamma^{16} + 156133967087099004714\gamma^{18}\\
& +157409522222069149956\gamma^{20} + 75314811652151245182\gamma^{22}\\
& +41695659944998900 92\gamma^{24} - 2015557868654141981 4\gamma^{26}\\
& -15882244595493905700\gamma^{28} - 6818058573715824678\gamma^{30}\\
& -1850109425790483978\gamma^{32} - 301146961631290581\gamma^{34}\\
& -12297992828350350\gamma^{36} + 8047908867120815\gamma^{38} + 2104984082966110\gamma^{40}\\
& +224322029461865\gamma^{42} - 66037811138170\gamma^{44} - 14404146643895\gamma^{46}\\
& +4307153529500\gamma^{48} + 103033012500\gamma^{50}
\end{aligned}$$

and the resultant with respect to the variable γ : $Resultant[Ra53, Ra43, \gamma] \neq 0$. Therefore, the system $\{\Delta_3 = 0, \Delta_4 = 0, \Delta_5 = 0\}$ is incompatible in rapport with the variables a and γ.

In what follows, in this subsection we will consider $\delta_0\mathcal{R}_0 \neq 0$ and using the resultants we solve the system of polynomials equations in a, f, γ : $\delta_3 = 0, \delta_4 = 0, \delta_5 = 0$. The system $f = 0, \delta_3 = 0, \delta_4 = 0, \delta_5 = 0, \delta_0 \neq 0$ has not real solutions. Eliminating nonzero factors such as $f, \gamma, 1+\gamma^2, 3+\gamma^2, 1+3\gamma^2, 2+\gamma^2, 403+1741\gamma^2+522\gamma^4, 72361+91494574\gamma^2+2288596483\gamma^4+5776070816\gamma^6+5787798803\gamma^8+2797313270\gamma^{10}+652276861\gamma^{12}+63507100\gamma^{14}+2102500\gamma^{16}, 61009+540628\gamma^2+1380646\gamma^4+606612\gamma^6+165105\gamma^8$, we calculate the resultants:

$$\begin{aligned}
&\mathcal{R}a43 = Resultant[\delta_4, \delta_3, a],\ \mathcal{R}a53 = Resultant[\delta_5, \delta_3, a],\\
&\mathcal{R}a54 = Resultant[\delta_5, \delta_4, a], \mathcal{R}af_1 = Resultant[\mathcal{R}a53, \mathcal{R}a43, f],\\
&\mathcal{R}af_2 = Resultant[\mathcal{R}a54, \mathcal{R}a43, f].
\end{aligned}$$

The polynomials in γ : $\mathcal{R}af_1, \mathcal{R}af_2$ have only the following common real solutions $\gamma = \pm 1,\ \gamma = \pm\sqrt{5}$. If $\gamma = \pm 1$, then

$$\mathcal{R}a43 = 0,\ \mathcal{R}a53 = 0, \mathcal{R}a54 = 0 \tag{39}$$

$\Rightarrow b = 0$, and if $\gamma = \pm\sqrt{5}$, then (39) has not real solutions. □

4.4. Centers in conditions (14).

Lemma 4.5. *In the conditions* (14) *the system* (3) *has a center at the origin* $(0, 0)$ *if and only if the first three Lyapunov quantities vanish* $L_1 = L_2 = L_3 = 0$.

Proof. Let $\alpha = \gamma - \beta$. For the system $\{(3),\ (14)\}$ we calculate at $(0, 0)$ the first three Lyapunov quantities L_1, L_2 and L_3. The Lyapunov quantity L_1 looks as $L_1 = f_0 f_1$, where

$$\begin{aligned}
&f_0 = 2(c+2)^2\beta\gamma + (c+2)(1 + d\beta - f\beta + 4f\beta\gamma^2) + f\beta\gamma(d - 2f + 2f\gamma^2),\\
&f_1 = 1 + d\beta + f\beta + 3\beta\gamma + c\beta\gamma - \gamma^2 + f\beta\gamma^2.
\end{aligned}$$

If $f_0 = 0$, then Lemma $\{3.2, (21)\}$. Let $f_1 = 0$. Then

$$d = -(1 + f\beta + 3\beta\gamma + c\beta\gamma - \gamma^2 + f\beta\gamma^2)/\beta \Rightarrow L_2 = A_\beta\beta + B_\beta,$$

where

$$A_\beta = 10 + 9c + 2c^2 - 2f^2 + 17f\gamma + 8cf\gamma + 6f^2\gamma^2,\ B_\beta = \beta + 2\gamma + c\gamma + f\gamma^2 - f.$$

If $A_\beta = 0,\ B_\beta = 0$, then $c = -(5 + 3\gamma^2)/(2(1 + \gamma^2)),\ f = -\gamma/(2(1 + \gamma^2)) \Rightarrow L_3 = \beta - \gamma = 0 \Rightarrow$ Lemma $\{3.5, (33)\}$.

Let now $A_\beta \neq 0$. Then $L_2 = 0 \Rightarrow \beta = -B_\beta/A_\beta \Rightarrow L_3 = \varphi_1\varphi_2\varphi_3$, where

$$\begin{aligned}
&\varphi_1 = 2f + \gamma + 2f\gamma^2,\ \varphi_2 = 4 + 4c + c^2 - f^2 + 8f\gamma + 4cf\gamma + 3f^2\gamma^2,\\
&\varphi_3 = 6 + 5c + c^2 - f^2 + 9f\gamma + 4cf\gamma + 3f^2\gamma^2.
\end{aligned}$$

If $\varphi_1 = 0$, then Lemma $\{3.5, (34)\}$. Denote $\gamma(u) = (1 - 6u^2 + u^4)/(4u(u^2 - 1))$. Then $\varphi_2 = \varphi_{21}\varphi_{22}/(16u^2(u^2-1)^2)$, where $\varphi_{21} = f - 8u - 4cu - 14fu^2 + 8u^3 + 4cu^3 + fu^4$, $\varphi_{22} = 3f - 8u - 4cu - 10fu^2 + 8u^3 + 4cu^3 + 3fu^4$. If $\varphi_{21} = 0$, then Lemma $\{3.6, (36)\}$. The case $\{\gamma(u), \varphi_{22} = 0\}$ is reduced by transformation $u = (v - 1)/(1 + v)$ to the case $\{\gamma(v), \varphi_{21}|_{u=v} = 0\}$.

Let now $\varphi_3 = 0$ and put $\gamma = (f - \delta - f\delta^2)/(2f\delta)$. Then $\varphi_3 = \varphi_{31}\varphi_{32}/(4\delta^2)$, where $\varphi_{31} = f + 3\delta + 2c\delta - 3f\delta^2$ and $\varphi_{32} = 3f + 3\delta + 2c\delta - f\delta^2$. From $\varphi_{31} = 0$ we calculate $c:\ c = (3f\delta^2 - 3\delta - f)/(2\delta) \Rightarrow \beta = -(f + \delta + f\delta^2)/(2f\delta) \Rightarrow f = -\delta/(1 + 2\beta\delta + \delta^2) \Rightarrow$ Lemma $\{3.4,\ (29)\}$. In the case $\varphi_{32} = 0$ we have $c = (f\delta^2 - 3f - 3\delta)/(2\delta) \Rightarrow \beta = (f - \delta + f\delta^2)/(2f\delta) \Rightarrow f = \delta/(1 - 2\beta\delta + \delta^2) \Rightarrow$ Lemma $\{3.4,\ (30)\}$.

□

The statement of the Main Theorem follows from Lemmas 4.1 − 4.5.

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Center problem for quartic differential systems with an affine invariant straight line of maximal multiplicity

Olga Vacaraş

Abstract. In this paper the quartic differential systems with a center-focus critical point and non-degenerate infinity are examined. We show that in this family the maximal multiplicity of an affine invariant straight line is six. Modulo the affine transformation and time rescaling the classes of systems with an affine invariant straight line of multiplicity two, three,..., six are determined. In the cases when the quartic systems has an affine invariant straight line of maximal multiplicity the problem of the center is solved.

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Keywords: quartic differential system, multiple invariant line, center-focus critical point.

Problema centrului pentru sistemele diferenţiale cuartice cu o dreaptă invariantă afină de multiplicitate maximală

Rezumat. În această lucrare sunt examinate sistemele diferenţiale cuartice cu punct critic de tip centru-focar şi infinitul nedegenerat. Se arată că în această familie de sisteme multiplicitatea maximală a unei drepte invariante afine este egală cu şase. Cu exactitatea unei transformări afine de coordonate şi rescalarea timpului sunt determinate clasele de sisteme cu o dreaptă invariantă afină de multiplicitatea doi, trei, ..., şase. În cazurile când sistemele cuartice au o dreaptă invarinată de multiplicitate maximală problema centrului este rezolvată.

Cuvinte-cheie: sistem diferenţial cuartic, dreaptă invariantă multiplă, punct critic de tip centru-focar.

1. Introduction

We consider the real polynomial differential systems

$$\dot{x} = p(x, y), \quad \dot{y} = q(x, y), \tag{1}$$

where $\dot{x} = dx/dt,\ \dot{y} = dy/dt$.

Let $n = max\{deg(p),\ deg(q)\}$. If $n = 2$ (respectively, $n = 3$, $n = 4$) then the system (1) is called quadratic (respectively, cubic, quartic). Via an affine transformation of coordinates and time rescaling each non-degenerate quartic system with a non-degenerate infinity and a center-focus critical point, i.e. a critical point with pure imaginary eigenvalues, can be written in the form

$$\begin{cases} \dot{x} = y + p_2(x, y) + p_3(x, y) + p_4(x, y) \equiv p(x, y), \\ \dot{y} = -(x + q_2(x, y) + q_3(x, y) + q_4(x, y)) \equiv q(x, y), \end{cases} \tag{2}$$

where $p_i(x, y) = \sum_{j=0}^{i} a_{i-j,j}x^{i-j}y^j$, $q_i(x, y) = \sum_{j=0}^{i} b_{i-j,j}x^{i-j}y^j$, $i = 2, 3, 4$ are homogeneous polynomials in x and y of degree i with real coefficients.

The critical point $(0, 0)$ of system (2) is either a focus or a center. The problem of distinguishing between a center and a focus is called the center problem.

Suppose that

$$yp_4(x, y) - xq_4(x, y) \not\equiv 0, \gcd(p, q) = 1, \tag{3}$$

i.e. at infinity the system (2) has at most five distinct singular points and the right-hand sides of (2) do not have the common divisors of degree greater than 0.

Denote $\mathbb{X} = p\ (x, y)\ \frac{\partial}{\partial x} + q\ (x, y)\ \frac{\partial}{\partial y}$.

An algebraic curve $f(x, y) = 0,\ f \in \mathbb{C}[x, y]$ (a function $f = \exp(g/h), g, h \in \mathbb{C}[x, y]$) is called invariant algebraic curve (exponential factor) of the system (1) if there exists a polynomial $K_f \in \mathbb{C}[x, y]$, $\deg(K) \le n-1$ such that the identity $\mathbb{X}(f) \equiv f(x, y)K_f(x, y)$, $(x, y) \in \mathbb{R}^2$ $((x, y) \in \mathbb{R}^2 \setminus \{(x, y) \in \mathbb{R}^2 \,|\, h(x, y) = 0\})$ holds. In particular, a straight line $l \equiv \alpha x + \beta y + \gamma = 0,\ \alpha, \beta, \gamma \in \mathbb{C}$ is invariant for (1) if there exists a polynomial $K_l \in \mathbb{C}[x, y]$ such that the identity $\alpha P(x, y) + \beta Q(x, y) \equiv (\alpha x + \beta y + \gamma)K_l(x, y)$, $(x, y) \in \mathbb{R}^2$ holds.

The invariant straight line $\alpha x + \beta y + \gamma = 0$ has *multiplicity* ν if ν is the greatest positive integer such that $(\alpha x + \beta y + \gamma)^\nu$ divides $\mathbb{E} = p \cdot \mathbb{X}(q) - q \cdot \mathbb{X}(p)$ [1].

The quartic differential systems of the form (2) with multiple line at infinity were examined in [8]. In this paper, we establish that in the class of systems (2) the maximal multiplicity of an affine invariant line is six. The coefficient conditions when (2) has an affine invariant line of multiplicity two, three, four, five and six are determined and in the cases of multiplicity six, the center problem is solved.

2. Classification of the quartic systems with a multiple affine invariant straight line

Let the quartic system (2) have an affine real invariant straight line l_1. By a transformation of the form

$$x \to \nu \cdot (x\cos\varphi + y\sin\varphi),\ \ y \to \nu \cdot (y\cos\varphi - x\sin\varphi),\ \ \nu \neq 0$$

we can do l_1 to be described by the equation $x = 1$. Then,

$$\begin{aligned} &a_{40} = -(a_{20} + a_{30}), a_{31} = -(1 + a_{11} + a_{21}),\\ &a_{22} = -(a_{02} + a_{12}), a_{13} = -a_{03}, a_{04} = 0, \end{aligned} \tag{4}$$

and (2) is reduced to the system

$$\begin{cases} \dot{x} = (1-x)(y + a_{20}x^2 + xy + a_{11}xy + a_{02}y^2 + a_{20}x^3 + a_{30}x^3 + x^2y+ \\ \qquad +a_{11}x^2y + a_{21}x^2y + a_{02}xy^2 + a_{12}xy^2 + a_{03}y^3) \equiv p(x,y), \\ \dot{y} = -(x + b_{20}x^2 + b_{11}xy + b_{02}y^2 + b_{30}x^3 + b_{21}x^2y + b_{12}xy^2+ \\ \qquad +b_{03}y^3 + b_{40}x^4 + b_{31}x^3y + b_{22}x^2y^2 + b_{13}xy^3 + b_{04}y^4)) \equiv q(x,y). \end{cases} \tag{5}$$

Next, we will determine the conditions when the invariant line $x = 1$ for the system (5) has maximal multiplicity.

For (5) we have

$$\mathbb{E} = (x-1)\Big(Y_2(y) + Y_3(y)\cdot(x-1) + \cdots + Y_{10}(y)\cdot(x-1)^{10}\Big),$$

where $Y_j(y),\ j = 2, \ldots 10$, are polynomial in y. The invariant line $x-1 = 0$ has multiplicity at least j if the system of identity $\{Y_2(y) \equiv 0, \ldots, Y_j(y) \equiv 0\}$ holds. In particular, the line $x - 1 = 0$ has multiplicity at least two if $Y_2(y) \equiv 0$. The polynomial $Y_2(y)$ look as: $Y_2(y) = Y_{21}(y) \cdot Y_{22}(y)$, where

$Y_{21}(y) = 1+b_{20}+b_{30}+b_{40}+(b_{11}+b_{21}+b_{31})y+(b_{02}+b_{12}+b_{22})y^2+(b_{03}+b_{13})y^3+b_{04}y^4,$

$Y_{22}(y) = 3 + 2a_{11} + 4a_{20}^2 + a_{21} + 4a_{20}a_{30} + a_{30}^2 - 2a_{20}b_{11} - a_{30}b_{11} + 3b_{20} + 2a_{11}b_{20} + a_{21}b_{20} - 2a_{20}b_{21} - a_{30}b_{21} + 3b_{30} + 2a_{11}b_{30} + a_{21}b_{30} - 2a_{20}b_{31} - a_{30}b_{31} + 3b_{40} + 2a_{11}b_{40} + a_{21}b_{40} + 2(2a_{02} + a_{12} + 6a_{20} + 4a_{11}a_{20} + 2a_{20}a_{21} + 3a_{30} + 2a_{11}a_{30} + a_{21}a_{30} - 2a_{20}b_{02} - a_{30}b_{02} - 2a_{20}b_{12} - a_{30}b_{12} + 2a_{02}b_{20} + a_{12}b_{20} - 2a_{20}b_{22} - a_{30}b_{22} + 2a_{02}b_{30} + a_{12}b_{30} + 2a_{02}b_{40} + a_{12}b_{40})y + (9 + 3a_{03} + 12a_{11} + 4a_{11}^2 + 8a_{02}a_{20} + 4a_{12}a_{20} + 6a_{21} + 4a_{11}a_{21} + a_{21}^2 + 4a_{02}a_{30} + 2a_{12}a_{30} - 3b_{02} - 2a_{11}b_{02} - a_{21}b_{02} - 6a_{20}b_{03} - 3a_{30}b_{03} + 2a_{02}b_{11} + a_{12}b_{11} - 3b_{12} - 2a_{11}b_{12} - a_{21}b_{12} - 6a_{20}b_{13} - 3a_{30}b_{13} + 3a_{03}b_{20} + 2a_{02}b_{21} + a_{12}b_{21} - 3b_{22} - 2a_{11}b_{22} - a_{21}b_{22} + 3a_{03}b_{30} + 2a_{02}b_{31} + a_{12}b_{31} + 3a_{03}b_{40})y^2 + 2(6a_{02} + 4a_{02}a_{11} + 3a_{12} + 2a_{11}a_{12} + 2a_{03}a_{20} + 2a_{02}a_{21} + a_{12}a_{21} + a_{03}a_{30} - 3b_{03} - 2a_{11}b_{03} - a_{21}b_{03} - 4a_{20}b_{04} - 2a_{30}b_{04} + a_{03}b_{11} - 3b_{13} - 2a_{11}b_{13} - a_{21}b_{13} + a_{03}b_{21} + a_{03}b_{31})y^3 + (4a_{02}^2 + 6a_{03} + 4a_{03}a_{11} +$

$4a_{02}a_{12}+a_{12}^2+2a_{03}a_{21}+a_{03}b_{02}-2a_{02}b_{03}-a_{12}b_{03}-9b_{04}-6a_{11}b_{04}-3a_{21}b_{04}+a_{03}b_{12}-2a_{02}b_{13}-a_{12}b_{13}+a_{03}b_{22})y^4+2(2a_{02}+a_{12})(a_{03}-b_{04})y^5+a_{03}(a_{03}-b_{04})y^6$.

If $Y_{21}(y)\equiv 0$, then the system (5) is degenerate, i.e. $\deg\Big(\gcd(p,q)\Big)>0$, therefore we require $Y_{22}(y)$ to be identically equal to zero. Solving the identity $Y_{22}(y)\equiv 0$ we obtain the following result:

Lemma 2.1. *The invariant straight line $x=1$ has for quartic system* (5) *the multiplicity at least two if and only if the coefficients of* (5) *verify the following five series of conditions:*

$$\begin{aligned}&a_{03}=0, b_{04}=0, a_{12}=-2a_{02}, b_{13}=-b_{03}, a_{21}=-3-2a_{11},\\ &b_{22}=-b_{02}-b_{12}, b_{31}=2a_{20}+a_{30}-b_{11}-b_{21};\end{aligned}\tag{6}$$

$$\begin{aligned}&a_{03}=0, b_{04}=0, a_{12}=-2a_{02}, b_{13}=-b_{03}, b_{22}=3+2a_{11}+a_{21}-b_{02}-b_{12},\\ &b_{40}=(-3-2a_{11}-4a_{20}^2-a_{21}-4a_{20}a_{30}-a_{30}^2+2a_{20}b_{11}+a_{30}b_{11}-3b_{20}-\\ &-2a_{11}b_{20}-a_{21}b_{20}+2a_{20}b_{21}+a_{30}b_{21}-3b_{30}-2a_{11}b_{30}-a_{21}b_{30}+2a_{20}b_{31}+\\ &+a_{30}b_{31})/(3+2a_{11}+a_{21});\end{aligned}\tag{7}$$

$$\begin{aligned}&a_{03}=0,\ b_{04}=0,\ b_{13}=2a_{02}+a_{12}-b_{03},\ \ b_{31}=(-9-12a_{11}-4a_{11}^2+\\ &+4a_{02}a_{20}+2a_{12}a_{20}-6a_{21}-4a_{11}a_{21}-a_{21}^2+2a_{02}a_{30}+a_{12}a_{30}+3b_{02}+\\ &+2a_{11}b_{02}+a_{21}b_{02}-2a_{02}b_{11}-a_{12}b_{11}+3b_{12}+2a_{11}b_{12}+a_{21}b_{12}-\\ &-2a_{02}b_{21}-a_{12}b_{21}+3b_{22}+2a_{11}b_{22}+a_{21}b_{22})/(2a_{02}+a_{12}),\\ &b_{40}=-(2a_{02}+a_{12}+6a_{20}+4a_{11}a_{20}+2a_{20}a_{21}+3a_{30}+2a_{11}a_{30}+\\ &a_{21}a_{30}-2a_{20}b_{02}-a_{30}b_{02}-2a_{20}b_{12}-a_{30}b_{12}+2a_{02}b_{20}+a_{12}b_{20}-\\ &-2a_{20}b_{22}-a_{30}b_{22}+2a_{02}b_{30}+a_{12}b_{30})/(2a_{02}+a_{12});\end{aligned}\tag{8}$$

$$a_{03}=0, a_{12}=-2a_{02}, a_{21}=-3-2a_{11}, a_{30}=-2a_{20};\tag{9}$$

$$\begin{aligned}&b_{04}=a_{03},\ \ b_{22}=(-4a_{02}^2+3a_{03}+2a_{03}a_{11}-4a_{02}a_{12}-a_{12}^2+a_{03}a_{21}-\\ &-a_{03}b_{02}+2a_{02}b_{03}+a_{12}b_{03}-a_{03}b_{12}+2a_{02}b_{13}+a_{12}b_{13})/a_{03},\\ &b_{31}=(-6a_{02}-4a_{02}a_{11}-3a_{12}-2a_{11}a_{12}+2a_{03}a_{20}-2a_{02}a_{21}-a_{12}a_{21}+\\ &a_{03}a_{30}+3b_{03}+2a_{11}b_{03}+a_{21}b_{03}-a_{03}b_{11}+3b_{13}+2a_{11}b_{13}+a_{21}b_{13}-\\ &-a_{03}b_{21})/a_{03},\quad b_{40}=-(a_{03}+4a_{02}a_{20}+2a_{12}a_{20}+2a_{02}a_{30}+a_{12}a_{30}-\\ &-2a_{20}b_{03}-a_{30}b_{03}-2a_{20}b_{13}-a_{30}b_{13}+a_{03}b_{20}+a_{03}b_{30})/a_{03}.\end{aligned}\tag{10}$$

The multiplicity of the invariant straight line $x=1$ is at least three if $\{Y_2(y)\equiv 0,\ Y_3(y)\equiv 0\}$. Taking into account the condition (3) the identity $Y_3(y)\equiv 0$ gives, in each of the five cases (6)-(10) of Lemma 2.1 the following series of conditions:

1) (6) $\Rightarrow$

$$a_{30}=-2a_{20},\ a_{02}=0,\ a_{11}=-3;\tag{11}$$

$$\begin{aligned}
&b_{03} = a_{02},\ b_{12} = (2a_{02} + 3a_{20} + a_{11}a_{20} - 2a_{20}b_{02} + a_{02}b_{20} - a_{02}b_{40})/a_{20},\\
&b_{21} = (6 + 2a_{11} + a_{20}^2 - 2a_{20}b_{11} + 3b_{20} + a_{11}b_{20} - 3b_{40} - a_{11}b_{40})/a_{20},\\
&b_{30} = (3a_{20} + 2a_{30} + a_{20}b_{20} + a_{30}b_{20} - 3a_{20}b_{40} - a_{30}b_{40})/a_{20};
\end{aligned} \tag{12}$$

$$\begin{aligned}
&b_{03} = a_{02},\ b_{12} = (3a_{02} + 3a_{30} + a_{11}a_{30} - 2a_{30}b_{02} + 2a_{02}b_{20} + a_{02}b_{30})/a_{30},\\
&b_{21} = 9 + 3a_{11} - 2a_{30}b_{11} + 6b_{20} + 2a_{11}b_{20} + 3b_{30} + a_{11}b_{30})/a_{30},\\
&a_{20} = 0,\ b_{40} = b_{20} + 2;
\end{aligned} \tag{13}$$

2) (7) $\Rightarrow$

$$\begin{aligned}
&b_{03} = a_{02},\ b_{12} = (9 + 9a_{11} + 2a_{11}^2 + 2a_{02}a_{20} + 3a_{21} + a_{11}a_{21} + a_{02}a_{30}-\\
&-6b_{02} - 4a_{11}b_{02} - 2a_{21}b_{02})/(3 + 2a_{11} + a_{21}),\\
&b_{21} = (15a_{20} + 8a_{11}a_{20} + 3a_{20}a_{21} + 6a_{30} + 3a_{11}a_{30} + a_{21}a_{30} - 6b_{11}-\\
&-4a_{11}b_{11} - 2a_{21}b_{11})/(3 + 2a_{11} + a_{21}),\\
&b_{31} = (-3a_{20} + a_{20}a_{21} + a_{11}a_{30} + a_{21}a_{30} + 3b_{11} + 2a_{11}b_{11}+\\
&+a_{21}b_{11})/(3 + 2a_{11} + a_{21});
\end{aligned} \tag{14}$$

$$\begin{aligned}
&b_{03} = a_{02},\quad b_{12} = (9 + 6a_{11} + a_{11}^2 + a_{02}a_{20} + a_{02}a_{30} - 6b_{02} - 2a_{11}b_{02}+\\
&+a_{02}b_{11} - a_{02}b_{31})/(3 + a_{11}),\\
&b_{21} = (9a_{20} + 4a_{11}a_{20} + a_{20}a_{21} + 6a_{30} + 3a_{11}a_{30} + a_{21}a_{30} + a_{11}b_{11}+\\
&+a_{21}b_{11} - 6b_{31} - 3a_{11}b_{31} - a_{21}b_{31})/(3 + a_{11}),\\
&b_{30} = (9a_{20} + 4a_{11}a_{20} + a_{20}a_{21} + 6a_{30} + 3a_{11}a_{30} + a_{21}a_{30} + a_{11}b_{11}+\\
&+a_{21}b_{11} - 6b_{31} - 3a_{11}b_{31} - a_{21}b_{31})/(3 + a_{11});
\end{aligned} \tag{15}$$

$$\begin{aligned}
&b_{03} = a_{02},\ b_{12} = (a_{02}a_{20} - 6b_{02} + 2a_{21}b_{02} - 2a_{02}b_{11} - a_{02}b_{21})/(3 - a_{21}),\\
&a_{11} = -3,\quad b_{31} = a_{20} + a_{30} + b_{11},\quad b_{30} = (-9 + 3a_{20}^2 + 3a_{21} + a_{20}a_{30}-\\
&-6a_{20}b_{11} - 2a_{30}b_{11} - 6b_{20} + 2a_{21}b_{20} - 3a_{20}b_{21} - a_{30}b_{21})/(3 - a_{21});
\end{aligned} \tag{16}$$

3) (8) $\Rightarrow$

$$\begin{aligned}
&b_{03} = a_{02},\quad b_{12} = (3a_{02} + 4a_{02}a_{11} + a_{11}a_{12} + 3a_{02}a_{21} + a_{12}a_{21}+\\
&+a_{02}b_{02} + a_{12}b_{02} - 3a_{02}b_{22} - a_{12}b_{22})/a_{02},\\
&b_{30} = -(3a_{02} - 3a_{11}a_{20} - 3a_{20}a_{21} - a_{11}a_{30} - a_{21}a_{30} - 3a_{20}b_{02}-\\
&-a_{30}b_{02} + 2a_{02}b_{20} + 3a_{20}b_{22} + a_{30}b_{22})/a_{02},\\
&b_{21} = (6a_{11} + 3a_{11}^2 + a_{02}a_{20} + 6a_{21} + 4a_{11}a_{21} + a_{21}^2 + 6b_{02}+\\
&+3a_{11}b_{02} + a_{21}b_{02} - 2a_{02}b_{11} - 6b_{22} - 3a_{11}b_{22} - a_{21}b_{22})/a_{02};
\end{aligned} \tag{17}$$

$$\begin{aligned}
&b_{03} = a_{02},\quad b_{12} = (3a_{02} + 4a_{02}a_{11} + a_{11}a_{12} + 3a_{02}a_{21} + a_{12}a_{21}+\\
&a_{02}b_{02} + a_{12}b_{02} - 3a_{02}b_{22} - a_{12}b_{22})/a_{02},\\
&b_{30} = -(6a_{02}^4 + 18a_{02}^2a_{11} + 3a_{02}^2a_{11}^2 + 3a_{02}^3a_{12} - 9a_{02}a_{11}^2a_{12} - a_{02}a_{11}^3a_{12}+
\end{aligned} \tag{18}$$

$a_{11}^3a_{12}^2 + 3a_{02}^3a_{20} - 2a_{02}^3a_{11}a_{20} - 2a_{02}^2a_{11}a_{12}a_{20} + 18a_{02}^2a_{21} - 6a_{02}^2a_{11}a_{21}-$
$a_{02}^2a_{11}^2a_{21} - 18a_{02}a_{11}a_{12}a_{21} + 3a_{11}^2a_{12}^2a_{21} - 3a_{02}^3a_{20}a_{21} - 2a_{02}^2a_{12}a_{20}a_{21}-$
$9a_{02}^2a_{21}^2 - 9a_{02}a_{12}a_{21}^2 + 3a_{02}a_{11}a_{12}a_{21}^2 + 3a_{11}a_{12}^2a_{21}^2 + a_{02}^2a_{21}^3 + 2a_{02}a_{12}\cdot$
$a_{21}^3 + a_{12}^2a_{21}^3 + 18a_{02}^2b_{02} - 15a_{02}^2a_{11}b_{02} - 2a_{02}^2a_{11}^2b_{02} - 18a_{02}a_{11}a_{12}b_{02}+$
$2a_{02}a_{11}^2a_{12}b_{02} + 3a_{11}^2a_{12}^2b_{02} - 4a_{02}^3a_{20}b_{02} - 2a_{02}^2a_{12}a_{20}b_{02} - 27a_{02}^2a_{21}b_{02}+$
$a_{02}^2a_{11}a_{21}b_{02} - 18a_{02}a_{12}a_{21}b_{02} + 10a_{02}a_{11}a_{12}a_{21}b_{02} + 6a_{11}a_{12}^2a_{21}b_{02}+$
$5a_{02}^2a_{21}^2b_{02} + 8a_{02}a_{12}a_{21}^2b_{02} + 3a_{12}^2a_{21}^2b_{02} - 18a_{02}^2b_{02}^2 + 2a_{02}^2a_{11}b_{02}^2 - 9a_{02}\cdot$
$a_{12}b_{02}^2 + 7a_{02}a_{11}a_{12}b_{02}^2 + 3a_{11}a_{12}^2b_{02}^2 + 8a_{02}^2a_{21}b_{02}^2 + 10a_{02}a_{12}a_{21}b_{02}^2+$
$3a_{12}^2a_{21}b_{02}^2 + 4a_{02}^2b_{02}^3 + 4a_{02}a_{12}b_{02}^3 + a_{12}^2b_{02}^3 - 6a_{02}^3b_{11} + 2a_{02}^2a_{11}a_{12}b_{11}+$
$2a_{02}^3a_{21}b_{11} + 2a_{02}^2a_{12}a_{21}b_{11} + 4a_{02}^3b_{02}b_{11} + 2a_{02}^2a_{12}b_{02}b_{11} + 4a_{02}^4b_{20}+$
$2a_{02}^3a_{12}b_{20} - 3a_{02}^3b_{21} + a_{02}^2a_{11}a_{12}b_{21} + a_{02}^3a_{21}b_{21} + a_{02}^2a_{12}a_{21}b_{21}+$
$2a_{02}^3b_{02}b_{21} + a_{02}^2a_{12}b_{02}b_{21} - 18a_{02}^2b_{22} + 15a_{02}^2a_{11}b_{22} + 2a_{02}^2a_{11}^2b_{22}+$
$18a_{02}a_{11}a_{12}b_{22} - 2a_{02}a_{11}^2a_{12}b_{22} - 3a_{11}^2a_{12}^2b_{22} + 4a_{02}^3a_{20}b_{22} + 2a_{02}^2\cdot$
$a_{12}a_{20}b_{22} + 27a_{02}^2a_{21}b_{22} - a_{02}^2a_{11}a_{21}b_{22} + 18a_{02}a_{12}a_{21}b_{22} - 10a_{02}\cdot$
$a_{11}a_{12}a_{21}b_{22} - 6a_{11}a_{12}^2a_{21}b_{22} - 5a_{02}^2a_{21}^2b_{22} - 8a_{02}a_{12}a_{21}^2b_{22} - 3a_{12}^2\cdot$
$a_{21}^2b_{22} + 36a_{02}^2b_{02}b_{22} - 4a_{02}^2a_{11}b_{02}b_{22} + 18a_{02}a_{12}b_{02}b_{22} - 14a_{02}a_{11}a_{12}\cdot$
$b_{02}b_{22} - 6a_{11}a_{12}^2b_{02}b_{22} - 16a_{02}^2a_{21}b_{02}b_{22} - 20a_{02}a_{12}a_{21}b_{02}b_{22} - 6a_{12}^2\cdot$
$a_{21}b_{02}b_{22} - 12a_{02}^2b_{02}^2b_{22} - 12a_{02}a_{12}b_{02}^2b_{22} - 3a_{12}^2b_{02}^2b_{22} - 4a_{02}^3b_{11}b_{22}-$
$2a_{02}^2a_{12}b_{11}b_{22} - 2a_{02}^3b_{21}b_{22} - a_{02}^2a_{12}b_{21}b_{22} - 18a_{02}^2b_{22}^2 + 2a_{02}^2a_{11}b_{22}^2-$
$9a_{02}a_{12}b_{22}^2 + 7a_{02}a_{11}a_{12}b_{22}^2 + 3a_{11}a_{12}^2b_{22}^2 + 8a_{02}^2a_{21}b_{22}^2 + 10a_{02}a_{12}\cdot$
$a_{21}b_{22}^2 + 3a_{12}^2a_{21}b_{22}^2 + 12a_{02}^2b_{02}b_{22}^2 + 12a_{02}a_{12}b_{02}b_{22}^2 + 3a_{12}^2b_{02}b_{22}^2-$
$4a_{02}^2b_{22}^3 - 4a_{02}a_{12}b_{22}^3 - a_{12}^2b_{22}^3))/(a_{02}^3(2a_{02} + a_{12})),$
$a_{30} = (3a_{02}a_{11} - a_{11}^2a_{12} - 2a_{02}^2a_{20} + 3a_{02}a_{21} - a_{02}a_{11}a_{21} - 2a_{11}a_{12}\cdot$
$a_{21} - a_{02}a_{21}^2 - a_{12}a_{21}^2 + 3a_{02}b_{02} - 2a_{02}a_{11}b_{02} - 2a_{11}a_{12}b_{02} - 3a_{02}\cdot$
$a_{21}b_{02} - 2a_{12}a_{21}b_{02} - 2a_{02}b_{02}^2 - a_{12}b_{02}^2 - 3a_{02}b_{22} + 2a_{02}a_{11}b_{22}+$
$2a_{11}a_{12}b_{22} + 3a_{02}a_{21}b_{22} + 2a_{12}a_{21}b_{22} + 4a_{02}b_{02}b_{22} + 2a_{12}b_{02}b_{22}-$
$2a_{02}b_{22}^2 - a_{12}b_{22}^2)/a_{02}^2;$

$b_{03} = a_{02},\quad a_{02} = 0,\quad b_{22} = a_{11} + a_{21} + b_{02},$
$b_{30} = (162 + 207a_{11} + 87a_{11}^2 + 12a_{11}^3 - 3a_{12}^2 - 9a_{12}a_{20} - 4a_{11}a_{12}a_{20}+$
$45a_{21} + 36a_{11}a_{21} + 7a_{11}^2a_{21} - a_{12}a_{20}a_{21} + 3a_{21}^2 + a_{11}a_{21}^2 - 198b_{02}-$
$174a_{11}b_{02} - 38a_{11}^2b_{02} + 4a_{12}a_{20}b_{02} - 42a_{21}b_{02} - 18a_{11}a_{21}b_{02} - 2a_{21}^2b_{02}+$
$72b_{02}^2 + 32a_{11}b_{02}^2 + 8a_{21}b_{02}^2 - 8b_{02}^3 + 12a_{12}b_{11} + 6a_{11}a_{12}b_{11} + 2a_{12}a_{21}b_{11}-$ (19)
$4a_{12}b_{02}b_{11} - 99b_{12} - 87a_{11}b_{12} - 19a_{11}^2b_{12} + 2a_{12}a_{20}b_{12} - 21a_{21}b_{12}-$
$9a_{11}a_{21}b_{12} - a_{21}^2b_{12} + 72b_{02}b_{12} + 32a_{11}b_{02}b_{12} + 8a_{21}b_{02}b_{12} - 12b_{02}^2b_{12}-$
$2a_{12}b_{11}b_{12} + 18b_{12}^2 + 8a_{11}b_{12}^2 + 2a_{21}b_{12}^2 - 6b_{02}b_{12}^2 - b_{12}^3 - 2a_{12}^2b_{20}+$

$$6a_{12}b_{21} + 3a_{11}a_{12}b_{21} + a_{12}a_{21}b_{21} - 2a_{12}b_{02}b_{21} - a_{12}b_{12}b_{21})/a_{12}^2,$$
$$a_{30} = -(18 + 15a_{11} + 3a_{11}^2 + 2a_{12}a_{20} + 3a_{21} + a_{11}a_{21} - 18b_{02}- 8a_{11}b_{02} - 2a_{21}b_{02} + 4b_{02}^2 - 9b_{12} - 4a_{11}b_{12} - a_{21}b_{12} + 4b_{02}b_{12}+ b_{12}^2)/a_{12};$$

$$\begin{aligned} &b_{03} = a_{02}, \quad a_{02} = 0, \quad b_{22} = a_{11} + a_{21} + b_{02},\\ &b_{30} = -(3a_{12} + 9a_{20} + 3a_{11}a_{20} + 3a_{30} + a_{11}a_{30} - 6a_{20}b_{02} - 2a_{30}b_{02}-\\ &-3a_{20}b_{12} - a_{30}b_{12} + 2a_{12}b_{20})/a_{12},\\ &b_{21} = (-18 - 15a_{11} - 3a_{11}^2 + a_{12}a_{20} - 3a_{21} - a_{11}a_{21} + 12b_{02} + 6a_{11}b_{02}+\\ &2a_{21}b_{02} - 2a_{12}b_{11} + 6b_{12} + 3a_{11}b_{12} + a_{21}b_{12})/a_{12}; \end{aligned} \tag{20}$$

4) (9) $\Rightarrow$

$$a_{02} = 0, \ a_{11} = -3, \ a_{20} = 0; \tag{21}$$

5) (10) $\Rightarrow$

$$\begin{aligned} &b_{13} = a_{12} + a_{02}, \quad b_{21} = (-6a_{02} - 3a_{02}a_{11} + a_{03}a_{20} - a_{02}a_{21} + 6b_{03}+\\ &+3a_{11}b_{03} + a_{21}b_{03} - 2a_{03}b_{11})/a_{03},\\ &b_{30} = -(3a_{03} + 3a_{02}a_{20} + a_{02}a_{30} - 3a_{20}b_{03} - a_{30}b_{03} + 2a_{03}b_{20})/a_{03},\\ &b_{12} = (-3a_{02}^2 + 3a_{03} + a_{03}a_{11} - a_{02}a_{12} - 2a_{03}b_{02} + 3a_{02}b_{03} + a_{12}b_{03})/a_{03}; \end{aligned} \tag{22}$$

$$\begin{aligned} &b_{13} = a_{12} + a_{02}, \quad b_{21} = (9a_{02}^3 - 15a_{02}a_{03} - 6a_{02}a_{03}a_{11} + 6a_{02}^2a_{12}-\\ &3a_{03}a_{12} - a_{03}a_{11}a_{12} + a_{02}a_{12}^2 + a_{03}^2a_{20} - a_{02}a_{03}a_{21} + 6a_{02}a_{03}b_{02}+\\ &2a_{03}a_{12}b_{02} - 12a_{02}^2b_{03} + 9a_{03}b_{03} + 4a_{03}a_{11}b_{03} - 7a_{02}a_{12}b_{03} - a_{12}^2b_{03}+\\ &a_{03}a_{21}b_{03} - 2a_{03}b_{02}b_{03} + 3a_{02}b_{03}^2 + a_{12}b_{03}^2 - 2a_{03}^2b_{11} + 3a_{02}a_{03}b_{12}+\\ &a_{03}a_{12}b_{12} - a_{03}b_{03}b_{12})/a_{03}^2,\\ &b_{30} = (12a_{02}^4 + 3a_{02}^2a_{03} - 9a_{03}^2 - 3a_{03}^3 + 5a_{02}^2a_{03}a_{11} - 9a_{03}^2a_{11} - 2a_{03}^2a_{11}^2+\\ &7a_{02}^3a_{12} + a_{02}a_{03}a_{11}a_{12} + a_{02}^2a_{12}^2 - a_{02}a_{03}^2a_{20} + 4a_{02}^2a_{03}a_{21} - 3a_{03}^2a_{21}-\\ &a_{03}^2a_{11}a_{21} + a_{02}a_{03}a_{12}a_{21} + 6a_{02}^2a_{03}b_{02} + 6a_{03}^2b_{02} + 4a_{03}^2a_{11}b_{02} + 2a_{02}a_{03}\cdot\\ &a_{12}b_{02} + 2a_{03}^2a_{21}b_{02} - 31a_{02}^3b_{03} - 3a_{02}a_{03}b_{03} - 6a_{02}a_{03}a_{11}b_{03} - 16a_{02}^2a_{12}\cdot\\ &b_{03} - a_{03}a_{11}a_{12}b_{03} - 2a_{02}a_{12}^2b_{03} + a_{03}^2a_{20}b_{03} - 5a_{02}a_{03}a_{21}b_{03} - a_{03}a_{12}a_{21}\cdot\\ &b_{03} - 8a_{02}a_{03}b_{02}b_{03} - 2a_{03}a_{12}b_{02}b_{03} + 27a_{02}^2b_{03}^2 + a_{03}a_{11}b_{03}^2 + 11a_{02}a_{12}\cdot\\ &b_{03}^2 + a_{12}^2b_{03}^2 + a_{03}a_{21}b_{03}^2 + 2a_{03}b_{02}b_{03}^2 - 9a_{02}b_{03}^3 - 2a_{12}b_{03}^3 + b_{03}^4+\\ &3a_{02}^2a_{03}b_{12} + 3a_{03}^2b_{12} + 2a_{03}^2a_{11}b_{12} + a_{02}a_{03}a_{12}b_{12} + a_{03}^2a_{21}b_{12}-\\ &4a_{02}a_{03}b_{03}b_{12} - a_{03}a_{12}b_{03}b_{12} + a_{03}b_{03}^2b_{12} - 2a_{03}^3b_{20})/a_{03}^3,\\ &a_{30} = -(3a_{02}^3 + 3a_{02}a_{03} + 2a_{02}a_{03}a_{11} + a_{02}^2a_{12} + 2a_{03}^2a_{20} + a_{02}a_{03}a_{21}-\\ &7a_{02}^2b_{03} - 3a_{03}b_{03} - 2a_{03}a_{11}b_{03} - 2a_{02}a_{12}b_{03} - a_{03}a_{21}b_{03} + 5a_{02}b_{03}^2+\\ &a_{12}b_{03}^2 - b_{03}^3)/a_{03}^2. \end{aligned} \tag{23}$$

Lemma 2.2. *The invariant straight line* $x = 1$ *has for quartic system* (5) *the multiplicity at least three if and only if the coefficients of* (5) *verify the following series of conditions:*
1) $\{(6), (11)\}$; *2)* $\{(6), (12)\}$; *3)* $\{(6), (13)\}$; *4)* $\{(7), (14)\}$;
5) $\{(7), (15)\}$; *6)* $\{(7), (16)\}$; *7)* $\{(8), (17)\}$; *8)* $\{(8), (18)\}$;
9) $\{(8), (19)\}$; *10)* $\{(8), (20)\}$; *11)* $\{(9), (21)\}$; *12)* $\{(10),(22)\}$;
13) $\{(10), (23)\}$.

