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on the occasion of his 80th birthday

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PROFESSOR VASILE NEAGU ON HIS 80TH BIRTHDAY

Vasile Neagu, Full Professor, Doctor Habilitatus in Mathematics and Physics. He is a remarkable mathematician from the Moldovan school of functional analysis, who has made a major contribution to the development of the theory of integral equations and to the training of new generations of highly qualified specialists. His scientific results have been noted and cited by many well-known mathematicians, specialists in the field of integral equations. On November 1, 2025, Professor Vasile Neagu celebrated his 80th anniversary.

He was born in the village of Vărzărești, Nisporeni District, Republic of Moldova. In 1959, he graduated from primary school in Vărzărești; then, in 1963, he graduated with a silver medal from high school in Nisporeni. In the same year he became a student at the State University of Chișinău, Faculty of Physics and Mathematics. After a three-year interruption due to the compulsory military service (1964-1967), he graduated with honors in 1971 from the Faculty of Physics and Mathematics of State University of Chișinău.

In the 1960s, the prominent Moldovan school of functional analysis had already emerged, recognized far beyond the country's borders, founded by the world-renowned professor Israel Gohberg. The courses "Normal Solvable Operators" and "Singular Integral Operators," among others, taught by Professors Israel Gohberg and Nahum Krupnik, as well as their scientific publications, had a decisive impact on his career as a scientific researcher. In his fifth year of study, Professor I. Gohberg formulated for him a scientific research topic that proved to be highly promising. The topic concerned singular integral operators with Cauchy kernel in the case of an integration contour with angular points - a subject that continues to be central to Professor V. Neagu's research.

After graduating from university in 1971, he became a doctoral student at the Department of Mathematical Analysis of the State University of Chișinău, under the scientific supervision of Professor Nahum Krupnik (as Academician I. Gohberg later emigrated to Israel). The research topic had originally been formulated by Professor I. Gohberg. In 1974, Vasile Neagu defended (ahead of schedule) his PhD thesis in physics and mathematics in the specialty 01.01.02 (Differential and Integral Equations), entitled "On Certain Problems of Singular Integral Operators in the Case of Contours with Angular Points". In 1976, he obtained the scientific-pedagogical title of Associate Professor. In 2003, he defended his Doctor Habilitatus thesis in mathematics in the specialty 01.01.02 (Mathematical Analysis), entitled "Noetherian Conditions and Symbols of Singular Integral Operators in the Case of Piecewise Lyapunov and Unbounded Contours" and in 2004, he was awarded the academic title of Full Professor.

From 1974, V. Neagu was employed as a lecturer at the Department of Mathematical Analysis of the State University. In 1976–2004, he worked as Associate Professor, and from 2004 until 2023 as Full Professor. In 1989–1994, he held the position of Vice-Dean of the Faculty of Mathematics and Computer Science of Moldova State University. Since 2023, he has been a Coordinating Scientific Researcher at Moldova State University. During 1983 -1987, he worked at the University of Oran, Algeria. There he taught complex analysis, functional analysis, and specialized courses to third-year students, as well as integral equations and Fredholm-type operators to master's and doctoral students. Under Professor V. Neagu's supervision at the University of Oran, two doctoral students defended their PhD theses in mathematics. Since 1991, Professor V. Neagu has also been working (concurrently) at the Department of Mathematical Analysis and Differential Equations of Tiraspol State University, which merged in 2022 with "Ion Creangă" State Pedagogical University of Chișinău.

Professor V. Neagu's contribution to the training of new generations of highly qualified mathematicians is significant. He has supervised 6 PhD theses in mathematical sciences, approximately 90 bachelor's theses, and 40 master's theses. He is co-author of a textbook for students [3], author (and co-author) of 12 didactic and scientific works for students of faculties of [4], [5], [6], and co-author of textbooks and collections of mathematical problems for 11th- and 12th-grade students.

Throughout his activity, Professor Vasile Neagu has established himself as a formidable researcher. His scientific activity is related to singular integral equations, Riemann–Hilbert boundary value problems for analytic functions, Carleman-type equations in various functional spaces, as well as conditions imposed on the integration contour.

He is the author of more than 120 scientific publications, published in prestigious journals in Moldova, Russia, Ukraine, Romania, the United States, Georgia, Belarus, and Algeria, including two monographs [1], [2]. The scientific results obtained by Professor V. Neagu have been presented at numerous national and international scientific conferences and symposia in Moldova, Romania, Russia, Ukraine, Kazakhstan, Belarus, and Algeria, as well as at scientific seminars at universities in Moldova, Odesa, Tbilisi, Kyiv, Kharkiv, Alma-Ata, Rostov-on-Don, Bucharest, Minsk, Cluj-Napoca, the Steklov Institute of Mathematics (Moscow), and others. We present the main results obtained by Professor V. Neagu, most of which are contained in his monographs [1], [2].

He defined and constructed the symbol of singular integral operators with piecewise continuous coefficients in the case of piecewise Lyapunov and unbounded contours. This symbol essentially reflects the particularities of the contour. In particular, it depends explicitly on the magnitudes of the angles at the angular points and is defined in a special way at the point at infinity; moreover, the symbol depends on the space and on the jumps of the operator coefficients at their points of discontinuity. He established that the necessary and sufficient Noetherian conditions for singular operators (with and without translation), as well as the formula for their index, are expressed through the operator's symbol.

He calculated the exact constants in the theorems on the continuity of singular operators in weighted spaces and of matrix singular operators in spaces. Using these results, Vasile Neagu was the first to establish certain sufficient conditions under which singular equations (and systems of such equations) with measurable and bounded coefficients are Noetherian in the case of piecewise Lyapunov contours with self-intersection points. Under fairly general conditions imposed on the integration contour, he obtained a method that makes it possible to reduce the study of singular integral equations and the problem of factorization of matrices of functions in weighted Lebesgue spaces to the study of similar problems in unweighted spaces. With the help of the proposed method, and by applying I. Simonenko's local principle, Professor Vasile Neagu established the necessary and sufficient conditions under which singular integral operators with arbitrary measurable and bounded coefficients are Noetherian in weighted spaces. He studied a class of linear and bounded (non-compact) operators that represent admissible perturbations for

Noetherian singular integral operators. These results play an important role in the study of singular operators with discontinuous translations. The difficulties encountered in the study of singular operators with translation in the case of unbounded contours were overcome by Vasile Neagu by passing to a space with a special weight in which the studied operators form an algebra. This approach proved decisive for obtaining the final results. As a result, the study of singular operators with translation was reduced not to characteristic singular operators (as in the case of Lyapunov contours), but to singular operators perturbed by integral operators with homogeneous kernels. For such perturbed operators he established the Noetherian criteria.

He studied various (non-commutative) algebras of singular operators with translation and demonstrated their equivalence to algebras of singular operators without translation. On the basis of these results, he defined the notion of symbol for these algebras and established the Noetherian and invertibility criteria. The matrix symbol of non-commutative algebras of singular operators constructed by Professor Vasile Neagu possesses the same properties as the symbol in I. Gelfand's theory concerning maximal ideals of commutative Banach algebras. He demonstrated that both for Carleman-type singular integral equations with translation and complex conjugation, and for Riemann–Hilbert boundary value problems, the Noetherian conditions and the indices of the equations depend essentially on the presence of angular points on the integration contour. He showed that both the Noetherian conditions and the indices can be expressed through the determinant of the symbol of these equations.

We note that some ideas and results from the works of Professor Vasile Neagu and joint works with Professor N. Krupnik have been used in the realization of several PhD theses in mathematics both in Moldova and in other countries. References to some of his works can be found in the monographs of well-known mathematicians: B. Hvedelidze, G. Litvinciuk, N. Krupnik and I. Gohberg, A. Soldatov, N. Karapetyants, and S. Samko, as well as in numerous scientific publications. They are also used by students and master's students of the faculties of mathematics and computer science in the preparation of bachelor's and master's theses.

Although he is a renowned scientist, Professor Vasile Neagu remains a modest and kind person. He is always attentive to colleagues and students. He is distinguished by his open and friendly nature, generosity of spirit, and fidelity to his word. He possesses an astonishing ability to respect and even consider the viewpoints of others.

The special appreciation of his scientific and pedagogical work has brought him numerous diplomas, titles, and medals, such as: member of the Assembly of the Academy of Sciences of the Republic of Moldova; the "Nicolae Milescu Spătaru" Medal (2020);

the Medal of the Academy of Sciences of the Republic of Moldova; the Medal of Moldova State University; the Prize of the Academy of Sciences of the Republic of Moldova for scientific works (2008). He is also an honorary citizen of the village of Vărzărești, Nisporeni District (2017), and holds a first-category title in chess.

The present volume is dedicated to Professor Vasile Neagu at the age of 80, very active in the academic community, full of vigor and optimism, who has made a significant contribution to the development of mathematics in the Republic of Moldova.

On the occasion of his 80th anniversary, we congratulate Professor Vasile Neagu and wish him good health, continued success and fruitful teaching and research activities.

Happy Anniversary, dear Professor Vasile Neagu!

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On some applications of the essential norms of singular integral operators

Diana Bîclea and Petru Moloşnic

Abstract. In this paper, we calculate the norms and essential norms of singular operators with the Cauchy kernel. We also determine the conditions under which these quantities coincide. Using the essential norms, conditions are obtained under which singular integral operators with bounded coefficients are Noetherian in the weighted space L_2. In the absence of a weights, the presented conditions coincide with the corresponding conditions of I. Simonenko.

2020 Mathematics Subject Classification: 45E05.

Keywords: singular integral operators, Notherian operators, Lyapunov contur, essential norms.

Asupra unor aplicații ale normelor esențiale ale operatorilor integrali singulari

Rezumat. În această lucrare se calculează normele și normele esențiale ale operatorilor singulari cu nucleu Cauchy. Se determină condițiile în care aceste mărimi coincid. Folosind normele esențiale, se obțin condiții în care operatorii integrali singulari cu coeficienți mărginiți sunt noetherieni în spațiul L_2 cu ponderi. În absența ponderii, aceste condiții coincid cu condițiile respective ale lui I. Simonenko.

Cuvinte-cheie: operatori integrali singulari, operatori noetherieni, contur de tip Lyapunov, norme esențiale.

1. Introduction

Let Γ be a Lyapunov contour on the $\mathbb{C}$ plane, and S_Γ be the operator of singular integration along Γ:

$$(S_\Gamma\varphi)(t) = \frac{1}{\pi i}\int_\Gamma \frac{\varphi(\tau)}{\tau - t}\,d\tau \qquad (t \in \Gamma).$$

Here, the integral is understood in the sense of the principal value. We agree on the following notation:

$$\|\varphi\|_{p,\rho} = \left(\int_\Gamma |\varphi(t)|^p \rho(t)\,|dt|\right)^{1/p} \tag{1}$$

the norm in the space $L_p(\Gamma, \rho)$; $\|\varphi\|_p = \|\varphi\|_{p\,1}$; if $A \in L(E)$, then

$$||A|| = \inf \|A + T\| \quad (T \in \mathfrak{T}(E)), \tag{2}$$

the indices p, ρ of the norms $\|A\|$ and $|A|$ will emphasize the space $L_p(\Gamma, \rho)$ in which the operator A acts as follows

$$\gamma(p) = \begin{cases} ctg\dfrac{\pi}{2p}, & \text{if } 1 < p \le 2, \\ tg\dfrac{\pi}{2p}, & \text{if } 2 \le p < \infty. \end{cases} \tag{3}$$

M. Riesz [1] proved the boundedness of the operator S_{Γ_0} in the space $L_p(\Gamma_0)$, where Γ_0 is the unit circle and $1 < p < \infty$. Then, Hardy and Littlewood [2], and Babenko [3] transferred this result to the space $L_p(R, |x|^{\beta})$ $(1 < p < \infty,\ 0 < \beta < 1)$. In the paper [4], the boundedness of the operator S_Γ in the space $L_p(\Gamma, \rho)$ was proved for the case when Γ is a Lyapunov contour and

$$\rho(t) = \prod_{k=1}^{n} |t - t_k|^{\beta_k} \quad (t_k \in \Gamma,\ -1 < \beta_k < p - 1).$$

The condition $-1 < \beta_k < p - 1$ is also necessary for the boundedness of the operator S_Γ in $L_p(\Gamma, \rho)$. This is confirmed by

Lemma 1.1. *Let $S_\Gamma \in L(L_p(\Gamma, \rho))$, then $\rho \in l(L_p(\Gamma))$ and $\rho^{-1} \in L_q(\Gamma)$* $\left(q = \dfrac{p}{p-1}\right)$.

Proof. The boundedness of the operator S_Γ in the space $L_p(\Gamma, \rho)$ implies the boundedness in $L_p(\Gamma)$ of the operator $M = \pi i \rho (S_\Gamma R - R S_\Gamma) \rho^{-1}$, where $(R\varphi)(t) = t\varphi(t)$. But

$$(M\varphi)(t) = \rho(t) \int_\Gamma \rho^{-1}(t) \varphi(t)\, dt,$$

therefore $\rho \in L_p(\Gamma)$, $\rho^{-1} \in L_q(\Gamma)$. □

It was later discovered that the norms and essential norms of singular operators in weighted L_p spaces are of great theoretical and applied interest. In particular, the problem of using these quantities to find criteria for the Noetherian property of singular integral operators with bounded coefficients should be noted. This problem has been investigated by many mathematicians, especially after the appearance of Simonenko's works [5], [6]. In these papers, in particular, the norm of the operator S_Γ is used to obtain conditions for the Noetherian property of a singular operator in L_2. Naturally, the problem of generalizing these results to weighted L_2 spaces arose. In the present paper, we prove a theorem that allows us to express the conditions for the Noetherian property of a characteristic singular operator in a weighted space through the conditions for the

Noetherian property of another singular operator in an unweighted space. Using this theorem and the results of Simonenko's work [6], conditions for the Noetherian property of singular operators in the space L_2 with weight are obtained. These conditions are not only sufficient but also necessary for the singular operator to be Noetherian. The present work is devoted to the presentation of these results.

2. Relationship between the norm and the essential norm of the operator S_Γ

Let Γ consist of a finite number of simple closed disjoint Lyapunov curves, $t_1, t_2, \dots, t_n \in \Gamma$ and

$$\rho(t) = \prod_{k=1}^{n} |t - t_k|^{\beta_k}, \quad p > 1,\ -1 < \beta_k < p - 1,\ r_0 = p,\ r_k = p(1 + \beta_k)^{-1},$$

$$(k = 1, 2, \dots, n),\ \bar{r}_k = \max(r_k, r_k(r_k - 1)^{-1}),\ r = \max(\bar{r}_0, \bar{r}_1, \dots, \bar{r}_n).$$

In the works [7], [8], it is shown that

$$|S_\Gamma|_{p,\rho} \geq ctg\frac{\pi}{2r}. \tag{4}$$

We present one sufficient condition under which the equality $\|A\| = |A|$ holds (see [7], [8], [9]).

Theorem 2.1. *Let for some operator $A \in L(E)$ there exists a sequence of operators $B_n \in L(E)$ such that*

(1) $\|B_n\varphi\| = \|\varphi\|,\ \varphi \in E$;

(2) $AB_n = B_nA,\ \forall n \in \mathbb{N}$;

(3) B_n weakly converges to zero.

Then $\|A\| \geq |A|$.

Proof. It suffices to prove that, for every compact operator T, the inequality

$$\|A + T\| \geq \|A\| \tag{5}$$

holds.

Let $\varepsilon > 0$ be arbitrary and φ be some vector in E, satisfying the conditions $\|\varphi\| = 1$, $\|A\varphi\| > \|A\| - \varepsilon$. We set $\varphi_n = B_n\varphi$. Since $B_n \to 0$ weakly, we have $\|TB_n\varphi\| \to 0$. We choose n such that $\|TB_n\varphi\| < \varepsilon$, then

$$\|(A + T)\varphi_n\| \geq \|AB_n\varphi\| - \varepsilon = \|A\varphi\| - \varepsilon \geq (\|A\| - 2\varepsilon)\|\varphi_n\|,$$

hence $\|A + T\| \geq \|A\|$. The theorem is proved. □

Denote the unit circle by Γ_0, $\Gamma_0 = \{t : |t| = 1\}$.

Theorem 2.2. *For each $p \in (1, \infty)$, the equality*

$$|S_0|_p = \|S_0\|_p = \gamma(p) \qquad (S_0 = S_{\Gamma_0}). \tag{6}$$

Proof. In [10], the norm of the operator was calculated

$$(H\varphi)(t) = \frac{1}{2\pi} \int_0^{2\pi} ctg \frac{\tau - t}{2} \, \varphi(\tau)\, d\tau,$$

in the real space $L_p(0, 2\pi)$. It turned out that $\|H\|_p = \gamma(p)$. After some time, in [11], it was proved that this equality is also valid in the complex space $L_p(0, 2\pi)$. Let α be an isometry $L_p(\Gamma_0) \to L_p(0, 2\pi)$, defined by the equality $(\sigma\varphi)(\theta) = \varphi(\exp(i\theta))$. It is directly verified that $\sigma H \sigma^{-1} = i(K_0 - S_0)$, where K_0 is a one-dimensional operator,

$$(K_0\varphi)(t) = \frac{1}{2\pi} \int_{\Gamma_0} \varphi(\tau)\, |dt|.$$

It follows that $\|S_0 - K_0\|_p = \gamma(p)$ and $|S_0|_p \le \gamma(p)$. Taking into account inequality (4), we obtain the equality $|S_0|_p = \gamma(p)$. Consider the sequence of operators $(B_n\varphi)(t) = t^n \varphi(t^n)$. It is directly verified that the sequence (B_n) and the operator S_0 satisfy the conditions of Theorem 2.1 in the space $L_p(\Gamma_0)$ and, consequently, $|S_0|_p = \|S_0\|_p$. The theorem is proved. □

Remark 2.1. *Let Γ be a simple closed Lyapunov contour. If $|S_\Gamma|_2 = \|S_\Gamma\|_2$, then Γ is a circle.*

Indeed, let Γ not be a circle, then in [11] $S_\Gamma^* \ne S_\Gamma$. Since $S_\Gamma^2 = I$, then $S_\Gamma S_\Gamma^* \ne I$, hence the spectrum of the operator $S_\Gamma S_\Gamma^*$ contains a point $\lambda \ne 1$. From the equality $S_\Gamma S_\Gamma^* - \lambda I = S_\Gamma^*(I - S_\Gamma^* S_\Gamma \lambda) S_\Gamma$ it follows that the spectrum of the operator $S_\Gamma S_\Gamma^*$ contains a point $\lambda_0 > 1$. Hence, $|S_\Gamma|_2 > 1$. On the other hand, (see [11]), $|S_\Gamma|_2 = |S_0|_2 = 1$. The resulting contradiction proves that Γ is a circle.

Let us consider two more examples of calculating the norms of the operator S_Γ.

Example 2.1. Let $\Gamma = \mathbb{R}$, $\rho(x) = |x|^\beta$ $(-1 < \beta < p - 1)$; $A = aI + bS_\mathbb{R}$, $(a, b \in \mathbb{C})$ and

$$(B_n\varphi)(x) = n^{\frac{1+\beta}{p}} \varphi(nx).$$

It is directly verified that the operators A and B_n satisfy the conditions of Theorem 2.1, therefore,

$$\|aI + bS_\mathbb{R}\|_{p,\rho} = |aI + bS_\mathbb{R}|_{p,\rho}.$$

In particular,

$$\|S_\mathbb{R}\|_{p,|x|^\beta} = |S_\mathbb{R}|_{p,|x|^\beta}. \tag{7}$$

Similar equalities can be written for projectors

$$P_{\mathbb{R}} = \frac{1}{2}(I + S_{\mathbb{R}}), \quad Q_{\mathbb{R}} = \frac{1}{2}(I - S_{\mathbb{R}}).$$

Example 2.2. Let us show that

$$\|aI + bS_0\|_{p,\hat{\rho}} = |aI + bS_0|_{p,\hat{\rho}}, \tag{8}$$

where $a, b \in \mathbb{C}$ and $\hat{\rho}(t) = |t-1|^{p-2-\beta} \cdot |t+1|^{\beta}$.

Consider the operator B, defined by the equality

$$(B\varphi)(t) = \frac{2^{1/p}}{t-1}\,\varphi\left(i\,\frac{t+1}{t-1}\right)$$

and let us prove that it isometrically maps the space $L_p(\mathbb{R}; |x|^{\beta})$ onto $L_p(\Gamma_0; |t-1|^{p-2-\beta} \cdot |t+1|^{\beta})$. Furthermore, let us also prove that the equality $BS_{\mathbb{R}}B^{-1} = -S_0$ holds. Indeed, let $\varphi \in L_p(\mathbb{R}; |x|^{\beta})$, then

$$(BS_{\mathbb{R}}\varphi)(t) = \frac{1}{\pi i}\left(B\int_{-\infty}^{+\infty} \frac{\varphi(x)}{x-y}\,dx\right)(t) = \frac{2^{1/p}}{\pi i}\,\frac{1}{t-1}\int_{-\infty}^{+\infty} \frac{\varphi(x)}{x - i\,\frac{t+1}{t-1}}\,dx.$$

In the last integral we will make a change of variables

$$x = i\,\frac{\tau+1}{\tau-1}, \quad dx = -i\,\frac{2}{(\tau-1)^2}\,dt, \quad t \in \Gamma_0,$$

then

$$(BS_{\mathbb{R}}\varphi)(t) = -\frac{2^{1/p}}{\pi i}\,\frac{1}{t-1}\int_{\Gamma_0} \frac{\varphi\left(i\,\frac{\tau+1}{\tau-1}\right)}{\frac{\tau+1}{\tau-1} - \frac{t+1}{t-1}} \cdot \frac{2}{(\tau-1)^2}\,d\tau = -\frac{1}{\pi i}\int_{\Gamma_0} \frac{(B\varphi)(\tau)}{\tau - t}\,d\tau.$$

Thus, we obtained the equality $BS_{\mathbb{R}}\varphi = -S_0B\varphi$, i.e., $BS_{\mathbb{R}}B^{-1} = -S_0$. Let us prove the isometry. We have

$$\|\varphi\|^p_{L_p(\mathbb{R};|x|^{\beta})} = \int_{-\infty}^{+\infty} |\varphi(x)|^p |x|^{\beta}\,dx = \int_{\Gamma_0} \left|\varphi\left(i\,\frac{t+1}{t-1}\right)\right|^p \left|\frac{t+1}{t-1}\right|^{\beta} \left|\frac{2}{(t-1)^2}\right| dt$$

$$= \int_{\Gamma_0} |(B\varphi)(t)|^p\,|t-1|^{p-2-\beta} \cdot |t+1|^{\beta}\,dt = \|B\varphi\|^p_{L_p(\Gamma_0;|t-1|^{p-2-\beta}\cdot|t+1|^{\beta})}.$$

From the obtained properties of the operator B and from equality (7) it follows

$$\|aI + bS_0\|_{p,\hat{\rho}} = |aI + bS_0|_{p,\hat{\rho}}, \quad \text{where } \hat{\rho}(t) = |t-1|^{p-2-\beta} \cdot |t+1|^{\beta}. \tag{8}$$

3. Noetherian conditions for the operator $aP + Q$ in the space $L_2(\Gamma)$

Let Γ be a closed contour of Lyapunov type and A be the operator defined by the equality

$$A = aP + Q, \tag{9}$$

where a is a measurable and bounded function on the contour Γ,

$$P = \frac{1}{2}(I + S), \quad Q = I - P$$

and S is the operator

$$(S\varphi)(t) = \frac{1}{\pi i}\int_\Gamma \frac{\varphi(\tau)}{\tau - t}\,d\tau \qquad (t \in \Gamma).$$

In this Section we will use Simonenko's local principle and the essential norms of the operator S to determine the Noetherian conditions for the operator (9). First, we will reduce the study of the operator A to the study of a simpler operator. We consider the condition $\operatorname{ess\,inf}|a(t)| > 0$ to be fulfilled, otherwise the operator A is not (see [11]) Noetherian. We express the function a in the form

$$a(t) = |a(t)| \cdot e^{i\varphi(t)}, \quad \varphi(t) = \arg a(t).$$

As the function $a(t)$ $(\operatorname{ess\,inf}|a(t)| > 0)$ admits a factorization (see [11])

$$|a(t)| = a_+(t) \cdot a_-(t),\ a_+(t) = \exp(P \ln |a(t)|),$$

$$a_-(t) = \exp(Q \ln |a(t)|),\ a_+^{\pm 1} \in L_\infty^+(\Gamma),\ a_-^{\pm 1} \in L_\infty^-(\Gamma),$$

the operator $A = aP + Q$ can be written in the form

$$A = a_-\big(e^{i\varphi}P + Q\big)\big(a_+P + a_-^{-1}Q\big), \tag{10}$$

where the operators a_-I and $a_+P + a_-^{-1}Q$ are invertible in $L_p(\Gamma)$:

$$(a_-I)^{-1} = a_-^{-1}I, \qquad (a_+P + a_-^{-1}Q)^{-1} = a_+^{-1}P + a_-Q.$$

Therefore, from equality (10), the following theorem follows.

Theorem 3.1. *The operator $A = aP+Q$ is Noetherian in $L_p(\Gamma)$ if and only if the operator $e^{i\varphi}P + Q$ is Noetherian in this space. Moreover,*

$$\operatorname{Ind}(aP + Q) = \operatorname{Ind}(e^{i\varphi}P + Q).$$

Thus, Theorem 3.1 reduces the study of operators of the form $aP + Q$ to operators of the form $e^{i\varphi(t)}P + Q$, where $\varphi(t)$ is real, bounded, and measurable. We also mention an important property of the operators $A = aP + Q,\ a(t) \geq \delta > 0,\ \forall t \in \Gamma$.

Theorem 3.2. *The operator* $A = bP + Q$, $b \in L_\infty(\Gamma)$ *is Noetherian in* $L_p(\Gamma)$ *if and only if the operator* $C = abP + Q$, $a(t) \geq \delta > 0$, $t \in \Gamma$ *is Noetherian and* $\operatorname{Ind} B = \operatorname{Ind} C$.

The statement of the theorem follows immediately from the fact that the function a admits a factorization, $a = a_+ \cdot a_-$ and from the equality

$$C = a_-(bP + Q)(a_+ P + a_-^{-1} Q).$$

We denote by $\mathfrak{S}_2(\Gamma)$ the set of all functions $a \in L_\infty(\Gamma)$ that verify the condition: for any $\tau \in \Gamma$ there is a neighborhood $u(\tau) \subset \Gamma$ and two functions $g_\tau^\pm \in L_\infty^\pm(\Gamma)$, such that the set of values of the function $g_\tau^-(t)a(t)g_\tau^+(t)$, for $t \in u(\tau)$, lies in a closed half-plane Λ_τ that does not contain the origin of coordinates (Figure 1).

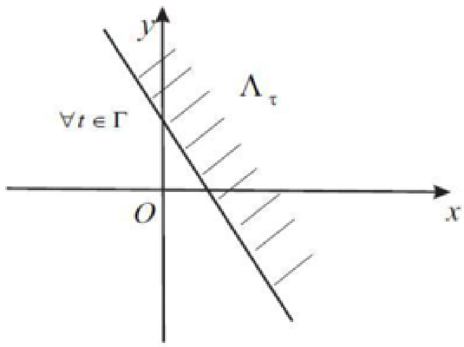

Figure 1. The set Λ_τ

The simplest function that satisfies these conditions is the function $a(t) = e^{i\varphi(t)}$, where $\varphi(t)$ is a real function and $\sup|\varphi(t)| < \frac{\pi}{2}$. The above condition is satisfied for this function with $g_\tau^\pm(t) \equiv 1$. The set of values of this function lies in the right half-plane, $\{z \mid Re z > 0\}$.

Theorem 3.3. *If* $a \in \mathfrak{G}_2(\Gamma)$, *then the operator* $A = aP + Q$ *is Noetherian in the space* $L_2(\Gamma)$.