The multiplicity of the invariant straight line $x = 1$ is at least four if $\{Y_2(y) \equiv 0,\ Y_3(y) \equiv 0,\ Y_4(y) \equiv 0\}$. Taking into account the condition (3) the identity $Y_4(y) \equiv 0$ gives, in each of the cases 1)-13) of Lemma 2.2 the following series of conditions:

1) $\{(6), (11)\} \Rightarrow$

$$b_{03} = 0,\ \ b_{12} = -2b_{02},\ \ b_{40} = -1 - 2a_{20}^2 + 2a_{20}b_{11} - b_{20} + a_{20}b_{21} - b_{30}; \tag{24}$$

2) $\{(6), (12)\} \Rightarrow$

$$b_{02} = (3a_{02} + a_{20} + a_{02}b_{20})/a_{20},\ \ b_{11} = (4 + a_{11})(3 + b_{20})/a_{20},\ b_{40} = -1; \tag{25}$$

3) $\{(6), (13)\} \Rightarrow$

$$b_{02} = (-3a_{02} + a_{30} + a_{02}b_{30})/a_{30},\ b_{11} = (4 + a_{11})(-3 + b_{30})/a_{30},\ b_{20} = -3; \tag{26}$$

4) $\{(7), (14)\} \Rightarrow$

$$\begin{aligned} b_{02} &= (3 + 2a_{11} + 2a_{02}a_{20} + a_{21} + a_{02}a_{30})/(3 + 2a_{11} + a_{21}),\\ b_{11} &= (4 + a_{11})(2a_{20} + a_{30})/(3 + 2a_{11} + a_{21}),\\ b_{20} &= -(9 + 6a_{11} - 2a_{20}^2 + 3a_{21} - a_{20}a_{30})/(3 + 2a_{11} + a_{21}),\\ b_{30} &= (9 + 6a_{11} + 2a_{20}^2 + 3a_{21} + 3a_{20}a_{30} + a_{30}^2)/(3 + 2a_{11} + a_{21}); \end{aligned} \tag{27}$$

5) $\{(7), (15)\} \Rightarrow$

$$\begin{aligned} &b_{02} = 1 - a_{02}a_{20} - a_{02}a_{30} + a_{02}b_{31},\ \ b_{11} = (4 + a_{11})(b_{31} - a_{20} - a_{30}),\\ &b_{20} = -3 - a_{20}^2 - a_{20}a_{30} + a_{20}b_{31}; \end{aligned} \tag{28}$$

6) $\{(7), (16)\} \Rightarrow$

$$b_{02} = 1 + a_{02}b_{11},\ \ b_{21} = a_{20} - 5b_{11} + a_{21}b_{11},\ \ b_{20} = -3 + a_{20}b_{11}; \tag{29}$$

7) $\{(8), (17)\} \Rightarrow$

$$\begin{aligned} &b_{22} = 1 + a_{11} + a_{21},\ \ b_{11} = (4 + a_{11})(-1 + b_{02})/a_{02},\\ &b_{20} = -(3a_{02} + a_{20} - a_{20}b_{02})/a_{02}; \end{aligned} \tag{30}$$

$$\begin{aligned}
&b_{22} = (-6a_{02} - a_{02}a_{11} - 2a_{12} + 4a_{02}b_{02} + 2a_{12}b_{02})/a_{02},\\
&b_{11} = (4 + a_{11})(-1 + b_{02})/a_{02},\\
&a_{21} = (-7a_{02} - 2a_{02}a_{11} - 2a_{12} + 4a_{02}b_{02} + 2a_{12}b_{02})/a_{02},\\
&a_{30} = (2a_{02} + a_{12} - 2a_{02}^2a_{20} - 4a_{02}b_{02} - 2a_{12}b_{02} + 2a_{02}b_{02}^2 + a_{12}b_{02}^2)/a_{02}^2;
\end{aligned} \tag{31}$$

8) $\{(8), (18)\} \Rightarrow$

$$\begin{aligned}
&b_{22} = 1 + a_{11} + a_{21}, \quad b_{20} = -(3a_{02} + a_{20} - a_{20}b_{02})/a_{02},\\
&b_{21} = (2 - a_{11} + a_{02}a_{20} - a_{21} - 2b_{02} + a_{11}b_{02} + a_{21}b_{02})/a_{02},\\
&b_{11} = (4 + a_{11})(-1 + b_{02})/a_{02};
\end{aligned} \tag{32}$$

$$\begin{aligned}
&b_{22} = -(6a_{02} + a_{02}a_{11} + 2a_{12} - 4a_{02}b_{02} - 2a_{12}b_{02})/a_{02},\\
&b_{21} = (9a_{02} + a_{02}a_{11} + 2a_{12} + a_{02}^2a_{20} - 13a_{02}b_{02} - a_{02}a_{11}b_{02}-\\
&4a_{12}b_{02} + 4a_{02}b_{02}^2 + 2a_{12}b_{02}^2)/a_{02}^2, \quad b_{11} = (4 + a_{11})(-1 + b_{02})/a_{02},\\
&a_{21} = -(7a_{02} + 2a_{02}a_{11} + 2a_{12} - 4a_{02}b_{02} - 2a_{12}b_{02})/a_{02};
\end{aligned} \tag{33}$$

$$\begin{aligned}
&b_{22} = (-6a_{02} - a_{02}a_{11} - 2a_{12} + 4a_{02}b_{02} + 2a_{12}b_{02})/a_{02},\\
&b_{21} = (2a_{02}^2 - 2a_{02}^2a_{11} + 9a_{02}a_{12} + 2a_{12}^2 - 2a_{02}^3a_{20} - 6a_{02}^2b_{02}+\\
&2a_{02}^2a_{11}b_{02} - 16a_{02}a_{12}b_{02} - 4a_{12}^2b_{02} + 4a_{02}^2b_{02}^2 + 7a_{02}a_{12}b_{02}^2+\\
&2a_{12}^2b_{02}^2)/(a_{02}^2(2a_{02} + a_{12})),\\
&a_{21} = (-7a_{02} - 2a_{02}a_{11} - 2a_{12} + 4a_{02}b_{02} + 2a_{12}b_{02})/a_{02},\\
&b_{11} = (4a_{02} + a_{02}a_{11} + 3a_{02}^2a_{20} + a_{02}a_{12}a_{20} - 7a_{02}b_{02} - a_{02}a_{11}b_{02}-\\
&a_{12}b_{02} + 3a_{02}b_{02}^2 + a_{12}b_{02}^2)/(a_{02}(2a_{02} + a_{12}));
\end{aligned} \tag{34}$$

9) $\{(8), (19)\} \Rightarrow$

$$\begin{aligned}
&b_{02} = 1, \quad b_{21} = (2 + a_{11} - a_{11}^2 + a_{12}a_{20} - a_{21} - a_{11}a_{21} - 2b_{12}+\\
&a_{11}b_{12} + a_{21}b_{12})/a_{12}, \quad b_{20} = -(3a_{12} + a_{20} + a_{11}a_{20} - a_{20}b_{12})/a_{12},\\
&b_{11} = -(4 + a_{11})(1 + a_{11} - b_{12})/a_{12};
\end{aligned} \tag{35}$$

$$\begin{aligned}
&b_{02} = 1, \ b_{21} = 2(a_{20} - b_{11})(1 + a_{12}a_{20} - a_{12}b_{11}),\\
&a_{21} = -3 - 2a_{11} - 2a_{12}a_{20} + 2a_{12}b_{11}, \ b_{12} = 1 + a_{11} - a_{12}a_{20} + a_{12}b_{11};
\end{aligned} \tag{36}$$

$$\begin{aligned}
&b_{02} = 1, \ b_{21} = (7 + 10a_{11} + 3a_{11}^2 + a_{12}a_{20} - 9b_{12} - 5a_{11}b_{12} + 2b_{12}^2)/a_{12},\\
&a_{21} = -5 - 4a_{11} + 2b_{12}, \ b_{11} = -(4 + a_{11})(1 + a_{11} - b_{12})/a_{12};
\end{aligned} \tag{37}$$

10) $\{(8), (20)\} \Rightarrow$

$$\begin{aligned}
&b_{02} = 1, \ b_{11} = -(4 + a_{11})(1 + a_{11} - b_{12})/a_{12}, \ a_{21} = -5 - 4a_{11} + 2b_{12},\\
&a_{30} = (1 + 2a_{11} + a_{11}^2 - 2a_{12}a_{20} - 2b_{12} - 2a_{11}b_{12} + b_{12}^2)/a_{12};
\end{aligned} \tag{38}$$

$$\begin{aligned}
&b_{02} = 1, \ b_{11} = -(4 + a_{11})(1 + a_{11} - b_{12})/a_{12},\\
&b_{20} = -(3a_{12} + a_{20} + a_{11}a_{20} - a_{20}b_{12})/a_{12};
\end{aligned} \tag{39}$$

11) $\{(9),(21)\} \Rightarrow$

$$b_{04}=0,\ b_{13}=-b_{03},\ b_{22}=-b_{12}-b_{02},\ b_{40}=-b_{30}-b_{20}-1; \tag{40}$$

12) $\{(10),(22)\} \Rightarrow$

$$\begin{gathered}b_{02}=(-a_{02}^2+a_{03}+a_{02}b_{03})/a_{03}, \quad b_{20}=(16a_{02}^2-3a_{03}^2+4a_{02}^2a_{11}+\\ 4a_{02}a_{12}+a_{02}a_{11}a_{12}-a_{02}a_{03}a_{20}-24a_{02}b_{03}-6a_{02}a_{11}b_{03}-4a_{12}b_{03}-\\ a_{11}a_{12}b_{03}+a_{03}a_{20}b_{03}+8b_{03}^2+2a_{11}b_{03}^2+4a_{02}a_{03}b_{11}+a_{03}a_{12}b_{11}-\\ 2a_{03}b_{03}b_{11})/a_{03}^2, \quad a_{21}=(-7a_{02}^2-3a_{03}-2a_{03}a_{11}-2a_{02}a_{12}+\\ 10a_{02}b_{03}+2a_{12}b_{03}-3b_{03}^2)/a_{03}, \quad a_{30}=(4a_{02}^3+a_{02}^2a_{12}-2a_{03}^2a_{20}-\\ 10a_{02}^2b_{03}-2a_{02}a_{12}b_{03}+8a_{02}b_{03}^2+a_{12}b_{03}^2-2b_{03}^3)/a_{03}^2;\end{gathered} \tag{41}$$

$$\begin{gathered}b_{02}=(-a_{02}^2+a_{03}+a_{02}b_{03})/a_{03},\ b_{20}=-(3a_{03}+a_{02}a_{20}-a_{20}b_{03})/a_{03},\\ b_{11}=(4+a_{11})(b_{03}-a_{02})/a_{03};\end{gathered} \tag{42}$$

13) $\{(10),(23)\} \Rightarrow$

$$\begin{gathered}b_{02}=(-a_{02}^2+a_{03}+a_{02}b_{03})/a_{03},\ b_{20}=-(3a_{03}+a_{02}a_{20}-a_{20}b_{03})/a_{03},\\ b_{11}=-(4+a_{11})(a_{02}-b_{03})/a_{03}, \quad b_{12}=(-a_{02}^2+a_{03}+a_{03}a_{11}-\\ a_{02}a_{12}+a_{02}b_{03}+a_{12}b_{03})/a_{03};\end{gathered} \tag{43}$$

$$\begin{gathered}b_{02}=(-a_{02}^2+a_{03}+a_{02}b_{03})/a_{03}, \quad a_{21}=(-7a_{02}^2-3a_{03}-2a_{03}a_{11}-\\ 2a_{02}a_{12}+10a_{02}b_{03}+2a_{12}b_{03}-3b_{03}^2)/a_{03}, \quad b_{20}=(16a_{02}^2-3a_{03}^2+\\ 4a_{02}^2a_{11}+4a_{02}a_{12}+a_{02}a_{11}a_{12}-a_{02}a_{03}a_{20}-24a_{02}b_{03}-6a_{02}a_{11}b_{03}-\\ 4a_{12}b_{03}-a_{11}a_{12}b_{03}+a_{03}a_{20}b_{03}+8b_{03}^2+2a_{11}b_{03}^2+4a_{02}a_{03}b_{11}+\\ a_{03}a_{12}b_{11}-2a_{03}b_{03}b_{11})/a_{03}^2, \quad b_{12}=(-a_{02}^2+a_{03}+a_{03}a_{11}-a_{02}a_{12}+\\ a_{02}b_{03}+a_{12}b_{03})/a_{03};\end{gathered} \tag{44}$$

$$\begin{gathered}b_{02}=(-2a_{02}^2+2a_{03}+a_{03}a_{11}-a_{02}a_{12}+2a_{02}b_{03}+a_{12}b_{03}-a_{03}b_{12})/a_{03},\\ a_{21}=-(7a_{02}^2+3a_{03}+2a_{03}a_{11}+2a_{02}a_{12}-10a_{02}b_{03}-2a_{12}b_{03}+\\ 3b_{03}^2)/a_{03},\\ b_{20}=(24a_{02}^4-6a_{02}^2a_{03}+2a_{03}^2-3a_{03}^3-18a_{02}^2a_{03}a_{11}+2a_{03}^2a_{11}+\\ 32a_{02}^3a_{12}-7a_{02}a_{03}a_{12}-8a_{02}a_{03}a_{11}a_{12}+10a_{02}^2a_{12}^2-a_{03}a_{12}^2-\\ a_{03}a_{11}a_{12}^2+a_{02}a_{12}^3-50a_{02}^3b_{03}-4a_{02}a_{03}b_{03}+16a_{02}a_{03}a_{11}b_{03}-\\ 60a_{02}^2a_{12}b_{03}+2a_{03}a_{12}b_{03}+3a_{03}a_{11}a_{12}b_{03}-14a_{02}a_{12}^2b_{03}-a_{12}^3b_{03}+\\ 34a_{02}^2b_{03}^2+4a_{03}b_{03}^2-4a_{03}a_{11}b_{03}^2+35a_{02}a_{12}b_{03}^2+4a_{12}^2b_{03}^2-8a_{02}b_{03}^3-\\ 7a_{12}b_{03}^3+4a_{02}a_{03}^2b_{11}+a_{03}^2a_{12}b_{11}-2a_{03}^2b_{03}b_{11}+22a_{02}^2a_{03}b_{12}-\\ a_{03}^2b_{12}+a_{03}^2a_{11}b_{12}+8a_{02}a_{03}a_{12}b_{12}+a_{03}a_{12}^2b_{12}-23a_{02}a_{03}b_{03}b_{12}-\\ 3a_{03}a_{12}b_{03}b_{12}+7a_{03}b_{03}^2b_{12}-a_{03}^2b_{12}^2)/a_{03}^3,\end{gathered} \tag{45}$$

$a_{20} = (-6a_{02}^3 + 2a_{02}a_{03} + 4a_{02}a_{03}a_{11} - 6a_{02}^2 a_{12} + a_{03}a_{12} + a_{03}a_{11}a_{12} -$
$a_{02}a_{12}^2 + 10a_{02}^2 b_{03} - 2a_{03}a_{11}b_{03} + 9a_{02}a_{12}b_{03} + a_{12}^2 b_{03} - 4a_{02}b_{03}^2 - 3a_{12}b_{03}^2 -$
$5a_{02}a_{03}b_{12} - a_{03}a_{12}b_{12} + 3a_{03}b_{03}b_{12})a_{03}^2.$

It is easy to see that the set of conditions $\{(7), (16), (29)\}$ is a particular case for the set of conditions $\{(7), (15), (28)\}$ and the set of conditions $\{(8), (19), (35)\}$ is a particular case for $\{(8), (20), (39)\}$, the set of conditions $\{(10), (23), (43)\}$ is a particular case for the set of conditions $\{(10), (22), (42)\}$. The conditions $\{(8), (19), (37)\}$ and $\{(8), (20), (38)\}$ are the same.

Lemma 2.3. *The invariant straight line $x = 1$ has for quartic system* (5) *the multiplicity at least four if and only if the coefficients of* (5) *verify the following series of conditions:*
1) $\{(6), (11), (24)\}$*;* *2)* $\{(6), (12), (25)\}$*;* *3)* $\{(6), (13), (26)\}$*;*
4) $\{(7), (14), (27)\}$*;* *5)* $\{(7), (15), (28)\}$*;* *6)* $\{(8), (17), (30)\}$*;*
7) $\{(8), (17), (31)\}$*;* *8)* $\{(8), (18), (32)\}$*;* *9)* $\{(8), (18), (33)\}$*;*
10) $\{(8), (18), (34)\}$*;* *11)* $\{(8), (19), (36)\}$*;* *12)* $\{(8), (19), (37)\}$*;*
13) $\{(8), (20), (39)\}$*;* *14)* $\{(9), (21), (40)\}$*;* *15)* $\{(10), (22), (41)\}$*;*
16) $\{(10), (22), (42)\}$*;* *17)* $\{(10), (23), (44)\}$*;* *18)* $\{(10), (23), (45)\}$.

The multiplicity of the invariant straight line $x = 1$ is at least five if in each of the cases 1)–18) of Lemma 2.3 the identity $Y_5(y) \equiv 0$ holds. Proceeding as in the previous case and taking into account (3), we will examine each case separately.

1) $\{(6), (11), (24)\} \Rightarrow Y_5(y) \equiv 0 \Rightarrow$

$$\begin{aligned} b_{30} &= (3 - 2a_{20}^2 - 3b_{02} + a_{20}b_{11} + 2b_{20} - 2b_{02}b_{20})/(b_{02} - 1), \\ b_{21} &= (-5a_{20} + 2a_{20}b_{02} + 4b_{11} - 2b_{02}b_{11})/(b_{02} - 2); \end{aligned} \tag{46}$$

$$b_{21} = -a_{20}, \quad b_{11} = 2a_{20}, \quad b_{02} = 1. \tag{47}$$

2) $\{(6), (12), (25)\}$ − the identity $Y_5(y) \equiv 0$ and the conditions (3) are not compatible.

3) $\{(6), (13), (26)\} \Rightarrow Y_5(y) = -9 + 3a_{30}^2 + 6b_{30} - b_{30}^2 + 4a_{30}(3 - b_{30})y \not\equiv 0$, because in this case $a_{30} \neq 0$.

4) $\{(7), (14), (27)\} \Rightarrow Y_5(y) = (2a_{20} + a_{30} + (3 + 2a_{11} + a_{21})y)^2(2(3 + 2a_{11} + a_{21})^2 - (2a_{20} + a_{30})^2 - 3(3 + 2a_{11} + a_{21})(2a_{20} + a_{30})y)/(3 + 2a_{11} + a_{21})^2 \not\equiv 0$.

In the conditions 5) $\{(7), (15), (28)\}$, 6) $\{(8), (17), (30)\}$, 7) $\{(8), (17), (31)\}$, 8) $\{(8), (18), (32)\}$ − the identity $Y_5(y) \equiv 0$ and the conditions (3) are not compatible.

9) $\{(8), (18), (33)\} \Rightarrow Y_5(y) \equiv 0 \Rightarrow$

$$b_{02} = 1, \quad a_{20} = 0, \quad a_{12} = -(7a_{02} + 3a_{02}b_{20})/(2 + b_{20}). \tag{48}$$

10) $\{(8), (18), (34)\} \Rightarrow Y_5(y) \equiv 0 \Rightarrow$

$$a_{20} = 0, \quad b_{02} = 1, \quad b_{20} = -(7a_{02} + 2a_{12})/(3a_{02} + a_{12}). \tag{49}$$

In the conditions 11) $\{(8), (19), (36)\}$, 12) $\{(8), (19), (37)\}$ – the identity $Y_5(y) \equiv 0$ and the conditions (3) are not compatible.

13) $\{(8), (20), (39)\} \Rightarrow Y_5(y) \not\equiv 0$.

14) $\{(9), (21), (40)\} \Rightarrow Y_5(y) = (b_{11} + b_{21} + b_{31})y(-3 - 2b_{20} - b_{30} + (2b_{02} + b_{12})y^2 + 2b_{03}y^3) \equiv 0 \Rightarrow$

$$b_{03} = 0, \quad b_{12} = -2b_{02}, \quad b_{30} = -3 - 2b_{20}. \tag{50}$$

15) $\{(10), (22), (41)\} \Rightarrow Y_5(y) \equiv 0 \Rightarrow$

$$\begin{aligned}
&b_{11} = (-23a_{02}^3 + 5a_{02}a_{03}^2 - 10a_{02}^2a_{12} + a_{03}^2a_{12} - a_{02}a_{12}^2 + 46a_{02}^2b_{03} - \\
&3a_{03}^2b_{03} + 15a_{02}a_{12}b_{03} + a_{12}^2b_{03} - 29a_{02}b_{03}^2 - 5a_{12}b_{03}^2 + 6b_{03}^3)/a_{03}^2, \\
&a_{11} = (23a_{02}^3 - 3a_{02}a_{03} - 5a_{02}a_{03}^2 + 10a_{02}^2a_{12} - a_{03}^2a_{12} + a_{02}a_{12}^2 - \\
&46a_{02}^2b_{03} + 3a_{03}b_{03} + 3a_{03}^2b_{03} - 15a_{02}a_{12}b_{03} - a_{12}^2b_{03} + 29a_{02}b_{03}^2 + \\
&5a_{12}b_{03}^2 - 6b_{03}^3)/(a_{03}(a_{02} - b_{03})), \\
&a_{20} = (-24a_{02}^3 + 5a_{02}a_{03}^2 - 10a_{02}^2a_{12} + a_{03}^2a_{12} - a_{02}a_{12}^2 + 48a_{02}^2b_{03} - \\
&3a_{03}^2b_{03} + 15a_{02}a_{12}b_{03} + a_{12}^2b_{03} - 30a_{02}b_{03}^2 - 5a_{12}b_{03}^2 + 6b_{03}^3)/a_{03}^2;
\end{aligned} \tag{51}$$

$$b_{11} = 0, \quad a_{12} = -2a_{02}, \quad b_{03} = a_{02}. \tag{52}$$

16) $\{(10), (22), (42)\} \Rightarrow Y_5(y) \equiv 0 \Rightarrow$

$$b_{03} = a_{02}, \quad a_{12} = -2a_{02}, \quad a_{21} = -3 - 2a_{11}, \quad a_{30} = -2a_{20}. \tag{53}$$

17) $\{(10), (23), (44)\} \Rightarrow Y_5(y) \equiv 0 \Rightarrow$

$$b_{11} = 0, \quad b_{03} = a_{02}, \quad a_{12} = -2a_{02}; \tag{54}$$

$$\begin{aligned}
&b_{11} = (-23a_{02}^3 + 5a_{02}a_{03}^2 - 10a_{02}^2a_{12} + a_{03}^2a_{12} - a_{02}a_{12}^2 + 46a_{02}^2b_{03} - \\
&3a_{03}^2b_{03} + 15a_{02}a_{12}b_{03} + a_{12}^2b_{03} - 29a_{02}b_{03}^2 - 5a_{12}b_{03}^2 + 6b_{03}^3)/a_{03}^2, \\
&a_{20} = (-24a_{02}^3 + 5a_{02}a_{03}^2 - 10a_{02}^2a_{12} + a_{03}^2a_{12} - a_{02}a_{12}^2 + 48a_{02}^2b_{03} - \\
&3a_{03}^2b_{03} + 15a_{02}a_{12}b_{03} + a_{12}^2b_{03} - 30a_{02}b_{03}^2 - 5a_{12}b_{03}^2 + 6b_{03}^3)/a_{03}^2 \\
&a_{11} = (23a_{02}^3 - 3a_{02}a_{03} - 5a_{02}a_{03}^2 + 10a_{02}^2a_{12} - a_{03}^2a_{12} + \\
&a_{02}a_{12}^2 - 46a_{02}^2b_{03} + 3a_{03}b_{03} + 3a_{03}^2b_{03} - 15a_{02}a_{12}b_{03} - \\
&a_{12}^2b_{03} + 29a_{02}b_{03}^2 + 5a_{12}b_{03}^2 - 6b_{03}^3)/(a_{03}(a_{02} - b_{03})).
\end{aligned} \tag{55}$$

18) $\{(10), (23), (45)\} \Rightarrow Y_5(y) \equiv 0 \Rightarrow$

$$\begin{aligned}
&b_{03} = (5a_{02} + a_{12})/3, \\
&b_{11} = 2(2a_{02} + a_{12})(-2a_{02}^2 + 9a_{03} + 3a_{03}a_{11} - a_{02}a_{12})/(9a_{03}^2), \\
&b_{12} = (8a_{02}^2 - 3a_{03} + 3a_{03}a_{11} + 8a_{02}a_{12} + 2a_{12}^2)/(6a_{03});
\end{aligned} \tag{56}$$

$$b_{11} = 0,\ b_{03} = a_{02},\ a_{12} = -2a_{02},\ b_{12} = 1 + a_{11}; \tag{57}$$

$$\begin{aligned} &b_{11} = (-23a_{02}^3 + 5a_{02}a_{03}^2 - 10a_{02}^2a_{12} + a_{03}^2a_{12} - a_{02}a_{12}^2 + 46a_{02}^2b_{03} - \\ &3a_{03}^2b_{03} + 15a_{02}a_{12}b_{03} + a_{12}^2b_{03} - 29a_{02}b_{03}^2 - 5a_{12}b_{03}^2 + 6b_{03}^3)/a_{03}^2, \\ &a_{11} = (23a_{02}^3 - 3a_{02}a_{03} - 5a_{02}a_{03}^2 + 10a_{02}^2a_{12} - a_{03}^2a_{12} + a_{02}a_{12}^2 - \\ &46a_{02}^2b_{03} + 3a_{03}b_{03} + 3a_{03}^2b_{03} - 15a_{02}a_{12}b_{03} - a_{12}^2b_{03} + 29a_{02}b_{03}^2 + \\ &5a_{12}b_{03}^2 - 6b_{03}^3)/(a_{03}(a_{02} - b_{03})), \\ &b_{12} = (22a_{02}^3 - 2a_{02}a_{03} - 5a_{02}a_{03}^2 + 9a_{02}^2a_{12} - a_{03}^2a_{12} + a_{02}a_{12}^2 - \\ &44a_{02}^2b_{03} + 2a_{03}b_{03} + 3a_{03}^2b_{03} - 13a_{02}a_{12}b_{03} - a_{12}^2b_{03} + 28a_{02}b_{03}^2 + \\ &4a_{12}b_{03}^2 - 6b_{03}^3)(a_{03}(a_{02} - b_{03})). \end{aligned} \tag{58}$$

The sets of conditions {(10), (22), (41) (52)}, {(10), (22), (42) (53)}, {(10), (23), (44) (54)} and {(6), (11), (24), (46)}, {(10), (23), (44), (55)}, {(10), (23), (45), (58)} are the same. The set of conditions {(10), (23), (45), (57)} is a particular case for {(10), (22), (41), (52)}.

Lemma 2.4. *The invariant straight line $x = 1$ has for quartic system* (5) *the multiplicity at least five if and only if the coefficients of* (5) *verify the following series of conditions:*
1) {(6), (11), (24), (46)}*;* *2)* {(6), (11), (24), (47)}*;*
3) {(8), (18), (33), (48)}*;* *4)* {(8), (18), (34), (49)}*;*
5) {(9), (21), (40), (50)}*;* *6)* {(10), (22), (41), (51)}*;*
7) {(10), (22), (41), (52)}*;* *8)* {(10), (23), (45), (56)}.

The multiplicity of the invariant straight line $x = 1$ is at least six if in each of the cases 1)–8) of Lemma 2.4 the identity $Y_6(y) \equiv 0$ holds. Taking into account (3), we will examine each case separately:
1) {(6), (11), (24), (46)} $\Rightarrow \{Y_6(y) \equiv 0, \gcd(p, q) = 1\} \Rightarrow$

$$b_{02} = 3/2, \quad b_{11} = 2a_{20} \neq 0. \tag{59}$$

In the cases 2) {(6), (11), (24), (47)}, 3) {(8), (18), (33), (48)}, 4) {(8), (18), (34), (49)}, 6) {(10), (22), (41), (51)}, 7) {(10), (22), (41), (52)} − the identity $Y_6(y) \equiv 0$ and the conditions (3) are not compatible.

5) {(9), (21), (40), (50)} $\Rightarrow Y_6(y) \equiv 0 \Rightarrow$

$$b_{02} = 3, \quad b_{20} = -3. \tag{60}$$

8) {(10), (23), (45), (56)} $\Rightarrow Y_6(y) \not\equiv 0$.

Lemma 2.5. *The invariant straight line $x = 1$ has for quartic system* (5) *the multiplicity at least six if and only if the coefficients of* (5) *verify the following series of conditions:*
1) (6), (11), (24), (46), (59)*;* *2)* (9), (21), (40), (50), (60).

In the conditions 1) of Lemma 2.5 we have $Y_7(y) = a_{20}(7a_{20} + 2a_{20}b_{20} - 6y + 6a_{20}^2 y - 2b_{20}y - 6a_{20}y^2 + y^3) \not\equiv 0$, $a_{20} \neq 0$, otherwise ($a_{20} = 0$) the right-hand sides of (5) have the common divisors of degree greater than 0. Thus the multiplicity of the invariant straight line $x = 1$ is exactly six.

In the conditions 2) of Lemma 2.5 we have $Y_7(y) = -y(b_{11} + b_{21} + b_{31} + b_{11}y^2 + 2b_{21}y^2 + 3b_{31}y^2)$. The identity $Y_7(y) \equiv 0$ and the conditions (3) are not compatible (the right-hand sides of (5) have the common divisors of degree greater than 0), therefore the multiplicity of the invariant straight line $x = 1$ is exactly six.

In this way we have proved the following theorem.

Theorem 2.1. *In the class of quartic differential systems with a center-focus critical point and non-degenerate infinity the maximal multiplicity of an affine real invariant straight line is equal to six.*

3. Solution of the center problem for quartic systems with an affine invariant straight line of maximal multiplicity.

It is known that a critical point $(0, 0)$ is a center for (2) if and only if in a neighborhood of $(0, 0)$ the system has a nonconstant analytic first integral $F(x, y)$ (an analytic integrating factor of the form $\mu(x, y) = 1 + \sum \mu_j(x, y)$). If $F(x, y)$ ($\mu(x, y)$) has the form $f_1^{\alpha_1} \cdots f_s^{\alpha_s}$, where f_j, $1 \leq j \leq p$ are invariant straight lines and f_j, $p + 1 \leq j \leq s$ are exponential factors, then the system (2) is called Darboux integrable.

Let $F(x, y) = x^2 + y^2 + F_3(x, y) + F_4(x, y) + \cdots + F_n(x, y) + \cdots$, be a function such that

$$\frac{\partial F}{\partial x} p(x, y) + \frac{\partial F}{\partial y} q(x, y) \equiv \sum_{j=1}^{\infty} L_j (x^2 + y^2)^{j+1}, \tag{61}$$

where $F_k(x, y) = \sum_{i+j=k} f_{ij} x^i y^j$, $f_{0j} = 0$ if j is even. The L_j are polynomials in the coefficients of (2) and are called the Lyapunov quantities.

For example, the first two quantities look as

$L_1 = (a_{12} - a_{02}a_{11} - a_{11}a_{20} + 3a_{30} + 2a_{02}b_{02} - 3b_{03} + b_{02}b_{11} - 2a_{20}b_{20} + b_{11}b_{20} - b_{21})/4$,

$L_2 = (10a_{02}^3 a_{11} + 41a_{02}a_{03}a_{11} - 12a_{04}a_{11} - a_{02}a_{11}^3 - 10a_{02}^2 a_{12} - 21a_{03}a_{12} + a_{11}^2 a_{12} - 20a_{02}a_{13} + 124a_{02}^2 a_{11}a_{20} + 37a_{03}a_{11}a_{20} - a_{11}^3 a_{20} - 94a_{02}a_{12}a_{20} - 28a13a_{20} + 238a_{02}a_{11}a_{20}^2 - 112a_{12}a_{20}^2 + 124a_{11}a_{20}^3 + 19a_{02}a_{11}a_{21} - 15a_{12}a_{21} + 23a_{11}a_{20}a_{21} - 4a_{11}a_{22} - 90a_{02}^2 a_{30} - 27a_{03}a_{30} - 5a_{11}^2 a_{30} - 378a_{02}a_{20}a_{30} - 372a_{20}^2 a_{30} - 33a_{21}a_{30} - 12a_{02}a_{31} - 36a_{20}a_{31} + 20a_{11}a_{40} - 20a_{02}^3 b_{02} - 82a_{02}a_{03}b_{02} + 24a_{04}b_{02} - 39a_{02}a_{11}^2 b_{02} + 33a_{11}a_{12}b_{02} - 228a_{02}^2 a_{20}b_{02} - 32a_{03}a_{20}b_{02} - 37a_{11}^2 a_{20}b_{02} - 288a_{02}a_{20}^2 b_{02} - 24a_{20}^3 b_{02} + 2a_{02}a_{21}b_{02} + 40a_{20}a_{21}b_{02} - 16a_{22}b_{02} + 71a_{11}a_{30}b_{02} - 88a_{40}b_{02} + 158a_{02}a_{11}b_{02}^2 - 100a_{12}b_{02}^2 + 96a_{11}a_{20}b_{02}^2 - 248a_{30}b_{02}^2 -$

$152a_{02}b_{02}^3 + 24a_{20}b_{02}^3 + 30a_{02}^2b_{03} + 63a_{03}b_{03} + 9a_{11}^2b_{03} + 322a_{02}a_{20}b_{03} + 392a_{20}^2b_{03} + 21a_{21}b_{03} - 87a_{11}b_{02}b_{03} + 228b_{02}^2b_{03} + 80a_{02}b_{04} + 88a_{20}b_{04} - 37a_{02}^2a_{11}b_{11} - 8a_{03}a_{11}b_{11} + 37a_{02}a_{12}b_{11} + 8a13b_{11} - 138a_{02}a_{11}a_{20}b_{11} + 89a_{12}a_{20}b_{11} - 101a_{11}a_{20}^2b_{11} + 147a_{02}a_{30}b_{11} + 303a_{20}a_{30}b_{11} + 64a_{02}^2b_{02}b_{11} - 5a_{03}b_{02}b_{11} - 3a_{11}^2b_{02}b_{11} + 68a_{02}a_{20}b_{02}b_{11} - 144a_{20}^2b_{02}b_{11} - 7a_{21}b_{02}b_{11} + 29a_{11}b_{02}^2b_{11} - 76b_{02}^3b_{11} - 131a_{02}b_{03}b_{11} - 287a_{20}b_{03}b_{11} - 20b_{04}b_{11} + 27a_{02}a_{11}b_{11}^2 - 27a_{12}b_{11}^2 + 27a_{11}a_{20}b_{11}^2 - 81a_{30}b_{11}^2 + 3a_{02}b_{02}b_{11}^2 + 109a_{20}b_{02}b_{11}^2 + 77b_{03}b_{11}^2 - 23b_{02}b_{11}^3 - 21a_{02}a_{11}b_{12} + 21a_{12}b_{12} - 17a_{11}a_{20}b_{12} + 51a_{30}b_{12} + 2a_{02}b_{02}b_{12} - 40a_{20}b_{02}b_{12} - 39b_{03}b_{12} + b_{02}b_{11}b_{12} + 36b_{02}b13 - 29a_{02}a_{11}^2b_{20} + 29a_{11}a_{12}b_{20} + 60a_{02}^2a_{20}b_{20} + 18a_{03}a_{20}b_{20} - 27a_{11}^2a_{20}b_{20} + 252a_{02}a_{20}^2b_{20} + 248a_{20}^3b_{20} + 8a_{02}a_{21}b_{20} + 46a_{20}a_{21}b_{20} - 8a_{22}b_{20} + 59a_{11}a_{30}b_{20} - 80a_{40}b_{20} + 136a_{02}a_{11}b_{02}b_{20} - 86a_{12}b_{02}b_{20} + 28a_{11}a_{20}b_{02}b_{20} - 178a_{30}b_{02}b_{20} - 156a_{02}b_{02}^2b_{20} + 192a_{20}b_{02}^2b_{20} - 75a_{11}b_{03}b_{20} + 234b_{02}b_{03}b_{20} - 30a_{02}^2b_{11}b_{20} - 9a_{03}b_{11}b_{20} - 3a_{11}^2b_{11}b_{20} - 232a_{02}a_{20}b_{11}b_{20} - 350a_{20}^2b_{11}b_{20} - 3a_{21}b_{11}b_{20} + 42a_{11}b_{02}b_{11}b_{20} - 142b_{02}^2b_{11}b_{20} + 53a_{02}b_{11}^2b_{20} + 159a_{20}b_{11}^2b_{20} - 23b_{11}^3b_{20} - 8a_{02}b_{12}b_{20} - 50a_{20}b_{12}b_{20} + 5b_{11}b_{12}b_{20} + 12b13b_{20} + 30a_{02}a_{11}b_{20}^2 - 30a_{12}b_{20}^2 - 16a_{11}a_{20}b_{20}^2 - 30a_{30}b_{20}^2 - 60a_{02}b_{02}b_{20}^2 + 132a_{20}b_{02}b_{20}^2 + 90b_{03}b_{20}^2 + 13a_{11}b_{11}b_{20}^2 - 76b_{02}b_{11}b_{20}^2 + 20a_{20}b_{20}^3 - 10b_{11}b_{20}^3 + 30a_{02}^2b_{21} + 9a_{03}b_{21} + 3a_{11}^2b_{21} + 134a_{02}a_{20}b_{21} + 148a_{20}^2b_{21} + 3a_{21}b_{21} - 17a_{11}b_{02}b_{21} + 64b_{02}^2b_{21} - 53a_{02}b_{11}b_{21} - 105a_{20}b_{11}b_{21} + 23b_{11}^2b_{21} - 9b_{12}b_{21} - 13a_{11}b_{20}b_{21} + 46b_{02}b_{20}b_{21} + 10b_{20}^2b_{21} + 8a_{02}b_{22} + 16a_{20}b_{22} + 4b_{11}b_{22} - 15a_{02}a_{11}b_{30} + 15a_{12}b_{30} - 19a_{11}a_{20}b_{30} + 9a_{30}b_{30} + 30a_{02}b_{02}b_{30} + 32a_{20}b_{02}b_{30} - 45b_{03}b_{30} + 8a_{11}b_{11}b_{30} - 13b_{02}b_{11}b_{30} + 34a_{20}b_{20}b_{30} - 17b_{11}b_{20}b_{30} - 3b_{21}b_{30} - 8a_{11}b_{31} + 28b_{02}b_{31} + 20b_{20}b_{31} - 24a_{20}b_{40} + 12b_{11}b_{40})/96.$

The critical point $(0,0)$ is a center if all Lyapunov quantities L_j vanish. (see [2]).

In the following we will solve the center problem for the system (5) under the conditions 1) and 2) of Lemma 2.5, i.e. when the affine line $x-1=0$ is of maximal multiplicity.

The conditions 1) of Lemma 2.5 are

$$\begin{gathered}
a_{11} = -3,\ a_{02} = 0,\ a_{30} = -2a_{20},\ a_{21} = 3,\ a_{12} = 0,\ a_{03} = 0, \\
b_{11} = 2a_{20},\ b_{02} = 3/2,\ b_{30} = -2b_{20} - 3,\ b_{21} = 0,\ b_{12} = -3, b_{03} = 0, \\
b_{40} = 2a_{20}^2 + b_{20} + 2,\ b_{31} = -2a_{20},\ b_{22} = 3/2, b_{13} = 0,\ b_{04} = 0;\ a_{20} \neq 0.
\end{gathered} \tag{62}$$

The quartic system (5) takes the form:

$$\begin{aligned}
&\dot{x} = (x-1)^2(a_{20}x^2 + y - xy),\quad a_{20} \neq 0, \\
&\dot{y} = (-2x - 2b_{20}x^2 + 2(3+2b_{20})x^3 - 2(2 + 2a_{20}^2 + b_{20})x^4 - \\
&\quad -4a_{20}xy + 4a_{20}x^3y - 3y^2 + 6xy^2 - 3x^2y^2)/2.
\end{aligned} \tag{63}$$

We remark that the system (63) has the following integrating factor

$$\mu(x,y) = \frac{1}{(x-1)^6}.$$

The conditions 2) of Lemma 2.5 are

$$\begin{gathered}a_{20}=0,\ a_{11}=-3,\ a_{02}=a_{30}=0,\ a_{21}=3,\ a_{12}=a_{03}=0,\ b_{20}=-3,\\ b_{02}=3,\ b_{30}=3,\ b_{12}=-6,\ b_{03}=0,\ b_{40}=-1,\ b_{22}=3,\ b_{13}=0,\ b_{04}=0.\end{gathered}\tag{64}$$

The quartic system (5) takes the form:

$$\begin{aligned}&\dot{x}=-y(x-1)^3,\\ &\dot{y}=-x+3x^2-3x^3+x^4-b_{11}xy-b_{21}x^2y-b_{31}x^3y-\\ &\qquad -3y^2+6xy^2-3x^2y^2,\quad b_{11}^2+b_{21}^2+b_{31}^2\neq 0.\end{aligned}\tag{65}$$

The first two Lyapunov quantities of the system (65) are $L_1=-b_{21}/4$ and $L_2=b_{31}/2$. If $L_1=L_2=0$, i.e. $b_{21}=b_{31}=0$, then the system (65) has the following integrating factor

$$\mu(x,y)=\frac{1}{(x-1)^9}\exp\left(\frac{-b_{11}(b_{11}x^2(x^3-5x^2+10x-10)+20(x-1)^2y)}{20(x-1)^5}\right).$$

Theorem 3.1. *The quartic differential system* (5) *with an affine invariant straight line of maximal multiplicity six has a center at the origin* $(0,0)$ *if and only if its coefficients verify the following sets of conditions: 1)* (62); *2)* $\{(64),\ b_{21}=b_{31}=0\}$.

Theorem 3.2. *The quartic differential system* (5) *with an affine invariant straight line of maximal multiplicity six has a center at the origin* $(0,0)$ *if and only if the first two Lyapunov quantities vanish* $L_1=L_2=0$.

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On the symbol of singular operators in the case of contour with corner points

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Abstract. This paper proposes a method for constructing a symbol for singular integral operators in the case of a piecewise Lyapunov contour. The definition of the symbol function involves numbers that characterize the space in which the research is being carried out, as well as the values of the corner points of the contour, which makes it possible to obtain formulas for calculating the essential norms of singular operators and conditions for the solvability of singular equations with a shift and complex conjugation. In obtaining these results, we will essentially rely on the well-known results of I. Gelfand concerning maximal ideals of commutative Banach algebras [7]. In the absence of corner points on the integration contour, the results of this work are consistent with the results from [1].

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Keywords: singular operator, Banach algebras, piecewise Lyapunov contour, symbol, Noether conditions.

Asupra simbolului operatorilor singulari în cazul conturului cu puncte unghiulare

Rezumat. În această lucrare se propune o metodă de construire a simbolului operatorilor integrali singulari în cazul unui contur Lyapunov pe porţiuni. Definiţia funcţiei-simbol conţine parametrii, care caracterizează spaţiul în care se desfăşoară cercetarea, precum şi mărimile punctelor unghiulare ale conturului, ceea ce face posibilă obţinerea de formule de calcul a normelor esenţiale ale operatorilor singulari şi condiţiilor de rezolvabilitate a ecuaţiilor singulare cu translaţii şi conjugare complexă. În obţinerea acestor rezultate, ne vom baza în esenţă pe rezultatele binecunoscute ale lui I. Ghelfand privitoare la idealele maximale ale algebrelor Banach comutative [7]. În absenţa punctelor unghiulare pe conturul de integrare, rezultatele din această lucrare sunt în concordanţă cu rezultatele din [1].

Cuvinte-cheie: operator singular, algebre Banach, contur Lyapunov pe porţiuni, simbol, condiţii Noether.

1. Introduction

A great number of works are devoted to singular integral operators and Riemann boundary value problems in the case of a Lyapunov contour; it is enough to point out the monograph by I. Gokhberg and N. Krupnik [1], which contains an extensive bibliography on this issue. In papers [2], [3] and others, it was shown that the presence of corner points on the integration contour affects some properties of singular operators. In particular, if the integration contour contains one corner point with an angle equal to $\frac{\pi}{2}$, then the essential norm of the operator with the Cauchy kernel in the space L_2 is equal to $1 + \sqrt{2}$, and in the case of the Lyapunov contour this norm is equal to 1. The conditions for the Noetherian property of singular operators with shift or with complex conjugation also depend on the presence of corner points on the integration contour. As usual, by the Noether conditions of the operator A we mean, firstly, obtaining conditions under which the set of values of the operator A is a subspace, or the equality holds

$$ImA = \cap_{f \in KerA^*} Kerf,$$

and, secondly, the equations $Ax = 0$ and $A^*\varphi = 0$ have a finite number of linearly independent solutions. As it is known, a linear bounded Noetherian operator is true if and only if it has right and left regularizers. Obtaining the conditions for Noetherianity, as a rule, leads to the concept of an operator symbol, first introduced by S. Mikhlin, and which turned out to be fruitful in many branches of mathematics, including the construction of the Noetherian theory of singular integral operators [4], [5].

Note that Gelfand's theory of maximal ideals also played an important role in obtaining the criterion for the Noether property of one-dimensional singular integral operators with continuous coefficients, Wiener-Hopf operators, multidimensional singular operators, and Toeplitz matrices. The results presented in this paper are a generalization of known results to the case where the integration contour has corner points. Thus, in the case of the absence of corner points on the integration contour, the proposed results of this work agree with the results from [1].

Let us present some facts from the theory of Banach commutative algebras, which will be used below.

Definition 1.1. *A normed space X is called a normed algebra if it is an algebra with unity e and two more axioms are satisfied:*

$$\|e\| = 1; \quad \|xy\| \leq \|x\|\|y\| \quad \forall x, y \in X.$$

If the normed algebra X is also complete, then it is called a Banach algebra.

Let X be a commutative Banach algebra. An ideal M is called maximal if M is not contained in any other nontrivial ideal. Any ideal I (nontrivial) consists only of non-invertible elements. Any ideal is contained in a maximal ideal. According to I. Gelfand's Theorem [7], a Banach algebra over the field of complex numbers, which is a field, is isometrically isomorphic to the field $\mathbb{C}$.

A linear continuous functional f defined on a Banach algebra X is called multiplicative if for any x and y the equality holds

$$f(xy) = f(x) \cdot f(y).$$

The zero subspace of the functional f (i.e. the totality of those $x \in X$ for which $f(x) = 0$) is denoted by *Kerf* and is called the kernel of f.

Theorem 1.1. *The kernel Kerf for any multiplicative functional f is a maximal ideal.*

Theorem 1.2. *For any maximal ideal M, one can construct a unique multiplicative functional f such that Kerf=M.*

Conclusion. Thus, there is a one-to-one correspondence between the set of maximal ideals $\{M\}$ and the set of multiplicative functionals f defined on the algebra X. Therefore, the corresponding functionals are denoted f_M, $(f \leftrightarrow M)$.

Theorem 1.3. *(Gelfand (see* [7]. *An element $x \in X$ is invertible in X if and only if it is not contained in any maximal ideal (equivalent to $f(x) \neq 0$ for any multiplicative functional).*

Thus, the problem of invertibility in the algebra X can be reduced to determining all maximal ideals or to determining all multiplicative functionals defined on X.

2. Algebra $\mathcal{U}_{p\beta}$

Let $\mathcal{U}$ be some algebra (commutative or non-commutative). Recall that a set $\{f_M\}$ of multiplicative functionals is called sufficient if an element x is invertible in $\mathcal{U}$ if and only if $f_M(x) \neq 0$ for any M. According to I. Gelfand's theorem, every commutative Banach algebra has a sufficient set of multiplicative functionals. The set of functionals of the form $\{f_M\}$, where M runs over the set of maximal ideals, forms a sufficient set of functionals.

A simple example of a non-commutative Banach algebra that has a sufficient set of multiplicative functionals is the algebra of upper triangular numerical matrices

$$\mathcal{U} = \left\{ \begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix} \right\} \quad (a_{jk} \in C).$$

Two functionals $f_1(a) = a_{11}$ and $f_2(a) = a_{22}$ form a sufficient set.

Let $E = L_2(a, b)$ and let $\mathcal{U}$ be a subalgebra of $L(E)$, generated by one singular operator S:

$$\left(S_{\varphi}\right)(t) = \frac{1}{\pi i}\int_a^b \frac{\varphi(\tau)}{\tau - t}d\tau \quad (t \in [a, b]).$$

Since $S^* = S$, then $\mathcal{U}$ is a c^* subalgebra of $L(E)$ and, in particular, it is symmetric. The spectrum of the element S in the algebra $\mathcal{U}$ coincides with its spectrum in the algebra $L(E)$, i.e. with the segment $[-1, 1]$. Each multiplicative functional is defined by a point $\tau \in [-1, 1]$.

$$f_\tau\left(\sum_{k=0}^{n} \alpha_k S^k\right) = \sum_{k=0}^{n} \alpha_k \tau^k.$$

In particular, the operator $A = \alpha I + \beta S$ $(\alpha, \beta \in C)$ is invertible in $\mathcal{U}$ if and only if $\alpha + \beta\tau \neq 0, \forall\tau \in [-1, 1]$.

Consider the operator B, defined by the equality

$$(B\varphi)(t) = \alpha\varphi(t) + \frac{\beta}{\pi i}\int_a^b \frac{\varphi(\tau)}{\tau - t}d\tau + \frac{\gamma}{(\pi i)^2}\int_a^b Ln\frac{(b - t)(\tau - a)}{(t - a)(d - \tau)}\frac{\varphi(\tau)\,d\tau}{\tau - t}.$$

The operator B belongs to the algebra $\mathcal{U}$. Indeed, using the Poincaré-Bertrand formula, it is easy to deduce that $B = \alpha I + \beta S + \gamma\left(S^2 - I\right)$. This implies:

Theorem 2.1. *The operator B is invertible if and only if the inequality* $\gamma\tau^2+\beta\tau+(\alpha - \gamma) \neq 0$ *holds for all* $\tau \in [-1, 1]$.

Let us introduce the following notation. We denote by $L(\mathcal{B})$ the algebra of all linear bounded operators acting in a Banach space $\mathcal{B}$. Let $\mathcal{U}_{p\beta}$ be the smallest Banach subalgebra with algebra unit $L\left(L_p\left(R^+, t^\beta\right)\right)$ $(R^+ = [0, +\infty))$, containing the operator

$$\left(S_{\varphi}\right)(t) = \frac{1}{\pi i}\int_{R^+} \frac{\varphi(\tau)}{\tau - t}d\tau \quad \left(t \in R^+\right).$$

We will assume that $1 < p < \infty$ and $-1 < \beta < p - 1$. Let δ be a number from the interval $\left(0, \frac{1}{2}\right)$. Let us denote by $l(\delta)$ an arc of a circle containing points -1 and 1 having the following property: from point z $(z \neq \pm 1)$ of the arc $l(\delta)$ the segment $[-1, 1]$ is visible at an angle of $2\pi\delta$ and when going around the arc $l(\delta)$ from point -1 to 1 this segment remains to the left. For numbers δ from the interval $\left(\frac{1}{2}, 1\right)$ we set $l(\delta) = -l(1 - \delta)$. Let $l\left(\frac{1}{2}\right)$ denote the segment $[-1, 1]$. As, it is known [1], the spectrum of the operator S in the space $L_p\left(R^+, |t|^\beta\right)$ coincides with the arc $l\left(\frac{1+\beta}{p}\right)$. Since the algebra $\mathcal{U}_{p\beta}$ is generated by one element, then [1] takes place.

Theorem 2.2. *The set of maximal ideals of the algebra* $\mathcal{U}_{p\beta}$ *is homeomorphic to the arc* $l = l\left(\frac{1+\beta}{p}\right)$. *If* M_z *is the maximal ideal corresponding to the point* z $(\in l)$, *then the Gelfand transformation* $S\left(M_z\right) = z$.

This theorem can be significantly expanded (see [6]).

Theorem 2.3. *The algebra $\mathcal{U}_{p\beta}$ is an algebra without a radical with a symmetric involution $A \to \overleftarrow{A}$. In particular,*

$$\overleftarrow{S} = (\cos 2\pi\gamma S - i \sin 2\pi\gamma I)(\cos 2\pi\gamma I - i \sin 2\pi\gamma S)^{-1} \left(\gamma = \frac{1+\beta}{p}\right).$$

For $p = 2$, the Gelfand transformation $A(z) = A(M_z)$ satisfies the equality

$$\|A\| = \max_{z \in l(\gamma)} |A(z)|, \tag{1}$$

and for $p \neq 2$, the following estimates hold:

$$\max_{z \in l(\gamma)} |A(z)| \leq \|A\| \leq c \cdot \max\left(\max_{z \in l(\gamma)} |A(z)|,\ \max_{z \in l(\gamma)} \left| \left(1 - z^2\right) Ln \frac{1-z}{1+z} \frac{dA(z)}{dz} \right| \right) \tag{2}$$

where the constant c depends only on p and β.

Proof. Let $\gamma = \frac{1+\beta}{p}$. The operator B, defined by the equality $\left(B_\varphi\right)(t) = e^{\gamma t}\varphi\left(e^t\right)$, isometrically maps the space $L_p\left(R^+, t^\beta\right)$ onto $L_p(R)$. It is directly verified that the operator $\tilde{S} = BSB^{-1}$ has the form

$$\left(\tilde{S}_\varphi\right)(t) = \frac{1}{\pi i} \int_{-\infty}^{+\infty} \frac{e^{(t-s)\gamma}\varphi(s)}{1 - e^{t-s}} ds.$$

Thus, the algebra $\mathcal{U}_{p\beta}$, generated by one operator S, is isometric to some subalgebra of the convolution algebra and, therefore, has no radical [7]. Let $\pi i \hat{S}(\xi)$ be the Fourier transform of the function $\frac{e^{t\gamma}}{1-e^t}$. It can be shown (we will not go into details) that

$$\hat{S}(\xi) = \frac{e^{2\pi(\xi + i\gamma)} + 1}{e^{2\pi(\xi + i\gamma)} - 1} \ (-\infty \leq \xi \leq +\infty). \tag{3}$$

The set of values of the function $\hat{S}(\xi)$ runs along the arc $l(\gamma)$. We set $z = \hat{S}(\xi)$, then the operator $A \in \mathcal{U}_{p\beta}$ satisfies the equality

$$\boldsymbol{A}\left(\hat{S}(\xi)\right) = \left(FBAB^{-1}F^{-1}(\xi)\right),$$

where F is the Fourier transform. This, in particular, implies equality (1) for $p = 2$. For $p \neq 2$, a lower estimation for the norms of the operator A follows from Theorem 2.2. The upper estimation is obtained using theorem on multipliers of S. Mikhlin [4], in which it is established that

$$\|BAB^{-1}\| \leq \tilde{c}_p \cdot \max\left(\max_{\xi \in R} A\left(\hat{S}(\xi)\right)\right),\ \max_{\xi \in R} \left| \xi \cdot \frac{dA\left(\hat{S}(\xi)\right)}{d\xi} \right|,$$

where the number $\tilde{c}_p$ depends only on p. The theorem has been proven. □

Remark 2.1. *Let us define the functional over $L_2\left(R^+, t^\beta\right)$ by the equality*

$$f(\varphi)=\int_0^\infty \varphi(t)f(t)t^\beta dt,$$

then $S^*=t^{-\beta}St^\beta I$. It is directly verified that $FBS^*B^{-1}F^{-1}=FB\bar{S}B^{-1}F^{-1}$. Therefore, for $p=2$, we have $\bar{S}=S^*$.

Corollary 2.1. *Let the function f be differentiable at each point $z\in l(\gamma)\setminus\{-1,1\}$. If there exists a sequence of polynomials P_n such that*

$$\max_{z\in l(\gamma)}|P_n(z)-f(z)|\to 0;\quad \max_{z\in l(\gamma)}\left|(1-z^2)Ln\frac{1-z}{1+z}(P_n'(z)-f'(z)\right|\to 0$$

as $n\to\infty$, then $f(S)\in\mathcal{U}_{p\beta}$.

A more general corollary is the following.

Corollary 2.2. *Let $A_0\in\mathcal{U}_{p\beta}$ and let $\varphi(z)$ be the Gelfand's transform of operator A_0 and h be differentiable at each point $z\in l(\gamma)\setminus\{-1,1\}$. If there exists a sequence of polynomials P_n such that*

$$\max_{z\in l(\gamma)}|P_n(z)-h(z)|\to 0;\quad \max_{z\in l(\gamma)}\left|(1-z^2)\frac{d}{dz}Ln\frac{1-z}{1+z}(P_n(\varphi(z))-h(z)\right|\to 0$$

as $n\to\infty$, then $h(A_0)\in\mathcal{U}_{p\beta}$.

In what follows, we will need the following theorem.

Theorem 2.4. *Let $\omega=e^{\pi i\alpha}$, where α is some complex number. If $-1<Re\alpha<1$, then the operator N_ω, defined by the equality*

$$(N_\omega\varphi)(x)=\frac{1}{\pi i}\int_{R^+}\frac{\varphi(y)}{y+\omega x}dy,\ \left(x\in R^+\right),$$

belongs to the algebra $\mathcal{U}_{p\beta}$ and its Gelfand transformation has the form

$$N_\omega(z)=(z-1)^{\frac{1+\alpha}{2}}(z+1)^{\frac{1-\alpha}{2}}\ (z\in l(\gamma)).\tag{4}$$

The branch of this function is chosen so that at $z=-ictg\pi\gamma$ it takes the value

$$-\frac{iexp(-\pi i\gamma\alpha)}{\sin\pi\gamma}.$$

Proof. It is directly verified that

$$\pi iBN_\omega B^{-1}\varphi=\left(e^{\gamma t}\left(1+\omega e^t\right)\right)*\varphi.$$

It follows that

$$f_z(N_\omega)=\frac{1}{\pi i}\int_{-\infty}^{+\infty}\frac{e^{\gamma t-i\xi t}}{t+\omega e^t}dt=\frac{-ie^{i\xi-\gamma}}{\sin((\gamma-i\xi)\pi)}=(z-1)^{\frac{1+\alpha}{2}}(z+1)^{\frac{1-\alpha}{2}}.$$

Let us show that the function $h(z) = f_z(N_\omega)$ satisfies the conditions of Corollary 2.1 or 2.2. Let first $\left|\gamma - \frac{1}{2}\right| \leq \frac{1}{4}$, then $|z| \leq 1$. In this case, for any $\delta(Re\delta > 0)$, the function $(z+1)^\delta$ satisfies the condition of Corollary 2.2 (for example, partial sums of the Taylor series can be taken as the polynomials $P_n(z)$. If $|\gamma - \frac{1}{2}| > \frac{1}{4}$, then the function

$$f_z\left(N_\omega\right) = z\left(1 - z^{-1}\right)^{\frac{1+\alpha}{2}}\left(1 + z^{-1}\right)^{\frac{1-\alpha}{2}}$$

satisfies the conditions of Corollary 2.1. The role of the operator A_0 is played by the operator S^{-1}. The invertibility of the operator S follows from the condition $\gamma \neq \frac{1}{2}$. The theorem is proved. □

3. Symbol of the operator $aI + bS_\Gamma$

Let the contour Γ_α consist of two semi-axes starting from the point $z = 0$. We denote by α $(0 < \alpha \leq \pi)$ the angle formed by these half-lines. We will assume that one of these semi-straight lines coincides with the semi-axis $R^+ = [0, +\infty)$ and that the contour Γ_α is oriented in such a way that on $\Gamma_\alpha \cap R^+$ the orientation coincides with that on R^+.

Let $B = L_p\left(\Gamma_\alpha, |t|^\beta\right)$ $(-1 < \beta < p-1)$ and denote by $\lambda_0\left(\Gamma_\alpha\right)$ the set of constant functions on portions that receive two values on Γ_α: one value on R^+ and another value on $\Gamma_\alpha \backslash R^+$. If $h \in \lambda_0(\Gamma)$, then we write

$$h(t) = \begin{cases} h_1, & for \quad t \in R^+ \\ h_2, & for \quad t \in \Gamma_\alpha \backslash R^+ \end{cases}, \ h_j \in \mathbb{C}.$$

So, $h(0) = h_2$, $h(0+0) = h_1$, $h(\infty - 0) = h_1$, $h(\infty + 0) = h_2$.

We will consider the contour Γ_α compactified with a point at infinity, whose neighborhoods are complementary to the neighborhoods of $z_0 = 0$. Obviously, the contour Γ_α is homeomorphic to a bounded contour $\tilde{\Gamma}$, which has two angular points.

We denote by K_α the Banach algebra generated by the singular integration operator S_Γ and by all multiplication operators on the functions $h \in \lambda_0(\Gamma_\alpha)$. By K^+ we denote the subalgebra of the algebra $L\left(L_p\left(R^+, |t|^\beta\right)\right)$ generated by the singular integral operators $aI + bS$ $(S = S_{R^+})$ with constant coefficients on R^+. As K^+ is commutative, then it possesses [5] a sufficient system of multiplicative functionals. The operator ν,

$$(\nu\varphi)(x) = \left(\varphi(x), \varphi\left(e^{i\alpha}x\right)\right) \left(x \in R^+\right),$$

is linear and bounded and acts from the space $L_p\left(\Gamma_\alpha, |t|^\beta\right)$ to the space $L_p^2\left(R^+, t^\beta\right)$. Let $\varphi \in L_p\left(\Gamma_\alpha, |t|^\beta\right)$ and consider the equation

$$A\varphi = a\varphi + bS_{\Gamma_\alpha}\varphi = \psi,$$

$$a(t)=\begin{cases} a_1, & for \quad t\in R^+ \\ a_2, & for \quad t\in\Gamma_\alpha\backslash R^+ \end{cases}, \qquad b(t)=\begin{cases} b_1, & for \quad t\in R^+ \\ b_2, & for \quad t\in\Gamma_\alpha\backslash R^+ \end{cases}, \qquad a_j,b_j\in\mathbb{C}.$$

This equation can be written as a system of equations: in one equation $t\in R^+$, and in the second equation $t\in\Gamma_\alpha\backslash R^+$. We get,

$$\begin{cases} a(t)\varphi(t)+\frac{b(t)}{\pi i}\int_{R^+}\frac{\varphi(\tau)}{\tau-t}d\tau+\frac{b(\tau)}{\pi i}\int_{\Gamma_\alpha\backslash R^+}\frac{\varphi(\tau)}{\tau-t}d\tau=\psi(t), & t\in R^+, \\ a(t)\varphi(t)+\frac{b(t)}{\pi i}\int_{R^+}\frac{\varphi(\tau)}{\tau-t}d\tau+\frac{b(\tau)}{\pi i}\int_{\Gamma_\alpha\backslash R^+}\frac{\varphi(\tau)}{\tau-t}d\tau=\psi(t), & t\in\Gamma_\alpha\setminus R^+. \end{cases}$$

In the integral

$$\int_{\Gamma_\alpha\backslash R^+}\frac{\varphi(\tau)}{\tau-t}d\tau$$

we change the variable $\tau\longrightarrow e^{i\alpha}\tau$ and in the second equation of the obtained system, we change t by $e^{i\alpha}t$. Then, we obtain

$$\begin{cases} a_1\varphi_1(t)+\frac{b_1}{\pi i}\int_{R^+}\frac{\varphi_1(\tau)}{\tau-t}d\tau-\frac{b_1}{\pi i}\int_{R^+}\frac{\varphi_2(\tau)}{\tau-e^{-i\alpha}t}d\tau=\psi_1(t), & t\in R^+, \\ a_2\varphi_2(t)+\frac{b_2}{\pi i}\int_{R^+}\frac{\varphi_1(\tau)}{\tau-e^{i\alpha}t}d\tau-\frac{b_2}{\pi i}\int_{R^+}\frac{\varphi_2(\tau)}{\tau-t}d\tau=\psi_2(t), & t\in R^+. \end{cases}$$

in which the notations were used: $f_1(t)=f(t)$, $f_2(t)=f\left(e^{i\alpha}t\right)$ $(t\in R^+)$.

Thus, the operator $\nu A\nu^{-1}$ has the form

$$\nu A\nu^{-1}=\begin{Vmatrix} a_1I+b_1S, & -b_1M \\ b_2N, & a_2I-b_2S \end{Vmatrix},$$

where

$$(S\varphi)(t)=\frac{1}{\pi i}\int_{R^+}\frac{\varphi(\tau)}{\tau-t}d\tau, \quad (M\varphi)(t)=\frac{1}{\pi i}\int_{R^+}\frac{\varphi(\tau)}{\tau-e^{-i\alpha}t}d\tau,$$

$$(N\varphi)(t)=\frac{1}{\pi i}\int_{R^+}\frac{\varphi(\tau)}{\tau-e^{i\alpha}t}d\tau \quad \left(t\in R^+\right).$$

From Theorems 2.2 and 2.3 it follows that operators M and N belong to the algebra K^+ generated by the operator $S(=S_{R^+})$ and the multiplication operators to the constant functions. Therefore, $\nu K_\alpha\nu^{-1}\subset(K^+)^{2\times2}$. Let $\{\gamma_M\}$ be the homeomorphism system that determines the symbol on the algebra K^+. For any operator $A\in K_\alpha$ we put

$$\widetilde{\gamma}_M(A)=\left\|\gamma_M\left(A_{jk}\right)\right\|_{j,k=1}^2, \text{ where } \left\|A_{jk}\right\|_{j,k=1}^2=\nu A\nu^{-1}.$$

4. Conditions for Noetherianity

Theorem 4.1. *The operator $A\in K_\alpha$ is Noetherian in the space $L_p\left(\Gamma_\alpha,|t|^\beta\right)$ if and only if*

$$det\widetilde{\gamma}_M(A)\neq0.$$

Indeed, the factor algebra $\widehat{K}^{+}$ with respect to all compact operators in $L\left(L_p\left(R^{+},t^{\beta}\right)\right)$ is commutative, therefore, the elements of the matrix operator $\left\|A_{jk}\right\|_{j,k=1}^{2} = \nu A\nu^{-1}$ commute up to compact. Then, according to [5], the operator $\left\|A_{jk}\right\|_{j,k=1}^{2}$ is Noetherian in $L_p\left(R^{+},t^{\beta}\right)$, if and only if the operator $\Delta = det\left\|A_{jk}\right\|$ is Noetherian in $L_p\left(R^{+},t^{\beta}\right)$. But the operator $det\left\|A_{jk}\right\|$ is Noetherian if and only if $\gamma_M\left(det\left\|(A_{jk})\right\|\right)$. As $\gamma_M\left(det\left\|(A_{jk})\right\|\right) = det\left\|\gamma_M\left(A_{jk}\right)\right\|$, it follows that A is Noetherian if and only if $det\widetilde{\gamma}_M(A) \neq 0$.

The theorem is proved.

Conclusion. Theorem 4.1 allows us to define a symbol on the algebra K. Namely, it is natural to call the matrix $\widetilde{\gamma}_M(A)$ a symbol of the operators $A \in K$. Taking into account formulas (3) and (4), the symbol of the operators $H = hI$, $h \in \lambda_0(\Gamma)$ and S_Γ will have the form:

$$\widetilde{\gamma}_M(H)=\left\|\begin{matrix} h_1 & 0 \\ 0 & h_2 \end{matrix}\right\|,\ \widetilde{\gamma}_M(S_\Gamma)=\left\|\begin{matrix} z & (z-1)^{1-\frac{\alpha}{2\pi}}(z+1)^{\frac{\alpha}{2\pi}} \\ (z-1)^{\frac{\alpha}{2\pi}}(z+1)^{1-\frac{\alpha}{2\pi}} & -z \end{matrix}\right\|. \quad (5)$$

We will write the symbol of the operator S_Γ in a more convenient form. For this let us put

$$z = \frac{e^{2\pi(\xi+i\gamma)}+1}{e^{2\pi(\xi+i\gamma)}-1} = cth\left(\pi\left(\xi+i\gamma\right)\right)\ \left(-\infty \leq \xi \leq +\infty,\ \gamma = \frac{1+\beta}{p}\right).$$

Then

$$(z-1)^{1-\frac{\alpha}{2\pi}}(z+1)^{\frac{\alpha}{2\pi}} = 2\frac{e^{(\alpha-\pi)(\xi+i\gamma)}}{e^{\pi(\xi+i\gamma)}-e^{-\pi(\xi+i\gamma)}} = \frac{e^{(\alpha-\pi)(\xi+i\gamma)}}{sh\pi(\xi+i\gamma)},$$

$$(z-1)^{\frac{\alpha}{2\pi}}(z+1)^{1-\frac{\alpha}{2\pi}} = 2\frac{e^{(\pi-\alpha)(\xi+i\gamma)}}{e^{\pi(\xi+i\gamma)}-e^{-\pi(\xi+i\gamma)}} = \frac{e^{(\pi-\alpha)(\xi+i\gamma)}}{sh\pi(\xi+i\gamma)}.$$

Therefore the symbol of the operator S_Γ takes the form

$$\widetilde{\gamma}_M(S_\Gamma) = \left\|\begin{matrix} cth\left(\pi(\xi+i\gamma)\right) & \frac{e^{(\alpha-\pi)(\xi+i\gamma)}}{sh\pi(\xi+i\gamma)} \\ \frac{e^{(\pi-\alpha)(\xi+i\gamma)}}{sh\pi(\xi+i\gamma)} & -cth\left(\pi(\xi+i\gamma)\right) \end{matrix}\right\|. \quad (6)$$

Remark 4.1. *If $\alpha = \pi$, that is, the contour Γ satisfies the Lyapunov conditions at the point $z_0 = 0$, then the symbol of the operator $H = hI$ remains the same, and the symbol of the operator S_Γ has the form*

$$\widetilde{\gamma}_M(S_\Gamma) = \left\|\begin{matrix} z & \sqrt{z^2-1} \\ \sqrt{z^2-1} & -z \end{matrix}\right\| = \left\|\begin{matrix} cth\pi\left(\xi+i\gamma\right) & \left(sh\pi(\xi+i\gamma)\right)^{-1} \\ \left(sh\pi(\xi+i\gamma)\right)^{-1} & -cth\pi\left(\xi+i\gamma\right) \end{matrix}\right\|. \quad (7)$$

Now we have what it is needed to define the symbol of the singular integral operators with coefficients in $CP(\Gamma)$ in the case of the piecewise Lyapunov contour.

So, let Γ be a piecewise closed Lyapunov contour. We denote by $t_1, \dots, t_n$ all angular points with angles α_k $(0 < \alpha_k < \pi)$ $(k = 1, \dots, n)$ and

$$p(t) = \prod_{k=1}^{n} |t - t_k|^{\beta_k} \ (1 < p < \infty, \ -1 < \beta_k < p - 1) \, .$$

We denote by $\Sigma(\Gamma, p) (\subset L(L_p(\Gamma, p)))$ the algebra generated by the operators $(H\varphi)(t) = h(t)\varphi(t)$, $h(t) \in CP(\Gamma)$ and the operator S_Γ. We mention, that the ideal formed by the compact operators acting in the space $L_p(\Gamma, p)$ is contained in the algebra $\Sigma(\Gamma, p)$.

$$H(t, \xi) = \left\| \begin{array}{cc} h(t+0) & 0 \\ 0 & h(t-0) \end{array} \right\| . \tag{8}$$

We define the symbol $S_\Gamma(t, \xi)$ of the operator S_Γ as follows:

$$S(t, \xi) = \left\| \begin{array}{cc} cth\pi(\xi + i\gamma(t)) & -\frac{exp((\alpha(t)-\pi)(\xi+i\gamma(t)))}{sh\pi(\xi+i\gamma(t))} \\ \frac{exp((\pi-\alpha(t))(\xi+i\gamma(t)))}{sh\pi(\xi+i\gamma(t))} & -cth\pi(\xi + i\gamma(t)) \end{array} \right\| , \tag{9}$$

where

$$\alpha(t) = \begin{cases} \alpha_k, & if \quad t = t_k (k = 1, 2, \dots, n) \\ \pi, & if \quad t \in \Gamma \setminus \{t_1, t_2, \dots, t_n\} \end{cases}$$

and

$$\gamma(t) = \begin{cases} \frac{1+\beta_k}{p}, & if \quad t = t_k (k = 1, 2, \dots, n) \\ \frac{1}{p}, & if \quad t \in \Gamma \setminus \{t_1, t_2, \dots, t_n\} \end{cases}$$

Theorem 4.2. *Let and $A \in \Sigma(\Gamma, \rho)$ and $A(t, \xi)$ be its symbol. The operator A is Noetherian in the space $L_p(\Gamma, \rho)$ if and only if*

$$detA(t, \xi) \neq 0 \quad (t \in \Gamma, -\infty \leq \xi \leq +\infty).$$

The proof of Theorem 4.2 follows from Theorem 4.1, using the results from [8].