Proof. Let t be an arbitrary point on Γ, $g_\tau^\pm \in L_\infty^\pm(\Gamma)$ and $u(\tau)$ be the neighborhood of the point τ such that for $t \in u(\tau)$ the set of values of the function a lies in the half-plane Λ_τ. We denote by $a_\tau(t)$ the function

$$a_\tau(t) = \begin{cases} a(t), & \text{for } t \in u(\tau), \\ \alpha\,[g_\tau^-(t)g_\tau^+(t)]^{-1}, & \text{for } t \in \Gamma \setminus u(\tau), \end{cases}$$

where α is some complex number in Λ_τ. We will show that the operator $a_\tau P + Q$ is Noetherian in $L_2(\Gamma)$. Let $b(t) = g_\tau^-(t)a_\tau(t)g_\tau^+(t)$. The set of values of the function b for any $t \in \Gamma$ lies in Λ_τ and is bounded:

$$b(t) = \begin{cases} g_\tau^-(t)a(t)g_\tau^+(t), & \text{for } t \in u(\tau), \\ \alpha, & \text{for } t \in \Gamma \setminus u(\tau). \end{cases}$$

Then there is a number $\gamma \in \mathbb{C}$ such that

$$\sup_{t\in\Gamma} |\gamma b(t) - 1| < 1.$$

We write the operator $A_\tau = a_\tau P + Q$ in the form

$$A_\tau = (g_\tau^-)^{-1}\left[(\gamma b - 1)P + I\right]\left[(\gamma g_\tau^+)^{-1}P + g_\tau^- Q\right]. \tag{11}$$

In equality (11) the operators $(g_\tau^-)^{-1}I$ and $(\gamma g_\tau^+)^{-1}P + g_\tau^- Q$ are invertible, their inverses have the form $(g_\tau^-)I$ and, respectively, $\gamma g_\tau^+ P + (g_\tau^-)^{-1}Q$. From Theorem 2.2 we deduce that the essential norm of the operator P is equal to one, $|P|_{L_2(\Gamma)} = 1$. Then

$$|(\gamma b - 1)P|_{L_2(\Gamma)} = \inf_{T\in\mathcal{K}} \|(\gamma b - 1)P + T\|_{L_2(\Gamma)} \leq \sup_{t\in\Gamma} |\gamma b(t) - 1| < 1. \tag{12}$$

Thus, the operator $(\gamma b - 1)P + I$ is Noetherian, and consequently the operator A_τ, defined by equality (11), is Noetherian. At any point $\tau \in \Gamma$, the function a is equivalent to the function a_τ and the operator $a_\tau P + Q$ is Noetherian in $L_2(\Gamma)$. Based on the results of the work of Simonenko [6], we deduce that $A = aP + Q$ is also Noetherian in the space $L_2(\Gamma)$. Theorem 3.3 is proved. □

Remark 3.1. *We mention that in the proof of Theorem 3.3, an important role was played by the fact that we know the size of the essential norm of the operator P:* $|P|_{L_2(\Gamma)} = 1$. *This observation is also valid in what follows.*

4. Noetherian Criteria for the Operator $aP + Q$ in the Space L_p with Weights

To establish other Noetherian criteria for singular integral operators in the spaces L_p and L_p with weights, we recall some important notions from the theory of operators of the form $aP + bQ$ with measurable and bounded coefficients, notions that can be found in the works of I. Simonenko, I. Gohberg and N. Krupnik (see for example [5], [11]).

Definition 4.1. *A function $a \in L_\infty(\Gamma)$ is said to admit a factorization in the space $L_p(\Gamma, \rho)$ if it can be expressed in the form*

$$a(t) = a_+(t)t^\kappa a_-(t), \tag{13}$$

where κ is an integer, and the factors $a_\pm$ satisfy the following conditions:

(1) $a_- \in L_p^-(\Gamma, \rho)$, $a_+ \in L_q^+(\Gamma, \rho^{1-q})$, $a_-^{-1} \in L_q^-(\Gamma, \rho^{1-q})$ and $a_+^{-1} \in L_p^+(\Gamma, \rho)$;

(2) The operator $a_+^{-1}Pa_-^{-1}$ is bounded in the space $L_p(\Gamma, \rho)$.

In the same works [5], [11], it is shown that the operator $aP + bQ$ is Noetherian in $L_p(\Gamma, \rho)$ if and only if $\operatorname{ess\,inf} |a(t)| > 0$, $\operatorname{ess\,inf} |b(t)| > 0$, and the function $c(t) = a(t)b^{-1}(t)$ admits a factorization $c(t) = c_+(t)t^\chi c_-(t)$. Under these conditions,

$$\operatorname{Ind} A = -\chi.$$

The following theorem and the reasoning based on it will allow us to reduce the study of certain properties of singular operators in weighted spaces to the study of the corresponding properties of singular operators in unweighted spaces. Thus, in particular, we will conclude that it is sufficient to know Noetherian criteria for singular integral operators in unweighted L_p spaces.

Theorem 4.1. *Let the function $h \in L_\infty(\Gamma)$ admit a factorization, $h = h_+ \cdot h_-$, in the space $L_p(\Gamma, \rho)$. We denote $L_p(\Gamma, \rho) = \mathcal{B}_1$ and $\mathcal{B}_2 = L_p(\Gamma, \rho \cdot |h_+|^{-p})$. There are two invertible operators $B \in L(\mathcal{B}_2, \mathcal{B}_1)$ and $F \in L(\mathcal{B}_1, \mathcal{B}_2)$ such that for any functions $a, b \in L_\infty(\Gamma)$ the equality holds*

$$B(aP + bQ)F = haP + bQ. \tag{14}$$

Proof. We will define the operators F and B by the relations:

$$F = h_+P + h_-^{-1}Q, \qquad B = h_-. \tag{15}$$

We will prove that $B \in L(\mathcal{B}_2, \mathcal{B}_1)$ and $F \in L(\mathcal{B}_1, \mathcal{B}_2)$. Indeed, let $\varphi \in \mathcal{B}_2$, then

$$\|B\varphi\|_{\mathcal{B}_1}^p = \int_\Gamma |h_-(t)|^p |\varphi(t)|^p \rho(t)\,|dt| = \int_\Gamma |\varphi(t)|^p |h(t)|^p |h_+(t)|^{-p} \rho(t)\,|dt|$$

$$\leq \operatorname*{ess\,sup}_{t\in\Gamma} |h(t)|^p \, \|\varphi\|_{\mathcal{B}_2}^p.$$

Therefore, $\|B\| \leq \operatorname{ess\,sup}_{t\in\Gamma} |\omega(t)|$. We evaluate the norm of the operator F. Let $\psi \in \mathcal{B}_1$, then

$$\|F\psi\|_{\mathcal{B}_2}^p = \int_\Gamma |\omega_+(t)(P\psi)(t) + \omega_-^{-1}(t)(Q\psi)(t)|^p \, |\omega_+(t)|^{-p} \rho(t)\,|dt|$$

$$= \int_\Gamma |(P\psi)(t) + h(t)(Q\psi)(t)|^p \rho(t)\,|dt| = \|(P + hQ)\psi\|_{\mathcal{B}_1}^p.$$

From here it follows that $\|F\| = \|P + hQ\|_{\mathcal{B}_1}$. The operators B and F are invertible, their inverses are the operators

$$B^{-1} = h_-^{-1} I, \quad F^{-1} = h_+^{-1} P + h_- Q.$$

The equality (14) itself is based on the properties of the projectors P and Q and is easily verified. Theorem 4.1 is proved. □

From this theorem it follows

Corollary 4.1. *The operator $A = aP + bQ$ $(a, b \in L_\infty(\Gamma))$ is normally solvable in the space $L_p(\Gamma, \rho \cdot |h_+|^{-p})$ if and only if this property is also possessed in the space $L_p(\Gamma, \rho)$ by the operator*

$$A_h = haP + Q. \tag{16}$$

Corollary 4.2. *Under the conditions of Theorem 4.1, the following relations*

$$\dim\ker A|_{\mathcal{B}_2} = \dim\ker A_h|_{\mathcal{B}_1}, \quad \dim\ker A^*|_{\mathcal{B}_2^*} = \dim\ker A_h^*|_{\mathcal{B}_1^*}$$

hold.

Corollary 4.3. *The operator A admits a regularization (left, right or bilateral) in the space $L_p(\Gamma, \rho \cdot |h_+|^{-p})$ if and only if the operator A_h admits a similar regularization in the space $L_p(\Gamma, \rho)$. The regularizers M and M_h of the operators A and A_h are expressed by the relation*

$$M_h = FMB,$$

where B and F are the operators defined by equalities (15).

Corollary 4.4. *The operator A is invertible (left or right) in the space $L_p(\Gamma, \rho \cdot |h_+|^{-p})$ if and only if the operator A_h in the space $L_p(\Gamma, \rho)$ has this property. If A_h^{-1} is the inverse of the operator A_h, then the operator $FA_h^{-1}B$ is the inverse of the operator A.*

In what follows we will assume that the contour Γ consists of γ $(\gamma = 1, 2, \ldots)$ closed Liapunov lines on portions without common points (Figure 2).

Let

$$\rho(t) = \prod_{k=1}^{n} |t - t_k|^{\beta_k} \qquad (-1 < \beta_k < p - 1).$$

We denote by $\Gamma_k (\subset \Gamma)$ the simple closed curve containing the point t_k and define the function

$$h_k(t) = \begin{cases} (t - z_k)^{-\varepsilon\beta_k/p}, & \text{for } t \in \Gamma_k, \\ 1, & \text{for } t \in \Gamma \setminus \Gamma_k, \end{cases} \tag{17}$$

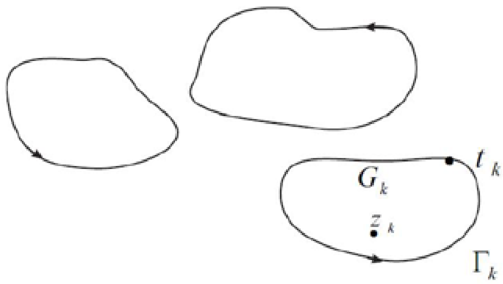

Figure 2. The composite contour Γ

where z_k is a point in the domain G_k, bounded by the curve Γ_k, $\varepsilon = 1$ ($\varepsilon = -1$) if the domain G_k, bounded by the oriented curve Γ_k, remains on the left (right), and $(t-z_k)^{-\varepsilon\beta_k/p}$ is the branch of the function $(z-z_k)^{-\varepsilon\beta_k}$ continuous at any point $t \in \Gamma\setminus\{t_k\}$ (Figure 2). The function $h_0(t) = h_1(t) \cdot h_2(t) \cdots h_\gamma(t)$ is continuous at any point $t \in \Gamma \setminus \{t_1, t_2, \dots, t_\gamma\}$ and

$$\frac{h_0(t_k - 0)}{h_0(t_k + 0)} = e^{-2\pi\beta_k/p}, \qquad k = (1, 2, \dots, \gamma).$$

As

$$\frac{2\pi}{p} - 2\pi < -\frac{2\pi\beta_k}{p} < \frac{2\pi}{p},$$

from the results of [8], [11], [12], it follows that the operator $h_0P + Q$ is Noetherian in the space $L_p(\Gamma, \rho)$. Therefore, the function $h_0(t)$ admits a factorization in $L_p(\Gamma, \rho)$: $h_0 = h_+ \cdot h_-$. It is easy to show that

$$|h_+(t)|^{-p} = \rho(t) \cdot g(t),$$

where $(g(t))^{\pm 1} \in C(\Gamma)$. Therefore, $L_p(\Gamma, |h_+|^{-p}) = L_p(\Gamma, \rho)$. From these arguments and from Corollary 4.3 it follows

Theorem 4.2. *Let $a, b \in L_\infty(\Gamma)$ and let $h_0(t) = h_1(t) \cdot h_2(t) \cdots h_\gamma(t)$, where h_k are the functions defined by equality (17). The operator $A = aP + bQ$ is Noetherian in the space $L_p(\Gamma, \rho)$ if and only if the operator $A_h = haP + bQ$ is Noetherian in the space $L_p(\Gamma)$. Moreover, the following equalities hold:*

$$\dim\ker A|_{L_p(\Gamma,\rho)} = \dim\ker A_h|_{L_p(\Gamma)}, \quad \dim\ker A^*|_{L_q(\Gamma,\rho^{1-q})} = \dim\ker A_h^*|_{L_q(\Gamma)}.$$

Theorem 4.2 allows us to extend to the space L_p with weight ρ some results of I. Simonenko, A. Devinatz, H. Poisson, N. Rabindranathan, I. Daniliuc, N. Krupnik which represent invertibility (or Noetherian) criteria of singular integral operators in the space L_p without weights.

We will focus on a result due to I. Simonenko. Let Γ consist of a finite number of Lyapunov-type curves without self-intersection points, and let $\rho(t) = \prod_{k=1}^{n} |t - t_k|^{\beta_k}$. From the results of I. Simonenko [5] with the help of Theorem 4.2, we obtain the following theorem.

Theorem 4.3. *Let the function $a \in L_\infty(\Gamma)$ satisfy the following three conditions:*

(1) $\operatorname{ess\,inf}_{t\in\Gamma} |a(t)| \neq 0$*;*

(2) for any point $t \in \Gamma \setminus \{t_1, \dots, t_n\}$ there is a neighborhood $u(\tau) \subset \Gamma$ and two functions g_τ^+, g_τ^- which belong to the sets $L_\infty^+(\Gamma)$, respectively $L_\infty^-(\Gamma)$ together with $(g_\tau^+)^{-1}$ and $(g_\tau^-)^{-1}$, such that, for $t \in u(\tau)$ the set of values of the function $g_\tau^+(t)a(t)g_\tau^-(t)$ is contained in the angle with the vertex at the point $z = 0$ and with the opening smaller than $\frac{2\pi}{\max(p,q)}$ $\quad (p^{-1} + q^{-1} = 1)$;

(3) there is a neighborhood $u(t_k)$ of the point t_k $(k = \overline{1,n})$ and two functions g_k^+, g_k^- from the sets $L_\infty^+(\Gamma)$ and $L_\infty^-(\Gamma)$ together with $(g_k^+)^{-1}$ and $(g_k^-)^{-1}$, such that, for $t \in u_k(t_k)$ and $t > t_k$ $(t \in u(t_k)$, and $t < t_k)$ the set of values of the function

$$g_k^+(t)a(t)g_k^\tau(t) \quad \left(g_k^+(t)a(t)g_-^k(t) \cdot e^{-2\pi i\beta_k/p}\right)$$

lies in the angle with the vertex at $z = 0$ and with the opening smaller than

$$\frac{2\pi}{\max(p,q)}.$$

Under these conditions the operator $aP + Q$ is Noetherian in the space $L_p(\Gamma, \rho)$.

We note that for $\rho(t) \equiv 1$ $(\beta_k = 0,\ k = \overline{1,n})$ this theorem is contained in the paper [5], and for $p = 2$ the conditions of Theorem 4.3 are necessary and sufficient in which the operator $aP + bQ$ is Noetherian. In addition, the results in the papers [5], [6] concerning the operators $aP + bQ$, where a and b are matrices of functions, can be extended using Theorem 4.3 to the space L_p^m with weights. Here we will not stop at these cases.

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On a quasigroup prolongation and its recursive differentiability

Elena Cuzneţov

Abstract. We consider a method of quasigroups prolongation by adding two new elements and by using two transversals that intersect in exactly one cell. In a Latin square L of order q, along with usual transversals we call "free transversals" any set of q cells taken by one from each row and each column of L. It is shown that a Latin square of order q can have at most $q-2$ (ordinary) transversals that intersect exactly in one cell, which is also the only common cell of every two transversals. It is proved that a latin square of order 4 has exactly 24 free transversals, 576 (ordinary) transversals, 96 pairs of free transversals that intersect exactly in one cell and 13824 pairs of ordinary transversals that intersect exactly in one cell. We also investigate the recursive 1-differentiability of considered prolongations of Latin squares of order 5, one of the transversals being on the main diagonal. The research done in this paper is applicable in cryptography and is important for data integrity and fault tolerance.

2020 Mathematics Subject Classification: 20N05, 05B15.

Keywords: quasigroup, transversal, free transversal, prolongation, recursively differentiable quasigroup, recursive derivative, Latin square.

Despre prelungirea quasigrupului şi derivabilitatea sa recursivă

Rezumat. Este considerată o metodă de prelungire a quasigrupurilor prin adjuncţia a două elemente noi şi utilizarea a două transversale care se intersectează într-o singură celulă. Într-un pătrat latin L de ordinul q, împreună cu transversalele obişnuite, sunt utilizate "transversale libere" - seturi de q celule luate câte una din fiecare linie şi fiecare coloană a lui L. Se demonstrează că un pătrat latin de ordinul q poate avea cel mult $q-2$ transversale (ordinare) ce se intersectează într-o singură celulă, care este şi singura celulă comună a fiecăror două transversale. Se demonstrează că un pătrat latin de ordinul 4 are exact 24 de transversale libere, 576 de transversale (ordinare), 96 de perechi de transversale libere care se intersectează exact într-o celulă şi 13824 de perechi de transversale ordinare care se intersectează exact într-o celulă. De asemenea, investigăm 1-derivabilitatea recursivă a prelungirilor considerate ale pătratelor latine de ordinul 5, una dintre transversale fiind pe diagonala principală. Cercetarea efectuată în această lucrare are aplicări în criptografie şi este importantă pentru integritatea datelor şi toleranţa la erori.

Cuvinte-cheie: quasigrup, transversală, transversală liberă, prelungire, quasigrup recursiv derivabil, derivată recursivă, pătrat latin.

1. Introduction

The concept of extending quasigroups was first examined by Bruck in 1944 for finite idempotent quasigroups [1]. For analogous constructions the term "prolongation" was introduced by Belousov in 1967 [2]. By "prolongation" of a finite quasigroup we mean a process of extending the quasigroup by adding one or more new elements and redefining the operation to obtain a new quasigroup of higher order.

Other methods of prolongations or modifications of the known ones have been proposed by Osborn, Yamamoto, Denes and Pasztor, Belousov and Belyavskaya, Belyavskaya, Deriyenko and Dudek, and others [3].

A method of quasigroups prolongation by adding two new elements and by using two transversals that intersect exactly in one cell is considered in the present work.

Estimations of the number of prolongations, using the proposed method, are given. One of our purposes is to study the recursive differentiability of the considered prolongations. In a Latin square L of order q, along with usual transversals, we call a "free transversal" any set of q cells taken by one from each row and each column of L. We show that a Latin square of order q can have at most $q-2$ (ordinary) transversals that intersect exactly in one cell, which is also the only common cell of every two transversals. We use the considered method of quasigroups prolongation mainly for Latin squares of order 4 and 5 (there does not exist Latin squares of order ≤ 3 which have two transversals intersecting exactly in one cell). It is proved that a Latin square of order 4 has exactly 24 free transversals, 576 (ordinary) transversals, 96 pairs of free transversals that intersect exactly in one cell, and 13824 pairs of ordinary transversals that intersect exactly in one cell. It is known that there does not exist recursively differentiable quasigroups of order 6, due to orthogonality of the recursive derivatives and the non existence of orthogonal Latin squares of order 6. Hence, the prolongations of Latin squares of order 4, by adding two new elements, are not recursively 1-differentiable. In the case of Latin squares of order 5 we prove that:

1. There exist exactly 48 different Latin squares of order 5 with two transversals intersecting in one cell, where one transversal is on the main diagonal.

2. The prolongations by adding two new elements and using two new transversals, that intersect exactly in one cell, one of which is on the main diagonal, and have the order of elements $\{1, 2, 3, 4, 5\}$ or $\{2, 3, 4, 5, 1\}$, are not recursively 1-differentiable.

The (ordinary) transversal of a Latin square of order n is a set of n cells taken by one from each row and each column, in which the elements are pairwise different. We call a free transversal of a Latin square L of order n any set of n cells taken exactly one from each row and each column of L.

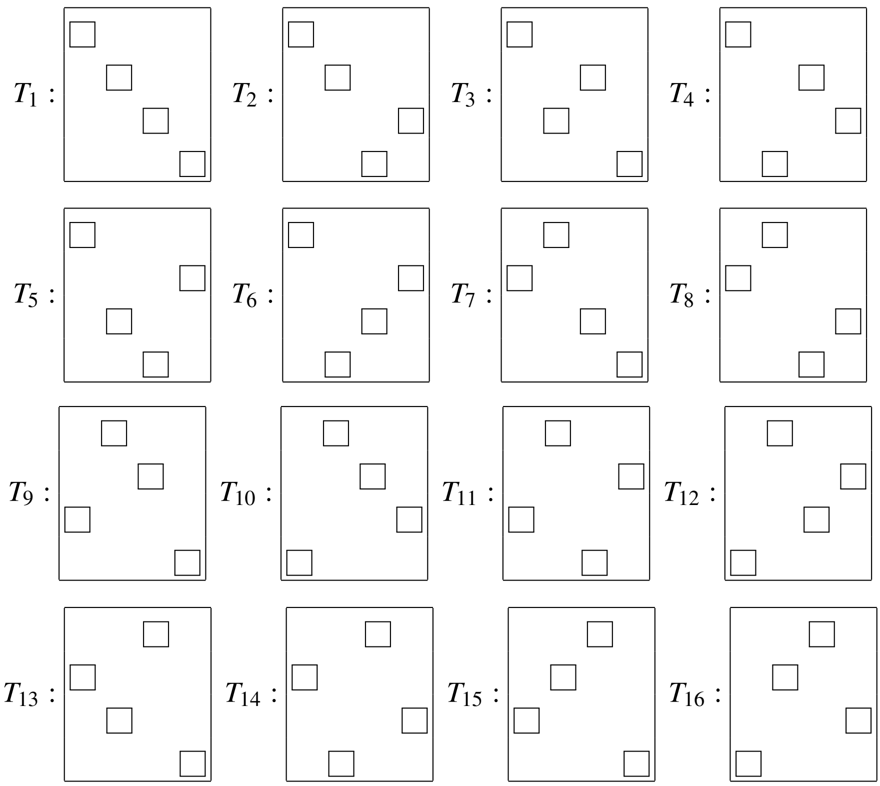
Proposition 1.1. *A Latin square of order n can have at most $n-2$ ordinary transversals that intersect exactly in one cell, if this is also the only common cell of every two transversals.*

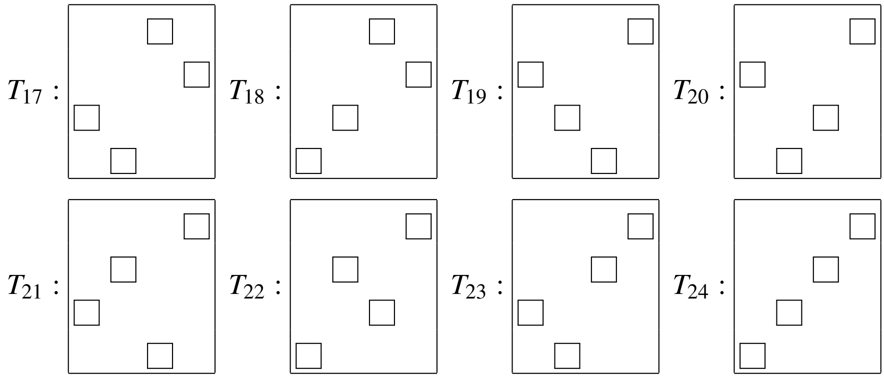
Proof. Let L be a Latin square of order n with the elements $1, 2, ..., n$ in its cells which has $n-2$ ordinary transversals that intersect exactly in one cell, which is also the only common cell of every two transversals. Let a be the element in the common cell of the $n-2$ transversals of the Latin square of order n. If L would have one more transversal with the same common cell, then this transversal must intersect a line which does not contain the common cell or in the cell with the element a or in the cell from the column (row) with the common cell, but both cases are impossible as the transversal already contains the element a in the common cell with the other $n-2$ transversals. $\square$

Corollary 1.1. *There does not exist Latin squares of order 3 with 2 transversals that intersect exactly in one cell.*

Proposition 1.2. *The Latin square of order four has exactly 24 free transversals.*

The proof follows immediately from the possibility of selecting cells from each row (column). These free transversals are given bellow:





Corollary 1.2. *Any Latin square of order four has exactly 576 (ordinary) transversals.*

Proposition 1.3. *The total number of pairs of free transversal that intersect exactly in one cell is 96.*

Proof. The proof follows from Proposition 1.2. The total number of pairs of free transversals that intersect exactly in one cell is 96 and these pairs are given bellow:

$$(T_1,T_4),(T_1,T_5),(T_2,T_3),(T_2,T_6),(T_3,T_6),(T_4,T_5),(T_7,T_{10}),(T_7,T_{11}),$$
$$(T_8,T_9),(T_8,T_{12}),(T_9,T_{12}),(T_{10},T_{11}),(T_{13},T_{16}),(T_{13},T_{17}),(T_{14},T_{15}),$$
$$(T_{14},T_{18}),(T_{15},T_{18}),(T_{16},T_{17}),(T_{19},T_{22}),(T_{19},T_{23}),(T_{20},T_{21}),$$
$$(T_{20},T_{24}),(T_{21},T_{24}),(T_{22},T_{23}),(T_7,T_{14}),(T_7,T_{19}),(T_8,T_{13}),(T_8,T_{20}),$$
$$(T_{13},T_{20}),(T_{14},T_{19}),(T_1,T_{16}),(T_1,T_{21}),(T_2,T_{15}),(T_2,T_{22}),(T_{15},T_{22}),$$
$$(T_{16},T_{21}),(T_3,T_{10}),(T_3,T_{23}),(T_4,T_9),(T_4,T_{24}),(T_9,T_{24}),(T_{10},T_{23}),$$
$$(T_5,T_{12}),(T_5,T_{17}),(T_6,T_{11}),(T_6,T_{18}),(T_{11},T_{18}),(T_{12},T_{17}),(T_9,T_{21}),$$
$$(T_9,T_{17}),(T_{11},T_{15}),(T_{11},T_{23}),(T_{15},T_{23}),(T_{17},T_{21}),(T_3,T_{18}),(T_3,T_{19}),$$
$$(T_5,T_{13}),(T_5,T_{24}),(T_{13},T_{24}),(T_{18},T_{19}),(T_1,T_{12}),(T_1,T_{20}),(T_6,T_7),$$
$$(T_6,T_{22}),(T_7,T_{22}),(T_{12},T_{20}),(T_2,T_{10}),(T_2,T_{14}),(T_4,T_8),(T_4,T_{16}),$$
$$(T_8,T_{16}),(T_{10},T_{14}),(T_{10},T_{18}),(T_{10},T_{22}),(T_{12},T_{16}),(T_{12},T_{24}),$$
$$(T_{16},T_{24}),(T_{18},T_{22}),(T_4,T_{17}),(T_4,T_{20}),(T_6,T_{14}),(T_6,T_{23}),(T_{14},T_{23}),$$
$$(T_{17},T_{20}),(T_2,T_{11}),(T_2,T_{19}),(T_5,T_8),(T_5,T_{21}),(T_8,T_{21}),(T_{11},T_{19}),$$
$$(T_1,T_9),(T_1,T_{13}),(T_3,T_7),(T_3,T_{15}),(T_7,T_{15}),(T_9,T_{13})$$

□

Corollary 1.3. *In any Latin square of order four there exist exactly 13824 pairs of ordinary transversals that intersect exactly in one cell.*

Proof. By fixing one of two transversals, we can fill its cells in 4! ways. Then the cells of the second transversal (which intersects the first in exactly one cell) can be filled in

3! ways. Therefore, there are a total of $96 \cdot 4! \cdot 3! = 13824$ pairs of regular transversals, which have only one cell in common. □

Let $(Q, \cdot)$ be a quasigroup. The recursive derivative of order k of the quasigroup $(Q, \cdot)$, denoted by $(\overset{k}{\cdot})$, $k \in N$, is defined as follows:

$$x \overset{0}{\cdot} y = x \cdot y;\ x \overset{1}{\cdot} y = y \cdot (x \cdot y);$$
$$x \overset{k}{\cdot} y = (x \overset{k-2}{\cdot} y) \cdot (x \overset{k-1}{\cdot} y), for\ all\ k \geq 2.$$

A binary quasigroup $(Q, \cdot)$ is recursively r-differentiable ($r \geq 0$) if each of its recursive derivatives $(\overset{k}{\cdot})$, where $k = 0, ..., r$, is a quasigroup operation.