Theorems 4.1 and 4.2 can be generalized to the case where the integration contour is complex. More precisely, let Γ consist of n rays: $\Gamma = \bigcup_{m=1}^{n} \Gamma_m$, where $\Gamma_m = (\varepsilon_m x : x \in R^+, \ \varepsilon_m \in \mathbb{C}, ||\varepsilon_m| = 1)$, $PC_0(\Gamma)$ is the set of functions continuous on $\Gamma \setminus \{0\}$ and having finite limits as $t \to 0$ and $t \to \infty$ along each ray Γ_m and $K_p (\subset L(L_p(\Gamma)))$ is the algebra generated by singular operators with coefficients from $PC_0(\Gamma)$. We assume that $\varepsilon_1 = 1$, i.e. that $\Gamma_1 = R^+$. Let μ denote the isometry $L_p(\Gamma) \to L_p^n(\Gamma_1)$, defined by the equality $\mu\varphi = (\varphi_1, \varphi_2, \dots, \varphi_n)$, where $\varphi_k(t) = \varphi(\varepsilon_k t)$ $(k = 1, 2, \dots, n; t \geq 0)$. In this case

$$\mu H \mu^{-1} = \left\| \begin{array}{cccc} H_1 & 0 & \dots & 0 \\ 0 & H_2 & \dots & 0 \\ . & . & . & . \\ 0 & 0 & \dots & H_n \end{array} \right\| , \quad \mu S_\Gamma \mu^{-1} = \left\| R_{jk} \right\|_{j,k=1}^{n} .$$

Here

$$(H\varphi)(t) = h(t)\varphi(t),\ (H_k\varphi)(t) = h(\varepsilon_k t)\varphi(t)\ (t \in \Gamma_1)$$

and

$$(R_{jk}\varphi) = \frac{1}{\pi i}\int_0^\infty \frac{\varphi(\tau)d\tau}{\tau - \varepsilon_j^{-1}\varepsilon_k t}.$$

It follows from Theorem 2.4 that $R_{jk} \in K^+$, hence $\mu K_p \mu^{-1} \subset (K^+)^{n\times n}$. As in Theorem 4.1, it can be shown that the operator $A \in K_p$ is Noetherian if and only if the condition

$$det \left\|\widetilde{\gamma}_M\left(A_{jk}\right)\right\|_{j,k=1}^n \neq 0,$$

where $\mu A \mu^{-1} = \left\|A_{jk}\right\|_{j,k=1}^n$. Thus, $\left\|\widetilde{\gamma}_M\left(A_{jk}\right)\right\|_{j,k=1}^n$ defines a matrix symbol on K_p.

5. Calculation of essential norms of singular operators

Recall (see [9]) that for any operator A from some Banach algebra $\mathcal{U}$ with symmetric symbol the following relation holds:

$$\inf_{T\in\mathcal{T}} \|A+T\| = \max_x S_1\left(\mathcal{A}(x)\right), \tag{10}$$

where $\mathcal{A}(x)$ is the symbol of the operator A, and $S_1^2(\mathcal{A}(x))$ denotes the largest eigenvalue of the matrix $\mathcal{A}(x)\cdot(\mathcal{A}(x))^*$. Equality (10) is equivalent to the following equality

$$\inf_{T\in\mathcal{J}} \|A+T\|^2 = \max_{\lambda\in\hat{\sigma}(AA^*)} \lambda, \tag{11}$$

where $\hat{\sigma}(AA^*)$ denotes the spectrum of the residue class $\{AA^* + T\}$ in the quotient algebra $\mathcal{U}/\mathcal{T}$. The set $\hat{\sigma}(AA^*)$ coincides with the set of numbers λ for which the operator $AA^* - \lambda I$ is not Noetherian.

Applying equality (11) to the operator S_{Γ_α}, taking into account formula (7), we obtain,

$$|S_{\Gamma_\alpha}|_\beta^2 = \lim_{\xi\in R}\left(f(\xi) + \sqrt{f^2(\xi)-1}\right), \tag{12}$$

where

$$f(\xi) = \frac{e^{4\pi\xi} + 2\left(e^{(4\pi-2\alpha)\xi} + e^{2\alpha\xi} - \cos\pi\beta e^{2\pi\xi}\right) + 1}{e^{4\pi\xi} + 2\cos\pi\beta e^{2\pi\xi} + 1}.$$

Let us give some examples. Suppose $\alpha = \pi$, i.e. Γ_α is the real axis $\mathbb{R}$, then from equality (12) we obtain

$$|S_{\Gamma_\alpha}|_\beta = \text{ctg}\,\frac{\pi(1-|\beta|)}{4}.$$

Assume that $\beta = 0$ and let

$$z = \frac{1-e^{2\pi\xi}}{1+e^{2\pi\xi}}\quad(-\infty \leq \xi \leq +\infty),$$

then from equality (12) follows the following formula for the essential norm of the operator S_{Γ_α}:

$$|S_{\Gamma_\alpha}|_0 = \operatorname{ctg}\left(\frac{\theta(\alpha)}{2}\right),$$

where

$$2\operatorname{ctg}\theta(\alpha) = \max_{-1\leq z\leq 1}\left|(1+z)\left(\frac{1-z}{1+z}\right)^{\frac{\alpha}{2\pi}} + (1-z)\left(\frac{1+z}{1-z}\right)^{\frac{\alpha}{2\pi}}\right|.$$

In particular, for $\alpha = \frac{\pi}{3}$, $\alpha = \frac{\pi}{2}$, we obtain $|S_{\Gamma_\alpha}|_0 = \frac{1+\sqrt{5}}{2}$, $|S_{\Gamma_\alpha}|_0 = \sqrt{2}$.

Thus, in the case of a contour with corner points, the essential norm of the singular operator also depends on the values of the angles formed by the contour at its corner points. We also note that for any α $(0 < \alpha \leq \pi)$, the inequalities hold

$$1 \leq |S_{\Gamma_\alpha}|_0 < 1 + \sqrt{2}. \tag{13}$$

Next, we will consider the case where the integration contour Γ has a finite number of corner points.

Let Γ be a piecewise Lyapunov contour, $\tau_1, \tau_2, \ldots, \tau_s$ be all corner points of the contour Γ, and $\alpha_1, \alpha_2, \ldots, \alpha_s$ be the angles between the one-sided tangents to Γ at the points $\tau_1, \tau_2, \ldots, \tau_s$, respectively. In the space $L_2(\Gamma)$, we will consider the operator A defined by the equality

$$A = S_\Gamma S_\Gamma^* - \lambda I.$$

The symbol of the operator A is the matrix function $A(t,\xi)$ $(t \in \Gamma, -\infty \leq \xi \leq \infty)$ of the second order, defined as follows:

At points t that do not coincide with any of the points $\tau_1, \tau_2, \ldots, \tau_s$, we have

$$A(t,\xi) = (1-\lambda)E_2, \tag{14}$$

where E_2 is the identity matrix of the second order. But, at the points τ_k $(k = 1, 2, \ldots, s)$ we obtain

$$A(\tau_k,\xi) = S_k(\xi)(S_k(\xi))^* - \lambda E_2, \tag{15}$$

where $S_k(\xi)$ coincides with the right-hand side of equality (9), in which $p = 2$ and $\beta_k = 0$.

Theorem 5.1. *An operator $A = S_\Gamma S_\Gamma^* - \lambda I$ is Noetherian in the space $L_2(\Gamma)$ if and only if the determinant of its symbol is nonzero:*

$$detA(t,\xi) \neq 0 (t \in \Gamma, -\infty \leq \xi \leq \infty).$$

To prove this theorem, we need the following lemma.

Lemma 5.1. *An operator $A_\alpha = S_\alpha S_\alpha^* - \lambda I$ ($S_\alpha = S_{\Gamma_\alpha}$), acting in the space $L_2(\Gamma_\alpha)$, is a local Noetherian operator* at $t = 0$ if and only if it is a local Noetherian operator at $t = \infty$.*

Proof. Let the operator A_α be local Noetherian at $t = 0$. This means (see [9]) that it has left and right local regularizers at this point, i.e. there exist operators R_1, R_2 and a neighborhood $U_0(\ni 0)$ such that. □

$$R_1 A_\alpha P_{U_0} = P_{U_0} + T_1, \quad P_{U_0} A_\alpha R_2 = P_{U_0} + T_2, \tag{16}$$

where T_1 and T_2 are compact operators and P_{U_0} is an operator acting according to the rule

$$\left(P_{U_0}\varphi\right)(t) = \begin{cases} \varphi(t), & if \quad t \in U_o \\ 0, & if \quad t \in U_0 \backslash \Gamma_\alpha \end{cases}.$$

Let us consider the operator M defined by the equality

$$\left(M_\varphi\right)(t) = \frac{e^{i\alpha}}{t}\varphi\left(\frac{e^{i\alpha}}{t}\right) \quad (t \in \Gamma_\alpha).$$

It is easy to prove that the operator M acts in the space $L_2(\Gamma_\alpha)$, $\|M\| = 1$ and the following equalities holds:

$$MS_\alpha M^{-1} = S_\alpha, \quad MS_\alpha^* M^{-1} = S_\alpha^*. \tag{17}$$

Applying the operator M to the equality (15) on the left and M^{-1} on the right and taking into account the equality (16), we obtain

$$\widetilde{R}_1 A_\alpha P_{U_\infty} = P_{U_\infty} + \widetilde{T}_1, \quad P_{U_\infty} A_\alpha \widetilde{R}_2 = P_{U_\infty} + \widetilde{T}_2, \tag{18}$$

where $\widetilde{R}_i = MP_iM^{-1}$ and $\widetilde{T}_i = MP_iM^{-1}$ $(i = 1, 2)$, and U_∞ is a neighborhood of the point $t = \infty$. The equality (18) means that the operator A_α is locally Noetherian at the point $t = \infty$. The converse statement of the lemma is proved similarly. The lemma is proved.

Proof of the Theorem 5.1. Let A be a Noetherian operator and U_τ be some neighborhood of a point $\tau(\in \Gamma)$ that does not contain points $\tau_k \neq \tau$. By φ_τ we denote a function defined on U_τ as follows. If $\tau \neq \tau_k$, then we set $\varphi_\tau(t) \equiv t$ $(t \in U_\tau)$. If $\tau = \tau_k$ $(k = 1, 2, \dots, s)$, then φ_{τ_k} is a function that maps one-to-one the neighborhood U_{τ_k} onto some neighborhood $V_k(\Gamma_{\alpha_k})$ of the point $t = 0$, where $\varphi_{\tau_k} = 0$ $(k = 1, 2, \dots, s)$. Since Γ is a piecewise Lyapunov contour, it is possible to achieve that the derivatives $\varphi'_{\tau_k}(t)(t \in U_{\tau_k})$ satisfy the Hölder. condition.

*For the definition of φ - equivalence, see [9] on page 576.

At each point $\tau \neq \tau_k$ the operator A is φ_τ equivalent to the operator $C = (1 - \lambda)I$ acting in the space $L_2(\Gamma)$. Since A is Noetherian, then (see [9] Theorem 1.4) the operator C is locally Noetherian at the point τ, hence $\lambda \neq 1$.

At the point τ_k, the operator A is φ_{τ_k} equivalent to the operator $A_k = S_{\alpha_k} S^*_{\alpha_k} - \lambda I$, acting in the space $L_2(\Gamma_{\alpha_k})$. It also follows that A_k is a local Noetherian operator at the point $t = 0$. By Lemma 5.1, A_k is a local Noetherian operator at the point $t = \infty$. At points $t \in \Gamma_{\alpha_k}$ other than zero and infinity, the operator A_k is equivalent to the operator $(1 - \lambda)I$. Since $\lambda \neq 1$, A_k is local Noetherian at these points as well. Hence, by Theorem 1.6, it follows from [9] that A_k is Noetherian in $L_2(\Gamma_{\alpha_k})$. It follows from Theorem 4.2 that $det A_k(t, \xi) \neq 0$ $(t \in \Gamma_{\alpha_k}, -\infty \leq \xi \leq \infty)$. It is easy to see that $A_k(0, \xi) = A(\tau_k, \xi)$. Therefore, $det A(t, \xi) \neq 0$ $(t \in \Gamma, -\infty \leq \xi \leq \infty)$.

The necessity of the theorem is proved.

Sufficiency. Let $det A(t, \xi) \neq 0$ $(t \in \Gamma, -\infty \leq \xi \leq \infty)$. Then $\lambda \neq 1$ and

$$det\,(S_k(0, \xi)(S_k(0, \xi))^* - \lambda E_2) \neq 0 \quad (k = 1, 2, \dots, s).$$

From this and Lemma 5.1 it follows that the operators $A_k (k = 1, 2, \dots, s)$ and $C = (1 - \lambda)I$ are Noetherian. Since the operator A at each point τ is φ_τ equivalent to one of these operators, it follows (see [9], Theorem 2.4) that A is Noetherian. The theorem is proved.

From Theorem 5.1 follows

Corollary 5.1. *The operator S^* does not belong to the algebra $\Sigma(\Gamma)$ generated by the operators aI $(a \in C(\Gamma))$ and S_Γ.*

Indeed, let us assume that S^* belongs to the algebra $\Sigma(\Gamma)$. Since the symbols of the operators from $\Sigma(\Gamma)$ commute, the symbol of the operator $R = \lambda I - (S^*_\Gamma S_\Gamma - S_\Gamma S^*_\Gamma)$ is equal to λ. Consequently, for all $\lambda \neq 0$ the operator R is Noetherian. It is easy to verify that this contradicts Theorem 5.1.

From Theorem 5.1 and equality (10) it is easy to deduce that the essential norm $|S_\Gamma|$ of the operator S_Γ in the space $L_2(\Gamma)$ is defined by the equality

$$|S_\Gamma| = \max_{1 \leq k \leq s} \left| S_{\alpha_k} \right|. \tag{19}$$

From this and from equality (12) we conclude that the essential norm of the operator S_Γ in the space $L_2(\Gamma)$ satisfies the conditions

$$1 \leq |S_\Gamma| < 1 + \sqrt{2}.$$

Note that similarly, using the symbol and equality (13), we can calculate the essential norms of the Riesz operators $P_\Gamma = (I + S_\Gamma)/2$ and $Q_\Gamma = (I - S_\Gamma)/2$. It turns out that for these operators the following relation holds:

$$|P_\Gamma| = |Q_\Gamma| = \frac{|S_\Gamma|^2 + 1}{2|S_\Gamma|^2}. \tag{20}$$

Remark 5.1. *The equality (20) confirms the following hypothesis of the mathematician S. Marcus: let B be some Banach space and L_1, L_2 subspaces from B such that $L_1 \bigcap L_2 = 0$ and $B = L_1 + L_2$, then equality*

$$|P| = |Q| = \frac{|S_\Gamma|^2 + 1}{2|S_\Gamma|^2}$$

takes place, where P and Q are projectors projecting the space B onto L_1, respectively, on L_2 and $S = P + Q$.

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On computation of the ordinary Hilbert series for Sibirsky graded algebras of differential system $s(3,5)$

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Abstract. The generalized and ordinary Hilbert series for Sibirsky graded algebras of comitants and invariants of autonomous polynomial differential systems are of particular importance for some problems of qualitative theory of differential systems. In the Republic of Moldova the computation of these series have their beginnings in the works of Professor M. N. Popa and his disciples. But the construction of these series for some complicated differential systems encounters insurmountable computational difficulties, especially, for the generalized Hilbert series, from which the ordinary Hilbert series can be easily obtained. In this paper, it is shown how the adaptation of Molien's formula address to the mentioned problem to overcome the enormous calculations, an ordinary Hilbert series were obtained for Sibirsky graded algebras of comitants and invariants for the differential system $s(3,5)$.

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Despre calcularea seriilor Hilbert obişnuite pentru algebrele graduate Sibirschi ale sistemului diferențial $s(3,5)$

Rezumat. Seriile Hilbert generalizate şi obişnuite pentru algebrele graduate Sibirschi ale comitanţilor şi invarianţilor sistemelor diferenţiale polinomiale autonome joacă un rol deosebit pentru unele probleme din teoria calitativă a acestor sisteme. În Republica Moldova calcularea seriilor Hilbert încep în lucrările profesorului M. N. Popa şi ale discipolilor săi. Totuşi construcţia acestor serii pentru unele sisteme diferenţiale complicate întâmpină dificultăţi enorme de calcul, în special pentru seria Hilbert generalizată din care se obţine cu uşurinţă seria Hilbert obişnuită. În această lucrare se arată cum se foloseşte adaptarea formulei lui Molien pentru a depăşi problema calculelor enorme. Au fost obţinute seriile Hilbert obişnuite pentru algebrele graduate Sibirschi ale comitanţilor şi invarianţilor pentru sistemul diferenţial $s(3,5)$.

Cuvinte-cheie: sisteme diferenţiale, algebre graduate Sibirschi, serii Hilbert, dimensiunea Krull.

1. Introduction

Consider a two-dimensional autonomous polynomial system of differential equations

$$\frac{dx}{dt} = \sum_{i=0}^{\ell} P_{m_i}(x, y), \quad \frac{dy}{dt} = \sum_{i=0}^{\ell} Q_{m_i}(x, y), \tag{1}$$

where $\Gamma = \{m_i\}_{i=0}^{\ell}$, $(\ell < \infty)$ is a some finite set of distinct non-negative integers, $P_{m_i}(x, y)$ and $Q_{m_i}(x, y)$ are homogeneous of degree m_i with respect to the phase variables x and y (i.e., $P_{m_i}(\alpha x, \alpha y) = \alpha^{m_i} P_{m_i}(x, y)$, $Q_{m_i}(\alpha x, \alpha y) = \alpha^{m_i} Q_{m_i}(x, y)$, $\alpha \in \mathbb{R}$). The coefficients and variables in the polynomials $P_{m_i}(x, y)$ and $Q_{m_i}(x, y)$ take values from the field of real numbers $\mathbb{R}$.

Hereafter for the system of the form (1) we will use the notation $s(m_0, m_1, ..., m_\ell)$ or $s(\Gamma)$ where $\Gamma = \{m_i\}_{i=0}^{\ell}$, m_i are degrees of homogeneities $P_{m_i}(x, y)$ and $Q_{m_i}(x, y)$ with respect to the phase variables x and y.

One of the methods to study the differential systems of the form (1) is "*The method of algebraic invariants in the theory of differential equations*", which is developed in the works of Academician K. S. Sibirsky [1, 2, 3] and his disciples.

This method generated applications of Lie groups and algebras, graded algebras of invariants and comitants, generating functions and Hilbert series to the study of the system (1) (see, for example [4, 5]).

One of the methods of computation of generalized and ordinary Hilbert series for differential systems is Silvester's generalized method known from [4]. This method for differential systems with high-degree polynomial on the right-hand sides is connected with cumbersome computations with application of supercomputers. In contrast to the mentioned above, using the residues method, were obtained the ordinary Hilbert series for Sibirsky graded algebras of comitants and invariants for the following differential systems $s(1, 3, 7)$ [6], $s(3, 7)$ [7], $s(1, 3, 5)$ [8], $s(1, 3, 5, 7)$ [9]. So, it is welcome to complete the set of computed Hilbert series with others, for example, the Hilbert series for the system $s(3, 5)$.

2. Graded algebras of comitants (invariants) of the system (1.1)

Let A be a set of coefficients of the system (1). Denote by $GL(2, \mathbb{R})$ a group of centro-affine transformations

$$q: \ \overline{x} = \alpha x + \beta y, \ \overline{y} = \gamma x + \delta y \ (\Delta = \begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix} \neq 0) \tag{2}$$

where $\alpha, \beta, \gamma, \delta$ takes value from the field of real numbers $\mathbb{R}$.

Definition 2.1. *The polynomial $k(x,y,A)$ of phase variables and coefficients of the system* (1) *is called a centro-affine comitant of this system, if the equality*

$$k(\overline{x},\overline{y},\overline{A}) = \Delta^{-g} k(x,y,A)$$

holds for any $q \in GL(2,\mathbb{R})$, any coefficients of the system (1)*, any phase variables. If the comitant k does not depend on the phase variables, then it its called the invariant* (*usually denoted by i*) *of the system* (1) *by the centro-affine group $GL(2,\mathbb{R})$.*

The number g is called the weight of the comitant k. If $g = 0$, then k is called the absolute comitant, otherwise the relative comitant.

Definition 2.2. *For any differential system $s(m_0, m_1, ..., m_\ell)$, a centro-affine comitant has a type*

$$(d) = (\delta, d_1, d_2, ..., d_\ell) \tag{3}$$

where d_i is the degree of homogeneity of comitant with respect to the coefficients of homogeneities $P_{m_i}(x,y)$ and $Q_{m_i}(x,y)$, δ is the degree of homogeneity of comitant with respect to the phase variables x, y. At the same time the number $d = \sum_{i=1}^{\ell} d_i$ (δ) *is called the degree* (*order*) *of comitant of the type* (3)*.*

Lemma 2.1. [4] *The set of centro-affine comitants of the system* (1) *of the same type* (3) *forms a finite-dimensional linear space, i.e., has a finite maximal system of linearly independent comitants* (*linear basis*) *of a given type through which all others are linearly expressed.*

Remark 2.1. *In* [4] *it is shown that the set of centro-affine comitants generate a finite-determined graded algebra of comitants* (*invariants*) *with respect to the unimodular group $SL(2,\mathbb{R}) \subset GL(2,\mathbb{R})$, which are denoted by*

$$S_\Gamma = \sum_{(d)} S_\Gamma^{(d)} \ \left(SI_\Gamma = \sum_{(d)} SI_\Gamma^{(d)}\right).$$

By $dim_{\mathbb{R}} S_\Gamma^{(d)}$ ($dim_{\mathbb{R}} SI_\Gamma^{(d)}$) *are denoted dimensions of linear spaces from the Lemma* 2.1*. These algebras in* [5] *were named Sibirsky graded algebra of comitants* (*invariants*)*, respectively.*

If for Sibirsky graded algebras $S_{m_0,m_1,...,m_\ell}$ and $SI_{m_0,m_1,...,m_\ell}$ we introduce a single notation A, then they can be written in the form

$$A =< a_1, a_2, ..., a_m | f_1 = 0, f_2 = 0, ..., f_n = 0 > \ (m, n < \infty) \tag{4}$$

where a_i are the generators of this algebra, f_j – defining relations (syzygies) between these generators.

One of the problems in studying these algebras is determining the numbers and expressions of its generators. For this, it is necessary to study the type of comitants and invariants that forms these generators. From the paper [10] on Modern Algebra, it follows that generating functions and Hilbert series play an essential role in solving this problem.

3. Hilbert series for Sibirsky graded algebras $S_{m_0,m_1,...,m_\ell}$ and $SI_{m_0,m_1,...,m_\ell}$

From [4] it is known

$$\varphi_\Gamma^{(0)}(u) = (1 - u^{-2})\psi_{m_0}^{(0)}(u)\psi_{m_1}^{(0)}(u)...\psi_{m_\ell}^{(0)}(u) \tag{5}$$

where

$$\psi_{m_i}^{(0)}(u) = \begin{cases} \frac{1}{(1-uz_i)(1-u^{-1}z_i)}, & \text{for } m_i = 0, \\ \frac{1}{(1-u^{m_i+1}z_i)(1-u^{-m_i-1}z_i)\prod_{k=1}^{m_i}(1-u^{m_i-2k+1}z_i)^2}, & \text{for } m_i \neq 0 \end{cases} \tag{6}$$

for each $\Gamma = \{m_i\}_{i=0}^{\ell}$.

The expressions (5)–(6) we will call *initial form of the generating function* for centro-affine comitants of the system (1).

In the paper [4], it is shown that if the function (5)–(6) is represented as

$$\varphi_\Gamma(u) - u^{-2}\varphi_\Gamma(u^{-1}) = \varphi_\Gamma^{(0)}(u), \tag{7}$$

then we can restrict ourselves to the study of only rational function $\varphi_\Gamma(u)$.

However, the question arises, how to obtain the function $\varphi_\Gamma(u)$ from (7) for more complicated Γ. This problem was solved by generalizing the Silvester's method by decomposition of the function $\varphi_\Gamma^{(0)}(u)$ in elementary fractions [4].

Following the paper [4], under a generalized Hilbert series of the algebra S_Γ, we will understand

$$H(S_\Gamma, u, z_0, z_1, ..., z_\ell) = \sum_{(d)} dim_{\mathbb{R}} S_\Gamma^{(d)} u^\delta z_0^{d_0} z_1^{d_1} ... z_\ell^{d_\ell},$$

and

$$H(S_\Gamma, u, z_0, z_1, ..., z_\ell) = \varphi_\Gamma(u) \tag{8}$$

where $\varphi_\Gamma(u)$ is from (7).

Note that (according to the same paper [4]) an ordinary Hilbert series is obtained in an obvious way from the generalized

$$H_{S_\Gamma}(u) = H(S_\Gamma, u, u, u, ..., u). \tag{9}$$

If we denote the algebra of invariants for a fixed Γ for the system (1) by SI_Γ, then for generalized Hilbert series of this algebra we have

$$H(SI_\Gamma, z_0, z_1, ..., z_\ell) = H(S_\Gamma, 0, z_0, z_1, ..., z_\ell) = \varphi_\Gamma(0), \tag{10}$$

and for the ordinary Hilbert series we obtain

$$H_{SI_\Gamma}(z) = H(SI_\Gamma, z, z, ..., z). \tag{11}$$

The computation of the generalized Hilbert series for differential systems with high-degree polynomial on the right-hand sides is connected with cumbersome computations with the application of supercomputers. This emphasizes the importance of the calculus of ordinary Hilbert series. From the papers [4], [5], it follows that, we can studying the structures of these algebras using the generalized and the ordinary Hilbert series of Sibirsky graded algebras for the differential system (1). The Hilbert series gives an information about the upper bound of degrees for generators of these algebras.

Remark 3.1. *According to* [4], [5] *the Krull dimension for finitely determined algebras is equal to the order of pole of corresponding ordinary Hilbert series at the unit. The Krull dimension gives us the maximal number of algebraically independent comitants and invariants of corresponding Sibirsky graded algebras of differential systems.*

4. Methods of computation of the generalized and ordinary Hilbert series for Sibirsky graded algebras of differential systems

From the paper [4], it is known the Silvester's generalized method for computation of the generalized and ordinary Hilbert series. Using this method, there were calculated Hilbers series for Sibirsky graded algebras for differential systems $s(1)$, $s(2)$, $s(0,2)$, $s(1,3)$, $s(2,3)$, $s(5)$ [4], $s(1,4)$, $s(1,5)$ [5]. An attempt to obtain the Hilbert series for relatively simple system $s(1,2,3)$ with this method was unsuccessful. For more complicated differential systems this method encounters insurmountable computational difficulties. Use of other methods is welcome.

Definition 4.1. [11] *For a graded vector space $V = \bigoplus_{d=k}^{\infty} V_d$ with V_d finite dimensional for all d we define the Hilbert series ov V as the formal Laurent series*

$$H(V,t) = \sum_{d=k}^{\infty} dim(V_d)t^d.$$

An important tool for computing invariants is the Hilbert series. The Hilbert series of a ring contains a lot of information about the ring itself. For example, the dimension and other geometric invariants can be read from the Hilbert series.

Theorem 4.1. (Molien's formula [11]) *Let G be a finite group acting on a finite-dimensional vector space V over a field K of characteristic not dividing $|G|$. Then*

$$H(K[V]^G, t) = \frac{1}{|G|} \sum_{\sigma \in G} \frac{1}{det_V^0(1 - t\sigma)}.$$

If K has the characteristic 0, then $det_V^0(1 - t\sigma)$ can be taken as $det_V(1 - t\sigma)$.

We recall the Residue Theorem in complex function theory. This theorem can be applied to compute the Hilbert series of invariant rings [11].

Theorem 4.2. (The Residue Theorem [11]) *Suppose that D is a connected, simply connected compact region in $\mathbb{C}$, whose border is ∂D, and $\gamma : [0, 1] \to \mathbb{C}$ is a smooth curve such that $\gamma([0, 1]) = \partial D$, $\gamma(0) = \gamma(1)$ and circles around D exactly once in the counter clockwise direction. Assume that f is a meromorphic function on $\mathbb{C}$ with no poles in ∂D. Then we have*

$$\frac{1}{2\pi i} \int_\gamma f(z)dz = \sum_{a \in D} Res(f, a).$$

There are only finitely many points in the compact region D such that f has non-zero residue there.

Theorem 4.3. [11]

$$H(K[V]^G, t) = \frac{1}{2\pi i} \int\limits_{S^1} \frac{1}{det(I - t_{\rho_V}(z))} \frac{dz}{z}$$

where $S^1 \subset \mathbb{C}$ is the unit circle $\{z : |z| = 1\}$.

Using the Residue Theorem and corresponding generating function [4] the last formula was adapted for computation of ordinary Hilbert series for Sibirsky graded algebras of comitants and invariants of differential systems as follows

Theorem 4.4. [5]

$$H_{SI_\Gamma}(t) = \frac{1}{2\pi i} \int\limits_{S^1} \frac{\varphi_\Gamma^{(0)}(z)}{z} dz$$

where $S^1 \subset \mathbb{C}$ is the unit circle $\{z : |z| = 1\}$, $\varphi_\Gamma^{(0)}(z)$ is the corresponding generating function (5)–(6).

5. Computation of the ordinary Hilbert series for Sibirsky graded algebras of differential system $s(3,5)$

Using Theorem 4.4, we obtain

Theorem 5.1. *For the differential system $s(3,5)$, the following ordinary Hilbert series for Sibirsky graded algebras of comitants $S_{3,5}$ and invariants $SI_{3,5}$ was obtained*

$$H_{S_{3,5}}(t) = \frac{1}{(1+t)^2(1-t^2)^4(1-t^4)^3(1-t^3)^7(1-t^5)^4(1-t^7)} \cdot (1 + 2t + 2t^2 +$$
$$+8t^3 + 49t^4 + 179t^5 + 533t^6 + 1382t^7 + 3301t^8 + 7356t^9 + 15353t^{10} +$$
$$+29865t^{11} + 54402t^{12} + 93137t^{13} + 150665t^{14} + 231125t^{15} + 337272t^{16} +$$
$$+468744t^{17} + 621438t^{18} + 786783t^{19} + 952653t^{20} + 1104296t^{21} + 1226739t^{22} +$$
$$+1306380t^{23} + 1334077t^{24} + 1306380t^{25} + 1226739t^{26} + 1104296t^{27} +$$
$$+952653t^{28} + 786783t^{29} + 621438t^{30} + 468744t^{31} + 337272t^{32} + 231125t^{33} +$$
$$+150665t^{34} + 93137t^{35} + 54402t^{36} + 29865t^{37} + 15353t^{38} + 7356t^{39} +$$
$$+3301t^{40} + 1382t^{41} + 533t^{42} + 179t^{43} + 49t^{44} + 8t^{45} + 2t^{46} + 2t^{47} + t^{48})$$

$$H_{SI_{3,5}}(t) = \frac{1}{(1-t)^4(1+t)^5(1-t^4)^4(1-t^3)^6(1-t^5)^3} \cdot (1 + t + t^2 + 7t^3 +$$
$$+36t^4 + 106t^5 + 290t^6 + 672t^7 + 1451t^8 + 2875t^9 + 5322t^{10} + 9053t^{11} +$$
$$+14398t^{12} + 21263t^{13} + 29463t^{14} + 38314t^{15} + 47076t^{16} + 54444t^{17} +$$
$$+59516t^{18} + 61259t^{19} + 59516t^{20} + 54444t^{21} + 47076t^{22} + 38314t^{23} +$$
$$+29463t^{24} + 21263t^{25} + 14398t^{26} + 9053t^{27} + 5322t^{28} + 2875t^{29} + 1451t^{30} +$$
$$+672t^{31} + 290t^{32} + 106t^{33} + 36t^{34} + 7t^{35} + t^{36} + t^{37} + t^{38})$$

Remark 5.1. *For the Sibirsky graded algebras $S_{3,5}$ ($SI_{3,5}$), the Krull dimensions is equal to* 19 (17) *respectively.*

Remark 5.2. *The Krull dimension gives us the maximal number of algebraically independent comitants and invariants of Sibirsky graded algebras $S_{3,5}$ and $SI_{3,5}$ of differential system $s(3,5)$.*

Remark 5.3. *Note that for Hilbert series of Sibirsky graded algebra of comitants of the system $s(\Gamma)$, where $0 \notin \Gamma$, the following equality holds $H_{S_\Gamma}(t) = H_{SI_{\Gamma\cup\{0\}}}(t)$.*

From [11], a method for computing the ordinary Hilbert series of invariants rings using the residues is known, that was adapted for the ordinary Hilbert series for Sibirsky graded algebras of comitants and invariants of differential systems. This method is more effective than the generalized Sylvester's method.

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Stability conditions of unperturbed motion governed by the ternary differential system of Lyapunov-Darboux type with nonlinearities of fifth degree

Natalia Neagu

Abstract. In this paper, there was studied Lyapunov stability of the unperturbed motion for the ternary differential system with nonlinearities of fifth degree on a center-affine variety. The Lyapunov series was constructed and the stability conditions of the unperturbed motion governed by this system were determined in the critical case.

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Keywords: differential system, stability of unperturbed motion, critical equation, non-critical equation, Lyapunov series.

Condiţiile de stabilitate a mişcării neperturbate guvernate de sistemul diferenţial ternar de tip Lyapunov-Darboux cu nelinearităţi de gradul cinci

Rezumat. În lucrare a fost studiată stabilitatea după Lyapunov a mişcării neperturbate pentru sistemul diferenţial ternar cu nelinearităţi de gradul cinci, pe o varietate centro-afină. A fost construită seria Lyapunov şi determinate condiţiile de stabilitate a mişcării neperturbate guvernate de acest sistem în cazul critic.

Cuvinte-cheie: sistem diferenţial, stabilitatea mişcării neperturbate, ecuaţie critică, ecuaţie necritică, serie Lyapunov.

1. Introduction

We examine the three-dimensional differential system $s^3(1,5)$ of unperturbed motion of the form

$$\frac{dx^j}{dt} = a^j_\alpha x^\alpha + a^j_{\alpha\beta\gamma\delta\mu} x^\alpha x^\beta x^\gamma x^\delta x^\mu \quad (j,\alpha,\beta,\gamma,\delta,\mu = \overline{1,3}), \tag{1}$$

where $a^j_{\alpha\beta\gamma\delta\mu}$ is a symmetric tensor in the lower indices, by which a total convolution is done.

Definition 1.1. *According to I. G. Malkin* [1], *we will say that the system (1) is critical if the characteristic equation of this system has one zero root, and all other roots of this equation have negative real parts.*

Lemma 1.1. *The three-dimensional differential system (1) is critical if and only if the following center-affine invariant conditions hold*

$$L_{1,3} > 0, \quad L_{2,3} > 0, \quad L_{3,3} = 0, \tag{2}$$

where

$$L_{1,3} = -\theta_1, \quad L_{2,3} = \frac{1}{2}(\theta_1^2 - \theta_2), \quad L_{3,3} = \frac{1}{6}(-\theta_1^3 + 3\theta_1\theta_2 - 2\theta_3), \tag{3}$$

and

$$\theta_1 = a_\alpha^\alpha, \quad \theta_2 = a_\beta^\alpha a_\alpha^\beta, \quad \theta_3 = a_\gamma^\alpha a_\alpha^\beta a_\beta^\gamma. \tag{4}$$

Lemma 1.2. *In the case of conditions* (2)*, by a center-affine transformation* [2]*, the system* (1) *can be brought to the critical Lyapunov form*

$$\begin{aligned} \frac{dx^1}{dt} &= a^j_{\alpha\beta\gamma\delta\mu}x^\alpha x^\beta x^\gamma x^\delta x^\mu, \\ \frac{dx^j}{dt} &= a^j_\alpha x^\alpha + a^j_{\alpha\beta\gamma\delta\mu}x^\alpha x^\beta x^\gamma x^\delta x^\mu \quad (j = 2,3; \alpha,\beta,\gamma,\delta,\mu = \overline{1,3}), \end{aligned} \tag{5}$$

where the first equation from (5) *is called the* ***critical equation*** *and the second one – the* ***non-critical equation****.*

2. The critical differential system of Lyapunov-Darboux type with nonlinearities of fifth degree

In the center-affine condition $\eta = a^\alpha_{\beta\gamma\delta\mu\nu}x^\beta x^\gamma x^\delta x^\mu x^\nu x^\tau y^\xi \varepsilon_{\alpha\tau\xi} \equiv 0$ [3], with notations

$$x^1 = x, \;\; x^2 = y, \;\; x^3 = z, \;\; a_1^2 = p, \;\; a_2^2 = q, \;\; a_3^2 = r, \;\; a_1^3 = s, \;\; a_2^3 = m, \;\; a_3^3 = n, \tag{6}$$

the system (5), it is a critical system of Lyapunov-Darboux type, of the form

$$\begin{aligned} \frac{dx}{dt} &= 5xR(x,y,z), \\ \frac{dy}{dt} &= px + qy + rz + 5yR(x,y,z), \\ \frac{dz}{dt} &= sx + my + nz + 5zR(x,y,z), \end{aligned} \tag{7}$$

where

$$\begin{aligned} R(x,y,z) &= a_1x^4 + a_2y^4 + a_3z^4 + 4a_4x^3y + 4a_5x^3z + 4a_6xy^3 + 4a_7xz^3 + 6a_8x^2y^2 + \\ &+ 6a_9x^2z^2 + 12a_{10}x^2yz + 12a_{11}xy^2z + 12a_{12}xyz^2 + 6a_{13}y^2z^2 + 4a_{14}y^3z + 4a_{15}yz^3, \end{aligned} \tag{8}$$

and $m, n, p, q, r, s, a_i (i = \overline{1,15})$ are real arbitrary coefficients.