Known estimations of the order of recursive differentiability of finite quasigroups are given in [4-7]. In particular, it is known that:

1. The maximum order r of recursive differentiability of a finite binary quasigroup of order q satisfies the inequality $r \leq q - 2$;
2. There exist finite binary quasigroups that are recursively $(q - 2)$ – differentiable for any primary order $q \geq 3$.

2. Methods

The method of construction of quasigroups prolongations by adding two new elements and by using two transversals that intersect exactly in one cell and the recursive differentiability of such prolongations for some particular transversals in Latin squares of order 5 were announced [8].

We examine the recursive derivability in a general case of such prolongations and study the recursive 1-differentiability of Latin squares of order 5 with a fixed transversal on the main diagonal bellow.

Let $(Q, \cdot)$ be a finite quasigroup of order q and let the Cayley table of the quasigroup $(Q, \cdot)$ contain 2 transversals T_1 and T_2, which have a single common cell, located at the intersection of the row of element x with the column of element y. We denote the element in the common cell of transversals T_1 and T_2 by a. We now consider the set $Q' = Q \cup \{\xi_1, \xi_2\}$, where $\xi_1, \xi_2 \notin Q$. On the set Q' we define the operation $\circ$ as follows: we complete the cells, different from the common cell, of one transversal with ξ_1, and of the other with ξ_2. We transfer the elements in the cells of the two transversals, except for the element in the common cell, preserving their order, into the cells of the row and, respectively, of the column of elements ξ_1 and ξ_2. The remaining cells in the row of element x, the column of element y, and at the intersection of the rows and columns of elements ξ_1 and ξ_2 can be filled in several possible ways, but following this method, there are exactly 12 possibilities for extending the quasigroup Q by adjointing two new elements and using 2 transversals that intersect exactly in a cell. These possibilities are illustrated in the diagrams below:

$$I.\quad \begin{array}{c|ccccc|cc} \circ & \dots & \dots & y_0 & \dots & \dots & \xi_1 & \xi_2 \\ \hline \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ x_0 & \dots & \dots & \xi_1 & \dots & \dots & a & \xi_2 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \hline \xi_1 & \dots & \dots & a & \dots & \dots & \xi_2 & \xi_1 \\ \xi_2 & \dots & \dots & \xi_2 & \dots & \dots & \xi_1 & a \end{array} \qquad II.\quad \begin{array}{c|ccccc|cc} \circ & \dots & \dots & y_0 & \dots & \dots & \xi_1 & \xi_2 \\ \hline \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ x_0 & \dots & \dots & \xi_1 & \dots & \dots & a & \xi_2 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \hline \xi_1 & \dots & \dots & \xi_2 & \dots & \dots & \xi_1 & a \\ \xi_2 & \dots & \dots & a & \dots & \dots & \xi_2 & \xi_1 \end{array}$$

$$III.\quad \begin{array}{c|ccccc|cc} \circ & \dots & \dots & y_0 & \dots & \dots & \xi_1 & \xi_2 \\ \hline \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ x_0 & \dots & \dots & \xi_1 & \dots & \dots & \xi_2 & a \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \hline \xi_1 & \dots & \dots & \xi_2 & \dots & \dots & a & \xi_1 \\ \xi_2 & \dots & \dots & a & \dots & \dots & \xi_1 & \xi_2 \end{array} \qquad IV.\quad \begin{array}{c|ccccc|cc} \circ & \dots & \dots & y_0 & \dots & \dots & \xi_1 & \xi_2 \\ \hline \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ x_0 & \dots & \dots & \xi_1 & \dots & \dots & \xi_2 & a \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \hline \xi_1 & \dots & \dots & a & \dots & \dots & \xi_1 & \xi_2 \\ \xi_2 & \dots & \dots & \xi_2 & \dots & \dots & a & \xi_1 \end{array}$$

$$V.\quad \begin{array}{c|ccccc|cc} \circ & \dots & \dots & y_0 & \dots & \dots & \xi_1 & \xi_2 \\ \hline \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ x_0 & \dots & \dots & \xi_2 & \dots & \dots & a & \xi_1 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \hline \xi_1 & \dots & \dots & a & \dots & \dots & \xi_1 & \xi_2 \\ \xi_2 & \dots & \dots & \xi_1 & \dots & \dots & \xi_2 & a \end{array} \qquad VI.\quad \begin{array}{c|ccccc|cc} \circ & \dots & \dots & y_0 & \dots & \dots & \xi_1 & \xi_2 \\ \hline \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ x_0 & \dots & \dots & \xi_2 & \dots & \dots & a & \xi_1 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \hline \xi_1 & \dots & \dots & \xi_1 & \dots & \dots & \xi_2 & a \\ \xi_2 & \dots & \dots & a & \dots & \dots & \xi_1 & \xi_2 \end{array}$$

$$VII.\quad \begin{array}{c|ccccc|cc} \circ & \dots & \dots & y_0 & \dots & \dots & \xi_1 & \xi_2 \\ \hline \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ x_0 & \dots & \dots & \xi_2 & \dots & \dots & \xi_1 & a \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \hline \xi_1 & \dots & \dots & \xi_1 & \dots & \dots & a & \xi_2 \\ \xi_2 & \dots & \dots & a & \dots & \dots & \xi_2 & \xi_1 \end{array} \qquad VIII.\quad \begin{array}{c|ccccc|cc} \circ & \dots & \dots & y_0 & \dots & \dots & \xi_1 & \xi_2 \\ \hline \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ x_0 & \dots & \dots & \xi_2 & \dots & \dots & \xi_1 & a \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \hline \xi_1 & \dots & \dots & a & \dots & \dots & \xi_2 & \xi_1 \\ \xi_2 & \dots & \dots & \xi_1 & \dots & \dots & a & \xi_2 \end{array}$$

$$IX.\quad \begin{array}{c|ccccc|cc} \circ & \dots & \dots & y_0 & \dots & \dots & \xi_1 & \xi_2 \\ \hline \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ x_0 & \dots & \dots & a & \dots & \dots & \xi_1 & \xi_2 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \hline \xi_1 & \dots & \dots & \xi_1 & \dots & \dots & \xi_2 & a \\ \xi_2 & \dots & \dots & \xi_2 & \dots & \dots & a & \xi_1 \end{array} \qquad X.\quad \begin{array}{c|ccccc|cc} \circ & \dots & \dots & y_0 & \dots & \dots & \xi_1 & \xi_2 \\ \hline \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ x_0 & \dots & \dots & a & \dots & \dots & \xi_1 & \xi_2 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \hline \xi_1 & \dots & \dots & \xi_2 & \dots & \dots & a & \xi_1 \\ \xi_2 & \dots & \dots & \xi_1 & \dots & \dots & \xi_2 & a \end{array}$$

$$XI.\quad \begin{array}{c|ccccc|cc} \circ & \dots & \dots & y_0 & \dots & \dots & \xi_1 & \xi_2 \\ \hline \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ x_0 & \dots & \dots & a & \dots & \dots & \xi_2 & \xi_1 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \hline \xi_1 & \dots & \dots & \xi_2 & \dots & \dots & \xi_1 & a \\ \xi_2 & \dots & \dots & \xi_1 & \dots & \dots & a & \xi_2 \end{array} \qquad XII.\quad \begin{array}{c|ccccc|cc} \circ & \dots & \dots & y_0 & \dots & \dots & \xi_1 & \xi_2 \\ \hline \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ x_0 & \dots & \dots & a & \dots & \dots & \xi_2 & \xi_1 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \hline \xi_1 & \dots & \dots & \xi_1 & \dots & \dots & a & \xi_2 \\ \xi_2 & \dots & \dots & \xi_2 & \dots & \dots & \xi_1 & a \end{array}$$

As an example we consider a quasigroup of order 5 with two transversals – one marked in red and one marked in blue – that meet at position $(1,1)$. The prolongation by adding two new elements ξ_1 and ξ_2 and using the transversals $T_1 = \{(1,1),(2,3),(3,5),(4,2),(5,4)\}$ and $T_2 = \{(1,1),(2,2),(3,3),(4,4),(5,5)\}$ is given in the second Cayley table bellow:

$$\begin{array}{c|ccccc} \cdot & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & 2 & 4 & 1 & 3 & 5 \\ 2 & 1 & 3 & 5 & 2 & 4 \\ 3 & 5 & 2 & 4 & 1 & 3 \\ 4 & 4 & 1 & 3 & 5 & 2 \\ 5 & 3 & 5 & 2 & 4 & 1 \end{array} \longrightarrow \begin{array}{c|ccccc|cc} \circ & 1 & 2 & 3 & 4 & 5 & \xi_1 & \xi_2 \\ \hline 1 & \xi_1 & 4 & 1 & 3 & 5 & 2 & \xi_2 \\ 2 & 1 & \xi_1 & \xi_2 & 2 & 4 & 3 & 5 \\ 3 & 5 & 2 & \xi_1 & 1 & \xi_2 & 4 & 3 \\ 4 & 4 & \xi_2 & 3 & \xi_1 & 2 & 5 & 1 \\ 5 & 3 & 5 & 2 & \xi_2 & \xi_1 & 1 & 4 \\ \hline \xi_1 & \xi_2 & 1 & 5 & 4 & 3 & \xi_1 & 2 \\ \xi_2 & 2 & 3 & 4 & 5 & 1 & \xi_2 & \xi_1 \end{array}$$

Proposition 2.1. *Six extensions out of the 12 possible (Cases I, IV, VI, VII, X, XI), by the adjunction of two new elements and using two transversals that intersect exactly in a cell, are not recursively 1-differentiable.*

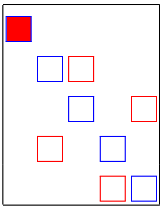
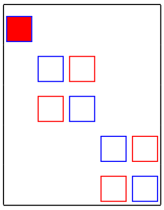
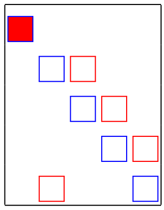
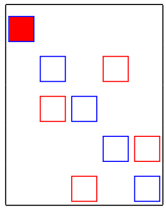
Proof. In the mentioned cases the recursive derivative $\overset{1}{\circ}$ is not a quasigroup operation as it is shown bellow:

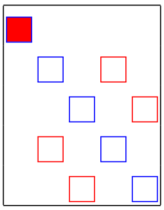
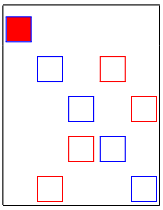
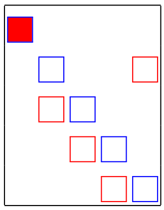
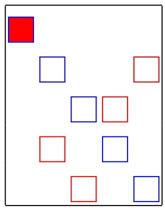
I. $\xi_1 \overset{1}{\circ} \xi_1 = \xi_1 \circ (\xi_1 \circ \xi_1) = \xi_1 \circ \xi_2 = \xi_1$, $\xi_1 \overset{1}{\circ} \xi_2 = \xi_2 \circ (\xi_1 \circ \xi_2) = \xi_2 \circ \xi_1 = \xi_1$.

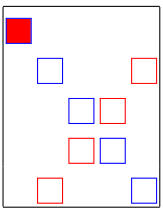
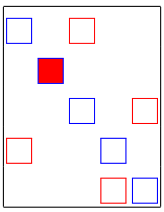
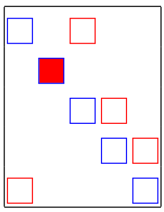
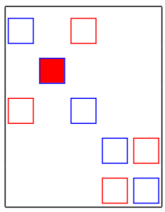
IV. $\xi_1 \overset{1}{\circ} \xi_1 = \xi_1 \circ (\xi_1 \circ \xi_1) = \xi_1 \circ \xi_1 = \xi_1$, $\xi_1 \overset{1}{\circ} \xi_2 = \xi_2 \circ (\xi_1 \circ \xi_2) = \xi_2 \circ \xi_2 = \xi_1$.
VI. $\xi_2 \overset{1}{\circ} \xi_1 = \xi_1 \circ (\xi_2 \circ \xi_1) = \xi_1 \circ \xi_1 = \xi_2$, $\xi_2 \overset{1}{\circ} \xi_2 = \xi_2 \circ (\xi_2 \circ \xi_2) = \xi_2 \circ \xi_2 = \xi_2$.
VII. $\xi_2 \overset{1}{\circ} \xi_1 = \xi_1 \circ (\xi_2 \circ \xi_1) = \xi_1 \circ \xi_2 = \xi_2$, $\xi_2 \overset{1}{\circ} \xi_2 = \xi_2 \circ (\xi_2 \circ \xi_2) = \xi_2 \circ \xi_1 = \xi_2$.
X. $x_0 \overset{1}{\circ} \xi_1 = \xi_1 \circ (x_0 \circ \xi_1) = \xi_1 \circ \xi_1 = a$, $x_0 \overset{1}{\circ} \xi_2 = \xi_2 \circ (x_0 \circ \xi_2) = \xi_2 \circ \xi_2 = a$.
XI. $x_0 \overset{1}{\circ} \xi_1 = \xi_1 \circ (x_0 \circ \xi_1) = \xi_1 \circ \xi_2 = a$, $x_0 \overset{1}{\circ} \xi_2 = \xi_2 \circ (x_0 \circ \xi_2) = \xi_2 \circ \xi_1 = a$. □

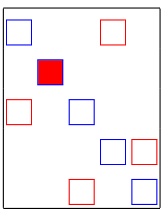
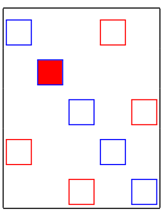
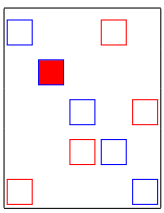
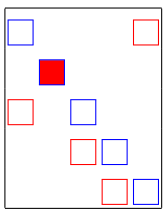
Proposition 2.2. *In a Latin square of order 5, the total number of pairs of free transversals, one of which is on the main diagonal that intersect exactly in one cell, is 45.*

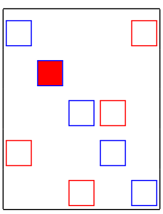
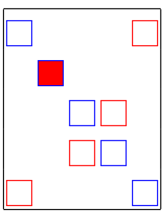
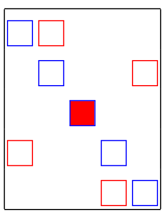
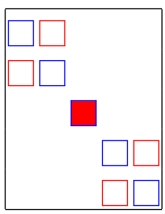
Proof. The 45 diagrams illustrating Latin squares with two free transversals, one of which is on the main diagonal, that intersect exactly in one cell are the following:

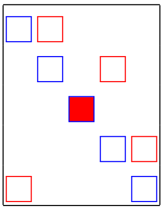
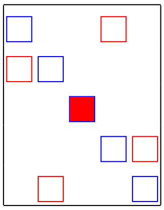
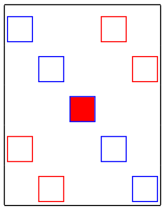
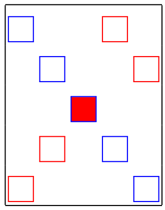
1.  2.  3.  4. 

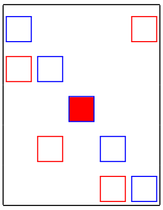
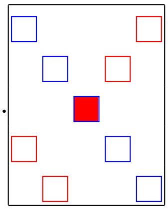
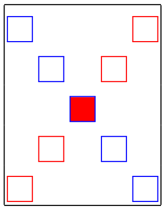
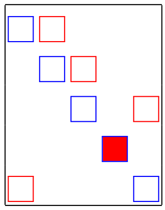
5.  6.  7.  8. 

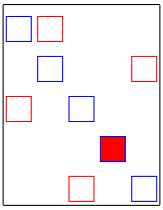
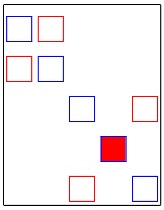
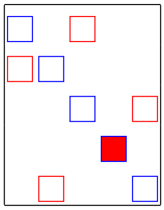
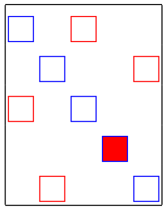
9.  10.  11.  12. 

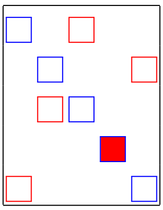
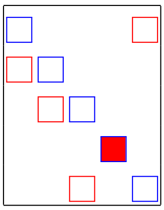
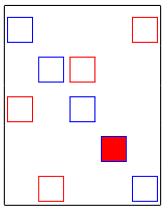
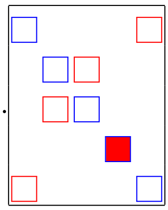
13.  14.  15.  16. 

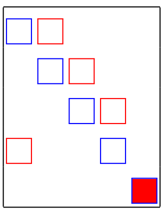
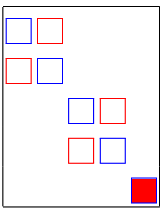
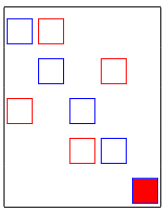
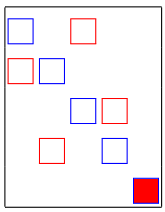
17.  18.  19.  20. 

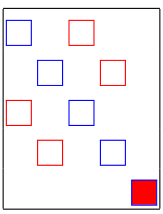
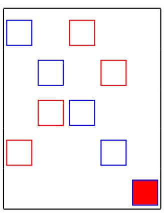
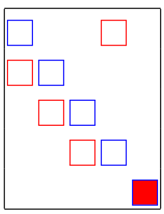
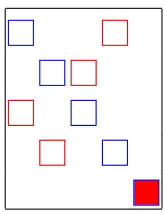
21. 
22. 
23. 
24. 

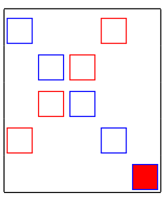
25. 
26. 
27. 
28. 

29. 
30. 
31. 
32. 

33. 
34. 
35. 
36. 

37. 
38. 
39. 
40. 

41. 
42. 
43. 
44. 

45. 

□

Proposition 2.3. *There exist exactly 48 different Latin squares of order 5 with two transversals intersecting in one cell, where one transversal is on the main diagonal with the fixed order of elements.*

Proof. The 48 Latin squares of order 5 with two transversals, one of which is on the main diagonal with the fixed order of elements $\{1,2,3,4,5\}$, that intersect exactly in one cell are:

$$L_1 = \begin{array}{|ccccc|}\hline 1&3&5&2&4\\ 3&2&4&5&1\\ 5&4&3&1&2\\ 2&5&1&4&3\\ 4&1&2&3&5\\ \hline\end{array}\; L_2 = \begin{array}{|ccccc|}\hline 1&3&5&2&4\\ 5&2&4&1&3\\ 4&1&3&5&2\\ 3&5&2&4&1\\ 2&4&1&3&5\\ \hline\end{array}\; L_3 = \begin{array}{|ccccc|}\hline 1&5&4&3&2\\ 4&2&5&1&3\\ 2&1&3&5&4\\ 5&3&2&4&1\\ 3&4&1&2&5\\ \hline\end{array}\; L_4 = \begin{array}{|ccccc|}\hline 1&4&2&5&3\\ 4&2&5&3&1\\ 2&5&3&1&4\\ 5&3&1&4&2\\ 3&1&4&2&5\\ \hline\end{array}$$

$$L_5 = \begin{array}{|ccccc|}\hline 1&4&5&2&3\\ 3&2&4&5&1\\ 2&5&3&1&4\\ 5&3&1&4&2\\ 4&1&2&3&5\\ \hline\end{array}\; L_6 = \begin{array}{|ccccc|}\hline 1&3&2&5&4\\ 5&2&4&1&3\\ 4&5&3&2&1\\ 3&1&5&4&2\\ 2&4&1&3&5\\ \hline\end{array}\; L_7 = \begin{array}{|ccccc|}\hline 1&4&5&3&2\\ 3&2&4&5&1\\ 2&5&3&1&4\\ 5&1&2&4&3\\ 4&3&1&2&5\\ \hline\end{array}\; L_8 = \begin{array}{|ccccc|}\hline 1&3&2&5&4\\ 5&2&4&3&1\\ 4&5&3&1&2\\ 2&1&5&4&3\\ 3&4&1&2&5\\ \hline\end{array}$$

$$L_9 = \begin{array}{|ccccc|}\hline 1&5&4&2&3\\ 3&2&5&1&4\\ 2&4&3&5&1\\ 5&3&1&4&2\\ 4&1&2&3&5\\ \hline\end{array}\; L_{10} = \begin{array}{|ccccc|}\hline 1&3&2&5&4\\ 4&2&5&1&3\\ 5&4&3&2&1\\ 3&5&1&4&2\\ 2&1&4&3&5\\ \hline\end{array}\; L_{11} = \begin{array}{|ccccc|}\hline 1&5&4&3&2\\ 3&2&5&1&4\\ 2&4&3&5&1\\ 5&1&2&4&3\\ 4&3&1&2&5\\ \hline\end{array}\; L_{12} = \begin{array}{|ccccc|}\hline 1&3&2&5&4\\ 4&2&5&3&1\\ 5&4&3&1&2\\ 2&5&1&4&3\\ 3&1&4&2&5\\ \hline\end{array}$$

$$L_{13} = \begin{array}{|ccccc|}\hline 1&4&5&2&3\\ 5&2&4&3&1\\ 2&1&3&5&4\\ 3&5&1&4&2\\ 4&3&2&1&5\\ \hline\end{array}\; L_{14} = \begin{array}{|ccccc|}\hline 1&5&2&3&4\\ 5&2&4&1&3\\ 2&4&3&5&1\\ 3&1&5&4&2\\ 4&3&1&2&5\\ \hline\end{array}\; L_{15} = \begin{array}{|ccccc|}\hline 1&3&4&5&2\\ 4&2&5&3&1\\ 5&1&3&2&4\\ 2&5&1&4&3\\ 3&4&2&1&5\\ \hline\end{array}\; L_{16} = \begin{array}{|ccccc|}\hline 1&3&4&5&2\\ 3&2&5&1&4\\ 4&5&3&2&1\\ 5&1&2&4&3\\ 2&4&1&3&5\\ \hline\end{array}$$

$$L_{17} = \begin{array}{|ccccc|}\hline 1&4&2&5&3\\ 5&2&1&3&4\\ 4&5&3&2&1\\ 3&1&5&4&2\\ 2&3&4&1&5\\ \hline\end{array}\; L_{18} = \begin{array}{|ccccc|}\hline 1&5&4&3&2\\ 3&2&1&5&4\\ 5&4&3&2&1\\ 2&1&5&4&3\\ 4&3&2&1&5\\ \hline\end{array}\; L_{19} = \begin{array}{|ccccc|}\hline 1&4&5&2&3\\ 5&2&1&3&4\\ 4&1&3&5&2\\ 3&5&2&4&1\\ 2&3&4&1&5\\ \hline\end{array}\; L_{20} = \begin{array}{|ccccc|}\hline 1&4&5&2&3\\ 3&2&1&5&4\\ 4&5&3&1&2\\ 5&3&2&4&1\\ 2&1&4&3&5\\ \hline\end{array}$$

$$L_{21} = \begin{array}{|ccccc|}\hline 1&5&2&3&4\\ 4&2&1&5&3\\ 5&4&3&1&2\\ 2&3&5&4&1\\ 3&1&4&2&5\\ \hline\end{array}\; L_{22} = \begin{array}{|ccccc|}\hline 1&5&4&3&2\\ 4&2&1&5&3\\ 5&1&3&2&4\\ 2&3&5&4&1\\ 3&4&2&1&5\\ \hline\end{array}\; L_{23} = \begin{array}{|ccccc|}\hline 1&5&4&2&3\\ 5&2&1&3&4\\ 4&1&3&5&2\\ 2&3&5&4&1\\ 3&4&2&1&5\\ \hline\end{array}\; L_{24} = \begin{array}{|ccccc|}\hline 1&4&5&3&2\\ 4&2&1&5&3\\ 5&1&3&2&4\\ 3&5&2&4&1\\ 2&3&4&1&5\\ \hline\end{array}$$

$$L_{25} = \begin{array}{|ccccc|}\hline 1&3&4&5&2\\ 4&2&5&1&3\\ 5&1&3&2&4\\ 3&5&2&4&1\\ 2&4&1&3&5\\ \hline\end{array} \quad L_{26} = \begin{array}{|ccccc|}\hline 1&5&4&2&3\\ 3&2&1&5&4\\ 5&4&3&1&2\\ 2&3&5&4&1\\ 4&1&2&3&5\\ \hline\end{array} \quad L_{27} = \begin{array}{|ccccc|}\hline 1&4&5&3&2\\ 3&2&1&5&4\\ 4&5&3&2&1\\ 5&1&2&4&3\\ 2&3&4&1&5\\ \hline\end{array} \quad L_{28} = \begin{array}{|ccccc|}\hline 1&3&5&2&4\\ 5&2&4&3&1\\ 4&1&3&5&2\\ 2&5&1&4&3\\ 3&4&2&1&5\\ \hline\end{array}$$

$$L_{29} = \begin{array}{|ccccc|}\hline 1&3&2&5&4\\ 5&2&4&3&1\\ 4&5&3&1&2\\ 2&1&5&4&3\\ 3&4&1&2&5\\ \hline\end{array} \quad L_{30} = \begin{array}{|ccccc|}\hline 1&4&2&5&3\\ 5&2&4&3&1\\ 2&5&3&1&4\\ 3&1&5&4&2\\ 4&3&1&2&5\\ \hline\end{array} \quad L_{31} = \begin{array}{|ccccc|}\hline 1&4&5&3&2\\ 5&2&4&1&3\\ 2&1&3&5&4\\ 3&5&2&4&1\\ 4&3&1&2&5\\ \hline\end{array} \quad L_{32} = \begin{array}{|ccccc|}\hline 1&4&2&5&3\\ 5&2&1&3&4\\ 4&5&3&1&2\\ 2&3&5&4&1\\ 3&1&4&2&5\\ \hline\end{array}$$

$$L_{33} = \begin{array}{|ccccc|}\hline 1&5&4&2&3\\ 4&2&5&3&1\\ 2&1&3&5&4\\ 5&3&1&4&2\\ 3&4&2&1&5\\ \hline\end{array} \quad L_{34} = \begin{array}{|ccccc|}\hline 1&5&2&3&4\\ 4&2&1&5&3\\ 5&4&3&2&1\\ 3&1&5&4&2\\ 2&3&4&1&5\\ \hline\end{array} \quad L_{35} = \begin{array}{|ccccc|}\hline 1&3&5&2&4\\ 3&2&4&5&1\\ 4&5&3&1&2\\ 5&1&2&4&3\\ 2&4&1&3&5\\ \hline\end{array} \quad L_{36} = \begin{array}{|ccccc|}\hline 1&3&4&5&2\\ 3&2&5&1&4\\ 5&4&3&2&1\\ 2&5&1&4&3\\ 4&1&2&3&5\\ \hline\end{array}$$

$$L_{37} = \begin{array}{|ccccc|}\hline 1&3&4&5&2\\ 5&2&1&3&4\\ 4&5&3&2&1\\ 2&1&5&4&3\\ 3&4&2&1&5\\ \hline\end{array} \quad L_{38} = \begin{array}{|ccccc|}\hline 1&3&5&2&4\\ 5&2&4&1&3\\ 2&4&3&5&1\\ 3&5&1&4&2\\ 4&1&2&3&5\\ \hline\end{array} \quad L_{39} = \begin{array}{|ccccc|}\hline 1&5&4&2&3\\ 3&2&5&1&4\\ 4&1&3&5&2\\ 5&3&2&4&1\\ 2&4&1&3&5\\ \hline\end{array} \quad L_{40} = \begin{array}{|ccccc|}\hline 1&5&2&3&4\\ 3&2&4&5&1\\ 5&4&3&1&2\\ 2&1&5&4&3\\ 4&3&1&2&5\\ \hline\end{array}$$

$$L_{41} = \begin{array}{|ccccc|}\hline 1&4&2&5&3\\ 4&2&5&3&1\\ 5&1&3&2&4\\ 3&5&1&4&2\\ 2&3&4&1&5\\ \hline\end{array} \quad L_{42} = \begin{array}{|ccccc|}\hline 1&4&5&3&2\\ 4&2&1&5&3\\ 2&5&3&1&4\\ 5&3&2&4&1\\ 3&1&4&2&5\\ \hline\end{array} \quad L_{43} = \begin{array}{|ccccc|}\hline 1&3&4&5&2\\ 4&2&5&3&1\\ 2&5&3&1&4\\ 5&1&2&4&3\\ 3&4&1&2&5\\ \hline\end{array} \quad L_{44} = \begin{array}{|ccccc|}\hline 1&3&5&2&4\\ 4&2&1&5&3\\ 5&4&3&1&2\\ 3&5&2&4&1\\ 2&1&4&3&5\\ \hline\end{array}$$

$$L_{45} = \begin{array}{|ccccc|}\hline 1&4&5&3&2\\ 3&2&4&5&1\\ 5&1&3&2&4\\ 2&5&1&4&3\\ 4&3&2&1&5\\ \hline\end{array} \quad L_{46} = \begin{array}{|ccccc|}\hline 1&4&2&5&3\\ 3&2&5&1&4\\ 4&5&3&2&1\\ 5&3&1&4&2\\ 2&1&4&3&5\\ \hline\end{array} \quad L_{47} = \begin{array}{|ccccc|}\hline 1&5&2&3&4\\ 5&2&4&1&3\\ 4&1&3&5&2\\ 2&3&5&4&1\\ 3&4&1&2&5\\ \hline\end{array} \quad L_{48} = \begin{array}{|ccccc|}\hline 1&5&4&2&3\\ 5&2&1&3&4\\ 2&4&3&5&1\\ 3&1&5&4&2\\ 4&3&2&1&5\\ \hline\end{array}$$

The corresponding 240 second transversals for the transversal given on the main diagonal of the Latin squares $L_1 - L_{48}$, are the following:

L_1: $T_1 = \{(1,1),(2,3),(3,5),(4,2),(5,4)\}$, $T_2 = \{(1,1),(2,4),(3,2),(4,5),(5,3)\}$,
$T_3 = \{(1,3),(2,2),(3,4),(4,5),(5,1)\}$, $T_4 = \{(1,5),(2,2),(3,1),(4,3),(5,4)\}$,
$T_5 = \{(1,4),(2,5),(3,3),(4,2),(5,1)\}$, $T_6 = \{(1,5),(2,4),(3,3),(4,1),(5,2)\}$,
$T_7 = \{(1,2),(2,5),(3,1),(4,4),(5,3)\}$, $T_8 = \{(1,3),(2,1),(3,5),(4,4),(5,2)\}$,
$T_9 = \{(1,2),(2,3),(3,4),(4,1),(5,5)\}$, $T_{10} = \{(1,4),(2,1),(3,2),(4,3),(5,5)\}$.

L_2: $T_{11} = \{(1,1),(2,3),(3,5),(4,2),(5,4)\}$, $T_{12} = \{(1,1),(2,5),(3,4),(4,3),(5,2)\}$,
$T_{13} = \{(1,3),(2,2),(3,1),(4,5),(5,4)\}$, $T_{14} = \{(1,5),(2,2),(3,4),(4,1),(5,3)\}$,
$T_{15} = \{(1,4),(2,1),(3,3),(4,5),(5,2)\}$, $T_{16} = \{(1,5),(2,4),(3,3),(4,2),(5,1)\}$,
$T_{17} = \{(1,2),(2,1),(3,5),(4,4),(5,3)\}$, $T_{18} = \{(1,3),(2,5),(3,2),(4,4),(5,1)\}$,
$T_{19} = \{(1,2),(2,4),(3,1),(4,3),(5,5)\}$, $T_{20} = \{(1,4),(2,3),(3,2),(4,1),(5,5)\}$.

L_3: $T_{21} = \{(1,1),(2,3),(3,5),(4,2),(5,4)\}$, $T_{22} = \{(1,1),(2,5),(3,4),(4,3),(5,2)\}$,
$T_{23} = \{(1,3),(2,2),(3,4),(4,5),(5,1)\}$, $T_{24} = \{(1,4),(2,2),(3,5),(4,1),(5,3)\}$,
$T_{25} = \{(1,2),(2,1),(3,3),(4,5),(5,4)\}$, $T_{26} = \{(1,5),(2,4),(3,3),(4,1),(5,2)\}$,
$T_{27} = \{(1,2),(2,5),(3,1),(4,4),(5,3)\}$, $T_{28} = \{(1,5),(2,3),(3,2),(4,4),(5,1)\}$,
$T_{29} = \{(1,3),(2,4),(3,1),(4,2),(5,5)\}$, $T_{30} = \{(1,4),(2,1),(3,2),(4,3),(5,5)\}$.

L_4: $T_{31} = \{(1,1),(2,3),(3,5),(4,2),(5,4)\}$, $T_{32} = \{(1,1),(2,4),(3,2),(4,5),(5,3)\}$,
$T_{33} = \{(1,4),(2,2),(3,5),(4,3),(5,1)\}$, $T_{34} = \{(1,5),(2,2),(3,4),(4,1),(5,3)\}$,
$T_{35} = \{(1,2),(2,5),(3,3),(4,1),(5,4)\}$, $T_{36} = \{(1,4),(2,2),(3,3),(4,5),(5,2)\}$,
$T_{37} = \{(1,3),(2,5),(3,2),(4,4),(5,1)\}$, $T_{38} = \{(1,5),(2,3),(3,1),(4,4),(5,2)\}$,
$T_{39} = \{(1,2),(2,4),(3,1),(4,3),(5,5)\}$, $T_{40} = \{(1,3),(2,1),(3,4),(4,2),(5,5)\}$.

L_5: $T_{41} = \{(1,1),(2,3),(3,2),(4,5),(5,4)\}$, $T_{42} = \{(1,1),(2,4),(3,5),(4,2),(5,3)\}$,

L_6: $T_{43} = \{(1,1),(2,3),(3,2),(4,5),(5,4)\}$, $T_{44} = \{(1,1),(2,5),(3,4),(4,3),(5,2)\}$,

L_7: $T_{45} = \{(1,1),(2,3),(3,2),(4,5),(5,4)\}$, $T_{46} = \{(1,1),(2,4),(3,5),(4,3),(5,2)\}$,
$T_{47} = \{(1,3),(2,2),(3,4),(4,5),(5,1)\}$, $T_{48} = \{(1,4),(2,2),(3,5),(4,1),(5,3)\}$,
$T_{49} = \{(1,2),(2,5),(3,3),(4,1),(5,4)\}$, $T_{50} = \{(1,5),(2,4),(3,3),(4,2),(5,1)\}$,
$T_{51} = \{(1,3),(2,5),(3,1),(4,4),(5,2)\}$, $T_{52} = \{(1,5),(2,1),(3,2),(4,4),(5,3)\}$,
$T_{53} = \{(1,2),(2,1),(3,4),(4,3),(5,5)\}$, $T_{54} = \{(1,4),(2,3),(3,1),(4,2),(5,5)\}$.

L_8: $T_{55} = \{(1,1),(2,3),(3,2),(4,5),(5,4)\}$, $T_{56} = \{(1,1),(2,4),(3,5),(4,3),(5,2)\}$,
$T_{57} = \{(1,4),(2,2),(3,1),(4,5),(5,3)\}$, $T_{58} = \{(1,5),(2,2),(3,4),(4,3),(5,1)\}$,
$T_{59} = \{(1,4),(2,5),(3,3),(4,1),(5,2)\}$, $T_{60} = \{(1,5),(2,1),(3,3),(4,2),(5,4)\}$,
$T_{61} = \{(1,2),(2,1),(3,5),(4,4),(5,3)\}$, $T_{62} = \{(1,3),(2,5),(3,2),(4,4),(5,1)\}$,
$T_{63} = \{(1,2),(2,3),(3,4),(4,1),(5,5)\}$, $T_{64} = \{(1,3),(2,4),(3,1),(4,2),(5,5)\}$.

L_9: $T_{65} = \{(1,1),(2,3),(3,2),(4,5),(5,4)\}$, $T_{66} = \{(1,1),(2,5),(3,4),(4,2),(5,3)\}$,
$T_{67} = \{(1,3),(2,2),(3,5),(4,1),(5,4)\}$, $T_{68} = \{(1,5),(2,2),(3,4),(4,3),(5,1)\}$,
$T_{69} = \{(1,2),(2,4),(3,3),(4,5),(5,1)\}$, $T_{70} = \{(1,4),(2,5),(3,3),(4,1),(5,2)\}$,
$T_{71} = \{(1,2),(2,1),(3,5),(4,4),(5,3)\}$, $T_{72} = \{(1,5),(2,3),(3,1),(4,4),(5,2)\}$,
$T_{73} = \{(1,3),(2,4),(3,1),(4,2),(5,5)\}$, $T_{74} = \{(1,4),(2,1),(3,2),(4,3),(5,5)\}$.

L_{10}: $T_{75} = \{(1,1),(2,3),(3,2),(4,5),(5,4)\}$, $T_{76} = \{(1,1),(2,5),(3,4),(4,2),(5,3)\}$,
$T_{77} = \{(1,4),(2,2),(3,5),(4,1),(5,3)\}$, $T_{78} = \{(1,5),(2,2),(3,1),(4,3),(5,4)\}$,
$T_{79} = \{(1,2),(2,4),(3,3),(4,5),(5,1)\}$, $T_{80} = \{(1,5),(2,4),(3,3),(4,2),(5,1)\}$,
$T_{81} = \{(1,2),(2,3),(3,5),(4,4),(5,1)\}$, $T_{82} = \{(1,3),(2,5),(3,1),(4,4),(5,2)\}$,
$T_{83} = \{(1,2),(2,1),(3,4),(4,3),(5,5)\}$, $T_{84} = \{(1,3),(2,4),(3,2),(4,1),(5,5)\}$.

L_{11}: $T_{85} = \{(1,1),(2,3),(3,2),(4,5),(5,4)\}$, $T_{86} = \{(1,1),(2,5),(3,4),(4,3),(5,2)\}$,

L_{12}: $T_{87} = \{(1,1),(2,3),(3,2),(4,5),(5,4)\}$, $T_{88} = \{(1,1),(2,4),(3,5),(4,2),(5,3)\}$,

L_{13}: $T_{89} = \{(1,1),(2,3),(3,4),(4,5),(5,2)\}$, $T_{90} = \{(1,1),(2,4),(3,5),(4,2),(5,3)\}$,
$T_{91} = \{(1,3),(2,2),(3,5),(4,1),(5,4)\}$, $T_{92} = \{(1,5),(2,2),(3,4),(4,3),(5,1)\}$,
$T_{93} = \{(1,2),(2,1),(3,3),(4,5),(5,4)\}$, $T_{94} = \{(1,4),(2,5),(3,3),(4,2),(5,1)\}$,
$T_{95} = \{(1,3),(2,5),(3,1),(4,4),(5,2)\}$, $T_{96} = \{(1,5),(2,1),(3,2),(4,4),(5,3)\}$,
$T_{97} = \{(1,2),(2,4),(3,1),(4,3),(5,5)\}$, $T_{98} = \{(1,4),(2,3),(3,2),(4,1),(5,5)\}$.

L_{14}: $T_{99} = \{(1,1),(2,3),(3,4),(4,5),(5,2)\}$, $T_{100} = \{(1,1),(2,5),(3,2),(4,3),(5,4)\}$,
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$T_{107} = \{(1,3),(2,4),(3,2),(4,1),(5,5)\}$, $T_{108} = \{(1,4),(2,3),(3,1),(4,2),(5,5)\}$.

L_{15}: $T_{109} = \{(1,1),(2,3),(3,4),(4,5),(5,2)\}$, $T_{110} = \{(1,1),(2,4),(3,5),(4,2),(5,3)\}$,
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L_{16}: $T_{119} = \{(1,1),(2,3),(3,4),(4,5),(5,2)\}$, $T_{120} = \{(1,1),(2,5),(3,2),(4,3),(5,4)\}$,
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$T_{127} = \{(1,2),(2,4),(3,1),(4,3),(5,5)\}$, $T_{128} = \{(1,3),(2,1),(3,4),(4,2),(5,5)\}$.

L_{17}: $T_{129} = \{(1,1),(2,4),(3,2),(4,5),(5,3)\}$, $T_{130} = \{(1,1),(2,5),(3,4),(4,3),(5,2)\}$,
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$T_{133} = \{(1,2),(2,1),(3,3),(4,5),(5,4)\}$, $T_{134} = \{(1,4),(2,5),(3,3),(4,2),(5,1)\}$,
$T_{135} = \{(1,3),(2,1),(3,5),(4,4),(5,2)\}$, $T_{136} = \{(1,5),(2,3),(3,2),(4,4),(5,1)\}$,
$T_{137} = \{(1,2),(2,3),(3,4),(4,1),(5,5)\}$, $T_{138} = \{(1,3),(2,4),(3,1),(4,2),(5,5)\}$.

L_{18}: $T_{139} = \{(1,1),(2,4),(3,2),(4,5),(5,3)\}$, $T_{140} = \{(1,1),(2,5),(3,4),(4,3),(5,2)\}$,
$T_{141} = \{(1,3),(2,2),(3,1),(4,5),(5,4)\}$, $T_{142} = \{(1,4),(2,2),(3,5),(4,3),(5,1)\}$,
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$T_{145} = \{(1,2),(2,1),(3,5),(4,4),(5,3)\}$, $T_{146} = \{(1,5),(2,3),(3,1),(4,4),(5,2)\}$,
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L_{19}: $T_{149} = \{(1,1),(2,4),(3,5),(4,2),(5,3)\}$, $T_{150} = \{(1,1),(2,5),(3,4),(4,3),(5,2)\}$,

L_{20}: $T_{151} = \{(1,1),(2,4),(3,2),(4,5),(5,3)\}$, $T_{152} = \{(1,1),(2,5),(3,2),(4,3),(5,4)\}$,
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$T_{155} = \{(1,2),(2,4),(3,3),(4,5),(5,1)\}$, $T_{156} = \{(1,4),(2,5),(3,3),(4,1),(5,2)\}$,
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$T_{159} = \{(1,2),(2,1),(3,4),(4,3),(5,5)\}$, $T_{160} = \{(1,4),(2,3),(3,1),(4,2),(5,5)\}$.

L_{21}: $T_{161} = \{(1,1),(2,4),(3,5),(4,2),(5,3)\}$, $T_{162} = \{(1,1),(2,5),(3,2),(4,3),(5,4)\}$,
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L_{22}: $T_{171} = \{(1,1),(2,4),(3,5),(4,2),(5,3)\}$, $T_{172} = \{(1,1),(2,5),(3,4),(4,3),(5,2)\}$,

L_{23}: $T_{173} = \{(1,1),(2,4),(3,5),(4,3),(5,2)\}$, $T_{174} = \{(1,1),(2,5),(3,4),(4,2),(5,3)\}$,
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$T_{179} = \{(1,2),(2,3),(3,5),(4,4),(5,1)\}$, $T_{180} = \{(1,5),(2,1),(3,2),(4,4),(5,3)\}$,
$T_{181} = \{(1,3),(2,4),(3,2),(4,1),(5,5)\}$, $T_{182} = \{(1,4),(2,3),(3,1),(4,2),(5,5)\}$.

L_{24}: $T_{183} = \{(1,1),(2,4),(3,5),(4,3),(5,2)\}$, $T_{184} = \{(1,1),(2,5),(3,4),(4,2),(5,3)\}$,
$T_{185} = \{(1,3),(2,2),(3,5),(4,1),(5,4)\}$, $T_{186} = \{(1,4),(2,2),(3,1),(4,5),(5,3)\}$,
$T_{187} = \{(1,2),(2,4),(3,3),(4,5),(5,1)\}$, $T_{188} = \{(1,5),(2,1),(3,3),(4,2),(5,4)\}$,
$T_{189} = \{(1,3),(2,5),(3,2),(4,4),(5,1)\}$, $T_{190} = \{(1,5),(2,3),(3,1),(4,4),(5,2)\}$,
$T_{191} = \{(1,2),(2,3),(3,4),(4,1),(5,5)\}$, $T_{192} = \{(1,4),(2,1),(3,2),(4,3),(5,5)\}$.

L_{25}: $T_{193} = \{(1,3),(2,2),(3,1),(4,5),(5,4)\}$, $T_{194} = \{(1,4),(2,2),(3,5),(4,1),(5,3)\}$,

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L_{28}: $T_{199} = \{(1,3),(2,2),(3,1),(4,5),(5,4)\}$, $T_{200} = \{(1,5),(2,2),(3,4),(4,3),(5,1)\}$,

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L_{30}: $T_{203} = \{(1,4),(2,2),(3,5),(4,1),(5,3)\}$, $T_{204} = \{(1,5),(2,2),(3,4),(4,3),(5,1)\}$,

L_{31}: $T_{205} = \{(1,2),(2,1),(3,3),(4,5),(5,4)\}$, $T_{206} = \{(1,5),(2,4),(3,3),(4,2),(5,1)\}$,

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L_{33}: $T_{209} = \{(1,2),(2,1),(3,3),(4,5),(5,4)\}$, $T_{210} = \{(1,4),(2,5),(3,3),(4,1),(5,2)\}$,

L_{34}: $T_{211} = \{(1,2),(2,1),(3,3),(4,5),(5,4)\}$, $T_{212} = \{(1,5),(2,4),(3,3),(4,2),(5,1)\}$,

L_{35}: $T_{213} = \{(1,4),(2,5),(3,3),(4,1),(5,2)\}$, $T_{214} = \{(1,5),(2,4),(3,3),(4,2),(5,1)\}$,

L_{36}: $T_{215} = \{(1,4),(2,5),(3,3),(4,1),(5,2)\}$, $T_{216} = \{(1,5),(2,4),(3,3),(4,2),(5,1)\}$,

L_{37}: $T_{217} = \{(1,2),(2,1),(3,5),(4,4),(5,3)\}$, $T_{218} = \{(1,5),(2,3),(3,2),(4,4),(5,1)\}$,

L_{38}: $T_{219} = \{(1,2),(2,1),(3,5),(4,4),(5,3)\}$, $T_{220} = \{(1,3),(2,5),(3,1),(4,4),(5,2)\}$,

L_{39}: $T_{221} = \{(1,2),(2,1),(3,5),(4,4),(5,3)\}$, $T_{222} = \{(1,5),(2,3),(3,2),(4,4),(5,1)\}$,

L_{40}: $T_{223} = \{(1,2),(2,1),(3,5),(4,4),(5,3)\}$, $T_{224} = \{(1,3),(2,5),(3,1),(4,4),(5,2)\}$,

L_{41}: $T_{225} = \{(1,3),(2,5),(3,1),(4,4),(5,2)\}$, $T_{226} = \{(1,5),(2,3),(3,2),(4,4),(5,1)\}$,

L_{42}: $T_{227} = \{(1,3),(2,5),(3,1),(4,4),(5,2)\}$, $T_{228} = \{(1,5),(2,3),(3,2),(4,4),(5,1)\}$,

L_{43}: $T_{229} = \{(1,2),(2,1),(3,4),(4,3),(5,5)\}$, $T_{230} = \{(1,3),(2,4),(3,1),(4,2),(5,5)\}$,

L_{44}: $T_{231} = \{(1,2),(2,1),(3,4),(4,3),(5,5)\}$, $T_{232} = \{(1,4),(2,3),(3,2),(4,1),(5,5)\}$,

L_{45}: $T_{233} = \{(1,2),(2,1),(3,4),(4,3),(5,5)\}$, $T_{234} = \{(1,4),(2,3),(3,2),(4,1),(5,5)\}$,

L_{46}: $T_{235} = \{(1,2),(2,1),(3,4),(4,3),(5,5)\}$, $T_{236} = \{(1,3),(2,4),(3,1),(4,2),(5,5)\}$,

L_{47}: $T_{237} = \{(1,3),(2,4),(3,1),(4,2),(5,5)\}$, $T_{238} = \{(1,4),(2,3),(3,2),(4,1),(5,5)\}$,

L_{48}: $T_{239} = \{(1,3),(2,4),(3,1),(4,2),(5,5)\}$, $T_{240} = \{(1,4),(2,3),(3,2),(4,1),(5,5)\}$. $\square$

Proposition 2.4. *There does not exist recursively 1-differentiable prolongations of Latin squares of order 5, obtained by adding two new elements and two transversals, one of which is on the main diagonal with the fixed order of elements* $\{1, 2, 3, 4, 5\}$ *or* $\{2, 3, 4, 5, 1\}$*, that intersect exactly in one cell.*

Proof. The proof is obtained by direct calculation. For Latin squares with each of the 240 pairs of transversals, 6 types of prolongations according to extension schemes II, III, V, VIII, IX, XII were considered. □

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On some Noetherian criteria for singular integral operators with bounded coefficients

Vasile Neagu

Abstract. In this paper, Noetherian conditions are established for singular integral operators with measurable and bounded coefficients. The study is carried out in spaces of integrable functions with certain weights, and the integration contour is assumed to be piecewise smooth. The established Noetherian conditions depend on the space in which the research is carried out, on the coefficients of the operators, as well as on the sizes of the angles formed by the integration contour at its corner points. In the case of the Lyapunov-type contour, the presented Noetherian conditions are necessary and sufficient in which the operators studied are Noetherian.
2020 Mathematics Subject Classification: 45E05.
Keywords: singular operator, factorization of functions, essential norm, Noetherian conditions, weighted spaces.

Asupra unor criterii noetheriene pentru operatorii integrali singulari cu coeficienți mărginiți

Rezumat. În această lucrare sunt stabilite condiții noetheriene pentru operatorii integrali singulari cu coeficienți măsurabili și mărginiți. Studiul este realizat în spații de funcții integrabile cu anumite ponderi, conturul de integrare se presupune neted pe porțiuni. Condițiile noetheriene stabilite, depind de spațiu în care se desfășoară cercetarea, de coeficienții operatorilor, precum și de mărimile unghiurilor formate de conturul de integrare în punctele sale unghiulare. În cazul conturului de tip Lyapunov condițiile noetheriene prezentate sunt necesare și suficiente în care operatorii studiați sunt Noetherieni.
Cuvinte-cheie: operator singular, factorizarea funcțiilor, norma esențială, condiții noetheriene, spații cu ponderi.

1. Introduction

The previous theory of one-dimensional singular integral equations with continuous coefficients was systematically developed by N. Muskhelishvili, F. Gakhov, I. Vekua and others. Based on these classical developments, numerous studies were carried out in the case of singular integral equations with discontinuous coefficients. In particular,

I. Simonenko, I. Gohberg, and N. Krupnik established various local theories for singular integral equations, which reduce global problems to local ones. In an attempt to use these local theories, K. Clancey and I. Gohberg [1] investigated the concept of local factorization of functions. They showed that the existence of a global factorization is equivalent to the existence of a local factorization at each point of the integration contour, and the Noetherian conditions of singular integral operators are determined by the existence of factorization of the coefficients of the respective operators.

In this paper, the results of research in the theory of operators with local properties [2-4] are applied to obtain some sufficient conditions under which singular operators with scalar or matrix coefficients are Noetherian. These conditions are established for singular integral operators acting in the spaces $L_p(\Gamma, \rho)$, where Γ is a piecewise Lyapunov contour and ρ is the weight introduced by the mathematician B. Khvedelidze [5].

It is shown that the Noetherian conditions depend on the coefficients of the operator, on the space $L_p(\Gamma, \rho)$, but also on the sizes of the angles formed by the contour at its corner points. Thus, the Noetherian conditions at different points of the contour are different, and especially, at the self-intersection points of the contour. In the case, where the integration contour Γ does not contain corner points, the results presented in this paper also become necessary in which the operators under study are Noetherian.

2. Invertibility and local factorization

Let γ be a rectifiable arc in the complex plane $\mathbb{C}$. The notation $L_p(\gamma)$ $(1 \le p \le \infty)$ will be used for the usual Lebesgue spaces with respect to arc length measure on γ. In case γ is a piecewise Lyapunov arc and $1 < p < \infty$, we define the singular integral operator S on $L_p(\gamma)$ by

$$(S_\gamma \varphi)(t) = \frac{1}{\pi i} \int_\gamma \frac{\varphi(\tau)}{\tau - t}\, d\tau.$$

The fact that S_γ defines a bounded operator on $L_p(\gamma)$ $(1 < p < \infty)$, even if the arc γ has cusps, is established in [5], [6]. When the arc γ is a piecewise Lyapunov contour Γ, one defines the usual analytic projectors

$$(P_\gamma \varphi)(t) = \frac{\varphi(t)}{2} + \frac{1}{2\pi i} \int_\gamma \frac{\varphi(\tau)}{\tau - t}\, d\tau, \qquad (Q_\gamma \varphi)(t) = \varphi(t) - (P_\gamma \varphi)(t),$$

bounded in $L_p(\Gamma)$ $(1 < p < \infty)$. The range of the projector P_γ on $L_p(\Gamma)$ is denoted by $L_p^+(\Gamma)$, while $L_p^-(\Gamma)$ denotes the span of the range of Q_γ on $L_p(\Gamma)$ and the constant functions $(1 < p < \infty)$. Further, $L_\infty^\pm(\Gamma) = L_\infty(\Gamma) \cap L_2^\pm(\Gamma)$. Let a be a function in $L_\infty(\Gamma)$. We denote by T_a the singular integral operator $T_a = aP_\gamma + Q_\gamma$ acting on $L_p(\Gamma)$

$(1 < p < \infty)$. A general reference to the theory of operators of the form T_a is the monograph [7].

Let t_0 be a fixed point on the Lyapunov contour Γ and let φ belong to $L_p(\Gamma)$. In this section we will discuss the equivalence of various interpretations of the phrase *"the operator T_a on $L_p(\Gamma)$ is locally invertible at t_0"*.

Definition 2.1. *Let $a \in L_\infty(\Gamma)$, where Γ is a Lyapunov contour. The operator $T_a = aP+Q$ in $L_p(\Gamma)$ is called left-Fredholm, right-Fredholm or Fredholm operator at a point $t_0 \in \Gamma$ if there is a function $b \in L_\infty(\Gamma)$ and a neighborhood U of the point t_0 such that $a(t) = b(t)$ for $t \in U$, and the operator $T_b = bP + Q$ is left-Fredholm, respectively right-Fredholm or Fredholm.*

The next definition of *"local invertibility"* is due to Simonenko [2], see also the references in [3].

Definition 2.2. *We say that the operator $T_a = aP + Q$ in $L_p(\Gamma)$ admits a local right (left) regulator at a point $t_0 \in \Gamma$ if there exist an operator R, a compact operator K in $L_p(\Gamma)$, and a function u continuous on Γ and equal to one in a neighborhood of t_0, such that*

$$RT_aM_u = M_u + K; \qquad M_uT_aR = M_u + K,$$

where M_u denotes the operator in $L_p(\Gamma)$ of multiplication by the function u.

We say that the operator T_a admits a local regulator at t_0 if it admits both a right and a left local regularizer at t_0.

Let Γ be a Lyapunov contour bounding the domain G in the complex plane. It will be assumed that the point zero belongs to G. Let a be an element of $GL_\infty(\Gamma)$; here $GL_\infty(\Gamma)$ denotes the group of invertible elements in $L_\infty(\Gamma)$. The following is the standard definition of factorization of the function a relative to L_p.

Definition 2.3. *We say that a function a admits a factorization in the space L_p $(1 < p < \infty)$ if it can be represented in the form $a(t) = a_-(t)\, t^\kappa\, a_+(t),\ \ t \in \Gamma$, where κ is an integer and the following conditions hold:*

1) *the pair (a_+^{-1}, a_+) belongs to $L_p^+(\Gamma_+) \times L_q^+(\Gamma_+)$ and the pair (a_-, a^{-1}) belongs to $L_p^-(\Gamma) \times L_q^-(\Gamma)$ $(p^-1 + q^-1 = 1)$;*

2) *the operator $a_+^{-1}Sa_-^{-1}I$ is bounded in $L_p(\Gamma)$.*

The following definition of local factorization is similar to the previous definition and is contained in [2] (see also [1]).