Remark 2.1. *For the system* (7)*, we have*

$$L_{2,3} = nq - mr,$$

and according to conditions (2) *these values are greater than zero.*

Remark 2.2. *Under the conditions of Remark 2.1, without loss of generality, we can assume that* $nq \neq 0$*.*

Proof. We consider the center-affine substitution

$$\bar{x} = x, \quad \bar{y} = z, \quad \bar{z} = y. \tag{9}$$

It is easy to verify that in the case of substitution (9), we obtain that in system (7) the expression mr becomes nq. Taking into account Remark 2.1, we obtain that $nq \neq 0$. □

We analyze the noncritical equations

$$\begin{aligned} & px + qy + rz + 5y(a_1x^4 + a_2y^4 + a_3z^4 + 4a_4x^3y + 4a_5x^3z + 4a_6xy^3 + 4a_7xz^3 + \\ & \quad +6a_8x^2y^2 + 6a_9x^2z^2 + 12a_{10}x^2yz + 12a_{11}xy^2z + 12a_{12}xyz^2 + 6a_{13}y^2z^2 + \\ & \quad +4a_{14}y^3z + 4a_{15}yz^3) = 0, \\ & sx + my + nz + 5z(a_1x^4 + a_2y^4 + a_3z^4 + 4a_4x^3y + 4a_5x^3z + 4a_6xy^3 + 4a_7xz^3 + \\ & \quad +6a_8x^2y^2 + 6a_9x^2z^2 + 12a_{10}x^2yz + 12a_{11}xy^2z + 12a_{12}xyz^2 + 6a_{13}y^2z^2 + \\ & \quad +4a_{14}y^3z + 4a_{15}yz^3) = 0. \end{aligned} \tag{10}$$

Then from the first relation of (10) we express y, and from the second relation we express z

$$\begin{aligned} y = & -\frac{p}{q}x - \frac{r}{q}z - \frac{5}{q}y(a_1x^4 + a_2y^4 + a_3z^4 + 4a_4x^3y + 4a_5x^3z + 4a_6xy^3 + 4a_7xz^3 + \\ & +6a_8x^2y^2 + 6a_9x^2z^2 + 12a_{10}x^2yz + 12a_{11}xy^2z + 12a_{12}xyz^2 + 6a_{13}y^2z^2 + \\ & +4a_{14}y^3z + 4a_{15}yz^3), \\ z = & -\frac{s}{n}x - \frac{m}{n}y - \frac{5}{n}z(a_1x^4 + a_2y^4 + a_3z^4 + 4a_4x^3y + 4a_5x^3z + 4a_6xy^3 + 4a_7xz^3 + \\ & +6a_8x^2y^2 + 6a_9x^2z^2 + 12a_{10}x^2yz + 12a_{11}xy^2z + 12a_{12}xyz^2 + 6a_{13}y^2z^2 + \\ & +4a_{14}y^3z + 4a_{15}yz^3). \end{aligned} \tag{11}$$

We seek y and z as a holomorphic functions of x. Then we can write

$$y(x) = A_1x + A_2x^2 + A_3x^3 + A_4x^4 + A_5x^5 + \ldots,$$

$$z(x) = B_1x + B_2x^2 + B_3x^3 + B_4x^4 + B_5x^5 + \ldots \tag{12}$$

Substituting (12) into (11) we have

$$A_1x + A_2x^2 + A_3x^3 + A_4x^4 + A_5x^5 + \ldots = -\frac{p}{q}x - \frac{r}{q}(B_1x + B_2x^2 + B_3x^3 + B_4x^4 +$$
$$+ B_5x^5 + \ldots) - \frac{5}{q}(A_1x + A_2x^2 + A_3x^3 + A_4x^4 + A_5x^5 + \ldots)[a_1x^4 + a_2(A_1x + A_2x^2 +$$
$$+ A_3x^3 + A_4x^4 + A_5x^5 + \ldots)^4 + a_3(B_1x + B_2x^2 + B_3x^3 + B_4x^4 + B_5x^5 + \ldots)^4 +$$
$$+ 4a_4x^3(A_1x + A_2x^2 + A_3x^3 + A_4x^4 + A_5x^5 + \ldots) + 4a_5x^3(B_1x + B_2x^2 + B_3x^3 + B_4x^4 +$$
$$+ B_5x^5 + \ldots) + 4a_6x(A_1x + A_2x^2 + A_3x^3 + A_4x^4 + A_5x^5 + \ldots)^3 + 4a_7x(B_1x + B_2x^2 +$$
$$+ B_3x^3 + B_4x^4 + B_5x^5 + \ldots)^3 + 6a_8x^2(A_1x + A_2x^2 + A_3x^3 + A_4x^4 + A_5x^5 + \ldots)^2 +$$
$$+ 6a_9x^2(B_1x + B_2x^2 + B_3x^3 + B_4x^4 + B_5x^5 + \ldots)^2 + 12a_{10}x^2(A_1x + A_2x^2 + A_3x^3 +$$
$$+ A_4x^4 + A_5x^5 + \ldots)(B_1x + B_2x^2 + B_3x^3 + B_4x^4 + B_5x^5 + \ldots) + +12a_{11}x(A_1x + A_2x^2 +$$
$$+ A_3x^3 + A_4x^4 + A_5x^5 + \ldots)^2(B_1x + B_2x^2 + B_3x^3 + B_4x^4 + B_5x^5 + \ldots) +$$
$$+ 12a_{12}x(A_1x + A_2x^2 + A_3x^3 + A_4x^4 + A_5x^5 + \ldots)(B_1x + B_2x^2 + B_3x^3 + B_4x^4 +$$
$$+ B_5x^5 + \ldots)^2 + 6a_{13}(A_1x + A_2x^2 + A_3x^3 + A_4x^4 + A_5x^5 + \ldots)^2(B_1x + B_2x^2 + B_3x^3 +$$
$$+ B_4x^4 + B_5x^5 + \ldots)^2 + 4a_{14}(A_1x + A_2x^2 + A_3x^3 + A_4x^4 + A_5x^5 + \ldots)^3(B_1x + B_2x^2 +$$
$$+ B_3x^3 + B_4x^4 + B_5x^5 + \ldots) + 4a_{15}(A_1x + A_2x^2 + A_3x^3 + A_4x^4 + A_5x^5 + \ldots)(B_1x +$$
$$+ B_2x^2 + B_3x^3 + B_4x^4 + B_5x^5 + \ldots)^3],$$

$$B_1x + B_2x^2 + B_3x^3 + B_4x^4 + B_5x^5 + \ldots = -\frac{s}{n}x - \frac{m}{n}(A_1x + A_2x^2 + A_3x^3 +$$
$$+ A_4x^4 + A_5x^5 + \ldots) - \frac{5}{n}(B_1x + B_2x^2 + B_3x^3 + B_4x^4 + B_5x^5 + \ldots)[a_1x^4 + a_2(A_1x +$$
$$+ A_2x^2 + A_3x^3 + A_4x^4 + A_5x^5 + \ldots)^4 + a_3(B_1x + B_2x^2 + B_3x^3 + B_4x^4 + B_5x^5 + \ldots)^4 +$$
$$+ 4a_4x^3(A_1x + A_2x^2 + A_3x^3 + A_4x^4 + A_5x^5 + \ldots) + 4a_5x^3(B_1x + B_2x^2 + B_3x^3 +$$
$$+ B_4x^4 + B_5x^5 + \ldots) + 4a_6x(A_1x + A_2x^2 + A_3x^3 + A_4x^4 + A_5x^5 + \ldots)^3 + 4a_7x(B_1x +$$
$$+ B_2x^2 + B_3x^3 + B_4x^4 + B_5x^5 + \ldots)^3 + 6a_8x^2(A_1x + A_2x^2 + A_3x^3 + A_4x^4 + A_5x^5 + \ldots)^2 +$$
$$+ 6a_9x^2(B_1x + B_2x^2 + B_3x^3 + B_4x^4 + B_5x^5 + \ldots)^2 + 12a_{10}x^2(A_1x + A_2x^2 + A_3x^3 +$$
$$+ A_4x^4 + A_5x^5 + \ldots)(B_1x + B_2x^2 + B_3x^3 + B_4x^4 + B_5x^5 + \ldots) + 12a_{11}x(A_1x + A_2x^2 +$$
$$+ A_3x^3 + A_4x^4 + A_5x^5 + \ldots)^2(B_1x + B_2x^2 + B_3x^3 + B_4x^4 + B_5x^5 + \ldots) +$$
$$+ 12a_{12}x(A_1x + A_2x^2 + A_3x^3 + A_4x^4 + A_5x^5 + \ldots)(B_1x + B_2x^2 + B_3x^3 + B_4x^4 +$$
$$+ B_5x^5 + \ldots)^2 + 6a_{13}(A_1x + A_2x^2 + A_3x^3 + A_4x^4 + A_5x^5 + \ldots)^2(B_1x + B_2x^2 + B_3x^3 +$$
$$+ B_4x^4 + B_5x^5 + \ldots)^2 + 4a_{14}(A_1x + A_2x^2 + A_3x^3 + A_4x^4 + A_5x^5 + \ldots)^3(B_1x + B_2x^2 +$$

$$+B_3x^3+B_4x^4+B_5x^5+\ldots)+4a_{15}(A_1x+A_2x^2+A_3x^3+A_4x^4+A_5x^5+\ldots)(B_1x+$$
$$+B_2x^2+B_3x^3+B_4x^4+B_5x^5+\ldots)^3],$$

This implies that

$$A_1x+A_2x^2+A_3x^3+A_4x^4+A_5x^5+\ldots=-\frac{p+rB_1}{q}x-\frac{rB_2}{q}x^2-\frac{rB_3}{q}x^3-\frac{rB_4}{q}x^4-$$
$$-\frac{1}{q}(5a_1A_1+20a_4A_1^2+30a_8A_1^3+20a_6A_1^4+5a_2A_1^5+60a_{10}A_1^2B_1+60a_{11}A_1^3B_1+$$
$$+20a_{14}A_1^4B_1+20a_5A_1B_1+60a_{12}A_1^2B_1^2+30a_{13}A_1^3B_1^2+30a_9A_1B_1^2+20a_{15}A_1^2B_1^3+$$
$$+20a_7A_1B_1^3+5a_3A_1B_1^4+rB_5)x^5+\ldots$$

$$B_1x+B_2x^2+B_3x^3+B_4x^4+B_5x^5+\ldots=-\frac{mA_1+s}{n}x-\frac{mA_2}{n}x^2-\frac{mA_3}{n}x^3-\frac{mA_4}{n}x^4-$$
$$-\frac{1}{n}(5a_2A_1^4B_1+20a_4A_1B_1+20a_6A_1^3B_1+30a_8A_1^2B_1+60a_{10}A_1B_1^2+60a_{11}A_1^2B_1^2+$$
$$+20a_{14}A_1^3B_1^2+60a_{12}A_1B_1^3+30a_{13}A_1^2B_1^3+20a_{15}A_1B_1^4+mA_5+5a_1B_1+20a_5B_1^2+$$
$$+30a_9B_1^3+20a_7B_1^4+5a_3B_1^5)x^5+\ldots$$

From this identity we have

$$A_1=\frac{rs-np}{nq-mr},\quad B_1=\frac{mp-qs}{nq-mr};\quad A_2=B_2=A_3=B_3=A_4=B_4=0,$$

$$A_5=-\frac{5}{nq-mr}(a_1+4a_4A_1+6a_8A_1^2+4a_6A_1^3+a_2A_1^4+12a_{10}A_1B_1+$$
$$+12a_{11}A_1^2B_1+4a_{14}A_1^3B_1+4a_5B_1+12a_{12}A_1B_1^2+6a_{13}A_1^2B_1^2+6a_9B_1^2+4a_{15}A_1B_1^3+$$
$$+4a_7B_1^3+a_3B_1^4)(nA_1-rB_1),$$

$$B_5=\frac{5}{nq-mr}(a_1+4a_4A_1+6a_8A_1^2+4a_6A_1^3+a_2A_1^4+12a_{10}A_1B_1+$$
$$+12a_{11}A_1^2B_1+4a_{14}A_1^3B_1+4a_5B_1+12a_{12}A_1B_1^2+6a_{13}A_1^2B_1^2+6a_9B_1^2+4a_{15}A_1B_1^3+$$
$$+4a_7B_1^3+a_3B_1^4)(mA_1-qB_1),$$

$$A_6=B_6=A_7=B_7=A_8=B_8=0,$$

$$A_9=-\frac{5}{nq-mr}[(a_1+4a_4A_1+6a_8A_1^2+4a_6A_1^3+a_2A_1^4+12a_{10}A_1B_1+$$
$$+12a_{11}A_1^2B_1+4a_{14}A_1^3B_1+4a_5B_1+12a_{12}A_1B_1^2+6a_{13}A_1^2B_1^2+6a_9B_1^2+4a_{15}A_1B_1^3+$$
$$+4a_7B_1^3+a_3B_1^4)(nA_5-rB_5)+4(a_2A_1^3A_5+a_4A_5+3a_6A_1^2A_5+3a_8A_1A_5+$$
$$+3a_{10}A_5B_1+6a_{11}A_1A_5B_1+3a_{14}A_1^2A_5B_1+3a_{12}A_5B_1^2+3a_{13}A_1A_5B_1^2+$$
$$+a_{15}A_5B_1^3+3a_{10}A_1B_5+3a_{11}A_1^2B_5+a_{14}A_1^3B_5+a_5B_5+6a_{12}A_1B_1B_5+3a_{13}A_1^2B_1B_5+$$
$$+3a_9B_1B_5+3a_{15}A_1B_1^2B_5+3a_7B_1^2B_5+a_3B_1^3B_5)(nA_1-rB_1)],$$

$$
\begin{aligned}
B_9 = \frac{5}{nq-mr}[(&a_1 + 4a_4A_1 + 6a_8A_1^2 + 4a_6A_1^3 + a_2A_1^4 + 12a_{10}A_1B_1 + \\
&+12a_{11}A_1^2B_1 + 4a_{14}A_1^3B_1 + 4a_5B_1 + 12a_{12}A_1B_1^2 + 6a_{13}A_1^2B_1^2 + 6a_9B_1^2 + \\
&+4a_{15}A_1B_1^3 + 4a_7B_1^3 + a_3B_1^4)(mA_5 - qB_5) + 4(a_2A_1^3A_5 + a_4A_5 + \\
&+3a_6A_1^2A_5 + 3a_8A_1A_5 + 3a_{10}A_5B_1 + 6a_{11}A_1A_5B_1 + 3a_{14}A_1^2A_5B_1 + \\
&+3a_{12}A_5B_1^2 + 3a_{13}A_1A_5B_1^2 + a_{15}A_5B_1^3 + 3a_{10}A_1B_5 + 3a_{11}A_1^2B_5 + a_{14}A_1^3B_5 + \\
&+a_5B_5 + 6a_{12}A_1B_1B_5 + 3a_{13}A_1^2B_1B_5 + 3a_9B_1B_5 + 3a_{15}A_1B_1^2B_5 + \\
&+3a_7B_1^2B_5 + a_3B_1^3B_5)(mA_1 - qB_1)], \\
&A_{10} = B_{10} = A_{11} = B_{11} = 0, \dots
\end{aligned}
\tag{13}
$$

Substituting (12) into the right-hand sides of the critical differential equations (7), we get the following identity

$$
\begin{aligned}
C_1x + C_2x^2 + C_3x^3 + C_4x^4 + C_5x^5 + \dots = 5x(&a_1x^4 + a_2y^4 + a_3z^4 + 4a_4x^3y + 4a_5x^3z + \\
&+4a_6xy^3 + 4a_7xz^3 + 6a_8x^2y^2 + 6a_9x^2z^2 + 12a_{10}x^2yz + 12a_{11}xy^2z + 12a_{12}xyz^2 + \\
&+6a_{13}y^2z^2 + 4a_{14}y^3z + 4a_{15}yz^3),
\end{aligned}
$$

or in detailed form

$$
\begin{aligned}
&C_1x + C_2x^2 + C_3x^3 + C_4x^4 + C_5x^5 + \dots = \\
&= 5x[a_1x^4 + a_2(A_1x + A_2x^2 + A_3x^3 + A_4x^4 + A_5x^5 + \dots)^4 + a_3(B_1x + B_2x^2 + B_3x^3 + \\
&+B_4x^4 + B_5x^5 + \dots)^4 + 4a_4x^3(A_1x + A_2x^2 + A_3x^3 + A_4x^4 + A_5x^5 + \dots) + \\
&+4a_5x^3(B_1x + B_2x^2 + B_3x^3 + B_4x^4 + B_5x^5 + \dots) + 4a_6x(A_1x + A_2x^2 + A_3x^3 + A_4x^4 + \\
&+A_5x^5 + \dots)^3 + 4a_7xz(B_1x + B_2x^2 + B_3x^3 + B_4x^4 + B_5x^5 + \dots)^3 + 6a_8x^2(A_1x + A_2x^2 + \\
&+A_3x^3 + A_4x^4 + A_5x^5 + \dots)^2 + 6a_9x^2(B_1x + B_2x^2 + B_3x^3 + B_4x^4 + B_5x^5 + \dots)^2 + \\
&+12a_{10}x^2(A_1x + A_2x^2 + A_3x^3 + A_4x^4 + A_5x^5 + \dots)(B_1x + B_2x^2 + B_3x^3 + B_4x^4 + \\
&+B_5x^5 + \dots) + 12a_{11}x(A_1x + A_2x^2 + A_3x^3 + A_4x^4 + A_5x^5 + \dots)^2(B_1x + B_2x^2 + B_3x^3 + \\
&+B_4x^4 + B_5x^5 + \dots) + 12a_{12}x(A_1x + A_2x^2 + A_3x^3 + A_4x^4 + A_5x^5 + \dots)(B_1x + B_2x^2 + \\
&+B_3x^3 + B_4x^4 + B_5x^5 + \dots)^2 + 6a_{13}(A_1x + A_2x^2 + A_3x^3 + A_4x^4 + A_5x^5 + \dots)^2(B_1x + \\
&+B_2x^2 + B_3x^3 + B_4x^4 + B_5x^5 + \dots)^2 + 4a_{14}(A_1x + A_2x^2 + A_3x^3 + A_4x^4 + A_5x^5 + \\
&+\dots)^3(B_1x + B_2x^2 + B_3x^3 + B_4x^4 + B_5x^5 + \dots) + 4a_{15}(A_1x + A_2x^2 + A_3x^3 + A_4x^4 + \\
&+A_5x^5 + \dots)(B_1x + B_2x^2 + B_3x^3 + B_4x^4 + B_5x^5 + \dots)^3].
\end{aligned}
$$

From here, we obtain

$$
\begin{gathered}
C_1 = C_2 = C_3 = C_4 = 0, \\
C_5 = 5(a_1 + 4a_4A_1 + 6a_8A_1^2 + 4a_6A_1^3 + a_2A_1^4 + 12a_{10}A_1B_1 + 12a_{11}A_1^2B_1 + \\
+4a_{14}A_1^3B_1 + 4a_5B_1 + 12a_{12}A_1B_1^2 + 6a_{13}A_1^2B_1^2 + 6a_9B_1^2 + 4a_{15}A_1B_1^3 + \\
+4a_7B_1^3 + a_3B_1^4), \\
C_6 = 20(a_2A_1^3A_2 + a_4A_2 + 3a_6A_1^2A_2 + 3a_8A_1A_2 + 3a_{10}A_2B_1 + 6a_{11}A_1A_2B_1 + \\
+3a_{14}A_1^2A_2B_1 + 3a_{12}A_2B_1^2 + 3a_{13}A_1A_2B_1^2 + a_{15}A_2B_1^3 + 3a_{10}A_1B_2 + 3a_{11}A_1^2B_2 + \\
+a_{14}A_1^3B_2 + a_5B_2 + 6a_{12}A_1B_1B_2 + 3a_{13}A_1^2B_1B_2 + 3a_9B_1B_2 + 3a_{15}A_1B_1^2B_2 + \\
+3a_7B_1^2B_2 + a_3B_1^3B_2), \\
C_7 = 10(3a_2A_1^2A_2^2 + 2a_2A_1^3A_3 + 2a_4A_3 + 6a_6A_1A_2^2 + 6a_6A_1^2A_3 + 3a_8A_2^2 + \\
+6a_8A_1A_3 + 6a_{11}A_2^2B_1 + 6a_{14}A_1A_2^2B_1 + 6a_{10}A_3B_1 + 12a_{11}A_1A_3B_1 + \\
+6a_{14}A_1^2A_3B_1 + 3a_{13}A_2^2B_1^2 + 6a_{12}A_3B_1^2 + 6a_{13}A_1A_3B_1^2 + 2a_{15}A_3B_1^3 + 6a_{10}A_2B_2 + \\
+12a_{11}A_1A_2B_2 + 6a_{14}A_1^2A_2B_2 + 12a_{12}A_2B_1B_2 + 12a_{13}A_1A_2B_1B_2 + \\
+6a_{15}A_2B_1^2B_2 + 6a_{12}A_1B_2^2 + 3a_{13}A_1^2B_2^2 + 3a_9B_2^2 + 6a_{15}A_1B_1B_2^2 + 6a_7B_1B_2^2 + \\
+3a_3B_1^2B_2^2 + 6a_{10}A_1B_3 + 6a_{11}A_1^2B_3 + 2a_{14}A_1^3B_3 + 2a_5B_3 + 12a_{12}A_1B_1B_3 + \\
+6a_{13}A_1^2B_1B_3 + 6a_9B_1B_3 + 6a_{15}A_1B_1^2B_3 + 6a_7B_1^2B_3 + 2a_3B_1^3B_3), \ldots
\end{gathered} \tag{14}
$$

3. The stability conditions of unperturbed motion for the ternary differential system of Lyapunov-Darboux type with nonlinearities of fifth degree

We will introduce the following notation:

$$
\begin{gathered}
M = a_1 + 4a_4A_1 + 6a_8A_1^2 + 4a_6A_1^3 + a_2A_1^4 + 12a_{10}A_1B_1 + 12a_{11}A_1^2B_1 + \\
+4a_{14}A_1^3B_1 + 4a_5B_1 + 12a_{12}A_1B_1^2 + 6a_{13}A_1^2B_1^2 + 6a_9B_1^2 + 4a_{15}A_1B_1^3 + \\
+4a_7B_1^3 + a_3B_1^4,
\end{gathered} \tag{15}
$$

According to Lyapunov Theorem [4, §32], we have

Theorem 3.1. *Let the critical system* (7) *be given on the invariant variety. The stability of the unperturbed motion is described by one of the following three possible cases*:

I. *If $M > 0$, then the unperturbed motion is **unstable**;*

II. *If $M < 0$, then the unperturbed motion is **stable**;*

III. *If $M = 0$, then the unperturbed motion is **stable**.*

In the last case, the unperturbed motion belongs to some continuous series of stabilized motions. Moreover, this motion is asymptotically stable.

Proof. According to Lyapunov Theorem [4], we analyze the coefficients of the series (14). The stability or the instability of the unperturbed motion of the system (7) is determined by the sign of expression C_5, and we get Cases I and II.

Therefore, if $C_5 = 0$, then all $A_i = B_i = 0$ $(\forall i)$, so we get Case III of this theorem. □

Remark 3.1. *Theorem 3.1 was presented at the 31st Conference on Applied and Industrial Mathematics (CAIM-2024)* [6].

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On a method of constructing topological quasigroups obeying certain laws

Liubomir Chiriac, Natalia Josu, and Natalia Lupashco

Abstract. A new method of constructing non-associative topological quasigroups obeying certain laws is given. Also, in this paper we research T-quasigroups with Abel-Grassmann identity $(ab) \cdot c = (cb) \cdot a$.

2020 Mathematics Subject Classification: 34C14, 34C40.

Keywords: T-quasigroup, AG-quasigroup, GA-quasigroup, Manin quasigroup, Cote quasigroup, medial, semimedial, paramedial and bicommutative quasigroup, topological quasigroup.

Despre o metodă de construcţie a quasigrupurilor topologice care îndeplinesc anumite identităţi

Rezumat. În lucrarea dată este prezentată o nouă metodă de construcţie a quasigrupurilor topologice neasociative care respectă anumite legi. Totodată sunt cercetate T-quasigrupurile care satisfac identitatea Abel-Grassmann $(ab) \cdot c = (cb) \cdot a$.

Cuvinte-cheie: T-quasigrup, AG-quasigrup, GA-quasigrup, quasigrupul Manin, quasigrupul Cote, quasigrup medial, semimedial, paramedial şi bicomutativ, quasigrup topologic.

1. Introduction

In this paper, two central issues were examined.

Problem 1. Let $(Q, \cdot)$ be a $T-quasigroup$. Under which conditions the Q is a quasigroup (of its T - forms $(Q(+), \varphi, \psi, a))$ satisfying the identities P_i of some algebraic structure, where $i = 1, 2, ..., k$?

In the condition of problem formulated above we research $T - quasigroups$ with Abel-Grassmann identity $(ab) \cdot c = (cb) \cdot a$.

It is shown that if G is a $T - quasigroup$, then G is $AG - quasigroup$ if and only if for any of its $T - forms$ $(G(+), \varphi, \psi)$ is $\varphi^2(x) = \psi(x)$.

At the same time in this paper we examine the following problem.

Problem 2. Let $(G, +)$ be a commutative topological group. Under which conditions on the set $G \times G$ can be defined the binary operation $(\circ)$ such that $(G \times G, \circ)$ is a non-associative topological quasigroups obeying certain laws?

Our main goal is to prove a new method of constructing topological quasigroups.

The authors have used the concept of a special direct product of a topological Abelian group G and proved that a binary operation can be defined on the set $G \times G$, such that the new algebraic structure is a non-associative topological quasigroups obeying certain laws.

Thus, solving the problem formulated above, it was demonstrated that any commutative topological group can be "transformed" into non-associative topological quasigroup obeying certain laws using the method developed. Examples of quasigroups that satisfy the examined identities were constructed.

The results established are related to the results of L. Chiriac and N. Josu in [1, 2] and to the research papers [3, 4, 8, 9, 10].

2. Basic Notions

In this section we recall some fundamental definitions and notations [5, 6, 7, 11].

A non-empty set G is said to be a *groupoid* with respect to a binary operation denoted by $\{\cdot\}$, if for every ordered pair (a, b) of elements of G there is a unique element $ab \in G$.

A quasigroup is a binary algebraic structure in which one-sided multiplication is a bijection in that all equations of the form $ax = b$ and $ya = b$ have unique solutions.

A groupoid G is called a primitive groupoid with divisions, if there exist two binary operation $l : G \times G \to G$, $r : G \times G \to G$ such that $l(a, b) \cdot a = b, a \cdot r(a, b) = b$ for all $a, b \in G$. Thus, a primitive groupoid with divisions is a universal algebra with three binary operations.

A primitive groupoid G with divisions is called a quasigroup if the equations $ax = b$ and $ya = b$ have unique solutions. In a quasigroup G the divisions l, r are unique. If the multiplication operation in a quasigroup $(G, \cdot)$ with a topology is continuous, then G is called a semitopoligical quasigroup. If in a semitopological quasigroup G the divisions l and r are continuous, then G is called a topological quasigroup.

An element $e \in G$ is called an *identity* if $ex = xe = x$ every $x \in G$. A quasigroup $(G,)$ with an identity element $e \in G$ is called a loop.

A groupoid $(G, \cdot)$ is called *medial* if it satisfies the law $xy \cdot zt = xz \cdot yt$ for all $x, y, z, t \in G$. A groupoid $(G, \cdot)$ is called *paramedial* if it satisfies the law $xy \cdot zt = ty \cdot zx$ for all $x, y, z, t \in G$.

A groupoid $(G, \cdot)$ is called *bicommutative* if it satisfies the law $xy \cdot zt = tz \cdot yx$ for all $x, y, z, t \in G$.

A groupoid $(G, \cdot)$ is called *AD-groupoid* if it satisfies the law $a \cdot bc = c \cdot ba$ for all $a, b, c \in G$.

A groupoid $(G, \cdot)$ is called a *groupoid Abel-Grassmann* or *AG-groupoid* if it satisfies the left invertive law $(ab) \cdot c = (cb) \cdot a$ for all $a, b, c \in G$.

A groupoid $(G, \cdot)$ is called a *GA-groupoid* if it satisfies law $(ab) \cdot c = c \cdot ba$ for all $a, b, c \in G$.

A groupoid $(G, \cdot)$ is called a *groupoid Manin* or *CH-groupoid* if it satisfies the law $x(y \cdot xz) = (xx \cdot y)z$ for all $x, y, z \in G$.

A groupoid $(G, \cdot)$ is called a *groupoid Cote* if it satisfies the law $x(xy \cdot z) = (z \cdot xx)y$ for all $x, y, z \in G$.

Left semi-medial identity in a groupoid $(G, \cdot)$ has the following form: $xx \cdot zt = xy \cdot xz$ for all $x, y, z, t \in G$. R.H. Bruck [14] used this identity to define commutative Moufang loops in the class of loops.

3. T-quasigroups with Abel-Grassmann identity

In this section we study some aspects of charaterization of abelian groups isotopic to $T - quasigroups$.

Definition. Quasigroup $(G, \cdot)$ is a $T - quasigroup$ if and only if there exists an abelian group $(G, +)$, its automorphisms φ and ψ, and a fixed element $a \in G$ such that $x \cdot y = \varphi(x) + \psi(y) + a$ for all $x, y \in G$.

Under the conditions of Definition we shall say that the isotope $(G, \cdot)$ is generated by the automorphisms φ, ψ and a fixed element $a \in G$ of the abelian group $(G, +)$ and write $(G, \cdot) = g(G, +, \varphi, \psi, a)$.

We study the problem formulated below.

Problem 1. Let $(Q, \cdot)$ be a $T - quasigroup$. Under which conditions the Q is a quasigroup (of its T - forms $(Q(+), \varphi, \psi, a)$) satisfying the identities P_i of some algebraic structure, where $i = 1, 2, ..., k$?

Professor V. Shcherbacov and his students studied *"Schroder T-quasigroups of generalized associativity"* and *"T-quasigroups with Stein 2-nd and 3-rd identity"* in [12, 13].

We examine the T-quasigroups with Abel-Grassmann identity.

Theorem 3.1. *Let G be a $T-quasigroup$. Then G is $AG-quasigroup$ if and only if for any of its $T-forms$ $(G(+),\varphi,\psi))$, $\varphi^2(x)=\psi(x)$.*

Proof. We rewrite the identity of the $AG-quasigroup$,

$$(xy)\cdot z=(zy)\cdot x, \tag{1}$$

in the following form:

$$\varphi(xy)+\psi(z)=\varphi(zy)+\psi(x). \tag{2}$$

From (2) we have

$$\varphi(\varphi(x)+\psi(y))+\psi(z)=\varphi(\varphi(z)+\psi(y))+\psi(x), \tag{3}$$

$$\varphi^2(x)+\varphi\psi(y)+\psi(z)=\varphi^2(z)+\varphi\psi(y)+\psi(x). \tag{4}$$

If we substitute in equality (4) $x=y=0$, then we obtain

$$\psi(z)=\varphi^2(z). \tag{5}$$

Similarly, if we substitute in equality (4) $z=y=0$, then we obtain

$$\varphi^2(x)=\psi(x). \tag{6}$$

Converse. Substituting the expression $x\cdot y=\varphi(x)+\psi(y)$ in identity (1) then we get (4). Substituting in (4) equalities (5) and (6), $\psi(z)=\varphi^2(z)$ and $\varphi^2(x)=\psi(x)$, we obtain that the identity (4) is true. In this way, in this case, we have that identity (1) is true. The proof is complete. □

Example 3.1. *Examine the group $\mathbb{Z}_n$ of residues modulo n. Define the quasigroup $(G,\cdot)$. We define the binary operation $x\cdot y=3x+9y(mod18)$ for all $x,y\in G$. Then $(G,\cdot)$ is an $AG-quasigroup$. Check. Let $(x\cdot y)\cdot z=(z\cdot y)\cdot x$. Then, $3(3x+9y)+9z=3(3z+9y)+9x(mod18)$, $9x+27y+9z=9z+27y+9x(mod18)$, $0=0(mod18)$.*

Example 3.2. *Denote by $\mathbb{Z}_p=\mathbb{Z}/p\mathbb{Z}=\{0,1,...,p-1\}$ the cyclic group of order p. Let $(G,+)=(\mathbb{Z}_5,+)$, $\varphi(x)=2x$, $\psi(x)=4x$. Then $x\cdot y=2x+4y$ and $\varphi^2(x)=\psi(x)$. Hence, $(G,\cdot)=g(G,+,\varphi,\psi)$ is an AG-quasigroup.*

Below we have constructed the Cayley table for AG-quasigroup $(G, \cdot)$.

$(\cdot)$	0	1	2	3	4
0	0	4	3	2	1
1	2	1	0	4	3
2	4	3	2	1	0
3	1	0	4	3	2
4	3	2	1	0	4

Example 3.3. *Let* $(G,+) = (\mathbb{Z}_5,+)$, $\varphi(x) = 2x$, $\psi(x) = 3x$. *Then* $x \cdot y = 2x + 3y$ *and* $\varphi^2(x) \neq \psi(x)$. *Hence,* $(G,\cdot) = g(G,+,\varphi,\psi)$ *is not an AG-quasigroup.*

Below we have constructed the Cayley table for quasigroup $(G,\cdot)$, *where* $(ab)\cdot c = (cb)\cdot a$ *does not hold in* $(G,\cdot)$. *For example,* $(3\cdot 4)\cdot 2 \neq (2\cdot 4)\cdot 3$.

$(\cdot)$	0	1	2	3	4
0	0	3	1	4	2
1	2	0	3	1	4
2	4	2	0	3	1
3	1	4	2	0	3
4	3	1	4	2	0

4. On a method of constructing medial, paramedial and bicommutative topological quasigroups

In this section we examined $Problem 2$. In Section 4 we prove a new method of constructing medial, semimedial, paramedial, bicommutative, Manin, Cote and GA non-associative topological quasigroup.

Theorem 4.1. *Let* $(G,+,\tau)$ *be a commutative topological group where G is not a singleton. For* (x_1,y_1) *and* (x_2,y_2) *in* $G\times G$ *define*

$$(x_1,y_1)\circ(x_2,y_2) = (-x_1-x_2, y_1+y_2)$$

Then $(G\times G,\circ,\tau_G)$, *relative to the product topology* τ_G, *is a medial, semimedial, paramedial, bicommutative, Manin, Cote and GA non-associative topological quasigroup. Moreover, if* (G,τ) *is* $T_i-space$, *then* $(G\times G,\tau_G)$ *is* $T_i-space$ *too, where* $i = 1,2,3,3.5$.

Proof. **1.** We will prove that $(G\times G,\circ)$ is a quasigroup. To this end, we will show that the equations $y\circ a = b$ and $a\circ x = b$ have unique solutions in $(G\times G,\circ)$. Let $y = (y_1,y_2)$,

$x = (x_1, x_2)$, $a = (a_1, a_2)$ and $b = (b_1, b_2)$. Since, $y \circ a = b$ we have

$$(y_1, y_2) \circ (a_1, a_2) = (b_1, b_2). \tag{7}$$

According to the conditions of the Theorem

$$(y_1, y_2) \circ (a_1, a_2) = (-y_1 - a_1, y_2 + a_2). \tag{8}$$

From (7) and (8) we get

$$-y_1 - a_1 = b_1 \tag{9}$$

and

$$a_2 + y_2 = b_2. \tag{10}$$

From (10) and (9) we obtain

$$y_2 = b_2 - a_2. \tag{11}$$

and

$$y_1 = -a_1 - b_1. \tag{12}$$

Hence, $y_1 = -b_1 - a_1$ and $y_2 = b_2 - a_2$ are solutions of the equation $y \circ a = b$. It is easy to show that any other solutions of that equation coincide with y_1 and y_2.

In this case

$$l((a_1, a_2), (b_1, b_2)) = (-b_1 - a_1, b_2 - a_2)$$

and $l((a_1, a_2), (b_1, b_2)) \circ (a_1, a_2) = (b_1, b_2)$.

We will show that the equation $a \circ x = b$ have unique solutions in $(G \times G, \circ)$. Let $x = (x_1, x_2)$, $a = (a_1, a_2)$ and $b = (b_1, b_2)$. Since $a \circ x = b$ we have

$$(a_1, a_2) \circ (x_1, x_2) = (b_1, b_2). \tag{14}$$

According to the conditions of the Theorem

$$(a_1, a_2) \circ (x_1, x_2) = (-a_1 - x_1, a_2 + x_2). \tag{15}$$

From (14) and (15) we get

$$-x_1 - a_1 = b_1 \tag{16}$$

and

$$a_2 + x_2 = b_2. \tag{17}$$

From (17) and (16) we obtain

$$x_2 = b_2 - a_2. \tag{18}$$

and

$$x_1 = -a_1 - b_1. \tag{19}$$

Hence, $x_1 = -b_1 - a_1$ and $x_2 = b_2 - a_2$ are solutions of the equation $a \circ x = b$. It is easy to show that any other solutions of that equation coincide with x_1 and x_2. Then $x_1 = -b_1 - a_1$ and $x_2 = b_2 - a_2$ are unique solutions.

In this case

$$r((a_1, a_2), (b_1, b_2)) = (-b_1 - a_1, b_2 - a_2)$$

and $(a_1, a_2) \circ r((a_1, a_2), (b_1, b_2)) = (b_1, b_2)$.

Thus $(G \times G, \circ)$ is a quasigroup.

2. We will prove that associativity

$$((x_1, y_1) \circ (x_2, y_2)) \circ (x_3, y_3) = (x_1, y_1) \circ ((x_2, y_2) \circ (x_3, y_3)) \tag{20}$$

does not hold in $(G \times G, \circ)$.

Indeed, for the first side of the law (20) we obtain

$$((x_1, y_1) \circ (x_2, y_2)) \circ (x_3, y_3) = (-x_1 - x_2, y_1 + y_2) \circ (x_3, y_3) =$$

$$= (x_1 + x_2 - x_3, y_1 + y_2 + y_3). \tag{21}$$

Similarly, for the second side of the law (20) we have

$$(x_1, y_1) \circ ((x_2, y_2) \circ (x_3, y_3)) = (x_1, y_1) \circ (-x_2 - x_3, y_2 + y_3) =$$

$$= (-x_1 + x_2 + x_3, y_1 + y_2 + y_3). \tag{22}$$

From (21) and (22) it is clear that associativity does not hold in $(G \times G, \circ)$.

3. We will show that $(G \times G, \circ)$ is a medial quasigroup that is, the property $xy \cdot zt = = xz \cdot yt$ holds.