Definition 2.4. *A function a admits a local factorization in L_p at a point $t_0 \in \Gamma$ if there exists a neighborhood γ of the point t_0 such that $a(t) = a_-(t)a_+(t)$, where*
(i) *the pair of functions(a_+^{-1}, a_+) belongs to $L_p^+(\Gamma_+) \times L_q^+(\Gamma_+)$ and (a_-, a^{-1}) belongs to $L_p^+(\Gamma_-) \times L_q^+(\Gamma_-)$; here $\Gamma_\pm \subset G_+ \cup \Gamma$ are piecewise Lyapunov contours such that $\Gamma_\pm \cap \Gamma = \gamma$.*
(ii) *the operator $a_+^{-1} S a_-^{-1} I$ is bounded in $L_p(\gamma)$.*

In [1] it was established that a function a which admits a local factorization in L_p at every point $t_0 \in \Gamma$ also admits a (global) factorization in L_p. In the present work, alongside the norms and essential norms of the operator S_Γ, we will use the equivalence between the notions of local factorization of a function in $L_p(\Gamma)$ and local invertibility of the singular integral operator $T_a = aP + Q$. The main result that will be used is the following:

Theorem 2.1. *Let Γ be a Lyapunov contour. Assume $a \in GL_\infty(\Gamma)$, and let $t_0 \in \Gamma$.*

1) *Assume that $1 < p < \infty$. If a function a admits a local factorizationin L_p at a point t_0, then the operator $T_a = aP + Q$ in $L_p(\Gamma)$ has a local regulator at t_0.*

2) *If the operator $T_a = aP + Q$ in L_2 is locally invertible at t_0, then a admits a local factorization in L_p at t_0.*

3. Noetherian conditions for the operator $A = aP_\Gamma + bQ_\Gamma$ with scalar coefficients

Let a and b be two measurable and bounded functions on the contour Γ. Here we will establish sufficient conditions under which the operator

$$A = aP_\Gamma + bQ_\Gamma \tag{3.1}$$

is Noetherian in the space $L_2(\Gamma, \rho)$. If the contour Γ is of Lyapunov type and $\rho \equiv 1$, these conditions coincide with the conditions established by I. Simonenko [2] and, in addition, are also necessary under which the operator (3.1) is Noetherian in the space $L_2(\Gamma, \rho)$.

Henceforth, we will assume that the contour Γ satisfies the following conditions:

a) Γ has a finite number of self-intersection points $\tau_1, \tau_2, \ldots, \tau_m$ and a finite number of angular points τ_k $(k = r+1, \ldots, m)$;
b) at the points τ_k $(k = r+1, \ldots, m)$, the tangents to Γ are perpendicular;
c) the contour Γ can be completed up to a closed composite contour $\widetilde{\Gamma}$ of Lyapunov type by pieces, which divides the complex plane into two sets $F^+_{\widetilde{\Gamma}}$ and $F^-_{\widetilde{\Gamma}}$ and $\widetilde{\Gamma}$ is the boundary for each of these sets $F^\pm_{\widetilde{\Gamma}}$. Obviously, the contour $\widetilde{\Gamma}$ is determined unambiguously.

We denote by $\pi\alpha_0(t)$ $(0 < \alpha_0(t) < 2)$ the angle formed by the contour Γ at the point $\tau \neq \tau_k$ $(k = r+1, \dots, m)$. We will also need the functions $\alpha(t) = \min\{\alpha_0(t),\ 2-\alpha_0(t)\}$ $(0 < \alpha(t) < 1)$ and $\theta(t)$ $(t \in \Gamma \setminus \{\tau_1, \tau_2, \dots, \tau_m\})$ defined by the relation

$$ctg\theta(t) = \frac{1}{2}\max_{-1\le z\le 1}\left|(1+z)\left(\frac{1-z}{1+z}\right)^{\frac{\alpha(t)}{2}} - (1-z)\left(\frac{1+z}{1-z}\right)^{\frac{\alpha(t)}{2}}\right|. \tag{3.2}$$

We note that the set of values of the function $\theta(t)$ is contained on the interval $]\frac{\pi}{4}, \frac{\pi}{2}]$ and $\theta(t) = \frac{\pi}{2}$ at all points t of the contour Γ where the Lyapunov conditions are verified. In particular, if the contour Γ is simple and of Lyapunov type, then the set of values of the function $\theta(t)$ is contained on the interval $]\frac{\pi}{4}, \frac{\pi}{2}]$ and $\theta(t) = \frac{\pi}{2}$.

Let τ_0 be an arbitrary point of the contour Γ different from the points $\tau_1, \tau_2, \dots \tau_r$. We denote by $\Lambda(\tau_0)$ an angle with the vertex at the point $z = 0$ of size $2\theta(\tau_0)$, where $\theta(t)$ is defined in (3.2). By $\Lambda(\tau_k)$ $(k = 1, ..., r)$ we will denote an angle with the vertex at the point $z = 0$ of size $\theta = \frac{\pi}{2}$.

Let $\widetilde{\Gamma}(\supset \Gamma)$ be a completion of the contour Γ up to a closed composite contour, which satisfies condition c). If $a \in \Gamma_\infty(\Gamma)$, then we denote by $\widetilde{a}$ the function

$$\widetilde{a}(\tau) = \begin{cases} a(t), & \text{for} \quad t \in \Gamma, \\ 1, & \text{for} \quad t \in \widetilde{\Gamma} \setminus \Gamma. \end{cases}$$

Let us also denote by $M(\Gamma)$ the set of measurable and bounded functions on Γ that satisfy the conditions:

(i) $essinf[a(t)] > 0$;
(ii) for any point $\tau \in \widetilde{\Gamma}$ there exists a neighborhood $u(\tau) \subset \widetilde{\Gamma}$ of the point τ and two functions g_τ^+, g_τ^- which belong respectively to the sets[1] $L_\infty^+(\widetilde{\Gamma})$, $L_\infty^-(\widetilde{\Gamma})$ together with $(g_t^+)^{-1}$ and $(g_t^-)^{-1}$, such that for $t \in u(\tau)$ the set of values of the function $g_\tau^+(t)\, a(t)\, g_\tau^-(t)$ is contained in the angle $\Lambda(\tau_0)$.

It can be easily shown that the set $M(\Gamma)$ does not depend on the choice of the composite contour $\widetilde{\Gamma}$, if Γ satisfies conditions a) – c).

Theorem 3.1. *Let $a \in M(\Gamma)$. Then the operator $A = aP_\Gamma + Q_\Gamma$ is Noetherian in the space $L_2(\Gamma)$.*

To facilitate the proof of the stated theorem, we will need the following three auxiliary propositions. Let Γ_0 be a simple closed Lyapunov contour on portions with a single corner point t_0.

[1]By $L_\infty^+(\widetilde{\Gamma})$ $(\subset L_\infty\widetilde{\Gamma})$ we denote the set of bounded analytic functions in the domain $F_{\widetilde{\Gamma}}^+(F_{\widetilde{\Gamma}}^-)$.

Lemma 3.1. *If the real and measurable function $\varphi(t)$ $(t \in \Gamma_0)$ satisfies the condition*

$$\operatorname*{ess\,sup}_{t\in\Gamma_0} |\varphi(t)| < \theta(\tau_0),$$

then the operator $A_\varphi = e^{i\varphi} P_{\Gamma_0} + Q_{\Gamma_0}$ is Noetherian in the space $L_2(\Gamma_0)$.

Proof. Based on the results from [8], [9] we have

$$\inf_{T\in\mathfrak{T}} \|S_{\Gamma_0} + T\|_{L_2(\Gamma_0)} = ctg\frac{\theta(\tau_0)}{2},$$

where $\mathfrak{T}(= \mathfrak{T}(L_2(\Gamma_0)))$ is the set of compact operators in the space $L_2(\Gamma_0)$. The operator A_0 can be written in the form

$$A_0 = \frac{e^{i\varphi}+1}{2}\left(I + i\,tg\frac{\varphi}{2}\, S_{\Gamma_0}\right).$$

As

$$\inf_{T\in\mathfrak{T}} \left\| i\,tg\frac{\varphi}{2}\, S_{\Gamma_0} + T\right\|_{L_2(\Gamma_0)} \le \sup_{t\in\Gamma_0}\left| tg\frac{\varphi(t)}{2}\right| ctg\frac{\theta(\tau_0)}{2} < 1,$$

it follows that the operator A_0 is Noetherian (invertible) in the space $L_2(\Gamma_0)$. Lemma 3.1 is proved. □

Lemma 3.2. *Let $a \in L_\infty(\Gamma_0)$. If $\operatorname{ess\,inf}_{t\in\Gamma_0} |a(t)| > 0$ and its value set belongs to the angle $\Lambda(\tau_0)$, then the operator $B_a = aP_{\Gamma_0} + Q_{\Gamma_0}$ is Noetherian in the space $L_2(\Gamma_0)$.*

Proof. The function a can be represented in the form $a(t) = \gamma|a|\cdot e^{i\varphi}$, where $\varphi(t)$ satisfies the conditions of Lemma 3.1 and γ is a complex number ($|\gamma| = 1$). The function $b(t) = |a(t)|$ admits factorization (see [7]), $b = b_+ \cdot b_-$, where $b_+^{\pm 1} \in L_\infty^+(\Gamma_0)$ and $b_-^{\pm 1} \in L_\infty^-(\Gamma_0)$ and $b_+^{\pm 1} \in L_\infty^-(\Gamma_0)$. Then the operator B_0 can be represented in the form

$$B_0 = b_-\left(e^{i\varphi} P_{\Gamma_0} + Q_{\Gamma_0}\right)(\gamma b_+ P_{\Gamma_0} + b_-^{-1} Q_{\Gamma_0}). \tag{3.3}$$

By Lemma 3.1, the operator $e^{i\varphi} P_{\Gamma_0} + Q_{\Gamma_0}$ is Noetherian in the space $L_2(\Gamma_0)$. As the operators $b_- I$ and $\gamma b_+ P_{\Gamma_0} + b_-^{-1} Q_{\Gamma_0}$ are invertible in $L_2(\Gamma_0)$, from (3.3) it follows that B_0 is Noetherian in $L_2(\Gamma_0)$. The lemma is proved. □

Let $\widetilde{\Gamma}_0$ be a closed Lyapunov contour on portions with a single self-intersection point τ_0 and the tangents to $\widetilde{\Gamma}_0$ in τ_0 are perpendicular. As in the proof of Lemma 3.2, taking into account that in this case (see [8], [9])

$$\inf_{T\in\mathfrak{T}} \|S_{\Gamma_0} + T\|_{L_2(\Gamma_0)} = 1 + \sqrt{2},$$

the following lemma is also proved.

Lemma 3.3. *Let $a \in L_\infty(\tilde{\Gamma}_0)$. If $\operatorname{ess\,inf}_{t\in\Gamma_0} |a(t)| > 0$ and the set of values of the function a is in the angle with the vertex at the point $z = 0$ and of size $\frac{\pi}{2}$, then the operator $B_0 = aP_{\Gamma_0} + Q_{\Gamma_0}$ is Noetherian in the space $L_2(\tilde{\Gamma}_0)$.*

Proof. We give the proof of the Theorem 3.1. Let $\tau_{r+1}, \ldots, \tau_m$ denote all the angular points of the contour Γ. Let τ be an arbitrary point in $\widetilde{\Gamma}$, $g_\tau^\pm \in L_\infty^\pm(\tilde{\Gamma})$ and $u(\tau)$ be a neighborhood of the point τ, such that the set of values of the function $g_\tau^+(t)\widetilde{a}(t)g_\tau^-(t)$ lies in $\Lambda(\tau)$ for $t \in u(\tau)$. We can assume that $u(\tau)$ does not contain points in τ_k $(k = 1, 2, \ldots, m)$ different from τ. Let

$$\tilde{a}_\tau(t) = \begin{cases} \widetilde{a}(t), & \text{for } t \in u(\tau), \\ \delta\left(g_\tau^+(t)g_\tau^-(t)\right)^{-1}, & \text{for } t \in \widetilde{\Gamma} \setminus u(\tau), \end{cases}$$

where δ is a complex number in $\Lambda(\tau)$. The set of values of the function $\widetilde{b}_\tau(t) = g_\tau^+(t)\tilde{a}_\tau(t)g_\tau^-(t)$ lies in the angle $\Lambda(\tau)$. We will show that the operator $B_\tau = \widetilde{b}_\tau P_{\widetilde{\Gamma}} + Q_{\widetilde{\Gamma}}$ is Noetherian in the space $L_2(\Gamma)$. Indeed, let $\widetilde{\Gamma}_\tau(\supset u(\tau))$ be a piecewise Lyapunov contour that has a single (angular) self-intersection point for $\tau = \tau_k$ $(k = 1, 2, \ldots, r)$ (for $\tau = \tau_k$ $(k = r + 1, \ldots, m)$) and that has no such points for τ other than the points $\tau = \tau_k$ $(k = 1, 2, \ldots, m)$. We denote by $c_\tau(t)$ the function defined by the equality

$$c_\tau(t) = \begin{cases} \widetilde{b}_\tau(t), & \text{for } t \in u(\tau), \\ 1, & \text{for } t \in \widetilde{\Gamma}_\tau \setminus u(\tau). \end{cases}$$

As $\widetilde{b}_\tau(t) = 1$ for $t \in \widetilde{\Gamma} \setminus u(\tau)$, then the set of values of the function $c_\tau(t)$ $(t \in \widetilde{\Gamma}_\tau)$ also lies in the angle $\Lambda(\tau)$. From Theorem 1.1 Chapter VII of the work [7], we deduce that the operator $B_\tau = \widetilde{b}_\tau P_{\widetilde{\Gamma}} + Q_{\widetilde{\Gamma}}$ is Noetherian in the space $L_2(\widetilde{\Gamma})$ if and only if the operator $C_\tau = c_\tau P_{\widetilde{\Gamma}_\tau} + Q_{\widetilde{\Gamma}_\tau}$ is Noetherian in the space $L_2(\widetilde{\Gamma}_\tau)$. If $\tau = \tau_k$ $(k = r + 1, \ldots, m)$ or $\tau = \tau_k$ $(k = 1, \ldots, r)$, then, by virtue of Lemmas 3.2 and 3.3, respectively, it follows that the operator C_τ is Noetherian in the space $L_2(\widetilde{\Gamma}_\tau)$. However, if $\tau \in \widetilde{\Gamma} \setminus \{\tau_1, \ldots, \tau_m\}$, then from Theorem 5 of [10], it follows that C_τ is Noetherian in $L_2(\widetilde{\Gamma}_\tau)$.

Therefore, the operator $B_\tau = \widetilde{b}_\tau P_{\widetilde{\Gamma}} + Q_{\widetilde{\Gamma}}$ is Noetherian for any $\tau \in \widetilde{\Gamma}$. We consider the operator $\widetilde{A}_\tau = \tilde{a}_\tau P_{\widetilde{\Gamma}} + Q_{\widetilde{\Gamma}}$ and write it in the form

$$\widetilde{A}_\tau = (g_\tau^-)^{-1}\delta\big(\widetilde{b}_\tau P_{\widetilde{\Gamma}} + Q_{\widetilde{\Gamma}}\big)\big((g_\tau^+)^{-1}P_{\widetilde{\Gamma}} + \delta g_\tau^- Q_{\widetilde{\Gamma}}\big) \tag{3.4}$$

The operators $(g_\tau^-)^{-1}\delta I$ and $(g_\tau^-)^{-1}P_{\widetilde{\Gamma}} + \delta^{-1}g_\tau^- Q_{\widetilde{\Gamma}}$ are invertible in the space $L_2(\widetilde{\Gamma})$, and $B_\tau = \widetilde{b}_\tau P_{\widetilde{\Gamma}} + Q_{\widetilde{\Gamma}}$ is Noetherian in $L_2(\widetilde{\Gamma})$, so from (3.4) it follows that $\widetilde{A}_\tau$ is Noetherian.

Thus, the operator $\widetilde{A} = \tilde{a}P_{\widetilde{\Gamma}} + Q_{\widetilde{\Gamma}}$ at any point $\tau \in \widetilde{\Gamma}$ is equivalent to the Noetherian operator $\widetilde{A}_\tau$. Based on Theorem 1.6 of [2], it follows that $\widetilde{A}$ is Noetherian in $L_2(\widetilde{\Gamma})$.

Then from Theorem 1.1 of Chapter VII of the paper [7], it follows that the operator $A = aP_\Gamma + Q_\Gamma$ is Noetherian in $L_2(\Gamma)$. The theorem is proved. $\square$

Let

$$\rho(t) = \prod_{k=1}^{n} |t - t_k|^{\beta_k} \quad (-1 < \beta_k < p - 1,\ t_k = \tau_k,\ k = 1, \ldots, m,\ n \geq m)$$

and the function h defined in the paper [10]. From Theorem 3.1 and the results of the paper [11] it follows

Corollary 3.1. *Let the functions $a \in L_\infty(\Gamma)$. If $ah \in M(\Gamma)$, then the operator $A = aP_\Gamma + Q_\Gamma$ is Noetherian in the space $L_2(\Gamma, p)$.*

Corollary 3.2. *Let $a, b \in L_\infty(\Gamma)$ and $\operatorname{ess\,inf}_{t\in\Gamma} |b(t)| > 0$. If $ab^{-1}h \in M(\Gamma)$, then the operator $A = aP_\Gamma + bQ_\Gamma$ is Noetherian in the space $L_2(\Gamma, p)$.*

We note that the set $M(\Gamma)$ is quite large. Thus, if the contour Γ is of a simple closed Lyapunov type, then the class of functions $M(\Gamma)$ coincides with the class $\widetilde{A}(2, \Gamma)$, defined by I. Simonenko in the paper [12] and, in addition, the operator $aP_\Gamma + bQ_\Gamma$ is Noetherian in $L_2(\Gamma, p)$ if and only if $ab^{-1}h \in M(\Gamma)$.

Let the contour Γ be composite, closed and satisfy the conditions of Theorem 2.1. We denote by $\widetilde{M}(\Gamma) \subset L_\infty(\Gamma)$ the set of functions satisfying the conditions:

(i) $\operatorname{ess\,inf}_{t\in\Gamma} |a(t)| > 0$;

(ii) for any $\tau \in \Gamma \setminus \{\tau_1, \ldots, \tau_m\}$ there exists a neighbourhood $u(\tau) \subset \Gamma$ of the point τ and two functions g_τ^+, g_τ^- which belong respectively to the sets $L_\infty^+(\Gamma)$, $L_\infty^-(\Gamma)$ together with $(g_\tau^+)^{-1}$ and $(g_\tau^-)^{-1}$ such that for any $t \in u(\tau)$ the set of values of the function $g_\tau^+(t)a(t)g_\tau^-(t)$ lies in an angle with the vertex at $z = 0$ of measure π;

(iii) at the points τ_k $(k = 1, \ldots, r)$ there exists $a(\tau_k \pm 0)$, where the quantities $a(\tau_k \pm 0)$ are defined in the works [7], [8].

From Theorem 3.1 and the results of paper [11] it follows

Theorem 3.2. *For the operator $A = aP_\Gamma + bQ_\Gamma$ $(a, b \in L_\infty(\Gamma))$ to be Noetherian in the space $L_2(\Gamma)$ it is necessary and sufficient that the function ab^{-1} belongs to the set $\widetilde{M}(\Gamma)$.*

From Theorem 3.2 and Corollary 3.2, it also follows

Corollary 3.3. *The operator $A = aP_\tau + bQ_\tau$ is Noetherian in the space $L_2(\Gamma, p)$ if and only if the function $ab^{-1}h$ belongs to the set $\widetilde{M}(\Gamma)$.*

4. Noetherian conditions for systems of singular integral operators

We will study the operator $A = aP_\tau + bQ_\tau$ in the case where the coefficients of the operator are matrices of order m with elements in $L_\infty(\Gamma)$.

Let $\mathfrak{A}$ be a Banach space of functions, defined on the contour Γ. By $\mathfrak{A}^{m\times m}$ we denote the Banach algebra of matrix functions of order m with elements in $\mathfrak{A}$.

Definition 4.1. *Let the contour Γ be closed and $a \in GL^{l\times l}(\Gamma)$. We will say that the matrix a admits a factorization [13] in the space $L_p^l(\Gamma, p)$, if it can be represented in the form*

$$a = a_-\, d\, a_+, \tag{4.1}$$

where $d = \left(\delta_{jk}(t-z_0)^{k_j}\right)_{j,k=1}^{l}$, $z_0 \in F_\Gamma^+$ and k_l are integers ($k_1 \geq k_2 \geq \cdots \geq k_j$), and the matrices $a_\pm$ satisfy the conditions:

(i) $a_- \in^- L_p^{l\times l}(\Gamma, p)$, $\quad a_+ \in^+ L_q^{l\times l}(\Gamma, p^{1-q})$ and $a_-^{-1} \in^- L_q^{l\times l}(\Gamma, p^{1-q})$, $\quad a_+^{-1} \in^+ L_p^{l\times l}(\Gamma, p)$, $(p^{-1} + q^{-1} = 1)$;

(ii) the operator $a_+^{-1} P_\Gamma a_-^{-1} I$ is bounded in the space $L_p^l(\Gamma, p)$, where by P_Γ is denoted the operator $(\delta_{jk} P_\Gamma)_{j,k=1}^{l}$.

Theorem 4.1. *The matrix $a \in GL_\infty^{l\times l}(\Gamma)$ admits a factorization in the space $L_p^l(\Gamma, p)$ if and only if the operator*

$$A = aP_\tau + Q_\tau \qquad (Q_\tau = I - P_\tau)$$

is Noetherian in the space $L_p^l(\Gamma, p)$.

Theorem 4.2. *Let the matrix $a \in GL_\infty^{l\times l}(\Gamma)$ admits a factorization in the space $L_p^l(\Gamma, p)$, where*

$$k_1 \geq k_2 \geq \cdots \geq k_s \geq 0, \quad k_{s+1} \leq k_{s+2} \leq \cdots \leq k_j < 0 \quad (0 \leq s \leq l),$$

then

$$\dim\ker A = -\sum_{i=s+1}^{l} k_i, \qquad \dim\operatorname{coker} A = \sum_{i=1}^{s} k_i.$$

The proof of these theorems is done in a similar way to the one in Theorem 3.1 of [13]. Let the function h be defined in [11]. From Theorems 3.1 and 3.2 and from the above, it follows

Corollary 4.1. *The matrix $a \in GL_\infty^{l\times l}(\Gamma)$ admits a factorization in $L_p^l(\Gamma, p)$, if and only if the matrix $b = ha$ admits a factorization in the space $L_p^l(\Gamma)$. If $b = b_+ d b_-$, then $a = h_+^{-1} b_+ d b_- h_-^{-1}$ represents the factorization of the matrix a in $L_p^l(\Gamma, p)$.*

Corollary 4.2. *The operator $aP_\Gamma + Q_\Gamma$ is Noetherian in the space $L_p^l(\Gamma, p)$, if and only if the operator $A_h = haP_\Gamma + Q_\Gamma$ is Noetherian in the space $L_p^l(\Gamma)$. Furthermore,*

$$\operatorname{Ind} A\big|_{L_p^l(\Gamma,p)} = \operatorname{Ind} A_h\big|_{L_p^l(\Gamma)}.$$

As we mentioned, if $a, b \in L_\infty(\Gamma)$ and $b(t) \geq \delta > 0$, then the operators $A = aP_\Gamma + Q_\Gamma$ and $B = baP_\Gamma + Q_\Gamma$ are or are not Noetherian in the space $L_p(\Gamma, p)$ and $\operatorname{Ind} A = \operatorname{Ind} B$.

We will show that this property is not possessed by operators of the form A and B in the space $L_p^l(\Gamma, p)$. Moreover, we will show that for any p $(1 \leq p < \infty)$ and any natural l $(l \geq 2)$ there exist two matrices $a, b \in L_\infty^{l\times l}(\Gamma)$, such that sets[2] $b(t) \geq \delta > 0$, $baP_\Gamma + Q_\Gamma$ are Noetherian in $L_p^l(\Gamma, p)$, but the operator $aP_\Gamma + Q_\Gamma$ is not Noetherian in this space.

Based on Corollary 4.2, we can consider $p(t) = 1$. Obviously, it is enough to consider $l = 2$ and $\Gamma = \Gamma_0$, where Γ_0 is the unit circle. As the matrices a and b will have elements from $CP(\Gamma)$, then we can also consider $p = 2$. So, let $t = e^{i\theta}$ $(0 \leq \theta < 2\pi)$,

$$a(t) = \begin{pmatrix} 2 - \cos\dfrac{\theta}{2} & -1 \\ -4 + \sin\dfrac{\theta}{2} & 2 - \cos\dfrac{\theta}{2} \end{pmatrix}, \qquad b(t) = \begin{pmatrix} 3 + 2\cos\dfrac{\theta}{2} & 1 \\ 1 & 2 - \cos\dfrac{\theta}{2} \end{pmatrix}.$$

The matrices a and b are continuous at any point $t \in \Gamma_0 \setminus \{1\}$. Under these conditions the operator $aP_{\Gamma_0} + Q_{\Gamma_0}$ is Noetherian (see [7]) in the space $L_2^2(\Gamma_0)$, if and only if

$$\det\big[\mu a(t+0) + (1-\mu)a(t-0)\big] \neq 0 \quad (t \in \Gamma_0,\ 0 \leq \mu \leq 1).$$

It is directly verified that for $\mu_0 = \frac{1}{2}$ and $t_0 = 1$ we have

$$\det\big[\mu_0 a(t_0+0) + (1-\mu_0)a(t_0-0)\big] = 0.$$

Therefore, the operator $A = aP_{\Gamma_0} + Q_{\Gamma_0}$ is not Noetherian in the space $L_2^2(\Gamma_0)$. Let $c = ab$.

It is easy to show that

$$\det\big[\mu c(t+0) + (1-\mu)c(t-0)\big] \neq 0 \quad (t \in \Gamma_0,\ 0 \leq \mu \leq 1),$$

i.e. the operator $B = abP_{\Gamma_0} + Q_{\Gamma_0}$ is Noetherian in $L_2^2(\Gamma_0)$.

Theorem 4.3. *Let the matrix $a \in GL_\infty^{l\times l}(\Gamma)$ satisfy the following conditions:*

(i) for any $\tau \in \Gamma \setminus \{\tau_1, \ldots, \tau_m\}$ there exists a neighborhood $u(\tau)$ and a pair of matrices g_τ^+ and g_τ^-, $(g_\tau^+(t))^{\pm 1} \in^+ L_\infty^{l\times l}(\Gamma)$, $(g_\tau^-(t))^{\pm 1} \in^- L_\infty^{l\times l}(\Gamma)$, such that

$$\operatorname{Re}{}^3\big(g_\tau^+(t)a(t)g_\tau^-(t)\big) \geq \delta_\tau > 0 \quad (t \in u(\tau));$$

[2]Recall that $H \geq M$, where H and M are Hermitian matrices, means that $\sum_{i=1}^{l}\sum_{j=1}^{l}(c_{ij} - d_{ij})(x_i - \bar{x}_j) \geq 0$, where c_{ij} and d_{ij} are the elements of the matrices H și M, respectively.

[3]Recall that $\operatorname{Re} B = \frac{1}{2}(B + B^*)$.

(ii) there exist the limits $a_j(\tau_k \pm 0)$*;*

(iii) the spectrum of matrices $a^{-1}(\tau_k + 0) \cdot a(\tau_k - 0)$ $(k = 1, \dots, m)$ *have no common points with the semiaxis* $\mathbb{R}^-$.

Then the operator $A = aP_\Gamma + Q_\Gamma$ *is Noetherian in the space* $L_2^l(\Gamma)$.

The proof can be done using the results of the paper [13] and the local principle from [2] and [10].

If the matrix $a \in GL_\infty^{l\times l}(\Gamma)$ is piecewise continuous, then the conditions of Theorem 3.3 are also necessary for the operator $A = aP_\Gamma + Q_\Gamma$ to be Noetherian in the space $L_2^l(\Gamma)$. Indeed, let A be Noetherian in the space $L_2^l(\Gamma)$. Then from [13] we deduce that the spectrum of the matrix $a^{-1}(\tau + 0) \cdot a(\tau - 0)$ $(\tau \in \Gamma)$ has no common points with the semiaxis R^-. It remains to use the following lemma.

Lemma 4.1. *Let B and C be two non-degenerate numerical matrices. For there to exist numerical matrices M and N such that*

$$\mathrm{Re}(MBN) > 0 \quad \textit{and} \quad \mathrm{Re}(MCN) > 0 \tag{4.2}$$

it is necessary and sufficient that the spectrum of the matrix $B^{-1}C$ has no points in common with R^-.

Proof. Let X be the matrix that reduces the matrix $B^{-1}C$ to the Jordan form,

$$XB^{-1}CX^{-1} = \mathrm{diag}(G_1, \dots, G_s),$$

where

$$G_k = \begin{pmatrix} \lambda_k & \cdot & \cdot & 0 \\ 1 & \cdot & \cdot & 0 \\ . & . & . & . \\ 0 & \cdot & 1 & \lambda_k \end{pmatrix},$$

and $\mathrm{diag}(G_1, \dots, G_s)$ is the diagonal matrix with the elements G_k $(k = 1, \dots, s)$ on the diagonal.

As $\lambda_k \notin R^-$, there exists a number $\gamma_k \in \mathbb{C}$, such that $\mathrm{Re}\,\gamma_k > 0$ and $\delta_k = \mathrm{Re}(\gamma_k\lambda_k) > 0$. Let

$$F_k = \begin{pmatrix} \varepsilon_k^{l_k-1}\gamma_k & 0 & 0 & \cdots & 0 \\ 0 & \varepsilon_k^{l_k-2}\gamma_k & 0 & \cdots & 0 \\ 0 & 0 & \cdot & \cdots & 0 \\ . & . & . & \cdots & . \\ 0 & 0 & 0 & \cdots & \gamma_k \end{pmatrix},$$

where l_k is the order of the matrix G_k, ε_k is a positive number. Direct calculations show that the principal minor of order r of the matrix $\operatorname{Re}(G_k F_k)$ has the form $\varepsilon_k^{\omega_k} g(\varepsilon_k)$, where $\omega_k = (2l - r - 1)\frac{r}{2}$, and $g(\varepsilon_k) \to \delta_k^r$ for $\varepsilon_k \to 0$. It follows that we can choose $\varepsilon_k > 0$, such that $\operatorname{Re}(G_k F_k) > 0$. We denote $N = X^{-1}F$, where $F = \operatorname{diag}(F_1, \dots, F_s)$, and $M = XB^{-1}$. Based on what has been demonstrated, the matrices $MBN(= F)$ and $MCN = XB^{-1}CX^{-1}F$ satisfy conditions (4.2). The sufficiency is proved.

The necessity of conditions (4.2) can be demonstrated as follows. Consider on the unit circle Γ_0 a nondegenerate matrix a of continuous functions at any point $z \neq 1$ and $a(1+0) = B$, $a(1-0) = C$. If there are two matrices M and N such that $\operatorname{Re}(MBN) > 0$ and $\operatorname{Re}(MCN) > 0$, then in the space $L_2^l(\Gamma_0)$ the operator $aP_{\Gamma_0} + Q_{\Gamma_0}$ is locally Noetherian at the point $z = 1$. Since $\det a(z) \neq 0$ for $z \neq 1$ and a is continuous at any $z \in \Gamma_0 \setminus \{1\}$, it follows that the operator $aP_{\Gamma_0} + Q_{\Gamma_0}$ is locally Noetherian also at these points of the contour Γ_0. Therefore, the operator $aP_{\Gamma_0} + Q_{\Gamma_0}$ is Noetherian in $L_2^l(\Gamma_0)$. From Theorem 2 of [13], it follows that the spectrum of the matrix $B^{-1}C$ has no points in common with R^-. The lemma is proved. □

Next, to make the exposition easier, we will assume that the contour Γ is made up of two closed curves γ_1 and γ_2 of Liapunov type on portions with a single common point t_0. At the point t_0 the tangents to γ_1 and γ_2 form an angle of measure $\frac{\pi}{2}$.

In the following we will establish a class of matrices $L^{l\times l}(\Gamma)$ (which also contains the unitary matrices) such that the operator $aP_\Gamma + Q_\Gamma$ is Noetherian in the space $L_p^l(\Gamma, \rho)$, where

$$\rho(t) = \prod_0^n |t - t_k|^{\beta_k},$$

$t_0, t_1, \dots, t_n$ are distinct points on Γ in which all the angular points of the contour Γ are included and $-1 < \beta_k < p - 1$ $(k = 0, \dots, n)$.

To begin with, we will consider $p = 2$. Let $\tau \in \gamma_1 \setminus \{t_0\}$ $(\tau \in \gamma_2 \setminus \{t_0\})$ and z_1 (z_2) be fixed points of $F_{\gamma_1}^+$ $(F_{\gamma_2}^+)$. We denote by $h_\tau(t)$ the function $(t - z_1)^{-\beta(\tau)/2}((t - z_2)^{-\beta(\tau)/2})$, where $\beta(\tau) = \beta_k$ for $\tau = t_k$ $(k = 0, \dots, n)$ and $\beta(\tau) = 0$ at the other points of the contour Γ, and β_k and t_k respectively are the numbers and points that define the weight $\rho(t)$.

For $\tau = t_0$ we define the function $h_\tau(t)$ as follows:

$$h_{t_0}(t) = (t - z_1)^{\sigma_1}(t - z_2)^{\sigma_2},$$

where σ_1 and σ_2 are real numbers satisfying the condition $\sigma_1 + \sigma_2 = -\dfrac{\beta_0}{2}$. The functions $(t - z_j)^{-\beta(\tau)/2}$ $((t - z_j)^{\sigma_j})$ $(j = 1, 2)$ are defined so that they are continuous on the complex plane with the cut that joins the point z_j with ∞ and intersects the contour Γ at a single point τ (t_0).

Theorem 4.4. *The operator*

$$A = aP_\Gamma + bQ_\Gamma \quad (a, b \in L_\infty^{l\times l}(\Gamma))$$

in the space $L_2^l(\Gamma, \rho)$ is locally Noetherian at the point $\tau \in \Gamma$ if and only if this property is possessed by the operator

$$A_\tau = h_\tau a P_\Gamma + bQ_\Gamma$$

in the space $L_2^l(\Gamma)$.

The proof is based on the factorization of the function h_τ, applying Theorem 2.2, therefore we do not dwell on it.

Consider the following function (see [8])

$$\theta(t) = \begin{cases} \operatorname{arctg}\left[\dfrac{1}{2} \max\limits_{-1\le z\le 1} \left|(1+z)\left(\dfrac{1-z}{1+z}\right)^{\delta(t)} - (1-z)\left(\dfrac{1+z}{1-z}\right)^{\delta(t)}\right|\right], & t \in \Gamma \setminus \{t_0\}, \\ \dfrac{\pi}{4}, & t = t_0, \end{cases} \tag{4.3}$$

where $\delta(t) = \dfrac{\alpha(t)}{2\pi}$, $\alpha(t) = \min\bigl(\alpha_0(t),\ 2\pi - \alpha_0(t)\bigr)$, and $\alpha_0(t)$ is the angle formed by the contour Γ at the point $t \neq t_0$.

We denote by $M_p^l(\Gamma)$ the set of function matrices $a(t) = \bigl(a_{jk}(t)\bigr)_{j,k=1}^{l}$ $(t \in \Gamma)$ of order l with elements $a_{jk}(t) \in L_\infty(\Gamma)$ that satisfy the following conditions:

(i) $\operatorname{Vraiinf}\limits_{t\in\Gamma} \det|a(t)| > 0$;

(ii) for any point $\tau \in \Gamma$ except perhaps for a finite number of points τ_k $(k = 1, 2, \ldots, s)$, there exists a neighborhood $u(\tau) \subset \Gamma$ of the point τ and two function matrices $g_\tau^\pm$, $(g_\tau^+)^{\pm 1} \in^+ L_\infty^{l\times l} (g_\tau^-)^{\pm 1} \in^- L_\infty^{l\times l}$ such that for any $t \in u(\tau)$ the relation

$$\operatorname{Re}\bigl(g_\tau^+(t) h_\tau a(t) g_\tau^-(t)\bigr) > c(\tau)\cos\theta(\tau), \tag{4.4}$$

where $\theta(\tau)$ is the function defined by relation (3.3) and $c(\tau)$ is the norm of the operator $h_\tau a$ in the space $L_2^l(u(\tau))$. As, it is known, $c(\tau) = \sup_{t\in u(\tau)} s_1\bigl(h_\tau(t)a(t)\bigr)$, where $s_1(h_\tau a)$ is the largest eigenvalue of the matrix $(h_\tau a a^* h_\tau^*)$;

(iii) at the points τ_k $(k = 1, 2, \ldots, s)$ there exist the finite limits $a(\tau_k - 0)$ and $a(\tau_k + 0)$ of the matrix $a(t)$ and the spectrum of the matrix $e^{-\pi i\beta(\tau_k)} a^{-1}(\tau_k + 0)a(\tau_k - 0)$ does not intersect the negative semi-axis $\mathbb{R}^-$.

It is easy to see that if all points τ_k $(k = 1, 2, \ldots, s)$ where the limits $a(\tau \pm 0)$ exist are regular, then conditions (iii) are equivalent to conditions (ii). This can be deduced using Lemma 3.1.

Theorem 4.5. *Let $a \in M_p^l(\Gamma)$. Then the operator*

$$A = aP_\Gamma + Q_\Gamma$$

is Noetherian in the space $L_2^l(\Gamma, \rho)$.

We note that if Γ is a closed contour of Lyapunov type and $\rho(t) = 1$, then this theorem was proved by I. Simonenko (see [10], Theorem 8), and the set $M_1^l(\Gamma)$ coincides with the class $\widetilde{M}(2, \Gamma)$ defined in [10]. Given that Γ is a simple Lyapunov-type contour, Theorem 4.3 is contained in the work [14].

We will prove two lemmas from which we will immediately deduce Theorem 4.5 and which, at the same time, are also of interest in themselves.

Lemma 4.2. *Let the contour Γ_0 have a single corner point t_0 and $\rho_0(t) = |t - \tau_0|^\beta$ $(-1 < \beta < 1)$. If the matrix $a \in GL_\infty^{l\times l}(\Gamma)$ satisfies the condition*

$$\operatorname{Re}\big(h_{\tau_0}(t)a(t)\big) \geq \sigma > \|h_{\tau_0}a\| \cos\theta(\tau_0),$$

then the operator

$$A_0 = aP_{\Gamma_0} + Q_{\Gamma_0}$$

is Noetherian in the space $L_2^l(\Gamma, \rho)$ and its index are equal to zero.

Proof. Based on Corollary 3.3, it suffices to prove that the operator

$$h_0 aP_{\Gamma_0} + Q_{\Gamma_0} \qquad (h_0(t) = h_{\tau_0}(t))$$

is Noetherian in the space $L_2^l(\Gamma)$. Let $\varphi \in L_2^l(\Gamma_0, \rho)$ and $\|\varphi\| = 1$, then

$$\left\|\left(I - \frac{\sigma_{h_0} a}{\|h_0 aI\|^2}\right)\varphi\right\|^2 \leq 1 - \frac{2\sigma \operatorname{Re}(h_0 a)}{\|h_0 aI\|^2} + \frac{\sigma^2}{\|h_0 aI\|^2} \leq 1 - \frac{\sigma^2}{\|h_0 aI\|^2}.$$

Consequently,

$$\left\|I - \frac{\sigma h_0 a}{\|h_0 aI\|^2}\right\| \leq \sin\theta(\tau_0).$$

As (see [9]),

$$\inf_{T\in\mathfrak{T}} \|P_{\Gamma_0} + T\| = \frac{1}{\sin\theta(\tau_0)},$$

where $\mathfrak{T} = \mathfrak{T}(L_2^l(\Gamma_0))$ is the set of all compact operators in the space $L_2^l(\Gamma_0)$, then

$$\inf_{T\in\mathfrak{T}} \left\|\left(I - \frac{\sigma h_0 a}{\|h_0 aI\|^2}\right) P_{\Gamma_0} + T\right\| < 1.$$

It follows that the operator

$$I - \left(I - \frac{\sigma h_0 a}{\|h_0 aI\|^2}\right) P_{\Gamma_0}$$

is Noetherian in the space $L_2^l(\Gamma_0)$ and its index is equal to zero. We write the operator $h_0aP_{\Gamma_0} + Q_{\Gamma_0}$ in the form

$$h_0aP_{\Gamma_0} + Q_{\Gamma_0} = \Big(I - (I - \omega h_0 a)(\omega^{-1}P_{\Gamma_0} + Q_{\Gamma_0})\Big),$$

where $\omega = \sigma \|h_0 aI\|^{-2}$. The operator $\omega^{-1}P_{\Gamma_0} + Q_{\Gamma_0}$ is invertible, therefore the operator $h_0aP_{\Gamma_0} + Q_{\Gamma_0}$ is Noetherian and its indices are equal to zero. The lemma is proved. □

Lemma 4.3. *Let the matrix of functions $a \in GL^{l\times l}(\Gamma_0)$ satisfy the following conditions:*

(i) there are two matrices $g_\tau^\pm$ such that $(g^+)^{\pm 1} \in^+ L_\infty^{l\times l}(\Gamma_0)$, $(g^-)^{\pm 1} \in^- L_\infty^{l\times l}(\Gamma_0)$;

(ii) $\mathrm{Re}\big(g^+(t)h_0(t)a(t)g^-(t)\big) \geq \sigma \geq \|h_0 a\| \cos\theta(\tau_0)$,

then the operator $aP_{\Gamma_0} + Q_{\Gamma_0}$ is Noetherian in the space $L_2^l(\Gamma_0, \rho)$.

The proof of the lemma follows from Lemma 4.2, from the equality

$$h_0aP_{\Gamma_0} + Q_{\Gamma_0} = (g^-)^{-1}\big(g^+h_0ag^-P_{\Gamma_0} + Q_{\Gamma_0}\big)\big((g^+)^{-1}P_{\Gamma_0} + g^-Q_{\Gamma_0}\big),$$

and from the invertibility of the operators $(g^-)^{-1}I$ and $g^+P_{\Gamma_0} + g^-Q_{\Gamma_0}$.

Proof. We give the proof of Theorem 4.5. From conditions (i) and (iii) of the definition of the set $M_\rho^l(\Gamma)$, it follows that the operator

$$A = aP_\Gamma + Q_\Gamma$$

is locally Noetherian at the points τ_k $(k = 1, 2, \ldots, s)$. From conditions (ii) in the usual way, with the help of Lemma 4.3 and the local principle, it is established that the operator $aP_\Gamma + Q_\Gamma$ is locally Noetherian also at the points $\tau \in \Gamma \setminus \{\tau_1, \ldots, \tau_s\}$. Consequently the operator $aP_\Gamma + Q_\Gamma$ is Noetherian in the space $L_2^l(\Gamma, \rho)$. The theorem is proved. □

Corollary 4.3. *Let $a, b \in L_\infty^{l\times l}(\Gamma)$. If*

$$\mathrm{Vraiinf}_{t\in\Gamma} \det |b(t)| > 0$$

and $b^{-1}a \in M_\rho^l(\Gamma)$, then the operator $aP_\Gamma + bQ_\Gamma$ is Noetherian in the space $L_2^l(\Gamma, \rho)$.

We note that the set $M_\rho^l(\Gamma)$ is quite large. This follows from the following corollary.

Corollary 4.4. *Let the functions a and $b \in CP_l(\Gamma)$. Then the operator $aP_\Gamma + bQ_\Gamma$ is Noetherian in the space $L_2^l(\Gamma, \rho)$ if and only if $b^{-1}a \in M_\rho^l(\Gamma)$.*

An important role in establishing Theorems 4.2 and 4.3 was played by the knowledge of the essential norms in the space $L_2^l(\Gamma, \rho)$ of the operators S_Γ, P_Γ and of the multiplication operator on the matrix a. If in the space $L_p^l(\Gamma, \rho)$ the norm is defined by the equality

$$\|\varphi\|^p = \sum_{j=1}^{l} \|\varphi_j\|^p,$$

then for $p = 2$, the norm of the operator A_0 is calculated from the relation

$$\|A_0\|_{L_2^l(\Gamma,\rho)} = \operatorname{esssup}_{t\in\Gamma} s(t), \tag{4.5}$$

where $s(t)$ is the largest eigenvalue of the matrix $\left(a^*(t)a(t)\right)^{1/2}$. Relation (4.5) has an important role in determining the Noetherian conditions for singular operators. Simple examples show us, however, that for $p \neq 2$ relation (4.5) is false. In order to use the essential norm of the operators S_Γ and P_Γ also for $p \neq 2$, we will modify the norm in the space $L_p^l(\Gamma, \rho)$ in such a way that (4.5) holds.

Lemma 4.4. *If in the space $L_p^l(\Gamma, \rho)$ the norm is defined by the equality*

$$\|\varphi\|^p = \int_\Gamma \left(\sum_{j=1}^{l} |\varphi_j(t)|^2 \right)^{p/2} \rho(t)\,|dt|, \tag{4.6}$$

then the relation (4.5) holds.

Proof. Obviously, for any $t_0 \in \Gamma$, the norm of the operator $a(t_0)I$ in the Euclidean space $\mathbb{C}^l$ is equal to $s(t_0)$. Therefore the norm of A_0 is equal to $\operatorname{esssup}_{t\in\Gamma} s(t)$ and the lemma is proven. □

We will assume that the norm in $L_p^l(\Gamma, \rho)$ is defined by (4.6). In this case (see [15]), the equalities

$$\|S_{\Gamma_0}\|_{L_p^l(\Gamma_0,|t-1|^{p-2-\beta}|t+1|^\beta)} = \|S_{\Gamma_0}\|_{L_p(\Gamma_0,|t-1|^{p-2-\beta}|t+1|^\beta)}$$

where Γ_0 is the unit circle and $\beta \in (-1, p-1)$. From this, in particular, it follows that

$$\|S_{\Gamma_0}\|_{L_p^l(\Gamma_0)} = \begin{cases} ctg\dfrac{\pi}{2p}, & \text{for } p \geq 2, \\ tg\dfrac{\pi}{2p}, & \text{for } 1 < p \leq 2, \end{cases}$$

and

$$\|S_{\Gamma_0}\|_{L_2^l(\Gamma_0,|\frac{t+1}{t-1}|^\beta)} = ctg\pi\frac{(1-|\beta|)}{4} \qquad (|\beta| < 1).$$

Let Γ be a closed Lyapunov contour. We denote by G_δ the set of all function matrices a in $L_\infty^{l\times l}(\Gamma)$ that satisfy the following conditions:

(i) for any point $\tau \in \Gamma$ there exists a neighborhood $u(\tau)$ of this point and two matrices g_τ^+ and g_τ^-, such that

$$(g_\tau^+(t))^{\pm 1} \in^+ L_\infty^{l\times l}(\Gamma), \qquad (g_\tau^-(t))^{\pm 1} \in^- L_\infty^{l\times l}(\Gamma);$$

(ii) for $t \in u(\tau)$, the matrix $g_\tau^-(t)a(t)g_\tau^+(t)$ is unitary and $\sigma\big(g_\tau^-(t)a(t)g_\tau^+(t)\big)$ lies in an angle $\lambda_\tau(\delta)$ with the vertex at the point $z = 0$ and of measure δ.

Condition (ii) is equivalent to the fact that the set of values of the form

$$\big(g_\tau^-(t)a(t)g_\tau^+(t)\eta, \eta\big) \qquad (\eta \in \mathbb{C}^l,\ \|\eta\| = 1,\ t \in u(\tau))$$

lies inside the angle $\lambda_\tau(\delta)$.

Theorem 4.6. *Let $a \in G_\delta$, where*

$$tg\frac{\delta}{2} = \min\left(l^{\frac{2-p}{2p}}\, tg\frac{\pi}{2p}, l^{\frac{2-p}{2p}}\, ctg\frac{\pi}{2p}\right).$$

Then the operator $A = aP_+ + Q_-$ is Noetherian in the space $L_p^l(\Gamma)$.

The *proof* of this theorem is done in the usual way, using the local principle and the following lemma.

Lemma 4.5. *Let Γ_0 be the unit circle and b be a unitary matrix in $L_\infty^{l\times l}(\Gamma_0)$, and for any $t \in \Gamma_0$ its spectrum lies inside the angle $|\arg z| \le \alpha$, where*

$$tg\frac{\alpha}{2} = \min\left(l^{\frac{2-p}{2p}}\, tg\frac{\pi}{2p}, l^{\frac{2-p}{2p}}\, ctg\frac{\pi}{2p}\right).$$

Then the operator $B = bP_{\Gamma_0} + Q_{\Gamma_0}$ is Noetherian in the space $L_p^l(\Gamma_0)$.

Proof. The operator B can be written in the form

$$B = \frac{1}{2}(b+e)\big(I + (b+e)^{-1}(b-e)S_{\Gamma_0}\big),$$

where e is the unit matrix of order l. For $l = 1$ there exists [14] an operator K_0 of rank one, such that in the real space $L_p(\Gamma_0)$, we have

$$\|S_{\Gamma_0} - K_0\|_{L_p(\Gamma_0)} = \begin{cases} ctg\dfrac{\pi}{2p}, & \text{for } p \ge 2, \\ tg\dfrac{\pi}{2p}, & \text{for } 1 < p \le 2. \end{cases}$$

From [5] we deduce that this relation is also true in the complex space $L_p(\Gamma_0)$. As the norm in $L_p^l(\Gamma_0)$ is defined by (3.6), then it is easy to deduce the equality

$$\big\|S_{\Gamma_0} - K_0\big\|_{L_p^l(\Gamma_0)} = \begin{cases} l^{\frac{2-p}{2p}}\ ctg\dfrac{\pi}{2p}, & \text{for } p \ge 2, \\ l^{\frac{2-p}{2p}}\ tg\dfrac{\pi}{2p}, & \text{for } 1 < p \le 2, \end{cases} \tag{4.7}$$

By K_0 in the space $L_p^l(\Gamma_0)$ we mean the operator $\operatorname{diag}(K_0,\dots,K_0)$. Let $f = i(b+e)^{-1}(b-e)$. The matrix f is self-adjoint and for any $t \in \Gamma_0$ its spectrum lies on the segment $\left[-tg\frac{\alpha}{2}, tg\frac{\alpha}{2}\right]$. Let $\varphi = \{\varphi_j\}_{j=1}^l$ and $f\varphi = \{\psi_j\}_{j=1}^l$, then

$$\sum_{j=1}^{l} |\psi_j(t)| \le tg^2\frac{\alpha}{2}\sum_{j=1}^{l}|\varphi_j(t)|^2.$$

Therefore, $\|f\| \le tg\frac{\alpha}{2}$. From relations (4.7) and the assumptions of the lemma it follows that if $\|f(S_{\Gamma_0}-K_0)\| < 1$. Thus, the operator $I - ifS_{\Gamma_0}$ is Noetherian (invertible). Just as the multiplication operator on the matrix $b+e$ is invertible, it follows that the operator B is Noetherian. The lemma is proved. □

Let

$$\rho(t) = \prod_{k=1}^{n} |t-t_k|^{\beta_k} \qquad (t_k \in \Gamma - 1 < \beta_k < p-1).$$

We denote by $G_{\delta,\rho}$ the set of all matrices a in $L_\infty^{l\times l}(\Gamma)$ that satisfy the following conditions:

(i) for any $\tau \in \Gamma \setminus \{t_1,\dots,t_n\}$ there is a neighborhood $u(\tau)$ of this point and two g_τ^+ and g_τ^- such that $(g_\tau^+(t))^{\pm 1} \in^+ L_\infty^{l\times l}(\Gamma)$, $(g_\tau^-(t))^{\pm 1} \in^- L_\infty^{l\times l}(\Gamma)$;

(ii) for $\nabla t \in u(\tau)$ the matrix $g_\tau^-(t)a(t)g_\tau^+(t)$ is unitary and its spectrum lies in an angle $\lambda_\tau(\delta)$ with the vertex at the point $z = 0$ and of measure δ:

$$\sigma\big(g_\tau^-(t)a(t)g_\tau^+(t)\big) \subset \lambda_\tau(\delta);$$

(iii) there is a neighborhood $u(t_k)$ of the point t_k $(k = 1,\dots,n)$ and the pair of matrices g_k^+ and g_k^-, $(g_k^+(t))^{\pm 1} \in^+ L_\infty^{l\times l}(\Gamma)$, $(g_k^-(t))^{\pm 1} \in^- L_\infty^{l\times l}(\Gamma)$ such that, for any $\nabla t \in u(t_k)$, the matrix

$$g_\tau^-(t)a(t)g_\tau^+(t)\, g_\tau^-(t)a(t)g_\tau^+(t)$$

is unitary and

$$\sigma\big(g_\tau^-(t)a(t)g_\tau^+(t)\big) \subset \lambda_k(\delta) \quad (\textit{for } t \in u^+(t_k)),$$

$$\sigma\big(g_\tau^-(t)a(t)g_\tau^+(t)e^{-\frac{2\pi i\beta_k}{p}}\big) \subset \lambda_k(\delta) \quad (\textit{for } t \in u^-(t_k)),$$

where

$$u^+(t_k) = \{t : t \in u(t_k),\ t > t_k\}, \qquad u^-(t_k) = \{t : t \in u(t_k),\ t < t_k\}.$$

It follows from Theorem 4.4.

Theorem 4.7. *Let $a \in G_{\delta,\rho}$, where*

$$tg\frac{\alpha}{2} = \min\left(l^{\frac{2-p}{2p}}\, tg\frac{\pi}{2p},\, l^{\frac{2-p}{2p}}\, ctg\frac{\pi}{2p}\right).$$

Then the operator $A = aP_\Gamma + Q_\Gamma$ is Noetherian in the space $L_p^l(\Gamma,\rho)$.

5. Observations on the case of the unbounded contour. Generalizations

Let Γ be an unbounded and admissible contour, $0 \notin \Gamma$ and

$$\rho(z) = |z|^\beta \prod_{k=1}^n |z - z_k|^{\beta_k} \quad (z_k \in \Gamma,\ -1 < \beta_k < p-1,\ -1 < \beta + \sum_{k=1}^n \beta_k < p-1).$$

In the space $L_p^l(\Gamma,\rho)$, we consider the operator

$$A = aP_\Gamma + Q_\Gamma,$$

where $a, b \in L_\infty^{l\times l}(\bar\Gamma)$ $(\bar\Gamma = \Gamma \cup \infty)$. With the help of the invertible operator B, defined by the relation

$$(B\varphi)(z) = \frac{1}{z}\,\varphi\left(\frac{1}{z}\right), \tag{5.1}$$

the study of the operator A in the space $L_p^l(\Gamma,\rho)$ reduces to the study of the operator

$$\tilde A = \tilde a P_{\tilde\Gamma} + \tilde b Q_{\tilde\Gamma},$$

where $\tilde\Gamma = \{t : t = \frac{1}{z},\ z \in \bar\Gamma\}$, $\tilde a(t) = a\left(\frac{1}{t}\right)$, $\tilde b(t) = b\left(\frac{1}{t}\right)$, in the space $L_p^l(\tilde\Gamma,\tilde\rho)$ with the weight

$$\tilde\rho(z) = |z|^{\beta_0} \prod_{k=1}^n |z - z_k|^{\beta_k}, \qquad \left(\beta_0 = p - 2 - \beta - \sum_{k=1}^n \beta_k,\ t_k = \frac{1}{z_k}\right).$$

Thus, the theorems presented for the operator $\tilde A$ in the space $L_p^l(\tilde\Gamma,\tilde\rho)$ are easily transposed to the operator A in the space $L_p^l(\Gamma,\rho)$. Similar results to the Theorems 4.2 – 4.5 hold for the operator $A = aP_\Gamma + Q_\Gamma$ in the case of the unbounded contour, which we will not explain. We will focus on the case where the integration contour Γ consists of a finite number of bounded and unbounded curves.