Let $x = (x_1, y_1), y = (x_2, y_2), z = (x_3, y_3), t = (x_4, y_4)$, then

$$((x_1, y_1)\circ(x_2, y_2))\circ((x_3, y_3)\circ(x_4, y_4)) = ((x_1, y_1)\circ(x_3, y_3))\circ((x_2, y_2)\circ(x_4, y_7)). \tag{23}$$

According to the Theorem for the first side of the law (23) we have

$$((x_1, y_1) \circ (x_2, y_2)) \circ ((x_3, y_3) \circ (x_4, y_4)) =$$

$$= ((-x_1 - x_2, y_1 + y_2)) \circ ((-x_3 - x_4, y_3 + y_4)) =$$

$$= (x_1 + x_2 + x_3 + x_4, y_1 + y_2 + y_3 + y_4). \tag{24}$$

Similarly, for the other side of the law (23) we get

$$((x_1, y_1) \circ (x_3, y_3)) \circ ((x_2, y_2) \circ (x_4, y_4)) =$$

$$= (-x_1 - x_3, y_1 + y_3) \circ (-x_2 - x_4, y_2 + y_4) =$$

$$= (x_1 + x_3 + x_2 + x_4, y_1 + y_3 + y_2 + y_4). \tag{25}$$

From (24) and (25) we obtain that both sides are equal and $(G \times G, \circ)$ is a medial quasigroup. Similarly, it is shown that paramediality and bicommutative does hold in $(G \times G, \circ)$.

4. We will show that $(G \times G, \circ)$ is Manin quasigroup that is, the property $x(y \cdot xz) = (xx \cdot y)z$ holds.

Let $x = (x_1, y_1), y = (x_2, y_2), z = (x_3, y_3)$ then

$$(x_1, y_1)\circ((x_2, y_2)\circ((x_1, y_1)\circ(x_3, y_3))) = (((x_1, y_1)\circ(x_1, y_1))\circ(x_2, y_2))\circ(x_3, y_3)). \tag{26}$$

According to the Theorem for the first side of the law (26) we have

$$(x_1, y_1) \circ ((x_2, y_2) \circ ((x_1, y_1) \circ (x_3, y_3))) =$$

$$(x_1, y_1) \circ ((x_2, y_2) \circ (-x_1 - x_3, y_1 + y_3)) =$$

$$(x_1, y_1) \circ (-x_2 + x_1 + x_3, y_1 + y_2 + y_3) =$$

$$(-x_1 + x_2 - x_1 - x_3, 2y_1 + y_2 + y_3). \tag{27}$$

Similarly, for the other side of the law (26) we get

$$(((x_1, y_1) \circ (x_1, y_1)) \circ (x_2, y_2)) \circ (x_3, y_3)) =$$

$$((-x_1 - x_1, y_1 + y_1) \circ (x_2, y_2)) \circ (x_3, y_3)) =$$

$$(+x_1 + x_1 - x_2, y_1 + y_1 + y_2) \circ (x_3, y_3) =$$

$$= (-x_1 - x_1 + x_2 - x_3, 2y_1 + y_2 + y_3). \tag{28}$$

From (27) and (28) we obtain that both sides are equal and $(G \times G, \circ)$ is a Manin quasigroup. Similarly, it is shown that $(G \times G, \circ)$ is a Cote quasigroup.

5. Similarly, we will show that GA identity is fulfilled in $(G \times G, \circ)$. Let $x = (x_1, y_1), y = (x_2, y_2), z = (x_3, y_3)$ then

$$((x_1, y_1) \circ (x_2, y_2)) \circ ((x_3, y_3) = (x_3, y_3) \circ ((x_2, y_2)) \circ (x_1, y_1)) \tag{29}$$

Indeed, according to the Theorem for the first side of the law (29) we have

$$((x_1, y_1) \circ (x_2, y_2)) \circ (x_3, y_3) =$$

$$(-x_1 - x_2, y_1 + y_2) \circ (x_3, y_3) =$$

$$(x_1 + x_2 - x_3, y_1 + y_2 + y_3). \tag{30}$$

Similarly, for the other side of the law (29) we get

$$(x_3, y_3) \circ ((x_2, y_2)) \circ (x_1, y_1)) =$$

$$(x_3, y_3) \circ (-x_2 - x_1, y_2 + y_1) =$$

$$(-x_3 + x_2 + x_1, y_3 + y_2 + y_1). \tag{31}$$

From (30) and (31) we obtain that both sides are equal and $(G \times G, \circ)$ is a GA quasigroup.

Multiplication $(\circ)$ and divisions $l(a, b)$ and $r(a, b)$ are jointly continuous relative to the product topology.

Consequently, $(G \times G, \circ, \tau_G)$ is a non-associative, medial, semimedial, paramedial, bicommutative, Manin, Cote and GA topological quasigroup.

If (G, τ) is T_i-space, then according to Theorem 2.3.11 in [6], a product of T_i-spaces is a T_i-spaces, where $i = 1, 2, 3, 3.5$. The proof is complete. □

In [1] was proved the following Theorem.

Theorem 4.2. *Let $(G, +, \tau)$ be a commutative topological group where G is not a singleton. For (x_1, y_1) and (x_2, y_2) in $G \times G$ define*

$$(x_1, y_1) \circ (x_2, y_2) = (x_1 + y_1 - x_2, x_2 + y_2 - y_1).$$

Then $(G \times G, \circ, \tau_G)$, relative to the product topology τ_G, is a paramedial, non-medial and non-associative topological quasigroup. Moreover, if (G, τ) is $T_i - space$, then $(G \times G, \tau_G)$ is $T_i - space$ too, where $i = 1, 2, 3, 3.5$.

The following Theorem was proved in [2].

Theorem 4.3. *Let $(G, +, \tau)$ be a commutative topological group where G is not a singleton. For (x_1, y_1) and (x_2, y_2) in $G \times G$ define*

$$(x_1, y_1) \circ (x_2, y_2) = (-x_1 - y_1 + y_2, -x_2 - y_2 + x_1).$$

Then $(G \times G, \circ, \tau_G)$, relative to the product topology τ_G, is a non-associative, medial, AG and AD-topological quasigroup. Moreover, if (G, τ) is $T_i - space$, then $(G \times G, \tau_G)$ is $T_i - space$ too, where $i = 1, 2, 3, 3.5$.

Example 4.1. *Let $G = \{0, 1, 2\}$. We define the binary operation " + ".*

(+)	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

Then $(G,+)$ is a commutative group. Define a binary operation $(\circ)$ on the set $G \times G$ by $(x_1, y_1) \circ (x_2, y_2) = (-x_1 - x_2, y_1 + y_2)$ for all $x_1, y_1, x_2, y_2 \in G \times G$. If we label the elements as follows $(0,0) \leftrightarrow 0$, $(0,1) \leftrightarrow 1$, $(0,2) \leftrightarrow 2$, $(1,0) \leftrightarrow 3$, $(1,1) \leftrightarrow 4$, $(1,2) \leftrightarrow 5$, $(2,0) \leftrightarrow 6$, $(2,1) \leftrightarrow 7$, $(2,2) \leftrightarrow 8$, then obtain:

$(\circ)$	0	1	2	3	4	5	6	7	8
0	0	1	2	6	7	8	3	4	5
1	1	2	0	7	8	6	4	5	3
2	2	0	1	8	6	7	5	3	4
3	6	7	8	3	4	5	0	1	2
4	7	8	6	4	5	3	1	2	0
5	8	6	7	5	3	4	2	0	1
6	3	4	5	0	1	2	6	7	8
7	4	5	3	1	2	0	7	8	6
8	5	3	4	2	0	1	8	6	7

Then $(G \times G, \circ)$ is a medial, semimedial, paramedial, bicommutative, Manin, Cote, GA non-associative quasigroup.

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Affine invariant conditions for a class of differential polynomial cubic systems

Cristina Bujac

Abstract. In this article the affine invariant criteria constructed in terms of algebraic polynomials with coefficients $\tilde{a} \in \mathcal{R}^{20}$ for a class of cubic systems are established. We are focused on non-degenerate real cubic systems with 7 invariant straight lines, considering the line at infinity and their multiplicities and possesing four real singularities at infinity. Additionally, the only configurations of the type $(3,3)$ of mentioned systems are considered and we denote this class by $CSL^{4r\infty}_{(3,3)}$. In [5] the existence of exactly 14 configurations of invariant straight lines for systems in $CSL^{4r\infty}_{(3,3)}$ was proved. Here we complete this classification by determining necessary and sufficient conditions for the realization of each one of the 14 configurations in terms of affine invariant polynomials.
2020 Mathematics Subject Classification: 34C23, 34A34.
Keywords: polynomial cubic system, invariant straight line, finite/infinite singular point, configuration of invariant straight lines, affine invariant conditions.

Condiţii afin invariante pentru o clasă de sisteme polinomiale diferenţiale cubice

Rezumat. În acest articol sunt stabilite criterii invariante construite în termeni de polinoame algebrice cu coeficienţi $\tilde{a} \in \mathcal{R}^{20}$ pentru o clasă de sisteme cubice. Ne concentrăm pe sisteme cubice reale, nedegenerate, cu 7 drepte invariante, luând în consideraţie dreapta de la infinit şi multiplicităţile acesteia, care posedă patru singularităţi reale la infinit. În plus, sunt analizate doar configuraţiile de tipul $(3,3)$ ale sistemelor menţionate, iar această clasă este notată cu $CSL^{4r\infty}_{(3,3)}$. În [5] a fost demonstrată existenţa exact a 14 configuraţii de drepte invariante pentru sistemele din $CSL^{4r\infty}_{(3,3)}$. În acest articol, completăm această clasificare prin determinarea condiţiilor necesare şi suficiente afin-invarinate pentru realizarea fiecăreia dintre cele 14 configuraţii depistate.
Cuvinte-cheie: sistem cubic polinomial, dreaptă invariantă, punct singular finit/infinit, configuraţie de drepte invariante, condiţii afin-invariante.

1. Introduction and preliminary results

Consider the family $\mathbb{CS}$ of real cubic systems, i.e. systems of the form:

$$\begin{aligned}\dot{x} &= p_0 + p_1(x,y) + p_2(x,y) + p_3(x,y) \equiv P(a,x,y),\\ \dot{y} &= q_0 + q_1(x,y) + q_2(x,y) + q_3(x,y) \equiv Q(a,x,y)\end{aligned} \tag{1}$$

with variables x and y and real coefficients such that $\gcd(P,Q) = 1$ and $\max(\deg(P,Q)) = 3$. The polynomials $p_i(x,y)$ and $q_i(x,y)$ for $i = 0,1,2,3$ are homogeneous polynomials of degree i in variables x and y:

$$\begin{aligned}&p_0 = a_{00}, \quad p_1(x,y) = a_{10}x + a_{01}y,\\ &p_2(x,y) = a_{20}x^2 + 2a_{11}xy + a_{02}y^2,\\ &p_3(x,y) = a_{30}x^3 + 3a_{21}x^2y + 3a_{12}xy^2 + a_{03}y^3,\\ &q_0 = b_{00}, \quad q_1(x,y) = b_{10}x + b_{01}y,\\ &q_2(x,y) = b_{20}x^2 + 2b_{11}xy + b_{02}y^2,\\ &q_3(x,y) = b_{30}x^3 + 3b_{21}x^2y + 3b_{12}xy^2 + b_{03}y^3.\end{aligned}$$

Let $a \in R^{20}$, i.e. $a = (a_{00}, a_{10}, a_{01}, \dots, a_{03}, b_{00}, b_{10}, b_{01}, \dots, b_{03})$ be the 20-tuple of the coefficients of systems (1). We denote

$$\mathbb{R}[a,x,y] = \mathbb{R}[a_{00}, a_{10}, a_{01}, \dots, a_{03}, b_{00}, b_{10}, b_{01}, \dots, b_{03}, x, y].$$

The set $\mathbb{CS}$ of *cubic differential systems* (1) depends on 20 parameters, and therefore mathematicians began studying particular families of $\mathbb{CS}$. Among these families, there are cubic systems with invariant straight lines, and we denote such families of systems by $\mathbb{CSL}$.

A line $f(x,y) = w + ux + vy = 0$ over $\mathbb{C}$ *is an invariant* line for a system (1) if and only if there exists $K(x,y) \in \mathbb{C}[x,y]$, which satisfies the following identity in $\mathbb{C}[x,y]$:

$$uP(x,y) + vQ(x,y) = (w + ux + vy)K(x,y).$$

According to [1] the maximum number of the invariant straight lines (including the line at infinity $Z = 0$) for cubic differential systems with a finite number of infinite singularities is 9. In paper [17], all the possible configurations of invariant lines are obtained in the case, when the total multiplicity of these lines (including the line at infinity) equals nine. If the total multiplicity of these lines (including the line at infinity) equals eight, then all possible configurations of invariant lines are found in [7, 8, 9, 10, 11].

We continue our investigation on $\mathbb{CSL}$ with invariant lines of total multiplicity 7 (the line at infinity is considered). To each system in $\mathbb{CSL}$, we associate its *configuration of* *invariant lines*, i.e. the set of its invariant lines together with the real singular points of the system located on the union of these lines.

More precisely, we call *configuration of invariant straight lines* of a real planar polynomial differential system (1), the set of (complex) invariant straight lines (which may have real coefficients) including the line at infinity of the system, each endowed with its own multiplicity and together with all the real singular points of this system located on these invariant straight lines, each one endowed with its own multiplicity.

The notion of configuration of invariant lines for a polynomial differential system was first introduced in [15].

It is known that on $\mathbb{CS}$ (1), the group $Aff(2,\mathbb{R})$ of affine transformations of the plane acts [14]. For every subgroup $G \subseteq Aff(2,\mathbb{R})$ we have an induced action of G on $\mathbb{CS}$. We can identify the set $\mathbb{CS}$ of cubic systems (1) with a subset of $\mathbb{R}^{20}$ via the map $\mathbb{CS} \longrightarrow \mathbb{R}^{20}$, which associates to each cubic system (1) the 20-tuple $\tilde{a} = (a_{00}, a_{10}, a_{01}, \ldots, a_{03}, b_{00}, b_{10}, b_{01}, \ldots, b_{03})$ of its coefficients.

The definitions of an affine or GL-comitant or invariant as well as the definitions of a T-comitant and CT-comitant can be found in [15] (see also [2]).

Here, we construct the necessary invariant polynomials (T-comitants) that we need for detecting the existence of invariant lines for the family of cubic systems having four distinct real singularities and exactly seven invariant straight lines including the line at infinity and counting multiplicities.

We consider the polynomials

$$C_i(a,x,y) = yp_i(a,x,y) - xq_i(a,x,y) \in \mathbb{R}[a,x,y],\ i = 0,1,2,3,$$
$$D_i(a,x,y) = \frac{\partial}{\partial x}p_i(a,x,y) + \frac{\partial}{\partial y}q_i(a,x,y) \in \mathbb{R}[a,x,y],\ i = 1,2,3.$$

In [16] it was shown that the following polynomials

$$\big\{C_i(a,x,y),\ D_1(a),\ D_2(a,x,y),\ D_3(a,x,y), i = 0,1,2,3\big\} \tag{2}$$

of degree one in the coefficients of systems (1) are GL-comitants of these systems.

Notation 3. Let $f, g \in R[a,x,y]$ and

$$(f,g)^{(k)} = \sum_{h=0}^{k}(-1)^h \binom{k}{h} \frac{\partial^k f}{\partial x^{k-h}\partial y^h}\ \frac{\partial^k g}{\partial x^h \partial y^{k-h}}.$$

$(f,g)^{(k)} \in \mathbb{R}[a,x,y]$ is called *the transvectant of index k* of (f,g) (cf. [12, 18]).

To define the invariant polynomials, we first construct the comitants of the second degree, with respect to the coefficients of the initial systems (1), of the form:

$$
\begin{array}{lll}
S_1 = (C_0, C_1)^{(1)}, & S_{10} = (C_1, C_3)^{(1)}, & S_{19} = (C_2, D_3)^{(1)}, \\
S_2 = (C_0, C_2)^{(1)}, & S_{11} = (C_1, C_3)^{(2)}, & S_{20} = (C_2, D_3)^{(2)}, \\
S_3 = (C_0, D_2)^{(1)}, & S_{12} = (C_1, D_3)^{(1)}, & S_{21} = (D_2, C_3)^{(1)}, \\
S_4 = (C_0, C_3)^{(1)}, & S_{13} = (C_1, D_3)^{(2)}, & S_{22} = (D_2, D_3)^{(1)}, \\
S_5 = (C_0, D_3)^{(1)}, & S_{14} = (C_2, C_2)^{(2)}, & S_{23} = (C_3, C_3)^{(2)}, \\
S_6 = (C_1, C_1)^{(2)}, & S_{15} = (C_2, D_2)^{(1)}, & S_{24} = (C_3, C_3)^{(4)}, \\
S_7 = (C_1, C_2)^{(1)}, & S_{16} = (C_2, C_3)^{(1)}, & S_{25} = (C_3, D_3)^{(1)}, \\
S_8 = (C_1, C_2)^{(2)}, & S_{17} = (C_2, C_3)^{(2)}, & S_{26} = (C_3, D_3)^{(2)}, \\
S_9 = (C_1, D_2)^{(1)}, & S_{18} = (C_2, C_3)^{(3)}, & S_{27} = (D_3, D_3)^{(2)}.
\end{array}
$$

Next we determine the conditions for the existence of the couples of parallel invariant straight lines which a cubic system can have (see Theorem 1.1). For this we use the following invariant polynomials constructed in [17] and [8]:

$$
\begin{aligned}
\mathcal{V}_1(a,x,y) &= S_{23} + 2D_3^2, \\
\mathcal{V}_2(a,x,y) &= S_{26}, \\
\mathcal{V}_3(a,x,y) &= 6S_{25} - 3S_{23} - 2D_3^2, \\
\mathcal{V}_4(a,x,y) &= C_3\left[(C_3, S_{23})^{(4)} + 36\,(D_3, S_{26})^{(2)}\right], \\
\mathcal{V}_5(a,x,y) &= 6C_3(9A_5 - 7A_6) + 2D_3(4T_{16} - T_{17}) - 3T_3(3A_1 + 5A_2) + \\
&\quad + 3A_2T_4 + 36T_5^2 - 3T_{44}, \\
\mathcal{U}_1(a,x,y) &= S_{24} - 4S_{27}, \\
\mathcal{U}_2(a,x,y) &= 6(S_{23} - 3S_{25}, S_{26})^{(2)} - 3S_{23}(S_{24} - 8S_{27}) - \\
&\quad - 24S_{26}^2 + 2C_3(C_3, S_{23})^{(4)} + 24D_3(D_3, S_{26})^{(1)} + 24D_3^2S_{27}.
\end{aligned}
$$

In order to construct the needed affine invariant conditions, we will use the following polynomials:

$$
\begin{aligned}
\mathcal{H}_1 = {} & 48D_1^4 S_{24}\big[2D_1^2 + 3S_6\big] + 192D_1^5(S_{18}, D_2)^{(1)} + 12S_6^2 S_{24}\big[6D_1^2 + S_6\big] \\
& + 216S_3S_{24}\big[4D_1^4 - S_6^2 - 16S_3^2\big] - 108S_{24}(S_5, C_0)^{(1)}\big[8D_1^3 - 12D_1S_6 \\
& + 72D_1S_3 - 9(S_5, C_0)^{(1)}\big] - 216S_{24}(S_8, C_0)^{(1)}\big[4D_1^3 + 2D_1S_6 + 9(S_5, C_0)^{(1)}\big] \\
& - 192\left[(S_{18}, C_0)^{(1)}\right]^2\left[13D_1^2 + 9S_6 + 24S_3\right] - 24(S_{18}, C_0)^{(1)}(S_{14}, C_1)^{(2)}\big[66D_1^2 \\
& + 17S_6 - 72S_3\big] + 16(S_{18}, D_2)^{(1)}\big[12D_1^3S_6 + 3D_1S_6^2 + 104D_1^3S_3 - 45D_1S_3S_6 \\
& + 288D_1S_3^2 + 360D_1^2(S_5, C_0)^{(1)} + 189S_6(S_5, C_0)^{(1)} + 24S_6(S_8, C_0)^{(1)} - 144S_3(S_8, C_0)^{(1)}\big] \\
& + 216S_{24}\big((S_{11}, C_0)^{(1)}, C_0\big)^{(1)}\big[6D_1^2 - S_6 + 9S_3\big] + 36\big((S_{14}, C_0)^{(1)}, C_0\big)^{(1)} \\
& \times\big[15D_1^2S_{24} + 12S_3S_{24} + (S_{18}, D_2)^{(1)}\big] + 1152D_1(S_{18}, C_0)^{(1)}\big((S_{18}, C_1)^{(1)}, C_0\big)^{(1)} \\
& - 768\big[\big((S_{14}, C_0)^{(1)}, D_2\big)^{(1)}\big]^2 + 24\big((S_{18}, C_2)^{(1)}, C_1\big)^{(2)}\big[4D_1^4 + 4D_1^2S_6 + S_6^2 \\
& + 96D_1^2S_3 - 33D_1(S_8, C_0)^{(1)} - 63\big((S_{11}, C_0)^{(1)}, C_0\big)^{(1)}\big] + 3\big((S_{14}, C_2)^{(1)}, C_2\big)^{(3)}\times \\
& \big[4D_1^4 + 4D_1^2S_6 + S_6^2 + 32D_1^2S_3 - 16S_3S_6 - 32D_1(S_8, C_0)^{(1)} \\
& - 64\big((S_{14}, C_0)^{(1)}, C_0\big)^{(1)}\big] - 144\big[9D_1S_{24} + 16(S_{18}, D_2)^{(1)}\big]\times \\
& \big(((S_{17}, C_0)^{(1)}, C_0)^{(1)}, C_0\big)^{(1)} - 64\big(((S_{18}, C_2)^{(1)}, C_2)^{(2)}, C_0\big)^{(1)}\big[D_1^3 \\
& - 18(S_8, C_0)^{(1)}\big] + 243S_{24}\big((((S_{25}, C_0)^{(1)}, C_0)^{(1)}, C_0)^{(1)}, C_0\big)^{(1)};
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{H}_2 = {} & -3S_{24}\big[4D_1^3 - 18(S_5, C_0)^{(1)} + 9(S_8, C_0)^{(1)} + 2(S_{18}, D_2)^{(1)}\big[6D_1^2 + 16S_3 - 3S_6\big] \\
& + 18D_1S_{24}\big[3S_3 - S_6\big] - 12D_1\big((S_{18}, C_2)^{(1)}, C_1\big)^{(2)} + 32\big(((S_{18}, C_2)^{(1)}, C_2)^{(2)}, C_0\big)^{(1)};
\end{aligned}
$$

$$\mathcal{H}_3 = 72T_{136}(2307T_{140} - 607T_{141}) + T_{74}(13T_{144} + 264T_{145});$$

$$\mathcal{H}_4 = T_{74};$$

$$
\begin{aligned}
\mathcal{H}_5 = {} & 12D_1^4S_{24} - 18D_1S_6(S_{18}, D_2)^{(1)} + 128D_1S_3(S_{18}, D_2)^{(1)} \\
& - 48(S_8, C_0)^{(1)}(S_{18}, D_2)^{(1)} + 27S_{24}((S_{11}, C_0)^{(1)}, C_0)^{(1)} \\
& - 9S_{24}((S_{14}, C_0)^{(1)}, C_0)^{(1)} + 18D_1^2((S_{18}, C_2)^{(1)}, C_1)^{(2)} \\
& - 7S_6((S_{18}, C_2)^{(1)}, C_1)^{(2)} + 2D_1^2((S_{14}, C_2)^{(1)}, C_2)^{(3)} - S_6((S_{14}, C_2)^{(1)}, C_2)^{(3)} \\
& + 8S_3((S_{14}, C_2)^{(1)}, C_2)^{(3)} - 3S_6^2S_{24} - 16D_1(((S_{18}, C_2)^{(1)}, C_2)^{(2)}, C_0)^{(1)} \\
& + 54D_1^2S_3S_{24} + 27S_6S_3S_{24} - 36S_3^2S_{24} - 54D_1S_{24}(S_5, C_0)^{(1)} \\
& - 48(S_{18}, C_0)^{(1)})^2 + 60(S_{18}, C_0)^{(1)}(S_{14}, C_1)^{(2)} + 28D_1^3(S_{18}, D_2)^{(1)}.
\end{aligned}
$$

Here the polynomials

$$A_1 = S_{24}/288, \quad A_2 = S_{27}/72,$$
$$A_5 = (S_{23}, C_3)^{(4)}/2^7/3^5, \quad A_6 = (S_{26}, D_3)^{(2)}/2^5/3^3$$

are affine invariants and

$$T_3 = S_{23}/18, \quad T_4 = S_{25}/6, \quad T_5 = S_{26}/72,$$

$$\begin{aligned} T_6 =&(3C_1D_3^2 - 27C_1T_3 + 54C_1T_4 + 4C_3D_2^2 - 2C_3S_{14}+\\ &+ 16C_3S_{14} - 4C_2D_2D_3 + 2C_2S_{17} + 12C_2S_{21} - 4C_2S_{19})/2^4/3^2,\end{aligned}$$

$$\begin{aligned} T_{11} =&(D_3^2, C_2)^{(2)} - 9(T_3, C_2)^{(2)} + 18(T_4, C_2)^{(2)} - 6(D_3^2, D_2)^{(1)}+\\ &+ 54(T_3, D_2)^{(1)} - 108(T_4, D_2)^{(1)} + 12D_2S_{26} - 12(S_{26}, C_2)^{(1)}+\\ &+ 432C_2A_1 - 2160C_2A_2)/2^7/3^4,\end{aligned}$$

$$T_{16} = (S_{23}, D_3)^{(2)}/2^6 3^3, T_{17} = (S_{26}, D_3)^{(1)}/2^5/3^3,$$

$$\begin{aligned} T_{74} =&(2187T_3^2C_0 + 8748T_4^2C_0 + 20736T_{11}C_2^2 - 62208T_{11}C_1C_3+\\ &+ 108C_3D_1D_2D_3^2 - 8C_2D_2^2D_3^2 - 54C_2D_1D_3^3 + 6C_1D_2D_3^3+\\ &+ 27C_0D_3^4 - 54C_3D_3^2S_8 + 108C_3D_3^2S_9 + 27C_2D_3^2S_{11} - 27C_2D_3^2S_{12}+\\ &+ 4C_2D_3^2S_{14} - 32C_2D_3^2S_{15} + 54D_1D_3^2S_{16} - 3C_1D_3^2S_{17} + 6C_1D_3^2S_{19}-\\ &- 9T_3(54C_0(18T_4 + D_3^2) + 54C_3(2D_1D_2 - S_8 + 2S_9) - C_2(8D_2^2+\\ &+ 54D_1D_3 - 27S_{11} + 27S_{12} - 4S_{14} + 32S_{15}) + 54D_1S_{16} + 3C_1(2D_2D_3-\\ &-S_{17}+2S_{19}-6S_{21}))-576T_6(2D_2D_3-S_{17}+2S_{19}-6S_{21})-18C_1D_3^2S_{21}+\\ &+ 18T_4(6C_1D_2D_3 + 54C_0D_3^2 + 54C_3(2D_1D_2 - S_8 + 2S_9) - C_2(8D_2^2+\\ &+ 54D_1D_3 - 27S_{11} + 27S_{12} - 4S_{14} + 32S_{15}) + 54D_1S_{16} - 3C_1S_{17}+\\ &+ 6C_1S_{19} - 18C_1S_{21}))/2^8/3^4,\end{aligned}$$

$$T_{44} = ((S_{23}, C_3)^{(1)}, D_3)^{(2)}, \ T_{133} = \big(T_{74}, C_3\big)^{(1)}, \ T_{137} = \big(T_{74}, D_3\big)^{(1)}/6,$$

$$T_{136} = \big(T_{74}, C_3\big)^{(2)}/24, \ T_{140} = \big(T_{74}, D_3\big)^{(2)}/12,$$

$$T_{141} = \big(T_{74}, C_3\big)^{(3)}/36, \ T_{144} = \big(T_{133}, C_3\big)^{(4)}, \ T_{145} = \big(T_{137}, C_3\big)^{(3)}$$

are *T-comitants* of cubic systems (1) (see [15] for the definition of a T-comitant). We note that for the above invariant polynomials, we preserve the notations introduced in [8].

Using a different notation for the coefficients, we rewrite the cubic systems (1) as:

$$\begin{aligned}\dot{x} &= a+cx+dy+gx^2+2hxy+ky^2+px^3+3qx^2y+3rxy^2+sy^3 \equiv P(x,y),\\ \dot{y} &= b+ex+fy+lx^2+2mxy+ny^2+tx^3+3ux^2y+3vxy^2+wy^3 \equiv Q(x,y).\end{aligned} \tag{3}$$

Let $L(x,y) = W + Ux + Vy = 0$ be an invariant straight line of this family of cubic systems. Then, we get

$$UP(x,y) + VQ(x,y) = (W + Ux + Vy)(F + Dx + Ey + Ax^2 + 2Bxy + Cy^2),$$

and this identity yields the following equations:

$$\begin{aligned}
&Eq_1 = tV + (p - A)U = 0,\\
&Eq_2 = (3u - A)V + (3q - 2B)U = 0,\\
&Eq_3 = (3v - 2B)V + (3r - C)U = 0,\\
&Eq_4 = (s - C)U + Vw = 0,\\
&Eq_5 = lV + (g - D)U - AW = 0,\\
&Eq_6 = (2m - D)V + (2h - E)U - 2BW = 0,\\
&Eq_7 = (n - E)V + kU - CW = 0,\\
&Eq_8 = eV + (c - F)U - DW = 0,\\
&Eq_9 = (f - F)V + dU - EW = 0,\\
&Eq_{10} = bV + aU - FW = 0.
\end{aligned} \tag{4}$$

The infinite singularities (real or complex) of systems (3) are determined by the linear factors in the factorization over $\mathbb{C}$ of the polynomial

$$C_3 = yp_3(x,y) - xq_3(x,y).$$

All possible configurations of invariant lines, in the case, when the total multiplicity of these lines (including the line at infinity) equals seven possessing at infinity four distinct infinite singularities (all real, or two real and two complex), are determined in [5, 4, 3, 6]. In these papers, the author studied the above-mentioned systems according to the type of configurations of invariant straight lines. Additionally, the affine invariant conditions for the class of cubic systems possessing two real and two complex singularities at infinity was constructed.

In this paper, the class of cubic systems with four real distinct infinite singularities and invariant straight lines in the configuration of the type $(3,3)$ is considered. All possible configurations of invariant straight lines for this class were constructed in [5] (see *Figure 1*). Our goal is to determine the affine invariant conditions for the realization of each one of these 14 configurations.

According to [17] (see also [19]) we have the following results (Lemma 1.1, Lemma 1.2 and Theorem 1.1).

Lemma 1.1. *A cubic system $S \in \mathbb{CS}$ has 4 real distinct infinite singularities if and only if*

$$\mathcal{D}_1 > 0,\ \mathcal{D}_2 > 0,\ \mathcal{D}_3 > 0.$$

Lemma 1.2. *If a cubic system $S \in \mathbb{CS}$ has 4 real distinct infinite singularities, then this system could be brought via a linear transformation to the canonical form*

$$\begin{cases} x' = p_0 + p_1(x,y) + p_2(x,y) + (p+r)x^3 + (s+v)x^2y + qxy^2, \\ y' = q_0 + q_1(x,y) + q_2(x,y) + px^2y + (r+v)xy^2 + (q+s)y^3, \end{cases} \tag{5}$$

with $rs(r+s) \neq 0$ and $C_3 = xy(x-y)(rx+sy)$.

Theorem 1.1 ([3])**.** *Assume that a cubic system $S \in \mathbb{CS}$ possesses a given number of triplets or/and couples of invariant parallel lines real or/and complex. Then the following conditions are satisfied, respectively:*

(i) *two triplets* $\Rightarrow$ $\mathcal{V}_1 = \mathcal{V}_2 = \mathcal{U}_1 = 0$;

(ii) *one triplet and one couple* $\Rightarrow$ $\mathcal{V}_4 = \mathcal{V}_5 = \mathcal{U}_2 = 0$;

(iii) *one triplet* $\Rightarrow$ $\mathcal{V}_4 = \mathcal{U}_2 = 0$;

(iv) *3 couples* $\Rightarrow$ $\mathcal{V}_3 = 0$;

(v) *2 couples* $\Rightarrow$ $\mathcal{V}_5 = 0$.

According to [5] the following lemma is valid:

Lemma 1.3. *Assume the family of cubic system possessing 4 real distinct infinite singularities, i.e. the conditions $\mathcal{D}_1 > 0,\ \mathcal{D}_2 > 0,\ \mathcal{D}_3 > 0$ hold. We additionally consider that for this family the condition $\mathcal{V}_1 = \mathcal{V}_2 = 0$ is satisfied. Then:*

(A) *this family of cubic systems could be brought via an affine transformation and time rescaling to the systems*

$$\begin{aligned} \dot{x} &= a + cx + dy + 2hxy + ky^2 + x^3, \\ \dot{y} &= b + ex + fy + lx^2 + 2mxy + y^3; \end{aligned} \tag{6}$$

(B) *a cubic system* (6) *has invariant straight lines of total multiplicity 7 (including the line at infinity) in the configuration of the type* $(3,3)$ *if and only if the following conditions hold:*

$$k = d = h = e = l = m = 0,\ (c-f)^2 + (a^2 - b^2)^2 \neq 0. \tag{7}$$

So, according to (7) systems (6) became of the form

$$\dot{x} = a + cx + x^3, \quad \dot{y} = b + fy + y^3. \tag{8}$$

We denote

$$\xi_1 = -(27a^2 + 4c^3), \quad \xi_2 = -(27b^2 + 4f^3), \quad \nu_1 = a^2 + c^2, \quad \nu_2 = b^2 + f^2.$$

According to [5, Theorem 3.2, Subsection 3.1] we have the following lemma:

Lemma 1.4. *Assume that for a system* (8) *the conditions given below in terms of the polynomials* ξ_1, ξ_2, ν_1 *and* ν_2 *are satisfied. Then this system could be brought via an affine transformation and time rescaling to one of the presented below canonical systems* (9)–(17). *Moreover, this system possesses one of the configurations Config. 7.1a – 7.14a (see Figure 1) if and only if the conditions under the parameters a and b of the corresponding canonical system (when these conditions exist) are satisfied, respectively:*

$\xi_1\xi_2 > 0,\ \xi_1 + \xi_2 > 0 \quad \Rightarrow (9) \quad \Leftrightarrow$ *Config. 7.1a;*

$\xi_1\xi_2 > 0,\ \xi_1 + \xi_2 < 0 \quad \Rightarrow (10), \begin{cases} ab \neq 0 & \Leftrightarrow \textit{Config. 7.2a}; \\ ab = 0, a+b \neq 0 & \Leftrightarrow \textit{Config. 7. 3a}; \\ a = b = 0 & \Leftrightarrow \textit{Config. 7.4a}; \end{cases}$

$\xi_1\xi_2 < 0 \quad \Rightarrow (11)$ *with* $\begin{cases} b \neq 0 & \Leftrightarrow \textit{Config. 7.5a}; \\ b = 0 & \Leftrightarrow \textit{Config. 7.6a}; \end{cases}$

$\xi_1\xi_2 = 0,\ \xi_1 + \xi_2 > 0,\ \nu_1\nu_2 \neq 0 \quad \Leftrightarrow (12) \quad \Rightarrow$ *Config. 7.7a;*

$\xi_1\xi_2 = 0,\ \xi_1 + \xi_2 < 0,\ \nu_1\nu_2 \neq 0 \quad \Rightarrow (13)$ *with* $\begin{cases} b \neq 0 & \Leftrightarrow \textit{Config. 7.8a}; \\ b = 0 & \Leftrightarrow \textit{Config. 7.9a}; \end{cases}$

$\xi_1\xi_2 = 0,\ \xi_1 + \xi_2 > 0,\ \nu_1\nu_2 = 0 \quad \Rightarrow (14) \quad \Leftrightarrow$ *Config. 7.10a;*

$\xi_1\xi_2 = 0,\ \xi_1 + \xi_2 < 0,\ \nu_1\nu_2 = 0 \quad \Rightarrow (15)$ *with* $\begin{cases} b \neq 0 & \Leftrightarrow \textit{Config. 7.11a}; \\ b = 0 & \Leftrightarrow \textit{Config. 7.12a}; \end{cases}$

$\xi_1\xi_2 = 0,\ \xi_1 + \xi_2 = 0,\ \nu_1\nu_2 \neq 0 \quad \Rightarrow (16) \quad \Leftrightarrow$ *Config. 7.13a;*

$\xi_1\xi_2 = 0,\ \xi_1 + \xi_2 = 0,\ \nu_1\nu_2 = 0 \quad \Rightarrow (17) \quad \Leftrightarrow$ *Config. 7.14a.*

The canonical systems indicated in the statement of the above lemma are the following ones:

$$\dot{x} = x(x-1)(x-a), \quad \dot{y} = y(y-b)(y-c), \quad a(a+1)bc(b-c) \neq 0, \tag{9}$$

$$\dot{x} = x\big[(x+a)^2 + c\big], \quad \dot{y} = y\big[(y+b)^2 + f\big], \quad c > 0,\ f > 0. \tag{10}$$

$$\dot{x} = x(x-1)(x-a), \quad \dot{y} = y\big[(y+b)^2 + c\big], \quad a(a-1) \neq 0\ c > 0. \tag{11}$$

$$\dot{x} = x^2(x-1), \quad \dot{y} = y(y-b)(y-c), \quad bc(b-c) \neq 0. \tag{12}$$

$$\dot{x} = x^2(x-1), \quad \dot{y} = y\big[(y+b)^2 + c\big], \quad c > 0. \tag{13}$$

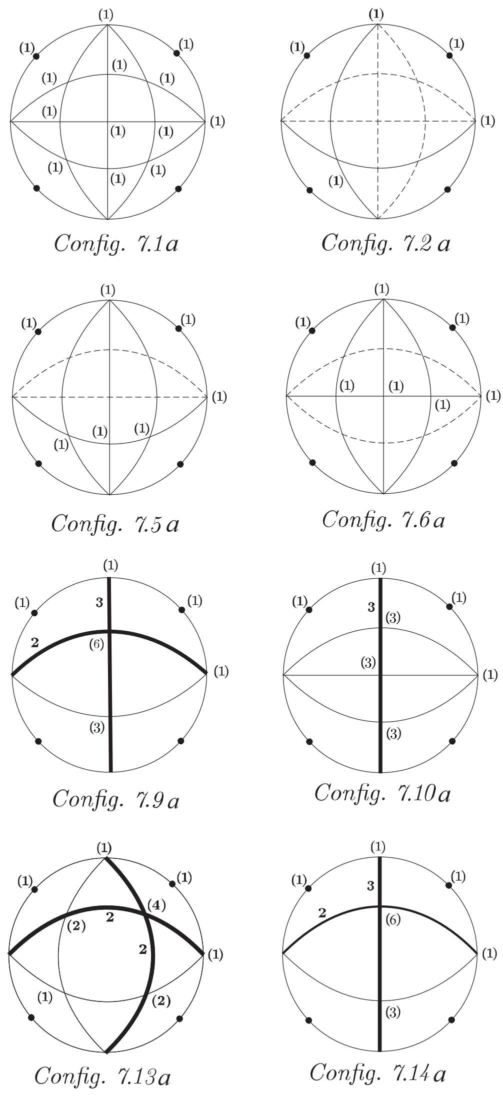

Figure 1. Configurations of invariant straght lines for systems in $CSL_{(3,3)}^{4r\infty}$

$$\dot{x} = x^3, \quad \dot{y} = y(y-1)(y-b), \;\; b(b-1) \neq 0. \tag{14}$$

$$\dot{x} = x^3, \quad \dot{y} = y\left[1 + (y+b)^2\right]. \tag{15}$$

$$\dot{x} = x^2(x-1), \quad \dot{y} = y^2(y-b), \;\; b \neq 0. \tag{16}$$

$$\dot{x} = x^3, \quad \dot{y} = y^2(y-1). \tag{17}$$

2. Invariant criteria for the realization of the configurations *Config. 7.1a* - *Config. 7.14a* of systems belonging to $CLS^{4r\infty}_{(3,3)}$

First we prove the following lemma.