So, let Γ be such a contour,

$$\Gamma = \bigcup_{j=0}^m \Gamma_j,$$

where Γ_0 is unbounded and admissible and Γ_j $(j = 1, \ldots, m)$ are bounded. For a start, we will assume that all the curves Γ_j are closed and disjoint. If

$$\rho(z) = |z|^{\beta} \prod_{k=1}^{n} |z - z_k|^{\beta_k} \quad (0 \notin \Gamma),$$

then by ρ_j $(j = 0, 1, \ldots, m)$ we will denote the restriction of the function $\rho(z)$ on Γ_j $(j = 0, 1, \ldots, m)$. We also denote by R_j the multiplication operator to the characteristic function of the contour Γ_j:

$$(R_j\varphi)(z) = \begin{cases} \varphi(z), & \text{for } z \in \Gamma_j, \\ 0, & \text{for } z \in \Gamma \setminus \Gamma_j. \end{cases}$$

We consider the operator

$$A = aP_\Gamma + Q_\Gamma,$$

where $a, b \in L_\infty^{l\times l}(\overline{\Gamma})$. As

$$\sum_{k=0}^{m} R_k = I,$$

then the operator A can be written in the form

$$A = \sum_{j=0}^{m} R_j A \sum_{k=0}^{m} R_k = \sum_{j,k=0}^{m} R_j A R_k. \tag{5.2}$$

The operators R_jAR_j represent the restriction of A on the contour Γ_j $(j = 0, i, \ldots, m)$ that were studied in the previous sections. As $\Gamma_j \cap \Gamma_k = \varnothing$ for $j \neq k$, the operator R_jAR_k $(j \neq k)$ is an integral operator with continuous kernel on $\Gamma_j \times \Gamma_k$ and, therefore, is compact in the space $L_p^l(\Gamma, \rho)$. From these arguments it follows the following assertion.

Theorem 5.1. *The operator $A = aP_\Gamma + Q_\Gamma$ is Noetherian in the space $L_p^l(\Gamma, \rho)$ if and only if the operators*

$$R_jAR_j = A = aP_{\Gamma_j} + Q_{\Gamma_j}$$

are Noetherian, respectively, in the spaces $L_p^l(\Gamma_j, \rho_j)$. In addition,

$$\operatorname{Ind} A\big|_{L_p^l(\Gamma,\rho)} = \sum_{j=0}^{m} \operatorname{Ind} A_j\big|_{L_p^l(\Gamma_j,\rho_j)}. \tag{5.3}$$

On the other hand, with the help of the results of the paper [6] we can extend the obtained results to the case where the contour Γ also contains open Lyapunov curves on portions, and by appealing to Theorems 4.2, 4.6 and the local principle, we can consider the case where the curves curbele Γ_j $(j = 0, 1, \ldots, m)$ have a finite number of intersection points, in particular the point $z = \infty$. In this paper we will not dwell on these details.

6. Conclusions

In the paper, the notions of equivalence and local inversibility were effectively applied to the determination of Noetherian conditions for singular integral operators with scalar or matrix coefficients, bounded and measurable. In certain cases, specified in the paper, the obtained leads are necessary and sufficient in which the studied operators are Noetherian. The Noetherian conditions, as well as the index of the operators, depend on the space in which the research is carried out, but also on the sizes of the angles formed by the contour at its corner points. At the same time, an important role in establishing the main theorems in this paper was played by the sizes of the essential norms of the Riesz projection operators, established by the author in the papers [8], [9, [11], as well as by the authors of the papers [7], [13], [14], [17]. The methods and ideas used, and the results presented in this paper can be used in the subsequent study of singular operators in order to expand the class of integration contours, research spaces, and operator coefficients.

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Lie algebra admitted by the differential system $s^3(3)$ and functional basis of invariants polynomials

Lidia Muşinschi

Abstract. The Lie algebra of the representation of the centro-affine group in the space of coefficients, phase variables and covariant vector coordinates for the ternary differential system with third-degree homogeneities $s^3(3)$ was obtained. Using this algebra, the number of elements in the functional basis of invariant centro-affine polynomials (invariants, comitants, contravariants, mixed comitants) was determined and irreductible elements from these bases were obtained, and for the comitants and contravariants a functional basis was constructed.
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Keywords: ternary differential systems, Lie algebras, functional bases, invariants, comitants, contravariants, mixed comitants.

Algebra Lie admisă de sistemul diferențial $s^3(3)$ şi baze funcţionale de polinoame invariante

Rezumat. A fost obţinută algebra Lie a grupului centroafin de reprezentare în spaţiul coeficienţilor, variabilelor de fază şi coordonatele vectorului covariant pentru sistemul diferenţial ternar cu omogenităţi de gradul trei $s^3(3)$. Cu ajutorul acestei algebre a fost determinat numărul de elemente din baza funcţională a polinoamelor centroafin invariante (invarianţi, comitanţi, contravarianţi, comitanţi micşti) şi obţinute elemente ireductibile din aceste baze, iar pentru comitanţi şi contravarianţi o bază funcţională a lor.
Cuvinte-cheie: sisteme diferențiale ternare, algebre Lie, baze funcționale, invarianți, comitanți, contravarianți, comitanți micști.

1. Introduction

We examine the ternary differential system $s^3(3)$ of the form

$$\frac{dx^j}{dt} = a^j_{\alpha\beta\gamma}x^\alpha x^\beta x^\gamma \equiv P^j \ (j, \alpha, \beta, \gamma = \overline{1, 3}), \tag{1}$$

where the coefficient tensor $a^j_{\alpha\beta\gamma}$ is symmetrical in lower indices in which the complete convolution holds. Let $GL(3, \mathbb{R})$ be the group of centro-affine transformations of the phase variables of the system (1).

Ternary differential systems are applied in diverse fields, including physics (rigid body dynamics, laser physics, geophysics, and circuit theory), biology (population dynamics and infectious disease models), economics, and engineering (missile defense systems and optimal control of physical processes like continuous casting).

Ternary cubic differential system has applications in many domains, for example: physic Kirchhoff equation for elastic rods with intrinsic curvature [1].

It is known [9] that the centro-affine group of transformations $GL(3,\mathbb{R})$ can be expressed as the product of the following elementary one-parameter transformations:

$$\begin{aligned}
&\overline{x}^1 = \lambda_1 x^1,\ \overline{x}^2 = x^2,\ \overline{x}^3 = x^3\ (\lambda_1 \in \mathbb{R}\backslash\{0\});\\
&\overline{x}^1 = x^1,\ \overline{x}^2 = \lambda_2 x^2,\ \overline{x}^3 = x^3\ (\lambda_2 \in \mathbb{R}\backslash\{0\});\\
&\overline{x}^1 = x^1,\ \overline{x}^2 = x^2,\ \overline{x}^3 = \lambda_3 x^3\ (\lambda_3 \in \mathbb{R}\backslash\{0\});\\
&\overline{x}^1 = x^1 + \mu_1 x_2,\ \overline{x}^2 = x^2,\ \overline{x}^3 = x^3;\\
&\overline{x}^1 = x^1 + \mu_2 x_3,\ \overline{x}^2 = x^2,\ \overline{x}^3 = x^3;\\
&\overline{x}^1 = x^1,\ \overline{x}^2 = \mu_3 x^1 + x^2,\ \overline{x}^3 = x^3;\\
&\overline{x}^1 = x^1,\ \overline{x}^2 = x^2 + \mu_4 x^3,\ \overline{x}^3 = x^3;\\
&\overline{x}^1 = x^1,\ \overline{x}^2 = x^2,\ \overline{x}^3 = \mu_5 x^1 + x^3;\\
&\overline{x}^1 = x^1,\ \overline{x}^2 = x^2,\ \overline{x}^3 = \mu_6 x^2 + x^3,
\end{aligned} \tag{2}$$

which are denoted by H_i with parameter λ_i $(i = \overline{1,3})$ and H_{j+3} with parameter μ_j $(j = \overline{1,6})$ in the space of the phase variables $E^3(x)$ of the system (1), where $x = (x^1, x^2, x^3)$.

We consider one-parameter linear transformations

$$\overline{x}^j = f^j(x,\alpha)\ (j = \overline{1,3}), \tag{3}$$

where α is a real parameter continuously changing over a certain interval of $\mathbb{R}$.

Applying the transformation (3) to the system (1) we obtain the system

$$\dot{\overline{x}}^j = \overline{a}^j_{\alpha\beta\gamma}\overline{x}^\alpha\overline{x}^\beta\overline{x}^\gamma\ (j,\alpha,\beta,\gamma = \overline{1,3}), \tag{4}$$

where the coefficients $\overline{a}^j_{\alpha\beta\gamma}$ are expressed linearly by the coefficients $a^j_{\alpha\beta\gamma}$.

Let us denote by a and $\overline{a}$ the sets of coefficients of the right-hand sides in (1) and (4), respectively. It is clear that each coefficient of the set $\overline{a}$ is a linear function of the elements of the set a with coefficients depending on the parameter α.

We write this as

$$\overline{a}^j_{j_1 j_2 j_3} = g^j_{j_1 j_2 j_3}(a,\alpha)\ (j, j_1, j_2, j_3 = \overline{1,3}). \tag{5}$$

Let us assume that the identity transformation in the group with formulas (3)) is achieved in the value of the parameter $\alpha = 0$. Having expanded the functions $\overline{x}^j$ $(j = \overline{1,3})$ and $\overline{a}^j_{j_1 j_2 j_3}$ $(k \in A)$ from (4), (5) into a Taylor series in the parameter α in some neighborhood of $\alpha = 0$, we obtain

$$\overline{x}^j = x^j + \xi^j(x)\alpha + o(\alpha)\ (j = \overline{1,3}), \tag{6}$$

$$\overline{a}^j_{j_1 j_2 j_3} = a^j_{j_1 j_2 j_3} + \eta^j_{j_1 j_2 j_3}(a)\alpha + o(\alpha)\ (k \in A), \tag{7}$$

where

$$\xi^j(x) = \frac{\partial \overline{x}^j(x,\alpha)}{\partial \alpha}|_{\alpha=0},\ \eta^j_{j_1 j_2 j_3}(a) = \frac{\partial \overline{a}^j_{j_1 j_2 j_3}}{\partial \alpha}|_{\alpha=0}\ (j, j_1, j_2, j_3 = \overline{1,3};\ k \in A). \tag{8}$$

In the case when we have a representation of r linear one-parameter groups of the form (3) in the space of variables and coefficients of the system (1), then their operators are written in the form

$$\upsilon_i = \xi^j_i(x)\frac{\partial}{\partial x^j} + \mathfrak{d}_i\ (i = \overline{1,r}; j = \overline{1,3}), \tag{9}$$

where

$$\mathfrak{d}_i = [\eta^j_{j_1 j_2 j_3 (a)}]_i \frac{\partial}{\partial a^j_{j_1, j_2, j_3}}\ (i = \overline{1,r};\ j, j_1, j_2, j_3 = \overline{1,3};\ k \in A). \tag{10}$$

To the representations of the groups (2) in the space of the coefficients and the phase variables $E^{33}(x,a)$ of the system (1) correspond the operators of the form

$$\begin{gathered}
X = \xi^1\frac{\partial}{\partial x^1} + \xi^2\frac{\partial}{\partial x^2} + \xi^3\frac{\partial}{\partial x^3} + \eta^1_{111}\frac{\partial}{\partial a^1_{111}} + \eta^1_{112}\frac{\partial}{\partial a^1_{112}} + \eta^1_{113}\frac{\partial}{\partial a^1_{113}} + \\
+\eta^1_{122}\frac{\partial}{\partial a^1_{122}} + \eta^1_{123}\frac{\partial}{\partial a^1_{123}} + \eta^1_{133}\frac{\partial}{\partial a^1_{133}} + \eta^1_{222}\frac{\partial}{\partial a^1_{222}} + \eta^1_{223}\frac{\partial}{\partial a^1_{223}} + \eta^1_{233}\frac{\partial}{\partial a^1_{233}} + \\
+\eta^1_{333}\frac{\partial}{\partial a^1_{333}} + \eta^2_{111}\frac{\partial}{\partial a^2_{111}} + \eta^2_{112}\frac{\partial}{\partial a^2_{112}} + \eta^2_{113}\frac{\partial}{\partial a^2_{113}} + \eta^2_{122}\frac{\partial}{\partial a^2_{122}} + \eta^2_{123}\frac{\partial}{\partial a^2_{123}} + \\
+\eta^2_{133}\frac{\partial}{\partial a^2_{133}} + \eta^2_{222}\frac{\partial}{\partial a^2_{222}} + \eta^2_{223}\frac{\partial}{\partial a^2_{223}} + \eta^2_{233}\frac{\partial}{\partial a^2_{233}} + \eta^2_{333}\frac{\partial}{\partial a^2_{333}} + \eta^3_{111}\frac{\partial}{\partial a^3_{111}} + \\
+\eta^3_{112}\frac{\partial}{\partial a^3_{112}} + \eta^3_{113}\frac{\partial}{\partial a^3_{113}} + \eta^3_{122}\frac{\partial}{\partial a^3_{122}} + \eta^3_{123}\frac{\partial}{\partial a^3_{123}} + \eta^3_{133}\frac{\partial}{\partial a^3_{133}} + \eta^3_{222}\frac{\partial}{\partial a^3_{222}} + \\
+\eta^3_{223}\frac{\partial}{\partial a^3_{223}} + \eta^3_{233}\frac{\partial}{\partial a^3_{233}} + \eta^3_{333}\frac{\partial}{\partial a^3_{333}},
\end{gathered} \tag{11}$$

where

$$\xi^j = \frac{\partial \overline{x}^j}{\partial \alpha}|_{\alpha=0},\ \eta^j_{\alpha\beta\gamma} = \frac{\partial \overline{a}^j_{\alpha\beta\gamma}}{\partial \alpha}|_{\alpha=0}. \tag{12}$$

2. Lie operators of representation of the centro-afine group $GL(3,\mathbb{R})$ in the space of phase variables and coefficients of the system $s^3(3)$

We analyze the representation of the group H_1 from (2) in the space of phase variables and coefficients $E^{33}(x,a)$ of the system (1).

The system (1) after transformation obtains the form

$$\begin{gathered}
\frac{d\overline{x}^1}{dt} = \overline{a^1_{111}}(\overline{x}^1)^3 + \overline{a^1_{112}}(\overline{x}^1)^2\overline{x}^2 + \overline{a^1_{122}}x^1(\overline{x}^2)^2 + \\
+\overline{a^1_{222}}(\overline{x}^2)^3 + \overline{a^1_{113}}(\overline{x}^1)^2\overline{x}^3 + \overline{a^1_{123}}\overline{x}^1\overline{x}^2\overline{x}^3 + \overline{a^1_{223}}(\overline{x}^2)^2\overline{x}^3 + \\
+\overline{a^1_{133}}\overline{x}^1(\overline{x}^3)^2 + \overline{a^1_{233}}\overline{x}^2(\overline{x}^3)^2 + \overline{a^1_{333}}(\overline{x}^3)^3, \\
\frac{d\overline{x}^2}{dt} = \overline{a^2_{111}}(\overline{x}^1)^3 + \overline{a^2_{112}}(\overline{x}^1)^2\overline{x}^2 + \overline{a^2_{122}}\overline{x}^1(\overline{x}^2)^2 + \\
+\overline{a^2_{222}}(\overline{x}^2)^3 + \overline{a^2_{3}}\overline{x}^3 + \overline{a^2_{113}}(\overline{x}^1)^2\overline{x}^3 + \overline{a^2_{123}}\overline{x}^1\overline{x}^2\overline{x}^3 + \overline{a^2_{223}}(\overline{x}^2)^2\overline{x}^3 + \\
+\overline{a^2_{133}}\overline{x}^1(\overline{x}^3)^2 + \overline{a^2_{233}}\overline{x}^2(\overline{x}^3)^2 + \overline{a^2_{333}}(\overline{x}^3)^3, \\
\frac{d\overline{x}^3}{dt} = \overline{a^3_{111}}(\overline{x}^1)^3 + \overline{a^3_{112}}(\overline{x}^1)^2\overline{x}^2 + \overline{a^3_{122}}\overline{x}^1(\overline{x}^2)^2 + \\
+\overline{a^3_{222}}(\overline{x}^2)^3 + \overline{a^3_{113}}(\overline{x}^1)^2\overline{x}^3 + \overline{a^3_{123}}\overline{x}^1\overline{x}^2\overline{x}^3 + \overline{a^3_{223}}(\overline{x}^2)^2\overline{x}^3 + \overline{a^3_{133}}\overline{x}^1(\overline{x}^3)^2 + \\
+\overline{a^3_{233}}\overline{x}^2(\overline{x}^3)^2 + \overline{a^3_{333}}(\overline{x}^3)^3,
\end{gathered} \tag{13}$$

where

$$\begin{gathered}
\overline{a^1_{111}} = \frac{a^1_{111}}{\lambda_1^2},\ \overline{a^1_{112}} = 3\frac{a^1_{112}}{\lambda_1},\ \overline{a^1_{122}} = 3a^1_{122}, \\
\overline{a^1_{222}} = \lambda_1 a^1_{222},\ \overline{a^1_{113}} = 3\frac{a^1_{113}}{\lambda_1},\ \overline{a^1_{123}} = 6a^1_{123},\ \overline{a^1_{223}} = 3\lambda_1 a^1_{223},\ \overline{a^1_{133}} = 3a^1_{133}, \\
\overline{a^1_{233}} = 3\lambda_1 a^1_{233},\ \overline{a^1_{333}} = \lambda_1 a^1_{333}, \\
\overline{a^2_{111}} = \frac{a^2_{111}}{\lambda_1^3},\ \overline{a^2_{112}} = 3\frac{a^2_{112}}{\lambda_1^2},\ \overline{a^2_{223}} = 3\frac{a^2_{122}}{\lambda_1},
\end{gathered}$$

$$\overline{a^2_{222}} = a^2_{222},\ \overline{a^2_3} = a^2_3,\ \overline{a^2_{113}} = 3\frac{a^2_{113}}{\lambda_1^2},\ \overline{a^2_{123}} = 6\frac{a^2_{123}}{\lambda_1},\ \overline{a^2_{223}} = 3a^2_{223},$$

$$\overline{a^2_{133}} = 3\frac{a^2_{133}}{\lambda_1},\ \overline{a^2_{233}} = 3a^2_{233},\ \overline{a^2_{333}} = a^2_{333},$$

$$\overline{a^3_{111}} = \frac{a^3_{111}}{\lambda_1^3},\ \overline{a^3_{112}} = 3\frac{a^3_{112}}{\lambda_1^2},\ \overline{a^3_{122}} = 3\frac{a^3_{122}}{\lambda_1},$$

$$\overline{a^3_{222}} = a^3_{222},\ \overline{a^3_{113}} = 3\frac{a^3_{113}}{\lambda_1^2},\ \overline{a^3_{123}} = 6\frac{a^3_{123}}{\lambda_1},\ \overline{a^3_{223}} = 3a^3_{223},\ \overline{a^3_{133}} = 3\frac{a^3_{133}}{\lambda_1},$$

$$\overline{a^3_{233}} = 3a^3_{233},\ \overline{a^3_{333}} = a^3_{333},\ |\lambda_1| = exp(\overline{\lambda_1}). \tag{14}$$

For $\overline{\lambda_1} = 0$ we obtain the following non-zero coordinates of the Lie operator corresponding to the group H_1:

$$\begin{gathered}
\xi^1 = x^1, \eta^1_{111} = -2a^1_{111}, \eta^1_{112} = -a^1_{112}, \eta^1_{113} = -a^1_{113},\\
\eta^1_{222} = a^1_{222}, \eta^1_{223} = a^1_{223}, \eta^1_{233} = a^1_{233}, \eta^1_{333} = a^1_{333}, \eta^2_{111} = -3a^2_{111},\\
\eta^2_{112} = -2a^2_{112}, \eta^2_{113} = -2a^2_{113}, \eta^2_{122} = -a^2_{122}, \eta^2_{123} = -a^2_{123}, \eta^2_{133} = -a^2_{133},\\
\eta^3_{111} = -3a^3_{111}, \eta^3_{112} = -2a^3_{112}, \eta^3_{113} = -2a^3_{113}, \eta^3_{122} = -a^3_{122},\\
\eta^3_{123} = -a^3_{123}, \eta^3_{133} = -a^3_{133}.
\end{gathered} \tag{15}$$

Thus, for the group H_1 we obtain the following operator:

$$v_1 = x^1\frac{\partial}{\partial x^1} + \mathfrak{d}_1, \tag{16}$$

where

$$\begin{gathered}
\mathfrak{d}_1 = -2a^1_{111}\frac{\partial}{\partial a^1_{111}} - a^1_{112}\frac{\partial}{\partial a^1_{112}} - a^1_{113}\frac{\partial}{\partial a^1_{113}} + a^1_{222}\frac{\partial}{\partial a^1_{222}} +\\
+a^1_{223}\frac{\partial}{\partial a^1_{223}} + a^1_{233}\frac{\partial}{\partial a^1_{233}} + a^1_{333}\frac{\partial}{\partial a^1_{333}} - 3a^2_{111}\frac{\partial}{\partial a^2_{111}} -\\
-2a^2_{112}\frac{\partial}{\partial a^2_{112}} - 2a^2_{113}\frac{\partial}{\partial a^2_{113}} - a^2_{122}\frac{\partial}{\partial a^2_{122}} - a^2_{123}\frac{\partial}{\partial a^2_{123}} -\\
-a^2_{133}\frac{\partial}{\partial a^2_{133}} - 3a^3_{111}\frac{\partial}{\partial a^3_{111}} - 2a^3_{112}\frac{\partial}{\partial a^3_{112}} - 2a^3_{113}\frac{\partial}{\partial a^3_{113}} -\\
-a^3_{122}\frac{\partial}{\partial a^3_{122}} - a^3_{123}\frac{\partial}{\partial a^3_{123}} - a^3_{133}\frac{\partial}{\partial a^3_{133}}.
\end{gathered} \tag{17}$$

Similarly for the groups H_i $(i = 2, 3)$ and H_{j+3} $(j = \overline{1, 6})$ we obtain the following operators:

$$H_2 \leftrightarrow \upsilon_2 = x^2 \frac{\partial}{\partial x^2} + \mathfrak{d}_2, \tag{18}$$

where

$$\begin{gathered}
\mathfrak{d}_2 = -a^1_{112} \frac{\partial}{\partial a^1_{112}} - 2a^1_{122} \frac{\partial}{\partial a^1_{122}} - a^1_{123} \frac{\partial}{\partial a^1_{123}} - 3a^1_{222} \frac{\partial}{\partial a^1_{222}} - \\
-2a^1_{223} \frac{\partial}{\partial a^1_{223}} - a^1_{233} \frac{\partial}{\partial a^1_{233}} + a^2_{111} \frac{\partial}{\partial a^2_{111}} + a^2_{113} \frac{\partial}{\partial a^2_{113}} - \\
-a^2_{122} \frac{\partial}{\partial a^2_{122}} + a^2_{133} \frac{\partial}{\partial a^2_{133}} - 2a^2_{222} \frac{\partial}{\partial^2_{222}} - a^2_{223} \frac{\partial}{\partial a^2_{223}} + \\
+a^2_{333} \frac{\partial}{\partial a^2_{333}} - a^3_{112} \frac{\partial}{\partial a^3_{112}} - 2a^3_{122} \frac{\partial}{\partial a^3_{122}} - a^3_{123} \frac{\partial}{\partial a^3_{123}} - \\
-3a^3_{222} \frac{\partial}{\partial a^3_{222}} - 2a^3_{223} \frac{\partial}{\partial a^3_{223}}.
\end{gathered} \tag{19}$$

$$H_3 \leftrightarrow \upsilon_3 = x^3 \frac{\partial}{\partial x^3} + \mathfrak{d}_3, \tag{20}$$

where

$$\begin{gathered}
\mathfrak{d}_3 = -a^1_{113} \frac{\partial}{\partial a^1_{113}} - a^1_{123} \frac{\partial}{\partial a^1_{123}} - 2a^1_{133} \frac{\partial}{\partial a^1_{133}} - a^1_{223} \frac{\partial}{\partial a^1_{223}} - \\
-2a^1_{233} \frac{\partial}{\partial a^1_{233}} - 3a^1_{333} \frac{\partial}{\partial a^1_{333}} - a^2_{113} \frac{\partial}{\partial a^2_{113}} - a^2_{123} \frac{\partial}{\partial a^2_{123}} - \\
-2a^2_{133} \frac{\partial}{\partial a^2_{133}} - a^2_{223} \frac{\partial}{\partial a^2_{223}} - 2a^2_{233} \frac{\partial}{\partial a^2_{233}} - 3a^2_{333} \frac{\partial}{\partial a^2_{333}} + \\
+a^3_{111} \frac{\partial}{\partial a^3_{111}} + a^3_{112} \frac{\partial}{\partial a^3_{112}} + a^3_{122} \frac{\partial}{\partial a^3_{122}} - \\
-a^3_{133} \frac{\partial}{\partial a^3_{133}} + a^3_{222} \frac{\partial}{\partial a^3_{222}} - a^3_{233} \frac{\partial}{\partial a^3_{233}} - 2a^3_{333} \frac{\partial}{\partial a^3_{333}}.
\end{gathered} \tag{21}$$

$$H_4 \leftrightarrow \upsilon_4 = x^2 \frac{\partial}{\partial x^1} + \mathfrak{d}_4, \tag{22}$$

where

$$\begin{gathered}
\mathfrak{d}_4 = a^2_{111}\frac{\partial}{\partial a^1_{111}} + (a^2_{112} - a^1_{111})\frac{\partial}{\partial a^1_{112}} + a^2_{113}\frac{\partial}{\partial a^1_{113}} + \\
+(a^2_{122} - 2a^1_{112})\frac{\partial}{\partial a^1_{122}} + (a^2_{123} - a^1_{113})\frac{\partial}{\partial a^1_{123}} + a^2_{133}\frac{\partial}{\partial a^1_{133}} + \\
+(a^2_{223} - 2a^1_{123})\frac{\partial}{\partial a^1_{223}} + (a^2_{233} - a^1_{133})\frac{\partial}{\partial a^1_{233}} + a^2_{333}\frac{\partial}{\partial a^1_{333}} - \\
-a^2_{111}\frac{\partial}{\partial a^2_{112}} - 2a^2_{112}\frac{\partial}{\partial a^2_{112}} - a^2_{113}\frac{\partial}{\partial a^2_{123}} - 3a^2_{122}\frac{\partial}{\partial a^2_{222}} - \\
-2a^2_{123}\frac{\partial}{\partial a^2_{223}} - a^2_{133}\frac{\partial}{\partial a^2_{233}} - a^3_{111}\frac{\partial}{\partial a^3_{112}} - 2a^3_{112}\frac{\partial}{\partial a^3_{122}} - \\
-a^3_{113}\frac{\partial}{\partial a^3_{123}} - 3a^3_{122}\frac{\partial}{\partial a^3_{222}} - 2a^3_{123}\frac{\partial}{\partial a^3_{223}} - a^3_{133}\frac{\partial}{\partial a^3_{233}}.
\end{gathered} \tag{23}$$

$$H_5 \leftrightarrow \upsilon_5 = x^2\frac{\partial}{\partial x^1} + \mathfrak{d}_5, \tag{24}$$

where

$$\begin{gathered}
\mathfrak{d}_5 = a^2_{111}\frac{\partial}{\partial a^1_{111}} + (a^2_{112} - a^1_{111})\frac{\partial}{\partial a^1_{112}} + a^2_{113}\frac{\partial}{\partial a^1_{113}} + \\
+(a^2_{122} - 2a^1_{112})\frac{\partial}{\partial a^1_{122}} + (a^2_{123} - a^1_{113})\frac{\partial}{\partial a^1_{123}} + a^2_{133}\frac{\partial}{\partial a^1_{133}} + \\
+(a^2_{223} - 2a^1_{123})\frac{\partial}{\partial a^1_{223}} + (a^2_{233} - a^1_{133})\frac{\partial}{\partial a^1_{233}} + a^2_{333}\frac{\partial}{\partial a^1_{333}} - \\
-a^2_{111}\frac{\partial}{\partial a^2_{112}} - 2a^2_{112}\frac{\partial}{\partial a^2_{112}} - a^2_{113}\frac{\partial}{\partial a^2_{123}} - 3a^2_{122}\frac{\partial}{\partial a^2_{222}} - \\
-2a^2_{123}\frac{\partial}{\partial a^2_{223}} - a^2_{133}\frac{\partial}{\partial a^2_{233}} - a^3_{111}\frac{\partial}{\partial a^3_{112}} - 2a^3_{112}\frac{\partial}{\partial a^3_{122}} - \\
-a^3_{113}\frac{\partial}{\partial a^3_{123}} - 3a^3_{122}\frac{\partial}{\partial a^3_{222}} - 2a^3_{123}\frac{\partial}{\partial a^3_{223}} - a^3_{133}\frac{\partial}{\partial a^3_{233}}.
\end{gathered} \tag{25}$$