Lemma 2.1. *An arbitrary non-degenerate cubic system belongs to the class* $CLS^{4r\infty}_{(3,3)}$ *if and only if the following conditions hold:*

$$\mathcal{D}_1 > 0, \mathcal{D}_2 > 0, \mathcal{D}_3 > 0, \mathcal{V}_1 = \mathcal{V}_2 = \mathcal{U}_1 = \mathcal{L}_1 = \mathcal{L}_8 = 0, \mathcal{L}_2^2 + \mathcal{K}_1^2 \neq 0. \tag{18}$$

Proof. According to Lemma 1.3 systems (6) could have two triplets of invariant straight line if and only if the conditions (7) are satisfied.

For systems (6) we calculate:

$$\mathcal{L}_1 = -20736(lx^3 + 2mx^2y - 2hxy^2 - ky^3).$$

Evidently $\mathcal{L}_1 = 0$ is equivalent to $l = m = h = k = 0$.

We define the new invariant polynomial

$$\mathcal{L}_8 = T_{15} - 2T_{14}$$

and evaluate it for systems (6) in the case $l = m = h = k = 0$:

$$\mathcal{L}_8 = 3ex^2 - 3dy^2.$$

It is evident that the condition $\mathcal{L}_8 = 0$ is equivalent to $d = e = 0$. Therefore, we have found out the invariant conditions which are equivalent with the first part of the conditions (7). So, applying $\mathcal{L}_1 = \mathcal{L}_8 = 0$ to systems (6) we arrive at systems (8) for which we calculate

$$\mathcal{L}_2 = -186624(c - f)xy, \quad \mathcal{K}_1 = 2^{17}3^{15}5^47^4 \cdot 817(a^2 - b^2)(x^2 - y^2).$$

Therefore, we deduce that the condition $\mathcal{L}_2^2 + \mathcal{K}_1^2 \neq 0$ is equivalent to

$$(c - f)^2 + (a^2 - b^2)^2 \neq 0.$$

The proof is complete. □

Next, we prove our main result.

Theorem 2.1. *Assume that for a generic cubic system* (3) *the conditions* (18) *are satisfied, i.e. this system belongs to the class* $CLS^{4r\infty}_{(3,3)}$. *Then this system has one of the configurations Config. 7.1 – 7.14 if and only if one of the following sets of conditions is satisfied, correspondingly:*

(A_1) $\mathcal{H}_1 > 0, \mathcal{H}_2 > 0$	$\Leftrightarrow$	*Config. 7.1a;*
(A_2) $\mathcal{H}_1 > 0, \mathcal{H}_2 < 0, \mathcal{H}_3 \neq 0$	$\Leftrightarrow$	*Config. 7.2a;*
(A_3) $\mathcal{H}_1 > 0, \mathcal{H}_2 < 0, \mathcal{H}_3 = 0, \mathcal{H}_4 \neq 0$	$\Leftrightarrow$	*Config. 7.3a;*
(A_4) $\mathcal{H}_1 > 0, \mathcal{H}_2 < 0, \mathcal{H}_3 = \mathcal{H}_4 = 0$	$\Leftrightarrow$	*Config. 7.4a;*
(A_5) $\mathcal{H}_1 < 0, \mathcal{H}_3 \neq 0$	$\Leftrightarrow$	*Config. 7.5a;*
(A_6) $\mathcal{H}_1 < 0, \mathcal{H}_3 = 0$	$\Rightarrow$	*Config. 7.5a or 7.6a;*
(A_7) $\mathcal{H}_1 = 0, \mathcal{H}_2 > 0, \mathcal{H}_5 \neq 0$	$\Leftrightarrow$	*Config. 7.7a;*
(A_8) $\mathcal{H}_1 = 0, \mathcal{H}_2 < 0, \mathcal{H}_5 \neq 0, \mathcal{H}_3 \neq 0$	$\Leftrightarrow$	*Config. 7.8a;*
(A_9) $\mathcal{H}_1 = 0, \mathcal{H}_2 < 0, \mathcal{H}_5 \neq 0, \mathcal{H}_3 = 0$	$\Leftrightarrow$	*Config. 7.9a;*
(A_{10}) $\mathcal{H}_1 = 0, \mathcal{H}_2 > 0, \mathcal{H}_5 = 0$	$\Leftrightarrow$	*Config. 7.10a;*
(A_{11}) $\mathcal{H}_1 = 0, \mathcal{H}_2 < 0, \mathcal{H}_5 = 0, \mathcal{K}_2 \neq 0$	$\Leftrightarrow$	*Config. 7.11a;*
(A_{12}) $\mathcal{H}_1 = 0, \mathcal{H}_2 < 0, \mathcal{H}_5 = 0, \mathcal{K}_2 = 0$	$\Leftrightarrow$	*Config. 7.12a;*
(A_{13}) $\mathcal{H}_1 = 0, \mathcal{H}_2 = 0, \mathcal{H}_3 \neq 0$	$\Leftrightarrow$	*Config. 7.13a;*
(A_{14}) $\mathcal{H}_1 = 0, \mathcal{H}_2 = 0, \mathcal{H}_3 = 0$	$\Leftrightarrow$	*Config. 7.14a.*

Proof. Consider a cubic system belonging to $CLS^{4r\infty}_{(3,3)}$. As it was prove earlier, such a system via an affine transformation an time rescaling could be brought to the canonical form (8). For these systems we calculate

$$\begin{aligned}\mathcal{H}_1 =& 2^{12}3^3(27a^2+4c^3)(27b^2+4f^3) = 2^{12}3^3\xi_1\xi_2;\\ \mathcal{H}_2 =& 2^7 3^4(27a^2+27b^2+4c^3+4f^3)(x^2+y^2) = 2^7 3^4(\xi_1+\xi_2)(x^2+y^2).\end{aligned} \tag{19}$$

The statement (A_1). According to Theorem 2.1, we have $\mathcal{H}_1 > 0, \mathcal{H}_2 > 0$ and by (19) these conditions are equivalent to $\xi_1\xi_2 > 0$ and $\xi_1+\xi_2 > 0$, respectively. As a result, according to Lemma 1.4, we get systems (9) for which we calculate:

$$\mathcal{H}_1 = 2^{12}3^3a^2b^2c^2(a-1)^2(b-c)^2.$$

Therefore, the condition $\mathcal{H}_1 > 0$ imply $a(a+1)bc(b-c) \neq 0$. Thus, according to Lemma 1.4 we arrive at configuration *Config. 7.1a*. This completes the proof of the statemant (A_1) of the theorem.

The statements $(A_2)-(A_4)$. We observe that the conditions $\mathcal{H}_1 > 0, \mathcal{H}_2 < 0$ are common for all these three statements. On the other hand these conditions are equivalent to $\xi_1\xi_2 > 0$ and $\xi_1+\xi_2 < 0$. By [5] via an affine transformation and time rescaling systems

(8) could be brought to the form (10) for which we calculate

$$\begin{aligned}\mathcal{H}_1 =& 2^{16}3^3 cf(a^2+c)^2(b^2+f)^2;\\ \mathcal{H}_2 =& -2^9 3^4[c(a^2+c)^2+f(b^2+f)^2)](x^2+y^2).\end{aligned}$$

It is evident that $\mathcal{H}_1 > 0$ implies $cf > 0$ and due to $\mathcal{H}_2 < 0$ we get $c > 0$ and $f > 0$.

For systems (10) we calculate:

$$\begin{aligned}\mathcal{H}_3 =& 2^{10}15^2 17ab(a^2+9c)(b^2+9f)x^6y^6(x^4-y^4);\\ \mathcal{H}_4 =& 2x^4y^4\big[b(b^2+9f)x-a(a^2+9c)y\big].\end{aligned}$$

Assume first that the condition $\mathcal{H}_3 \neq 0$ is satisfied. Since $c > 0$ and $f > 0$ we conclude that this condition is equivalent to $ab \neq 0$. So, according to Lemma 1.4 in this case we get the configuration *Config. 7.2a* and the statement (A_2) of our theorem is proved.

If $\mathcal{H}_3 = 0$ we get $ab = 0$ and we investigate two cases: $\mathcal{H}_4 \neq 0$ and $\mathcal{H}_4 = 0$.

The condition $\mathcal{H}_4 \neq 0$ implies $a^2 + b^2 \neq 0$. So, according to Lemma 1.4 in this case we get configurations *Config. 7.3a* and hence, the statement (A_3) is proved.

Assume finally $\mathcal{H}_3 = \mathcal{H}_4 = 0$. This implies $a = b = 0$ and by Lemma 1.4 we get *Config. 7.4a*. Thus, we proved the statement (A_4) of the theorem.

The statements $(A_5), (A_6)$. In this case we have $\xi_1\xi_2 < 0$ and according to [5] via an affine transformation and time rescaling systems (8) could be brought to the form (11) for which we calculate

$$\begin{aligned}\mathcal{H}_1 =& -2^{14}3^3(a-1)^2a^2c(b^2+c)^2,\\ \mathcal{H}_3 =& 2^9 3^2 5^2 17b(a-2)(a+1)(2a-1)(b^2+9c)x^6y^6(x^4-y^4).\end{aligned}$$

We observe that the condition $\mathcal{H}_1 < 0$ guarantees $a(a-1) \neq 0$ and $c > 0$, i.e. the conditions mentioned in (11) hold. At the same time due to $c > 0$ the condition $\mathcal{H}_3 \neq 0$ imply $b \neq 0$. So according to Lemma (1.4) the conditions $\mathcal{H}_1 < 0$ and $\mathcal{H}_3 \neq 0$ give us *Config. 7.5a*

Assume now $\mathcal{H}_3 = 0$. In this case, we get two possibilities: $b = 0$ or $b \neq 0$ and $(a-2)(a+1)(2a-1)$. In the first case, by Lemma 1.4 we have the configuration *Config. 7.6a*, whereas in the second case we arrive at the configuration *Config. 7.5a*. So we conclude that the statements (A_5) and (A_6) of Theorem 2.1 are valid.

We point out that the problem of determining of an invariant polynomial which gouverns the condition $b = 0$ remains open.

The statement (A_7)**.** In this case for systems (8) the conditions $\mathcal{H}_1 = 0$, $\mathcal{H}_2 > 0$ and $\mathcal{H}_5 \neq 0$ are satisfied. The first two conditions give us $\xi_1\xi_2 = 0$ and $\xi_1 + \xi_2 > 0$ for systems

(8) for which we have

$$\mathcal{H}_5 = -2^8 3^3 (c\xi_2 + f\xi_1). \qquad (20)$$

We claim that the condition $\mathcal{H}_5 \neq 0$ implies $\nu_1\nu_2 \neq 0$. Indeed, supposing the contrary that $\nu_1 = 0$ (respectively $\nu_2 = 0$) we get $a = c = 0$ (respectively $b = f = 0$) and this leads to the condition $\mathcal{H}_5 = 0$. This proves our claim and hence, we have the condition $\nu_1\nu_2 \neq 0$. Then according to Lemma 1.4 via an affine transformation and time rescaling systems (8) could be brought to the form (13) for which we calculate:

$$\mathcal{H}_5 = 2^8 3^2 b^2 (b-c)^2 c^2.$$

Evidently the condition $\mathcal{H}_5 \neq 0$ implies $bc(b-c) \neq 0$ and we get the condition required for systems (13).

So, accoridng to Lemma 1.4 in this case we get configuration *Config. 7.7a* and hence, the statement (A_7) is proved.

The statements (A_8), (A_9)**.** In both cases the conditions $\mathcal{H}_1 = 0$, $\mathcal{H}_2 < 0$ and $\mathcal{H}_5 \neq 0$ are satisfied. Then for systems (8) the first two conditions give us $\xi_1\xi_2 = 0$ and $\xi_1 + \xi_2 < 0$. Moreover, as it was shown in the case of the statement (A_7) the condition $\mathcal{H}_5 \neq 0$ implies $\nu_1\nu_2 \neq 0$. So according to Lemma 1.4 via an affine transformation and time rescaling systems (8) could be brought to the form (14) for which we calculate:

$$\mathcal{H}_2 = -\,2^9 3^4 c(b^2 + c)^2 (x^2 + y^2).$$

Clearly the condition $\mathcal{H}_2 < 0$ yields $c > 0$, i. e. we get the condition required for systems (14).

In order to distinguish the conditions $b \neq 0$ (the statement (A_8)) and $b = 0$ (the statement (A_9)) for systems (14) we evaluate the invariant polynomial $\mathcal{H}_3$:

$$\mathcal{H}_3 = 2^{10} 3^8 5^2 17 b(b^2 + 9c) x^6 (x-y) y^6 (x+y)(x^2+y^2).$$

Since $c > 0$, we obtain that the conditions $b = 0$ is equivalent to $\mathcal{H}_3 = 0$. Therefore by Lemma 1.4 we arrive at the configuration *Config. 7.8a* if $b \neq 0$ (i. e. $\mathcal{H}_3 \neq 0$) and *Config. 7.9a* if $b = 0$ (i. e. $\mathcal{H}_3 = 0$).

The statement (A_{10})**.** As earlier we determine that for systems (8) the conditions $\mathcal{H}_1 = 0$, $\mathcal{H}_2 > 0$ imply $\xi_1\xi_2 = 0$ and $\xi_1 + \xi_2 > 0$.

We claim that the condition $\mathcal{H}_5 = 0$ implies $\nu_1\nu_2 = 0$. Indeed, since $\xi_1\xi_2 = 0$, we may assume that $\xi_1 = 0$ due to the change $(x, y, a, b, c, f) \mapsto (y, x, b, a, f, c)$. Then we have $\xi_1 = -(25a^2 + 4c^3) = 0$. According to (20) the condition $\xi_1 = 0$ and $\xi_2 \neq 0$ implies $c = 0$ and then, we have $\xi_1 = 27a^2 = 0$, i. e. we get $a = 0$. Evidently we arrive at the codnition

$\nu_1 = 0$ and this complete the prove of our claim. Then according to Lemma 1.4 via an affine transformation and time rescaling systems (8) could be brought to the form (14) for which we calculate:

$$\mathcal{H}_2 = 2^7 3^4 b^2 (b-1)^2 (x^2 + y^2).$$

Evidently the condition $\mathcal{H}_2 > 0$ implies $b(b-1) \neq 0$, i. e we get the condition required for systems (14).

So, accoridng to Lemma 1.4 in this case we get configuration *Config. 7.10a* and hence, the statement (A_{10}) is proved.

The statements (A_{11}), (A_{12})**.** In both cases the conditions $\mathcal{H}_1 = 0$, $\mathcal{H}_2 < 0$ and $\mathcal{H}_5 = 0$ are satisfied. Simillary as in the case of statement (A_{10}) it can be proved that the condition $\mathcal{H}_5 = 0$ implies $\nu_1\nu_2 = 0$. So according to Lemma 1.4 via an affine transformation and time rescaling systems (8) could be brought to the form (15) for which we calculate:

$$\mathcal{H}_1 = \mathcal{H}_5 = 0,\ \mathcal{H}_2 = -2^9 3^4 (b^2 + 1)^2 (x^2 + y^2).$$

In order to distinguish the conditions $b \neq 0$ (the statement (A_{11})) and $b = 0$ (the statement (A_{12})) for systems (15) we evaluate the invariant polynomial $\mathcal{K}_2$:

$$\mathcal{K}_2 = 2b(9 + b^2)x^5 y^4.$$

We get that the conditions $b = 0$ is equivalent to $\mathcal{K}_2 = 0$. Therefore by Lemma (1.4) we arrive at the configuration *Config. 7.11a* if $b \neq 0$ (i. e. $\mathcal{K}_2 \neq 0$) and *Config. 7.12a* if $b = 0$ (i. e. $\mathcal{K}_2 = 0$).

The statements (A_{13}), (A_{14})**.** We observe that in both cases the conditions $\mathcal{H}_1 = 0$, $\mathcal{H}_2 = 0$ are satified and this is equivalent with $\xi_1\xi_2 = 0$ and $\xi_1 + \xi_2 = 0$. For systems (8) with $\xi_1 = \xi_2 = 0$, we set two new parameters u and v as follows: $a = 2u^3, b = 2v^3$ and then we get $c = -3u^2, f = -3v^2$. In this case we calculate:

$$\begin{aligned} \mathcal{H}_3 =& 2^{10} 3^2 5^2 17 u^3 v^3 x^6 y^6 (x^4 - y^4), \\ \mathcal{H}_4 =& -54 x^4 y^4 (v^3 x - u^3 y), \end{aligned} \tag{21}$$

and $\nu_1 = u^4(9 + 4u^2)$, $\nu_2 = v^4(9 + 4v^2)$. It is clear that $\nu_1\nu_2 \neq 0$ is equivalent to $\mathcal{H}_3 \neq 0$.

Thus by Lemma 1.4 via an affine transformation of coordinates and time rescaling systems (8) could be brought to the form (16) for which we calculate:

$$\mathcal{H}_3 = 2^{10} 3^2 5^2 17 b^3 x^6 y^6 (x^4 - y^4) \neq 0 \ \Rightarrow\ b \neq 0.$$

Therefore by Lemma 1.4 we arrive at the configuration *Config. 7.13a*.

Let now $\mathcal{H}_3 = 0$, i. e. $v_1 v_2 = 0$. This implies $uv = 0$ and we claim that $u^2 + v^2 \neq 0$ due to the condition $\mathcal{K}_1^2 + \mathcal{L}_2^2 \neq 0$. Indeed, for systems (8) with the parameters a, b, c, f given above, we calculate $\mathcal{L}_2 = 559872(u^2 - v^2)xy \neq 0$ and this proves our claim.

Thus we have $v_1 v_2 = 0$ and $v_1^2 + v_2^2 \neq 0$ and according to Lemma 1.4 via an affine transformation and time rescaling systems (8) could be brought to the form (17) and consequently we obtain the configuration *Config. 7.14a*. □

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User perception analysis of the developed AR applications: satisfaction and development directions

INGA TITCHIEV AND OLESEA CAFTANATOV

Abstract. This article explores the application of Augmented Reality (AR) in education, specifically focusing on the use of AR-based flashcards to support deep learning of mathematical concepts (geometry, Pi number) and vocabulary acquisition (metaphorical terms). AR flashcards offer an innovative solution by integrating dynamic, multimedia-rich content, which enhances understanding and engagement. Prototypes were tested with various user groups, including middle school and university students, who provided valuable feedback through surveys. The SWOT analysis revealed strengths such as clarity and usefulness, particularly in subjects like mathematics and biology, but also identified areas for improvement, such as technical issues and interface design. Based on user input, the design of animal-themed markers was refined to better align with user preferences by more relevant and specific imagery. The findings emphasize the importance of continuous refinement of AR applications to enhance their educational impact and accessibility.
2020 Mathematics Subject Classification: 68U15, 68T30.
Keywords: user perception, deep learning, augmented flashcards, user satisfaction.

Analiza percepției utilizatorilor asupra aplicațiilor AR dezvoltate: satisfacție și direcții pentru dezvoltare

Rezumat. Acest articol explorează aplicarea Realității Augmentate (AR) în educație, concentrându-se pe utilizarea cardurilor AR pentru a sprijini învățarea profundă a unor concepte matematice (geometrie, numărul pi) şi achiziția vocabularului (termeni metaforici). Cardurile AR oferă o soluție inovatoare prin integrarea de conținut dinamic și multimedia, care îmbunătățește înțelegerea și angajamentul. Prototipurile au fost testate cu diverse grupuri de utilizatori, inclusiv elevi de gimnaziu și studenți, care au furnizat un feedback valoros prin sondaje. Analiza SWOT a evidențiat punctele forte, cum ar fi claritatea și utilitatea, în special în domenii precum matematică și biologie, dar a identificat și zone de îmbunătățire, cum ar fi problemele tehnice și designul interfeței. Pe baza feedback-ului utilizatorilor, designul markerilor cu tematică animală a fost îmbunătățit pentru a se alinia mai bine la preferințele utilizatorilor prin imagini mai relevante și specifice. Rezultatele subliniază importanța rafinării continue a aplicațiilor AR pentru a spori impactul lor educațional și accesibilitatea.
Cuvinte-cheie: percepţia utilizatorilor, învăţare profundă, carduri augmentate, satisfacția utilizatorilor.

1. Introduction

In recent years, there is an increasing interest in applying Augmented Reality (AR) to create engaging, unique, and interactive educational environments [1, 2]. During the last three years, the research areas of our team are also related to the integration of Augmented Reality (AR) technologies in the educational field. This exploration has involved the design and implementation of various learning style scenarios aimed at enhancing user engagement with educational content [3]. Through this process, we have gained valuable insights into both the potential and the challenges associated with AR applications in education [4]. Our primary objective has been to develop AR applications that cater to diverse learning styles, thereby increasing user engagement. By creating immersive and interactive learning environments, we sought to make educational content more accessible and appealing to students. This approach has led to the development of scenarios that are not only visually engaging but also pedagogically effective, leveraging the unique capabilities of AR to provide enriched learning experiences.

We developed mobile applications to learn both math [5] and languages[6]. For mathematics learning, AR technology has been applied to topics related to geometry and exploring the world of Pi, offering a dynamic and immersive approach in the two developed applications: Learning Styles with AR and The Mysteries of Pi. In these applications, students can interact with various 2D and 3D geometric figures, as well as the number of Pi, in a creative and interactive way, developing a deeper understanding of these concepts.

For language, learning specifically includes apps like the Etymology app, which uses augmented flashcards, and Marker-Based approach. In the context of deep learning of a language, one of the most critical aspects of mastering a language is building a strong vocabulary. The size and depth of an individual's vocabulary significantly influence his ability to develop the four core Romanian skills: listening, speaking, reading, and writing. As teachers have consistently emphasized, expanding one's vocabulary accelerates overall language proficiency, improving comprehension and communication alike.

However, the process of vocabulary acquisition is not without challenges, particularly when it comes to learning metaphorical terms, which play a pivotal role in enriching language use and fostering nuanced understanding. Metaphorical expressions often carry meanings beyond their literal definitions, making them particularly difficult for learners to grasp. For students, deciphering and internalizing such terms can feel overwhelming because of cultural differences, abstract meanings, and limited exposure to contextual usage.

To address these challenges, innovative methods are required to support vocabulary learning. One promising approach is the use of augmented flashcards. Unlike traditional flashcards, augmented flashcards integrate dynamic, context-rich content-such as images, examples, and interactive exercises-helping learners connect abstract or metaphorical terms with concrete and memorable experiences. By combining technology with effective teaching strategies, augmented flashcards can transform vocabulary acquisition into a more engaging and accessible process, empowering students to overcome obstacles and expand their linguistic horizons.

In the early stages of our research, we successfully designed a series of augmented flashcard prototypes. These prototypes focus on various categories of metaphors, including mythological, sacred, anthropomorphic, chromatophore, and artifact metaphors, see Fig.1. Some flashcards are dedicated to conveying a single meaning, while others address multiple interpretations, resulting in tailored designs for each variation. To enhance comprehension and engagement, each flashcard is enriched with multimedia elements such as videos, GIFs, and images, providing vivid visual representations of the metaphorical terms.

Figure 1. Customized flashcards for learning the etymology of metaphorical terms

The customized cards feature homograph words that also represent metaphorical terms. When scanned using the developed mobile application, these cards generate augmented flashcards, as shown in Figure 2. For the sacred metaphor ”Romanița” (b), the front side of the flashcard displays the type of metaphorical term, the Moldovan-Romanian homograph word, and a video illustration of the term. Swiping the card reveals the back side, where a description with etymological explanations of the term is provided. These descriptions are translated into three languages, with the Spanish (a), Russian (b), and English (c) versions displayed in Figure 2.

However, to date, there is a lack of studies analyzing these applications that identify factors such as usability, satisfactions, advantages, limitations, effectiveness, challenges,

Figure 2. Screenshots from the augmented flashcard application

and features of augmented reality in educational settings. Personalization for promoting inclusive education using AR is also an area of growing interest. Thus, this article aims to shed light on some of these aspects.

In this regard, the applications have been subsequently tested with users of various age groups. The test was carried out with middle school students (grades 6–8) at IP Gimnaziul nr. 42, university students from the Technical University of Moldova under the supervision of Dr. Victoria Bobicev, and 70 students from Isen University in Cartagena and the University of Murcia in Spain, guided by Dr. Lucia Amaros. Student feedback has been collected through an opinion survey.

2. Analysis methodology

In the beginning, we will make a general presentation of the two sets of data that represent the answers provided by respondents from the Republic of Moldova and abroad. The first data set consisted of 70 respondents (from abroad), and the second data set consisted of 33 respondents (from Republic of Moldova). Both groups answered a set of 12 basic questions. The data processing algorithm consists of the following steps:

(1) Data cleaning - check for missing values and handle them appropriately, standardize text on key columns (e.g., capitalization, space trimming), check for missing data, and remove duplicate entries.
(2) Descriptive analysis summary (gender distribution, satisfaction distribution, usefulness distribution).
(3) Data is analyzed to identify trends - common themes or repeated keywords are identified, similar responses are grouped (for example, recommendations related to improving images or usability), key trends or frequent suggestions are highlighted.

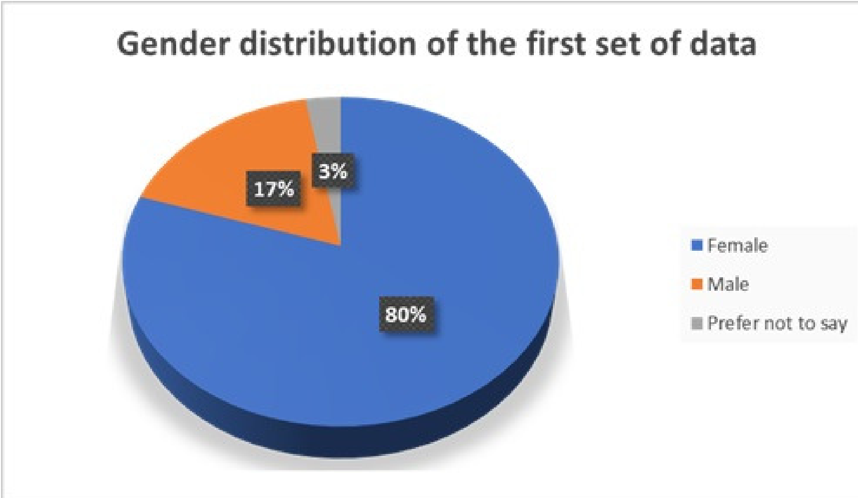


Figure 3. Gender distribution of the first set of data

(4) To determine the correlation between different data, the Chi-square test will be applied which is used to check if there is a relationship between two categorical variables or if an observed distribution differs significantly from an expected distribution.

2.1. For the first set of data

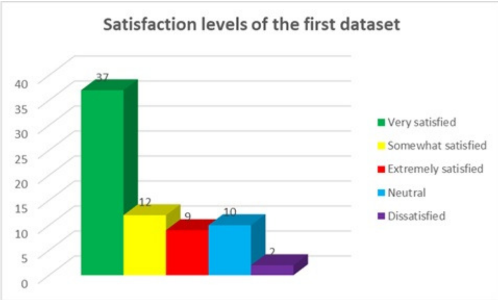
Descriptive analysis for the first data set consists of gender distribution (see Figure 3) of them, satisfaction level, and usefulness rating of users (see Figure 4).

In order to determine the correlation between gender and user satisfaction level, the chi-square test will be applied. Chi-square test results:

- Chi-square statistics (X^2): 121.31
- P-value: $6.27e - 09$ (≈ 0.00000000627)

The extremely low p-value (< 0.05) suggests that there is a significant relationship between gender and satisfaction level. This indicates that the distribution of satisfaction levels is not uniform across genders and that there may be a distinct pattern.

The Chi-square test indicated a significant relationship between gender and level of satisfaction ($p < 0.05$). Women had a majority turnout, which may influence the overall distribution. Specific issues which were mentioned by the users were of limited compatibility, large app size, and limited interactivity. Several recommendations were made, such as ensuring compatibility of applications with multiple platforms (multiple devices



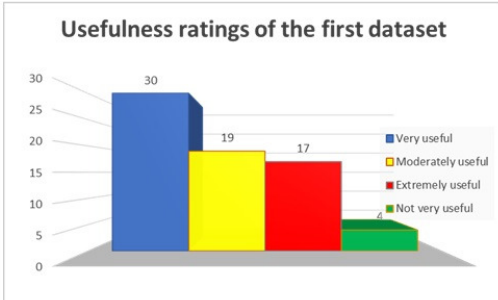


Figure 4. Satisfaction and usefulness for the first dataset

and operating systems), the possibility of using multiple languages, adding additional interactive features such as animations and sounds, especially for younger users, improving accessibility, reducing the size of the application for easier installation.

The relationship between utility and satisfaction was analyzed in order to identify whether the perception of utility influences the level of satisfaction.

Chi-square test results:

- Chi-square statistics (X^2): 253.33
- P-value: $5.10e - 22$

The extremely small p-value indicates a significant relationship between perceived usefulness and the level of satisfaction. The distributions suggest that perceived usefulness directly influences user satisfaction. Users who find the app extremely useful or very useful report higher levels of satisfaction. To the question of what they did not like about the application, the most respondents answer with ”Nothing”, indicating a general level of satisfaction. Dissatisfactions identified include limited compatibility, app size, and interactivity.

2.2. For the second set of data

The descriptive analysis for the second data set includes the gender distribution (see Figure 5), user satisfaction levels, and usefulness ratings provided by users (see Figure 6).

Chi-square test results for utility vs. satisfaction:

- Chi-square statistics (X^2): 27.92
- P-value: 0.063

There is no significant relationship at the 5% level ($p > 0.05$), but there is a notable trend. The distribution indicates that perceived usefulness can influence satisfaction.

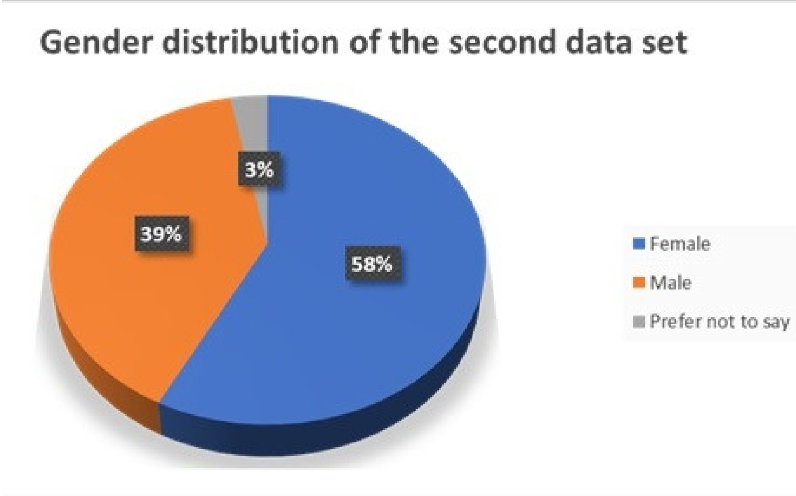


Figure 5. Gender distribution of the second set of data

Chi-square test results for clarity vs. satisfaction:

- Chi-square statistics (X^2): 17.29
- P-value: 0.836

The relationship between task clarity and satisfaction is not significant ($p > 0.05$). The distributions are more uniform, suggesting that satisfaction does not depend directly on perceived clarity.

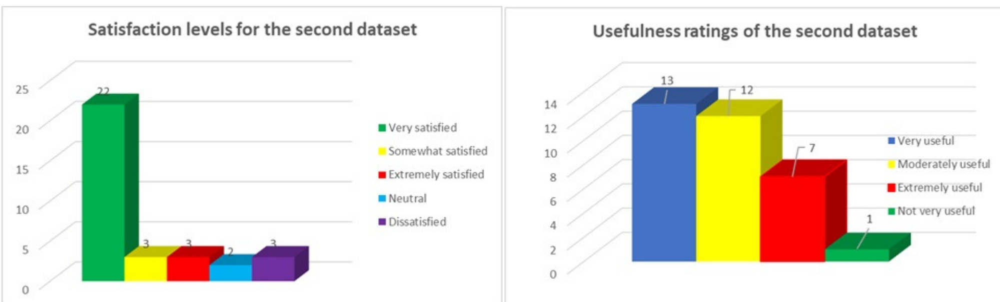


Figure 6. Satisfaction and usefulness for the second dataset

Participants who consider the app extremely useful or very useful tend to report higher levels of satisfaction. In contrast, perceptions of moderate usefulness are associated with varying levels of satisfaction, suggesting a partial correlation between perceived usefulness and overall satisfaction. Task clarity does not significantly influence overall satisfaction. The responses are evenly distributed for very clear or extremely clear scenarios, regardless of satisfaction. Increasing perceived usefulness of the apps may contribute to greater satisfaction. Task clarity is well rated but does not appear to be a determinant of satisfaction.

Recommendations: adding interactive functionality (e.g., animations, visual tutorials), developing more applicative scenarios that increase the practical value of the application, keeping current standards, given that participants already perceive them to be very clear, creating a continuous feedback mechanism to better understand user needs, and developing custom functionality for different levels of satisfaction. Cards and scenarios are the main elements that users found attractive. Feedback suggests requirements for more content, but also observations about what works well.

3. SWOT Analysis

Integrating augmented reality (AR) into education involves challenges such as infrastructure limitations, content creation complexity, teacher training, and ensuring equitable access for students. To overcome these obstacles and fully leverage AR's potential, conducting a SWOT analysis of user satisfaction survey data is essential. This analysis helps identify strengths, weaknesses of developed applications, opportunities for improvement, and highlight threats, providing a solid foundation for optimizing applications and creating a more effective educational experience.

- **Strengths**:

  Most respondents consider the tasks and scenarios very clear or extremely clear, and a significant number express high satisfaction with the applications. Many find these applications useful or very useful, particularly in educational fields such as mathematics, computer science, and biology. "Cards" and "Scenarios" are highly appreciated, especially those featuring metaphors and animations, while users also value interactive examples and 3D animations.
- **Weaknesses**:

  In some cases applications face technical issues, such as incompatibility with iOS or long loading times, while criticism has been directed at the text size on cards and the overall interface design. Additionally, the lack of clear navigation menus and the absence of integration into a single platform are noted as limitations.
- **Opportunities**:

  There is a desire to diversify scenarios, such as those related to geometry, robotics, physics, and biology, while also adapting applications to work seamlessly on all devices, including iOS. Personalization is another key focus, with the aim

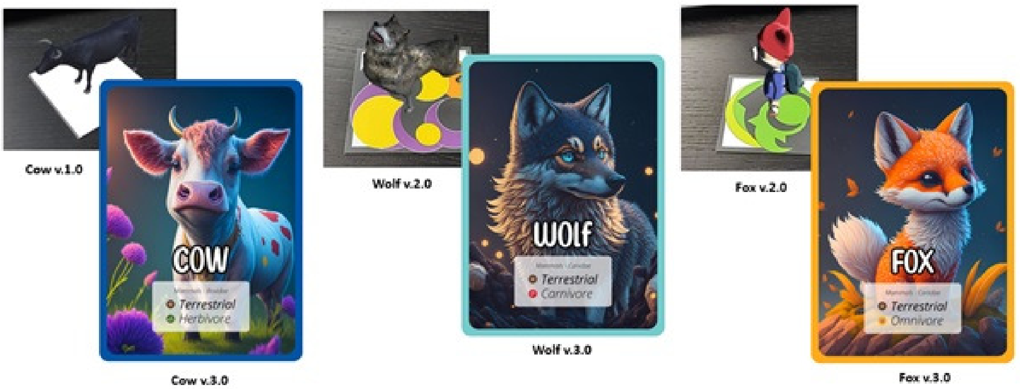


Figure 7. Marker designs improvements

of creating tailored scenarios based on user feedback, incorporating more animations with sounds and interactions. In addition, leveraging augmented reality is seen as a valuable opportunity to enhance lessons across multiple domains.

- **Threats**:

  The developed augmented reality applications do not collect personal data from students, making them relatively secure in terms of data privacy. However, several potential threats could arise. One is that users with limited engagement may find it difficult to provide detailed or meaningful feedback, which could hinder developers' efforts to enhance and adapt the platform based on user needs. Another challenge could stem from competition, as other AR learning platforms offering a wider range of features or personalized experiences might attract users, impacting the application's growth and user retention. In addition, compatibility and design issues present technical limitations that could negatively affect the overall perception of the applications.

4. Application improvements as a result of the data analysis from the questionnaire

After the presentation at Gimnaziul Nr. 42, where four of the developed applications were demonstrated to students for testing, they provided feedback on usability, functionality, and engagement, identified bugs, and suggested improvements to help refine the user experience. Although they preferred the card designs, they suggested that the markers feature more relevant and specific imagery. For instance, they would prefer to see real animals related to the cards rather than abstract designs.

Although they acknowledged that designing markers for math exercises can be challenging in terms of clarity and aesthetics, they felt that there is more flexibility for animal-themed applications. Based on this feedback, we have made several iterations to improve the marker designs (see Figure 7) for the animal app to better align with their preferences.

5. Conlusions

The integration of Augmented Reality (AR) into educational applications offers immense potential to enhance learning experiences. Through the development and testing of augmented flashcards, we have demonstrated that interactive and multimedia-rich tools can significantly increase engagement and effectiveness in deep learning.

Feedback from various user groups highlights both the strengths and challenges of AR-based educational applications. While most users expressed high satisfaction with the clarity and utility of the applications, issues such as technical compatibility, app size, and limited interactivity underline the need for ongoing improvements. The SWOT analysis further emphasizes opportunities for expanding the range of scenarios, improving accessibility, and incorporating personalized features to cater to diverse learning needs. Testing showed that the application was well-received, particularly in subjects like language learning, mathematics, and biology, although technical issues and interface design needed improvement. Feedback from users led to design improvements for animal-themed markers, aligning them with preferences for more relevant imagery. Continuous refinement based on user feedback is essential to maximize the application's educational value and accessibility.

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Center conditions for a cubic differential system with one invariant straight line and one invariant conic

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Abstract. In this work we find the center conditions for a cubic system of differential equations with a critical point of a center or a focus type having one invariant straight line and one invariant conic. The center-focus problem is studied by using the Darboux integrability and the rational reversibility methods.
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Keywords: cubic system of differential equations, the center-focus problem, invariant algebraic curve, Darboux integrability, rational reversibility.

Condiţii de existenţă a centrului pentru un sistem diferenţial cubic cu o dreaptă invariantă şi o conică invariantă

Rezumat. În lucrare se determină condiţii de existenţă a centrului pentru un sistem cubic de ecuaţii diferenţiale, cu punct critic de tip centru sau focar, care posedă o dreaptă invariantă şi o conică invariantă. Problema deosebirii centrului de focar se studiază aplicând integrabilitatea Darboux şi reversibilitatea raţională.
Cuvinte-cheie: sistem cubic de ecuaţii diferenţiale, problema centrului şi focarului, curbă algebrică invariantă, integrabilitatea Darboux, reversibilitate raţională.

1. Introduction

We consider the cubic system of differential equations

$$\begin{cases} \dot{x} = y + ax^2 + cxy + fy^2 + kx^3 + mx^2y + pxy^2 + ry^3 \equiv P(x,y), \\ \dot{y} = -(x + gx^2 + dxy + by^2 + sx^3 + qx^2y + nxy^2 + ly^3) \equiv Q(x,y), \end{cases} \tag{1}$$

where $P(x,y)$ and $Q(x,y)$ are relatively prime polynomials in the ring of real polynomials in the variables x, y and $\dot{x} = dx/dt, \dot{y} = dy/dt$. The origin of coordinates $O(0,0)$ is a critical point which is a center or focus (a fine focus) for (1).

The problem of distinguishing between a center and a focus (the center-focus problem) is open for cubic systems (1). It is completely solved for: quadratic systems, cubic symmetric systems, the Kukles system, and some families of polynomial differential systems of higher degree.

The center-focus problem was solved for some subclasses of cubic differential systems (1) with algebraic solutions: two parallel invariant straight lines [5], [25]; three invariant straight lines [8], [26], [27]; four invariant straight lines [8], [19], [22]; two invariant straight lines and one invariant cubic [13], [14]; two invariant straight lines and one invariant conic [10], [11], [12].

An approach to the center-focus problem is based on the theory of integrability. It means investigating the integrability of (1) in some neighborhood of the critical point $O(0,0)$. The integrability conditions were found for some subclasses of cubic differential systems (1) with invariant algebraic curves in [4], [6], [7], [16], [21]. It was found that every center in a cubic differential system (1) is provided by the Darboux integrability if the system has four invariant straight lines [19] or the system has two invariant straight lines and one invariant conic [8].

The Darboux integrability conditions were determined for: cubic systems (1) with two parallel invariant straight lines [5], a class of reversible cubic systems [1] and some complex cubic systems [20].

The purpose of this work is to find the center conditions for a cubic system (1) that has two invariant algebraic curves. The paper is structured as follows. In Section 2, we review established results related to the existence of invariant algebraic curves and the Darboux integrability. Section 3 examines the existence of Darboux first integrals that consist of an invariant straight line and an irreducible invariant conic. In Section 4, we apply the method of rational reversibility to determine the center conditions for a cubic system (1) that contains an invariant straight line and an invariant conic.

2. Invariant algebraic curves and Darboux integrability

Invariant algebraic curves play a crucial role in the study of the integrability of polynomial differential systems. They provide significant insights into the qualitative behavior of solutions and help in identifying the first integrals.

Definition 2.1. *An algebraic curve $\Phi(x,y)=0$ in $\mathbb{C}^2$ with $\Phi \in \mathbb{C}[x,y]$ is an invariant algebraic curve of a differential system (1) if there exists a polynomial $K(x,y) \in \mathbb{C}[x,y]$ such that*

$$\frac{\partial \Phi}{\partial x}P(x,y) + \frac{\partial \Phi}{\partial y}Q(x,y) = \Phi(x,y)K(x,y). \tag{2}$$

The polynomial $K(x,y)$ is called the cofactor of the invariant algebraic curve $\Phi(x,y)=0$.

It is a very hard problem to find invariant algebraic curves for a given system (1) because, in general, we do not have any evidence about the degree of a curve [24]. Not all polynomial differential systems admit invariant algebraic curves.

We analyze the center-focus problem for the cubic system (1) under the assumption that it possesses irreducible invariant algebraic curves in $\mathbb{C}[x, y]$. The notation $\mathbb{C}[x, y]$ denotes the ring of polynomials in two variables with complex coefficients [13].

Definition 2.2 ([8])**.** *The invariant algebraic curve $\Phi(x, y) = 0$ is said to be an algebraic solution of system (1) if and only if $\Phi(x, y)$ is an irreducible element of $\mathbb{C}[x, y]$.*

Knowledge of invariant algebraic curves is fundamental in the study of polynomial differential systems. They provide key information about integrability, phase portraits, stability, and global dynamics. The necessary and sufficient conditions for the existence of invariant algebraic curves in a cubic system (1) were determined when the curves are: straight lines [8], [18], [19], [27]; straight lines and conics [10], [11], [9], [8]; straight lines and cubics [13], [14]; conics [15]; cubics [17].

According to [7], [8], system (1) is considered integrable on an open set D of $\mathbb{R}^2$ if there exists a nonconstant analytic function $F : D \longrightarrow \mathbb{R}$ that remains constant along all solution curves $(x(t), y(t))$ within D, meaning that $F(x(t), y(t)) = C$ for all t where the solution is defined. This function F is called a *first integral* of the system on D.

Suppose that the function F exists in D. Then all the solutions of the cubic system (1) in D are known [24] and $F(x, y) = C$ gives every solution of (1) for some $C \in \mathbb{R}$. Clearly F is a first integral if and only if F solves the partial differential equation

$$P\frac{\partial F}{\partial x} + Q\frac{\partial F}{\partial y} \equiv 0. \tag{3}$$

For cubic system (1), we study the algebraic integrability which is called *the Darboux integrability* [2], [24]. Darboux's method provides a systematic way to construct a first integral or an integrating factor. Suppose that the curves $\Phi_j = 0,\ j = \overline{1, k}$ are invariant algebraic curves of (1) and $\alpha_j \in \mathbb{C}$. A first integral of the form

$$\Phi_1^{\alpha_1}\Phi_2^{\alpha_2}\cdots\Phi_k^{\alpha_k}, \tag{4}$$

is called a *Darboux first integral*.

We mention that for cubic systems (1), the conditions for the existence of integrating factors of the form $\mu = \Phi^\beta$ were obtained in [17] when $\Phi = 0$ is an invariant cubic and in [15] when $\Phi = 0$ is an invariant conic. First integrals and integrating factors of the form $l_1^{\alpha_1}\Phi^{\alpha_2}$, composed of one invariant straight line $l_1 = 0$ and one invariant cubic $\Phi = 0$, were determined in [7], [16]. In this paper, we study for cubic system (1) the problem of the existence of first integrals of the form

$$l_1^{\alpha}\Phi^{\beta} = C, \tag{5}$$

where $l_1 = 0$ is an invariant straight line and $\Phi = 0$ is an invariant conic.

It is known [24] that the origin will be a center for system (1) if and only if there exists a nonconstant analytic first integral

$$x^2 + y^2 + F_3(x, y) + \cdots + F_m(x, y) + \cdots = C$$

in some neighborhood of $O(0, 0)$, where F_m are homogeneous polynomials of degree m.

3. Cubic systems with two invariant algebraic curves

Assume that $Ax + By + 1 = 0$ is a real invariant straight line of the cubic differential system (1). Then, by a transformation of the form $x \to \omega(x \cos\alpha - y \sin\alpha)$, $y \to \omega(x \sin\alpha + y \cos\alpha)$, we can bring this line to the form $x = 1$. In [16] the following lemma was proved.

Lemma 3.1. *A straight line $x = 1$ is an invariant straight line for cubic system* (1) *if and only if the following set of conditions is satisfied*

$$r = 0,\ p = -f,\ m = -c - 1,\ k = -a. \tag{6}$$

When conditions (6) are satisfied, we obtain a cubic system of the form

$$\begin{cases} \dot{x} = (1 - x)(y + xy + ax^2 + cxy + fy^2) \equiv P(x, y), \\ \dot{y} = -(x + gx^2 + dxy + by^2 + sx^3 + qx^2y + nxy^2 + ly^3) \equiv Q(x, y). \end{cases} \tag{7}$$

Let us assume that the cubic differential system (7) has an irreducible invariant conic

$$\Phi(x, y) \equiv a_{20}x^2 + a_{11}xy + a_{02}y^2 + a_{10}x + a_{01}y + 1 = 0, \tag{8}$$

where $a_{01}, a_{10}, a_{02}, a_{11}, a_{20}$ are real parameters and $(a_{02}, a_{11}, a_{20}) \neq 0$. For every conic curve (8) the following quantities [8] are invariants

$$\begin{aligned} &I_1 = a_{02} + a_{20},\ I_2 = (4a_{20}a_{02} - a_{11}^2)/4, \\ &I_3 = (4a_{20}a_{02} - a_{20}a_{01}^2 + a_{11}a_{01}a_{10} - a_{10}^2a_{02} - a_{11}^2)/4 \end{aligned} \tag{9}$$

with respect to the rotation of axes. The conic (8) is: a parabola when $I_2 = 0$, an ellipse when $I_2 > 0$ and a hyperbola when $I_2 < 0$. If $I_3 = 0$, then the conic (8) is reducible into two straight lines.

By Definition 2.1, the curve (8) is an invariant conic for cubic system (7) if and only if there exists a cofactor $K(x, y) = c_{10}x + c_{01}y + c_{20}x^2 + c_{11}xy + c_{02}y^2$ such that

$$P(x, y)\frac{\partial\Phi}{\partial x} + Q(x, y)\frac{\partial\Phi}{\partial y} \equiv \Phi(x, y)K(x, y), \tag{10}$$

where $c_{10}, c_{01}, c_{20}, c_{11}, c_{02} \in \mathbb{R}$.

The identity (10) yields a system $\{F_{ij}=0,\ i+j=1,2,3,4\}$ of fourteen equations for the unknowns $c_{kl}, a_{kl},\ k+l=1,2$. When $i+j=1,2$, we find from (10) that

$$c_{10}=-a_{01},\ c_{01}=a_{10},\ c_{11}=a_{01}^2-da_{01}-a_{10}^2+ca_{10}-2a_{02}+2a_{20},$$
$$c_{20}=aa_{10}+a_{01}a_{10}-ga_{01}-a_{11},\ c_{02}=a_{11}-ba_{01}+fa_{10}-a_{01}a_{10}$$

and when $i+j=3,4$, we obtain from (10) the system of algebraic equations

$$\begin{aligned}
F_{40}\ &\equiv (a_{20}-s)a_{11}+(ga_{01}-aa_{10}-2a-a_{01}a_{10})a_{20}=0,\\
F_{31}\ &\equiv (2a_{20}-2s)a_{02}+(a_{10}^2-a_{01}^2+da_{01}-ca_{10}-2c-2)a_{20}+\\
&+(a_{11}+ga_{01}-aa_{10}-a-a_{01}a_{10}-q)a_{11}-2a_{20}^2=0,\\
F_{22}\ &\equiv (3a_{11}-aa_{10}-a_{01}a_{10}+ga_{01}-2q)a_{02}-3a_{11}a_{20}+\\
&+a_{11}(a_{10}^2-a_{01}^2+da_{01}-ca_{10}-c-n-1)+\\
&+((a_{10}+b)a_{01}-f(a_{10}+2))a_{20}=0,\\
F_{13}\ &\equiv (2a_{02}+a_{10}^2-a_{01}^2-2a_{20}+da_{01}-ca_{10}-2n)a_{02}-a_{11}^2+\\
&+((a_{10}+b)a_{01}-f(a_{10}+1)-l)a_{11}=0,\\
F_{04}\ &\equiv ((a_{10}+b)a_{01}-2l-fa_{10}-a_{11})a_{02}=0,\\
F_{30}\ &\equiv (a_{11}-a)a_{10}+(a_{01}+2a)a_{20}-ga_{11}+\\
&+ga_{01}a_{10}-aa_{10}^2-a_{01}a_{10}^2-sa_{01}=0,\\
F_{21}\ &\equiv a_{10}^3+(a-d+2a_{01})a_{11}-aa_{10}a_{01}+(2c-3a_{10})a_{20}+\\
&+2(a_{10}-g)a_{02}-ca_{10}^2-ca_{10}-a_{10}+da_{01}a_{10}+\\
&+ga_{01}^2-2a_{01}^2a_{10}-qa_{01}=0,\\
F_{12}\ &\equiv (3a_{01}-2d)a_{02}+(c-b-2a_{10})a_{11}+2(f-a_{01})a_{20}-a_{01}^3+\\
&+da_{01}^2+(2a_{10}+b-c)a_{01}a_{10}-na_{01}-fa_{10}^2-fa_{10}=0,\\
F_{03}\ &\equiv (f-a_{01})a_{11}-(a_{10}+2b)a_{02}+\\
&+(a_{01}a_{10}+ba_{01}-fa_{10}-l)a_{01}=0.
\end{aligned}\tag{11}$$

We shall study the consistency of (11) in $a_{10},\ a_{01},\ a_{20},\ a_{11},\ a_{02}$ and establish the conditions under which the system has one solution.

4. Cubic systems and first integrals

In this section, we study for cubic system (1) the problem of the existence of first integrals of the form

$$F(x,y)\equiv(x-1)^{\alpha}(a_{20}x^2+a_{11}xy+a_{02}y^2+a_{10}x+a_{01}y+1)^{\beta}=C,\tag{12}$$

where the invariant conic is irreducible and α,β are nonzero real exponents.

According to [8], the relation (12) is a first integral for the system (1) if and only if the identity (3) holds. We will use this identity to find the first integrals (12) of system (1).

Theorem 4.1. *The cubic differential system (1), where $P(x, y)$ and $Q(x, y)$ are relatively prime polynomials, does not possess Darboux first integrals in the form of* (12).

Proof. The identity (3) being applied to (12) yields the following system of equations

$$\{H_{ij} = 0, \ \ i + j = 1, 2, 3, 4\} \tag{13}$$

in the coefficients of (1) and the parameters $a_{10}, a_{01}, a_{20}, a_{11}, a_{02}, \alpha, \beta, \ \alpha\beta \neq 0$.

From equations $H_{01} = 0$ and $H_{10} = 0$ of the system (13) we obtain $\alpha = a_{10}\beta$ and $a_{01} = 0$. Then the equations $H_{02} = 0$, $H_{11} = 0$, and $H_{20} = 0$ of (13) yield the following

$$a_{11} = 0, a_{20} = (a_{10} + 2a_{02} + a_{10}^2)/2.$$

From equations $H_{ij} = 0, \ i + j = 3$ of the system (13) we find

$$a = 0, d = f, a_{10} = -2b, g = [(](b + c)a_{02} + b(2b^2 - 3b + 1)]/a_{02}.$$

Then the equations $H_{ij} = 0, \ i + j = 4$ of (13) imply

$$l = bf, a_{10} = -2b, q = [f(ba_{02} - a_{02} + 2b^3 - 3b^2 + b)]/a_{02},$$
$$n = bc + b, s = [(a_{02} - b + 2b^2)(b - 1)(c + 1)]/a_{02}.$$

We find that right-hand sides of (1) have a common factor $1 + (c + 1)x + fy = 0$ in contradictions to the assumption of Theorem. ∎

Remark 4.1. *There exists quadratic differential systems with first integrals containing one invariant straight line and one invariant conic. For example, in* [3] *it was shown that for quadratic system*

$$\dot{x} = -y - x^2 - y^2, \ \ \dot{y} = x(1 + y)$$

the straight line $y + 1 = 0$ *and the conic* $6x^2 + 3y^2 + 2y - 1 = 0$ *are invariants. This system has a first integral* $(y + 1)^2(6x^2 + 3y^2 + 2y - 1) = C$.

5. Cubic systems and rational reversibility

As established in [28], if the differential system (1) has a critical point $O(0, 0)$ of center or focus type and remains invariant under reflection with respect to the axis $X = 0$ and reversion of time, then $O(0, 0)$ is a center for system (1).

It is evident that the critical point $(O(0, 0)$ is a center for the system (1) if a diffeomorphism exists $H : U \longrightarrow V, \ H = \{X = g(x, y), \ Y = h(x, y)\}, \ H(0, 0) = (0, 0)$, which brings the system (1) to a system that has an axis of symmetry [28].

In this paper, we obtain centers by rational reversibility. We seek a rational transformation, invertible in a neighborhood of $O(0, 0)$, of the form [6], [23]

$$x = \frac{a_1X + b_1Y}{a_3X + b_3Y - 1}, \ y = \frac{a_2X + b_2Y}{a_3X + b_3Y - 1} \tag{14}$$

with $a_j, \ b_j \in \mathbb{R}, \ j = 1, 2, 3$, which maps the critical point $O(0, 0)$ to $X = Y = 0$.

Applying this transformation to system (7) we get a quartic system

$$\dot{X} = \sum_{i+j=0}^{4} A_{ij} X^i Y^j, \quad Y = \sum_{i+j=0}^{4} B_{ij} X^i Y^j, \tag{15}$$

where A_{ij}, B_{ij} are polynomials that depend on both the coefficients of system (1) and the parameters $a_1, a_2, a_3, b_1, b_2, b_3$ from the mapping (14).

We will show that the parameters in (14) can be found such that the system (15) is equivalent, in some neighborhood of $O(0,0)$, with a polynomial system [6]

$$\dot{X} = Y + M(X^2, Y), \quad \dot{Y} = -X(1 + N(X^2, Y)). \tag{16}$$

This system is symmetric with respect to the axis $X = 0$ and the critical point $O(0,0)$ is a center. The systems (15) and (16) are equivalent if the following conditions are fulfilled:

$$B_{40} = 0,\ A_{13} \equiv B_{04} = 0,\ A_{31} \equiv B_{22} = 0,\ A_{10} \equiv B_{01} = 0,\ A_{00} = B_{00} = 0,$$

and

$$\begin{aligned}
A_{30} &\equiv 2aa_3b_2a_1^2 + [2a_3(c-g) - (c+s+1)a_1 - a_2(q+f)]b_2a_2a_1 + \\
&\quad + a_2^3(lb_1 - nb_2) - ab_2a_1^3 + 2a_2^2a_3(bb_1 + (f-d)b_2) = 0,\\
A_{12} &\equiv b_1^3(qa_2 + 2a_3g) + [2(d+a)a_3 + a_2(2n - c - 3s - 1)]b_2b_1^2 + \\
&\quad + [a_2(3l - 2f + 3a - 2q) + 2a_3(c+b)]b_2^2b_1 + \\
&\quad + [2fa_3 - fa_1 - (n - 2c - 2)a_2]b_2^3 = 0,\\
A_{11} &\equiv [db_1 + b_2(c + 2b - 2g)]b_1a_2 + \\
&\quad + 3a_3 + b_2^2[ca_1 - a_2(d - 2f + 2a)] = 0,\\
A_{01} &\equiv b_2^2 + b_1^2 - 1 = 0,\ A_{10} \equiv b_2a_2 + b_1a_1 = 0,\\
B_{04} &\equiv [sb_1^4 + b_2b_1(b_1^2(q-a) + b_2b_1(n-c-1) + b_2^2(l-f))]a_3 = 0,\\
B_{22} &\equiv [a_2b_2^2a_1(-2f + 3a - q) + da_2b_1^2a_3 + na_2^2b_1^2 + ca_1b_2^2a_3 + \\
&\quad + (3l - f - 2q)a_2^2b_1b_2 + (3s - 2(n-c-1))a_2^2b_2^2 - (c+1)b_2^2a_1^2 + \\
&\quad + (c + 2b - 2g)a_2a_3b_1b_2 - (d + 2a - 2f)a_2b_2^2a_3 + a_3^2]a_3 = 0,\\
B_{03} &\equiv (-aa_2 - ga_3)b_1^3 - [(c+s+1)a_2 + (d+a)a_3]b_1^2b_2 + \\
&\quad + [(-f-q)a_2 - (c+b)a_3]b_1b_2^2 + [la_1 - na_2 - fa_3]b_2^3 = 0,\\
B_{21} &\equiv -fa_1^3b_2 + (2n - c - 3s - 1)a_1^2a_2b_2 + (d-a)a_1^2a_3b_2 - \\
&\quad - (2q - 3a - 3l + 2f)b_2a_2^2a_1 + [-fb_1 - (n + 2c + 2)b_2]a_2^3 + \\
&\quad + b_2a_3a_2a_1(2b - g) + [b_2(f - 2a) + b_1(c - b)]a_3a_2^2 = 0,\\
B_{02} &\equiv ab_1^2a_2 - (g-c)b_2a_2b_1 - b_2^2(da_2 - ba_1 - fa_2) - a_3 = 0,\\
B_{20} &\equiv 2a_3 + ga_1^3 + fa_2^3 + (d+a)a_2a_1^2 + (c+b)a_2^2a_1 = 0,\\
B_{10} &\equiv a_2^2 + a_1^2 - 1 = 0.
\end{aligned} \tag{17}$$

Theorem 5.1. *The cubic differential system* (1) *with two algebraic solutions* $x - 1 = 0$, $a_{20}x^2 + a_{11}xy + a_{02}y^2 + a_{10}x + a_{01}y + 1 = 0$ *is rationally reversible if one of the following conditions* (i), (ii), (iii) *holds:*

(i) $c = -3/2,\ b = m = 1/2,\ g = -1,\ f = l = p = r = 0,\ k = -a,\ q = (a-d)/2$, $s = -(2a + a_{01})(a_{01}^2 - da_{01} + n)/(2a_{01})$, $2na_{01}^2 - (a + 2dn)a_{01} + 2n^2 + n) = 0$;

(ii) $c = -3/2,\ b = m = 1/2,\ g = -1,\ l = f/2,\ k = -a,\ p = -f,\ r = 0$, $d = [(8a_{02} + 1)a_{01}]/(8a_{02}),\ q = (a-d)/2$, $n = (fa_{01} + 8a_{02}^2 - 2a_{02})/(8a_{02})$, $s = [(2a + a_{01})a_{01}]/(16a_{02})$;

(iii) $c = -3/2,\ b = m = 1/2,\ g = -1,\ l = f/2,\ k = -a,\ p = -f,\ q = (a-d)/2,\ a = [(a_{01}^2 - da_{01} + 2a_{20})(4a_{20} - 1)a_{01}]/(a_{01}^2 - 16a_{02}a_{20}),\ f = [4(d - a_{01})a_{01}^2a_{02} + 2a_{01}a_{02}(8a_{02} - 4a_{20} + 1) - 16da_{02}^2]/(a_{01}^2 - 16a_{02}a_{20}),\ n = [(d - a_{01}^3)a_{01} + a_{01}^2(8a_{02}a_{20} + 3a_{02} - 2a_{20}) - 2da_{01}a_{02}(4a_{20} + 1) + 4a_{02}a_{20}(4a_{20} - 4a_{02} + 1)]/(a_{01}^2 - 16a_{02}a_{20}),\ s = [a_{01}^2a_{20}(8a_{20} - 1) + 2da_{01}a_{20}(1 - 4a_{20}) + 4a_{20}^2(4a_{20} - 4a_{02} - 1)]/(a_{01}^2 - 16a_{02}a_{20}),\ r = 0$.

Proof. We study the consistency of systems {(17), (11)} considering two cases: $a_3 = 0$ and $a_3 \neq 0$. According to [6], the equations $A_{01} = 0$ and $B_{10} = 0$ from (17) can be parametrized as follows:

$$a_1 = \frac{2u}{u^2+1},\ a_2 = \frac{u^2-1}{u^2+1},\ b_1 = \frac{2v}{v^2+1},\ b_2 = \frac{v^2-1}{v^2+1}, \tag{18}$$

where u and v are real parameters. Then $A_{10} = 0$ becomes $A_{10} \equiv e_1e_2 = 0$, where

$$e_1 = u - v + uv + 1,\ e_2 = v - u + uv + 1.$$

Assume that $e_1 = 0$. Then the equation $e_1 = 0$ yields $v = (1+u)/(1-u)$ and $A_{10} \equiv 0$.

1. Let $a_3 = 0$. Then $B_{04} \equiv 0$ and $B_{22} \equiv 0$. When $u = 0$, the equations of (17) yield

$$r = q = p = l = k = f = d = a = 0,\ m = -1 - c.$$

In this case, the cubic system has two parallel invariant straight lines $1 - x = 0$, $1 + (c+1)x = 0$ and the center-focus problem was solved in [5], [25].

When $u = -1$, the equations of (17) imply

$$r = q = p = l = k = g = f = c = b = a = 0,\ m = -1.$$

The cubic system has the invariant straight lines $1 - x = 0$, $1 + x = 0$ and center-focus problem was solved in [5], [25].

If $u(u+1) \neq 0$, then the equations of (17) yield

$r = 0, p = -f, m = -c - 1, k = -a, a = [bu(20u^2 - 6u^4 - 6) + (f - d)(u^2(u^4 - 7u^2 + 7) - 1)]/[2(1-u^2)^3],\ c = [2b(6u^3 - u^5 - u) - f(u^6 - 1) + (4d - 7f)(u^2 - u^4)]/[2u(u^2-1)^2],\ g = [(f + d)(1 - u^6) + (f - 7d)(u^2 - u^4) + b(2u^5 + 2u -$

$12u^3)]/[4u(1-u^2)^2]$, $n = [f(1-u^{14})-4(b-1)(u+u^{13})-(15f+4d)(u^2-u^{12})+8(4b-11)(u^3+u^{11})-3(15f-4d)(u^4-u^{10})+4(95-7b)(u^5+u^9)+(61f-8d)(u^6-u^8)-16u^7(37+8b)]/[2u(1+u^2)^4(u^2-1)^2]$, $l = [f(u^8-10u^4-4u^2-4u^6+1)+(7+b)(4u^5-4u^3)+(1-b)(4u-4u^7)]/[(1+u^2)^4]$, $q = [6(124-15b)(u^5+u^9)-6(b-4)(u^{13}+u-10u^3-10u^{11})-24(44+13b)u^7-(31f+9d)(u^2-u^{12})+3(23f-15d)(u^4-u^{10})+(3d+5f)(1-u^{14})+3(35f-11d)(u^6-u^8)]/[2(u^2+1)^4(u^2-1)^3]$, $s = [(f+d)(9u^2+u^{18}-9u^{16}-1)-2(2+b)(u+u^{17})+64(u^3+u^{15})+4(9f+d)(u^{14}-u^4)+8(13b-46)(u^5+u^{13})+4(21f-19d)(u^6-u^{12})-64(2b-15)(u^7+u^{11})+2(65f-31d)(u^8-u^{10})-4u^9(326+115b)]/[4u(1-u^4)^4]$.

In this case the cubic system possesses two invariant straight lines $1-x=0$, $(1+u^2)^2-(1+u^4-6u^2)x+4(u^3-u)y=0$ and center-focus problem was solved in [6].

2. Let $a_3 \neq 0$. Then from the equation $B_{20}=0$ of (17) we get

$$a_3 = [u^2(3f-4d-4a)(u^2-1)-2u(c+b)(u^4+1)-f(u^6-1)+ \\ +4(c+b-2g)u^3]/[2(1+u^2)^3].$$

Assume that $u=0$. If $a=0$, then the equations of (17) yield

$$s=r=q=p=n=l=k=d=a=0,\ m=-c-1.$$

The cubic system has the invariant straight lines $1-x=0$, $1+(c+1)x=0$ and center-focus problem was solved in [5], [25].

Assume that $u=0$ and let $a \neq 0$. Then the equations of (17) yield

$$d=-3a,\ f=-2a,\ c=b-2,\ g=-1,\ l=-2ab,\ k=-a,\ m=1-b,\ n=2a^2, \\ p=q=2a,\ r=s=0.$$

The cubic system has three invariant straight lines $1-x=0$, $1-2ay=0$, $1-x-2ay=0$ and center-focus problem was solved in [26].

Assume that $u=-1$. If $g=-1$, then from the equations of (17) we get

$$c=-3/2,\ b=m=1/2,\ g=-1,\ l=f/2,\ k=-a,\ p=-f,\ q=(a-d)/2,\ r=0.$$

The equation $F_{04}=0$ of (11) implies two cases to be investigated: $a_{02}=0$ and $a_{02}\neq 0$.

Let $a_{02}=0$. If $a_{11}=0$, then $a_{01}a_{20}\neq 0$ and $F_{03}\equiv(a_{01}-f)(2a_{10}+1)=0$. When $a_{01}=f$, the equations $F_{22}\neq 0$ and when $a_{10}=(-1)/2$, we obtain that $F_{21}\neq 0$.

Assume that $a_{11}\neq 0$. We express s, n and a_{11} from the equations $F_{40}=0$, $F_{22}=0$ and $F_{13}=0$, respectively. In this case we have $F_{03}\equiv i_1 i_2 i_3=0$, where

$$i_1=a_{01}-f,\ i_2=2a_{10}+3,\ i_3=f.$$

If $i_1=0$, then $a_{01}=f$ and $F_{12}=0$ yields $a_{10}=-1-a_{20}$. In this case the conic is reducible.

If $i_1 \neq 0$ and $i_2 = 0$, then $a_{10} = (-3)/2$ and $F_{21} = 0$ implies $a_{20} = 1/2$. In this case the conic is also reducible.

If $i_1 i_2 \neq 0$ and $i_3 = 0$, then $f = 0$ and $F_{12} = 0$ implies $a_{10} = -1$. In this case we obtain the set of conditions (i) for the existence of an invariant conic

$$(a_{01}^2 - da_{01} + n)x^2 + a_{01}xy + 2(x - a_{01}y - 1) = 0.$$

Let now $a_{02} \neq 0$. We express a_{11}, s and n from equations $F_{04} = 0$, $F_{31} = 0$ and $F_{13} = 0$ of (11), respectively. Then $F_{03} \equiv j_1 j_2 = 0$, where $j_1 = a_{10} + 1$, $j_2 = fa_{01} - f^2 - a_{02}$.

Assume that $j_1 = 0$, then $a_{10} = -1$. If $a_{20} = a_{01}^2/(16a_{02})$, then $F_{22} = F_{40} = 0$ yields

$$d = [(8a_{02} + 1)a_{01}]/(8a_{02}).$$

In this case we get the set of conditions (ii) for the existence of an invariant conic

$$(a_{01}x - 4a_{02}y)^2 - 16a_{02}x + 16a_{01}a_{02}y + 16a_{02} = 0.$$

If $a_{20} \neq a_{01}^2/(16a_{02})$, then we express a and f from the equations $F_{40} = F_{30} = 0$ and $F_{12} = F_{22} = 0$ of (11). In this case we have the set of conditions (iii) for the existence of an invariant conic

$$2a_{20}x^2 - a_{01}xy + 2a_{02}y^2 - 2x + 2a_{01}y + 2 = 0.$$

Assume that $j_1 \neq 0$ and let $j_2 = 0$. Then $a_{10} = fa_{01} - f^2$ and $F_{21} = 0$ yields

$$d = (6a_{20} - 2a_{10}^2 - a_{10} + 4f^2 + 2af)/(2f).$$

If $a_{01} = 2f$, then $F_{22} = 0$ implies $a_{20} = a_{10}^2/4$ and the conic is reducible.

Let $a_{01} \neq 2f$. Then we express a from $F_{12} = 0$ and $F_{22} = 0$ yields $a_{20} = (-2a_{10} - 1)/4$. In this case the conic is also reducible.

Assume that $u = 0$ and let $g \neq -1$. Then the equations of (17) yield

$$r = p = l = k = f = a = 0,\ q = -d,\ m = 2,\ g = -2,\ c = -3,\ b = 1.$$

In this case the cubic system has the invariant straight lines $1 - x = 0$, $1 - 2x = 0$ and center-focus problem was solved in [5], [25].

Assume now $u(u + 1) \neq 0$. We express a, s, l, g, n, q, from the equations $A_{11} = 0$, $A_{12} = 0$, $A_{30} = 0$, $B_{02} = 0$, $B_{04} = 0$, $B_{21} = 0$ of (17), respectively. In this case we obtain that $B_{03} \equiv hf_1 = 0$, $B_{22} \equiv hf_2 = 0$, where

$$\begin{aligned}
h &= (7f - 4d)(u^2 - u^4) + f(u^6 - 1) + 2(c + b + 2)(u + u^5) - 4(c + 3b)u^3,\\
f_1 &= (2d - 3f)(1 + 14u^4 + u^8) - 2(11c + 25b + 4)(u^3 - u^5)+\\
&\quad + 2(5c + 7b + 4)(u - u^7) - 8(2d - 5f)(u^2 + u^6),\\
f_2 &= 2(b - 2 - c)(1 + u^8) + (2d - 15f)(u - u^7) - 8(6c + 16b + 1)u^4+\\
&\quad + 2(11c + 17b + 8)(u^2 + u^6) + (81f - 46d)(u^3 - u^5).
\end{aligned}$$

If $h = 0$, then the equation of (17) imply

$r = 0$, $p = -f$, $m = -c-1$, $k = -a$, $a = [f(1+7u^4-7u^2-u^6) - 2(c+b+2)(u+u^5)+4(3c-b+10)u^3]/[8(u^2-1)u^2]$, $d = 2a + [2(c+5-2b)u]/(1-u^2)$, $g = [f(15u^2-15u^4+u^6-1)+2u(2+c+b)(1+u^4)-4(3c+3b+14)u^3]/(16u^3)$, $l = [4bu(1-u^6)+8f(u^2+2u^4+u^6)+4(b-8)(u^3-u^5)]/(1+u^2)^4$, $n = [f(u^{12}-8u^{10}+32u^6+7u^8+7u^4+1-8u^2)-2(c+b+2)u-u^{11})-6(c-3b-2)(u^3-u^9)-4(c-5b+44)(u^5-u^7)]/[u(u^2-1)(1+u^2)^4]$, $q = [f(1+19u^{12}-19u^2+33u^4-33u^{10}+53u^6-53u^8-u^{14})+2(33c-46-47b)(u^5+u^9)+4(3c+5b+16)(u^3+u^{11})-2(c+b+2)(u+u^{13})+8u^7(13c+152-29b)]/[4(u^2-1)(1+u^2)^4u^2]$, $s = [f(1+u^4-19u^2+21u^6-u^{10}-21u^8+19u^{12}-u^{14})+2(17c-62-15b)(u^5+u^9)+4(c+5b+14)(u^3+u^{11})-2(c+b+2)(u+u^{13})+8u^7(7c+82-13b)]/[4u(1-u^2)^2(1+u^2)^4]$.

We have two invariant straight lines $1-x=0$, $(1+u^2)^2-8u^2x-4(u^2-1)uy=0$ and center-focus problem was solved in [6].

Assume that $h \neq 0$. We find the resultant of the polynomials f_1, f_2 with respect to d and obtain that $Res(f_1, f_2, d) = 0$, if

$b = [(c+2)(u^6-1)-f(6u^5-52u^3+6u)+(15c+22)u^2(1-u^2)]/[(1+u^2)^2(u^2-1)]$.

In this case we express d from the equations $B_{03} \equiv B_{22} = 0$ and the equations of (17) yield

$r = 0$, $p = -f$, $m = -c-1$,, $k = -a$, $a = [(6u^2-u^4-1)f]/[2(1-u^2)^2]$, $b = [(2+c)(u^6-1)-f(6u-52u^3+6u^5)+(22+15c)(u^2-u^4)]/[(1+u^2)^2(u^2-1)]$, $d = [f(3+3u^8-100u^2-100u^6+306u^4)+(3+2c)(12u^7+52u^3-52u^5-12u)]/[2(1-u^4)^2]$, $g = [1-u^6+2f(u-30u^3+u^5)+(23+16c)(u^4-u^2)]/[(1+u^2)^2(u^2-1)]$, $l = [(2+c)(u^6+7u^2-7u^4-1)+fu(20u^2-6u^4-6)][f(1-14u^2+u^4)+4(1+c)(u^3-u)]/[(1+u^2)^4(u^2-1)]$, $n = [4f(9+5c)(u^{11}-u)+2(32c^2+104c+72-49f^2)(u^2+u^{10})+92f(11+7c)(u^3-u^9)+(1391f^2-1152c-384c^2-768)(u^4+u^8)+8f(445+301c)(u^7-u^5)+4(160c^2+472c-791f^2+312)u^6+f^2(u^{12}+1)]/[2(1-u^2)^2(1+u^2)^4]$, $q = [4f(567+394c)u^6+2(87+127c+40c^2-61f^2)(u^9-u^3)-f(1103+720c)(u^4+u^8)+4(353f^2-223c-76c^2-147)(u^7-u^5)+2(f^2-3c-3)(u^{11}-u)-f(1+u^{12})+2f(33+14c)(u^2+u^{10})]/[(1-u^2)^2(1+u^2)^4]$, $s = u[f(1-36u^2+54u^4-36u^6+u^8)+8(1+c)(u^7+3u^3-3u^5-u)][f(2u^5+2u-28u^3)-(15+8c)(u^2-u^4)-u^6+1)]/[(1-u^4)^4]$.

The cubic system has three invariant straight lines $1-x=0$, $(fu^4+4cu^3-14fu^2-4cu+f)(2ux+u^2y-y)+(u^4x-14u^2x+x-8u^3y+8uy-u^4-2u^2-1)(1-u^2)=0$, $(fu^4-14fu^2+4cu^3-4cu+f)(2ux+u^2y-y)+(8u^2x+4u^3y-4uy+(1+u^2)^2)(u^2-1)=0$ and center-focus problem was solved in [26].

Assume that $e_2 = 0$. It is easy to see that $e_2(u, v) = e_1(-u, -v)$ and the case $e_2 = 0$ is equivalent to the case $e_1 = 0$. ∎

Theorem 5.2. *The critical point $O(0, 0)$ is a center for a cubic differential system* (1)*, with two algebraic solutions $x - 1 = 0$, $a_{20}x^2 + a_{11}xy + a_{02}y^2 + a_{10}x + a_{01}y + 1 = 0$, if one of the conditions* (i), (ii), (iii) *is satisfied.*

The proof of Theorem 5.2 follows directly from Theorem 5.1, if the cubic system (1) is rationally reversible, then the critical point $O(0, 0)$ is a center [23], [28].

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