$$H_6 \leftrightarrow \upsilon_6 = x^1\frac{\partial}{\partial x^2} + \mathfrak{d}_6, \tag{26}$$

where

$$\begin{aligned}
\mathfrak{d}_6 = &-3a^1_{112}\frac{\partial}{\partial a^1_{111}} - 2a^1_{122}\frac{\partial}{\partial a^1_{112}} - 2a^1_{123}\frac{\partial}{\partial a^1_{113}} - a^1_{222}\frac{\partial}{\partial a^1_{122}} - \\
&-a^1_{223}\frac{\partial}{\partial a^1_{123}} - a^1_{233}\frac{\partial}{\partial a^1_{133}} + (a^1_{111} - 3a^2_{112})\frac{\partial}{\partial a^2_{111}} + (a^1_{112} - 2a^2_{122})\frac{\partial}{\partial a^2_{112}} + \\
&+(a^1_{113} - 2a^2_{123})\frac{\partial}{\partial a^2_{113}} + (a^1_{122} - a^2_{222})\frac{\partial}{\partial a^2_{122}} + (a^1_{123} - a^2_{223})\frac{\partial}{\partial a^2_{123}} + \\
&+(a^1_{133} - a^2_{233})\frac{\partial}{\partial a^2_{133}} + a^1_{222}\frac{\partial}{\partial a^2_{222}} + a^1_{223}\frac{\partial}{\partial a^2_{223}} + a^1_{233}\frac{\partial}{\partial a^2_{233}} + \\
&+a^1_{333}\frac{\partial}{\partial a^2_{333}} - 3a^3_{112}\frac{\partial}{\partial a^3_{111}} - 2a^3_{122}\frac{\partial}{\partial a^3_{112}} - 2a^3_{123}\frac{\partial}{\partial a^3_{113}} - \\
&-a^3_{222}\frac{\partial}{\partial a^3_{122}} - a^3_{223}\frac{\partial}{\partial a^3_{123}} - a^3_{233}\frac{\partial}{\partial a^3_{133}}.
\end{aligned} \tag{27}$$

$$H_7 \leftrightarrow \upsilon_7 = x^3\frac{\partial}{\partial x^2} + \mathfrak{d}_7, \tag{28}$$

where

$$\begin{aligned}
\mathfrak{d}_7 = &-a^1_{112}\frac{\partial}{\partial a^1_{113}} - a^1_{122}\frac{\partial}{\partial a^1_{123}} - 2a^1_{123}\frac{\partial}{\partial a^1_{133}} - a^1_{222}\frac{\partial}{\partial a^1_{223}} - \\
&-2a^1_{223}\frac{\partial}{\partial a^1_{233}} - 3a^1_{233}\frac{\partial}{\partial a^1_{333}} + a^3_{111}\frac{\partial}{\partial a^2_{111}} + a^3_{112}\frac{\partial}{\partial a^2_{112}} + \\
&+(a^3_{113} - a^2_{112})\frac{\partial}{\partial a^2_{113}} + a^3_{122}\frac{\partial}{\partial a^2_{122}} + (a^3_{123} - a^2_{122})\frac{\partial}{\partial a^2_{123}} + (a^3_{133} - \\
&-2a^2_{123})\frac{\partial}{\partial a^2_{133}} + a^3_{222}\frac{\partial}{\partial a^2_{222}} + (a^3_{223} - a^2_{222})\frac{\partial}{\partial a^2_{223}} + \\
&+(a^3_{233} - 2a^2_{223})\frac{\partial}{\partial a^2_{233}} + (a^3_{333} - 3a^2_{233})\frac{\partial}{\partial a^2_{333}} - a^3_{112}\frac{\partial}{\partial a^3_{113}} - \\
&-a^3_{122}\frac{\partial}{\partial a^3_{123}} - 2a^3_{123}\frac{\partial}{\partial a^3_{133}} - a^3_{222}\frac{\partial}{\partial a^3_{223}} - 2a^3_{223}\frac{\partial}{\partial a^3_{233}} - 3a^3_{233}\frac{\partial}{\partial a^3_{333}}.
\end{aligned} \tag{29}$$

$$H_8 \leftrightarrow \upsilon_8 = x^1\frac{\partial}{\partial x^3} + \mathfrak{d}_8, \tag{30}$$

where

$$\begin{aligned}\mathfrak{d}_8 = &-3a^1_{113}\frac{\partial}{\partial a^1_{111}} - 2a^1_{123}\frac{\partial}{\partial a^1_{112}} - 2a^1_{133}\frac{\partial}{\partial a^1_{113}} - a^1_{223}\frac{\partial}{\partial a^1_{122}} - \\ &-a^1_{233}\frac{\partial}{\partial a^1_{123}} - a^1_{333}\frac{\partial}{\partial a^1_{133}} - 3a^2_{113}\frac{\partial}{\partial a^2_{111}} - 2a^2_{123}\frac{\partial}{\partial a^2_{112}} - \\ &-2a^2_{133}\frac{\partial}{\partial a^2_{113}} - a^2_{223}\frac{\partial}{\partial a^2_{122}} - a^2_{233}\frac{\partial}{\partial a^2_{123}} - a^2_{333}\frac{\partial}{\partial a^2_{133}} + \\ &+(a^1_{111} - 3a^3_{113})\frac{\partial}{\partial a^3_{111}} + (a^1_{112} - 2a^3_{123})\frac{\partial}{\partial a^3_{112}} + (a^1_{113} - 2a^3_{133})\frac{\partial}{\partial a^3_{113}} + \\ &+(a^1_{122} - a^3_{223})\frac{\partial}{\partial a^3_{122}} + (a^1_{123} - a^3_{233})\frac{\partial}{\partial a^3_{123}} + (a^1_{133} - a^3_{333})\frac{\partial}{\partial a^3_{133}} + \\ &+a^1_{222}\frac{\partial}{\partial a^3_{222}} + a^1_{223}\frac{\partial}{\partial a^3_{223}} + a^1_{233}\frac{\partial}{\partial a^3_{233}} + a^1_{333}\frac{\partial}{\partial a^3_{333}}.\end{aligned} \tag{31}$$

$$H_9 \leftrightarrow \upsilon_9 = x^2\frac{\partial}{\partial x^3} + \mathfrak{d}_9, \tag{32}$$

where

$$\begin{aligned}\mathfrak{d}_9 = &-a^1_{113}\frac{\partial}{\partial a^1_{112}} - 2a^1_{123}\frac{\partial}{\partial a^1_{122}} - a^1_{133}\frac{\partial}{\partial a^1_{123}} - 3a^1_{223}\frac{\partial}{\partial a^1_{222}} - \\ &-2a^1_{233}\frac{\partial}{\partial a^1_{223}} - a^1_{333}\frac{\partial}{\partial a^1_{233}} - a^2_{113}\frac{\partial}{\partial a^2_{112}} - 2a^2_{123}\frac{\partial}{\partial a^2_{122}} - a^2_{133}\frac{\partial}{\partial a^2_{123}} - \\ &-3a^2_{223}\frac{\partial}{\partial a^2_{222}} - 2a^2_{233}\frac{\partial}{\partial a^2_{223}} - a^2_{333}\frac{\partial}{\partial a^2_{233}} + a^2_{111}\frac{\partial}{\partial a^3_{111}} + \\ &+(a^2_{112} - a^3_{113})\frac{\partial}{\partial a^3_{112}} + a^2_{113}\frac{\partial}{\partial a^3_{113}} + (a^2_{122} - 2a^3_{123})\frac{\partial}{\partial a^3_{122}} + \\ &+(a^2_{123} - a^3_{133})\frac{\partial}{\partial a^3_{123}} + a^2_{133}\frac{\partial}{\partial a^3_{133}} + (a^2_{222} - 3a^3_{223})\frac{\partial}{\partial a^3_{222}} + \\ &+(a^2_{223} - 2a^3_{233})\frac{\partial}{\partial a^3_{223}} + (a^2_{233} - a^3_{333})\frac{\partial}{\partial a^3_{233}} + a^2_{333}\frac{\partial}{\partial a^3_{333}}.\end{aligned} \tag{33}$$

According to the work [9] we obtain

Theorem 2.1. *To the representation of the centro-affine group $GL(3,\mathbb{R})$ in the space of variables and coefficients of the system* (1) *correspond the Lie operators*

$$\begin{aligned} v_1 &= x^1\frac{\partial}{\partial x^1} + \mathfrak{d}_1, v_2 = x^2\frac{\partial}{\partial x^2} + \mathfrak{d}_2, v_3 = x^3\frac{\partial}{\partial x^3} + \mathfrak{d}_3, \\ v_4 &= x^2\frac{\partial}{\partial x^1} + \mathfrak{d}_4, v_5 = x^3\frac{\partial}{\partial x^1} + \mathfrak{d}_5, v_6 = x^1\frac{\partial}{\partial x^2} + \mathfrak{d}_6, \\ v_7 &= x^3\frac{\partial}{\partial x^2} + \mathfrak{d}_7, v_8 = x^1\frac{\partial}{\partial x^3} + \mathfrak{d}_8, v_9 = x^2\frac{\partial}{\partial x^3} + \mathfrak{d}_9, \end{aligned} \tag{34}$$

where $\mathfrak{d}_i$ are the operators of the representation of the group $GL(3,\mathbb{R})$ in the space of coefficients of the system (1) *and are from* (17)–(33).

For the system (1) the determining equations are [9]

$$\xi^i_{x^k}P^k = \xi^j P^i_{x^j} + \mathfrak{d}(P^i) \ (i,j,k = \overline{1,3}), \tag{35}$$

where $\xi^i_{x^k} = \frac{\partial \xi^i}{\partial x^k}$ şi $P^i_{x^j} = \frac{\partial P^i}{\partial x^j}$.

Remark 2.1. *The operators* (34) *satisfy the system of determining equations* (35) *for the system* (1) *and, consequently, are admitted by this system.* [9].

3. $\mathbf{GL(3,\mathbb{R})}$-Lie algebra admitted by the differential system $\mathbf{s^3(3)}$

Lemma 3.1. *The operators* (34) *form a nine-dimensional Lie algebra $L_9 < v_1, ..., v_9 >$ with structural equations*

$$\begin{gathered} [v_1,v_4] = -v_4, [v_1,v_5] = -v_5, [v_1,v_6] = v_6, [v_1,v_8] = v_8, [v_2,v_4] = v_4, \\ [v_2,v_6] = -v_6, [v_2,v_7] = -v_7, [v_2,v_9] = v_9, [v_3,v_5] = v_5, [v_3,v_7] = v_7, \\ [v_3,v_8] = -v_8, [v_3,v_9] = -v_9, [v_4,v_6] = v_2 - v_1, [v_4,v_7] = -v_5, \\ [v_4,v_8] = v_9, [v_5,v_6] = v_7, [v_5,v_8] = v_3 - v_1, [v_5,v_9] = -v_4, \\ [v_6,v_9] = v_8, [v_7,v_8] = -v_6, [v_7,v_9] = v_3 - v_2, \end{gathered} \tag{36}$$

(*the rest* $[v_i, v_j] = 0$).

If we denote by

$$\begin{gathered} \omega_1 = v_9,\ \omega_2 = v_1 - v_3,\ \omega_3 = v_1 - v_2,\ \omega_4 = v_4,\ \omega_5 = v_5,\ \omega_6 = v_6, \\ \omega_7 = v_7,\ \omega_8 = v_8,\ \omega_9 = v_1 + v_2 + v_3, \end{gathered} \tag{37}$$

then we obtain

Lemma 3.2. *$GL(3,\mathbb{R})$-Lie algebra $L_9 < \upsilon_1, ..., \upsilon_9 >$ with structural equations (36) is isomorphic to the Lie algebra $L_9 < \omega_1, ..., \omega_9 >$ with structural equations*

$$\begin{gathered}
[\omega_1,\omega_2] = -\omega_1,\ [\omega_1,\omega_3] = \omega_1,\ [\omega_1,\omega_5] = \omega_4,\ [\omega_1,\omega_6] = -\omega_8,\\
[\omega_1,\omega_7] = \omega_2 - \omega_3,\ [\omega_2,\omega_4] = -\omega_4,\ [\omega_2,\omega_5] = -2\omega_5,\ [\omega_2,\omega_6] = \omega_6,\\
[\omega_2,\omega_7] = -\omega_7,\ [\omega_2,\omega_8] = 2\omega_8,\ [\omega_3,\omega_4] = -2\omega_4,\ [\omega_3,\omega_5] = -\omega_5,\\
[\omega_3,\omega_6] = 2\omega_6,\ [\omega_3,\omega_7] = \omega_7,\ [\omega_3,\omega_8] = \omega_8,\ [\omega_4,\omega_6] = -\omega_6,\\
[\omega_4,\omega_7] = -\omega_5,\ [\omega_4,\omega_8] = \omega_1,\ [\omega_5,\omega_6] = \omega_7,\ [\omega_5,\omega_8] = -\omega_2,\\
[\omega_7,\omega_8] = -\omega_6, ([\omega_i,\omega_j] = 0 \textit{ otherwise}).
\end{gathered} \tag{38}$$

For the Lie algebra $L_9 < \omega_1, ..., \omega_9 >$, non-zero structural constants up to antisymmetry in lower indices are:

$$\begin{gathered}
C^1_{12} = -1,\ C^1_{13} = 1,\ C^4_{15} = 1,\ C^8_{16} = -1,\ C^2_{17} = 1, C^3_{17} = -1,\\
C^4_{24} = -1,\ C^5_{25} = -2,\ C^6_{26} = 1,\ C^7_{27} = -1,\ C^8_{28} = 2,\ C^4_{34} = -2,\\
C^5_{35} = -1,\ C^6_{36} = 2,\ C^7_{37} = 1,\ C^8_{38} = 1,\ C^3_{46} = -1,\ C^5_{47} = -1,\\
C^1_{48} = 1,\ C^7_{56} = 1,\ C^2_{58} = -1,\ C^6_{78} = -1.
\end{gathered} \tag{39}$$

According to (38) we have

$$L_9 < \omega_1, ..., \omega_9 >= L_8 < \omega_1, ..., \omega_8 > \oplus Z < \omega_9 >, \tag{40}$$

where $Z < \omega_9 >$ is a non-zero center of the Lie algebra $L_9 < \omega_1, ..., \omega_9 >$.

Next we determine the type of the algebra $L_8 < \omega_1, ..., \omega_8 >$.

The Killing form [2] of this algebra is

$$K < v, w >= K_{\alpha\beta} x^\alpha y^\beta,\ K_{\alpha\beta} = C^\tau_{\sigma\alpha} C^\sigma_{\tau\beta}\ (\alpha,\beta,\tau,\sigma = \overline{1,8}). \tag{41}$$

From (41) using (39) we obtain

$$\begin{gathered}
K < v, w >= 6x^7y^1 + 12x^2y^2 + 6x^3y^2 + 6x^2y^3 + 12x^3y^3 +\\
+6x^6y^4 + 6x^8y^5 + 6x^4y^6 + 6x^1y^7 + 6x^5y^8.
\end{gathered}$$

For it we have $det(K_{\alpha\beta}) \neq 0$, consequently, the Killing form is non-degenerate [2].

According to Cartan's criterion [2], Lie algebra $L_8 < \omega_1, ..., \omega_8 >$ is semisimple.

From (40) and definition of reductive Lie algebra we obtain

Theorem 3.1. *The algebra $L_9 < \upsilon_1, ..., \upsilon_9 >$ is a reductive Lie algebra and corresponds to the centro-affine group $GL(3,\mathbb{R})$.*

This algebra is isomorphic to the $GL(3,\mathbb{R})$-Lie algebra admitted by the differential system $s^3(1,2)$ [9].

4. On the number of center-affine invariant elements for the differential system $s^3(3)$

Based on the ideas from the papers [3]–[8] some ternary differential systems were examined in the works [8]–[10] under different aspects. The problem of constructing the functional bases of invariants, comitants, contravariants and mixed comitants for different ternary differential systems is a rather important one.

Consider the center-affine group $GL(3,\mathbb{R})$ given by transformations q:

$$\bar{x}^j = q^j_\alpha x^\alpha \quad (\Delta = det(q^j_\alpha) \neq 0) \quad (j, \alpha = \overline{1,3}). \tag{42}$$

Definition 4.1. *According to* [3] *the vector of phase variables* $x = (x^1, x^2, x^3)$ *of the system* (1)*, which changes by formulas* (42) *is usually called* ***contravariant****.*

Solving the equation (42) with respect to x^j we obtain

$$x^j = p^j_r \bar{x}^r \quad (r, j = \overline{1,3}), \tag{43}$$

where

$$p^r_j q^j_s = \delta^r_s \tag{44}$$

is the Kroniker's symbol, for which we have

$$\delta^r_s = \begin{cases} 1, & \text{for } r = s, \\ 0, & \text{for } r \neq s. \end{cases} \tag{45}$$

Definition 4.2. *According to* [3] *the vector of phase variables* $u = (u_1, u_2, u_3)$ *of the system* (1) *which changes according to the formulas*

$$\overline{u}_r = p^j_r u_j \quad (r, j = \overline{1,3}), \tag{46}$$

where p^j_r *are from* (43) *is called* ***covariant****, and it is also called* ***contragradient*** *with the vector* x*.*

Geometrically, the vector $x = (x^1, x^2, x^3)$ is a point in a three-dimensional space, and the vector $u = (u_1, u_2, u_3)$ - is a line.

Definition 4.3. *According to* [3] *we say that the polynomial* $\varkappa(x, u, a)$ *of the coefficients of the system* (1) *and of the coordinates of the vectors* x *and* u *is called a* ***mixed comitant*** *of the system* (1) *with respect to the center-affine group* $GL(3,\mathbb{R})$ *if the following identity holds*

$$s(\bar{x}, \bar{u}, \bar{a}) = \Delta^{-g} s(x, u, a) \tag{47}$$

for all q from $GL(3,\mathbb{R})$ and every coordinates of the vectors x and u, as well as all the coefficients a of the system (1)*, where g is an integer number called* ***the weight of comitant****.*

If the mixt comitant $s(x,u,a)$ does not depend on the coordinates of the vector u, then following the traditions of the academic C. Sibirschi school, we call it simply **comitant**, and we will note it by $k(x,u,a)$. If $s(x,u,a)$ does not depend on the coordinates of the vector x we call it **contravariant** [3] and we will note it by $r(x,u,a)$. If $s(x,u,a)$ does not depend on the coordinates of the vectors x and u, then we will call it **invariant** of the system (1) with respect to the $GL(3,\mathbb{R})$ group and we will note it by $j(x,u,a)$.

If the mixed comitant s has the weight $g = 0$, then it is called *absolute*, otherwise *relative*.

Definition 4.4. *We will say that the set of polynomial mixed comitants $\{s_k(x,u,a),\ k \in B\}$, of the system* (1)*, with respect to the $GL(3,\mathbb{R})$ group forms a functional basis of the comitants of the system* (1) *with respect to $GL(3,\mathbb{R})$ group, if any comitant $s_r(x,u,a)$ $r \notin B$ of the system* (1)*, with respect to the $GL(3,\mathbb{R})$ group, is expressed as a unique function from the comitants $s_k(x,u,a)$. Here B is a set of finite or infinite natural numbers.*

Definition 4.5. *The functional basis of the mixed comitants of the system* (1) *with respect to the $GL(3,\mathbb{R})$ group is called minimal, if excluding any comitant from the basis, it ceases to form a functional basis.*

Similary, it is defined the functional basis of the comitants, contravariants and invariants of the system (1) with respect to the $GL(3,\mathbb{R})$ group.

Definition 4.6. *The polynomial $s(x,u,a)$ is called irreducible, if it is not expressed polynomially by invariants, comitants, contravariants and mixed comitants, of the same or lower degree.*

Theorem 4.1. [9] For the polynomial $s(x,u,a)$ in the coefficients of the system (1) and the coordinates of the vectors x and u to be a mixed comitant of the system (1) with the weight g and respect to the $GL(3,R)$ group, it is necessary and sufficient that it satisfies the equations

$$\mathcal{V}_i(s) = -gs \ \ (i = \overline{1,3}),\ \ \mathcal{V}_j(s) = 0 \ \ (j = \overline{4,9}), \tag{48}$$

where $\mathcal{V}_i \quad (i = \overline{1,9})$ have the form

$$\begin{aligned}
&\mathcal{V}_1 = v_1 - u_1\frac{\partial}{\partial u_1},\ \mathcal{V}_2 = v_2 - u_2\frac{\partial}{\partial u_2},\ \mathcal{V}_3 = v_3 - u_3\frac{\partial}{\partial u_3},\\
&\mathcal{V}_4 = v_4 - u_1\frac{\partial}{\partial u_2},\ \mathcal{V}_5 = v_5 - u_1\frac{\partial}{\partial u_3},\ \mathcal{V}_6 = v_6 - u_2\frac{\partial}{\partial u_1},\\
&\mathcal{V}_7 = v_7 - u_2\frac{\partial}{\partial u_3},\ \mathcal{V}_8 = v_8 - u_3\frac{\partial}{\partial u_1},\ \mathcal{V}_9 = v_9 - u_3\frac{\partial}{\partial u_2},
\end{aligned} \tag{49}$$

where $v_i \quad (i = \overline{1,9})$ are from (34).

Note 4.1. *From the theory of systems of differential equations* (*see for example* [3]), *it follows that the maximum number of functionally independent solution* (*invariants j, comitants k, contravariants r and mixed comitants s*) *is equal to*

$$\mu = N - rankM + 1\ (g \neq 0),\ (\mu = N - rankM\ (g = 0)), \tag{50}$$

where N is the number of variables in the system of equations, M is the matrix constructed on the coordinates of the differential operators (34).

Theorem 4.2. *The number of the invariant center-affine polynomials of the differential system $s^3(3)$ that form the functional bases of invariants $\mu(j)$, comitants $\mu(k)$, contravariants $\mu(r)$ and mixed comitants $\mu(s)$, respectively, is given in the following table*

$\mu(j)$	$\mu(k)$	$\mu(r)$	$\mu(s)$
22	25	25	28

The number of this invariant elements is necessary for constructing their functional bases.

5. An elements from the functional bases of invariant polynomials of the differential system $s^3(3)$

According to the basic theorem of the theory of invariant tensors [4], of invariant centro-affine polynomials, these invariants are constructed using the product of the tensor coefficients, using the alternation operation (using the unit vector), followed by the total convolution operation (summation by an upper and a lower index).

In the case of the system $s^3(3)$ the general form of the mixed comitant is

$$\underbrace{a^{-}_{---}a^{-}_{---}\cdots a^{-}_{---}}_{d}\ \underbrace{x^{-}x^{-}\cdots x^{-}}_{\delta}\ \underbrace{u_{-}u_{-}\cdots u_{-}}_{\omega}\ \underbrace{\varepsilon^{\alpha_1\alpha_2\alpha_3}\cdots\varepsilon^{\omega_1\omega_2\omega_3}}_{g_1}\ \underbrace{\varepsilon_{\mu_1\mu_2\mu_3}\cdots\varepsilon_{v_1v_2v_3}}_{g_2}, \tag{51}$$

where d, δ, ω, g_1, g_2 are natural numbers, which indicates the number of elements in the brace.

Particular cases of the mixed comitant are the invariant ($\delta = \omega = 0$), the comitant ($\omega = 0$) and the contravariant ($\delta = 0$).

We will study the construction of invariant centro-affine polynomials of the system $s^3(3)$ of the form (51) through the prism of the type below

$$(\delta,\ d;\ \omega), \tag{52}$$

where δ is the degree with respect to the coordinates of the phase variable vector $x = (x_1, x_3, x_3)$, d - the degree with respect to the tensor coefficients $a^j_{\alpha\beta\gamma}$ of the system $s^3(3)$, and ω is the degree with respect to the coordinates of the covariant vector $u = (u_1, u_2, u_3)$.

Next, we will investigate the centro-affine invariants of the system $s^3(3)$ for which ($\delta = \omega = g_2 = 0$), and according to the form (51) will be written as

$$\underbrace{a^{-}_{---}a^{-}_{---}\cdots a^{-}_{---}}_{d}\ \underbrace{\varepsilon^{\alpha_1\alpha_2\alpha_3}\ldots\varepsilon^{\omega_1\omega_2\omega_3}}_{g_1}, \tag{53}$$

where each upper index meets once again lower. Following this property we obtain that $3d = d + 3g_1$. From here we have

$$2d = 3g_1, \tag{54}$$

where d and g_1 are natural numbers.

Then from the equality (54) it follows that

Remark 5.1. *For the product* (53) *to be a centro-affine invariant of the system* $s^3(3)$ *it is necessary and sufficient that d be a multiple of* 3 *and* g_1 *be a multiple of* 2.

We will mention that $\varepsilon^{\alpha_1\alpha_2\alpha_3}, \ldots, \varepsilon^{\omega_1\omega_2\omega_3}$ $(\varepsilon_{\mu_1\mu_2\mu_3}, \ldots, \varepsilon_{\upsilon_1\upsilon_2\upsilon_3})$ are called the unit trivectors with coordinates

$$\begin{gathered}
\varepsilon^{123} = -\varepsilon^{132} = \varepsilon^{312} = -\varepsilon^{321} = \varepsilon^{231} = -\varepsilon^{213} = 1, \\
\varepsilon^{\alpha_1\alpha_2\alpha_3} = 0 \text{ otherwise}, \\
(\varepsilon_{123} = -\varepsilon_{132} = \varepsilon_{312} = -\varepsilon_{321} = \varepsilon_{231} = -\varepsilon_{213} = 1, \\
\varepsilon_{\mu_1\mu_2\mu_3} = 0 \text{ otherwise}).
\end{gathered} \tag{55}$$

Next we will focus on the construction of the invariants of the system $s^3(3)$. In this case the type (52) of the invariants will be written as

$$(0, d; 0). \tag{56}$$

Therefore, the lowest degree of the invariant (51) of the system $s^3(3)$ has the type

$$(0, 3; 0).$$

In this case we can write unambiguously

$$a^{-}_{---}a^{-}_{---}a^{-}_{---}\varepsilon^{\alpha_1\alpha_2\alpha_3}\varepsilon^{\omega_1\omega_2\omega_3}. \tag{57}$$

The question arises, how will the indices of the expression (57) be trained in the total convolution?

In this situation, Aronhold's symbolic is welcome [3].

Invariants of the type (56) for the system $s^3(3)$ can be written using Aronhold's symbolic [3], from which it is easy to pass to the tensor form and vice versa.

Remark 5.2. *We will describe the work with Aronhold's symbolic. We introduce for the tensor coefficients of the system* $s^3(3)$ *the symbolic Aronhold equality*

$$a^j_{\alpha\beta\gamma} = a^j \cdot a_\alpha \cdot a_\beta \cdot a_\gamma \ (j,\alpha,\beta,\gamma = \overline{1,3}), \tag{58}$$

*which we will denote by the **ideal vector** a ($a^j, a_\alpha, a_\beta, a_\gamma$-its coordinates). According to the above mentioned, the invariants of the system* $s^3(3)$ *are a product of tensor coefficients. In this case, we will examine the removal of some misunderstandings in the product of tensors with Aronhold's symbolic.*

Suppose, for example, that for the tensor $a^j_{\alpha\beta\gamma}$ it is necessary to write the product

$$a^j_{\alpha\beta\gamma}a^k_{\delta\mu\nu}. \tag{59}$$

Taking into account (58) *with the **ideal vector** a (the symbols a^j, a_α, a_β, a_γ do not make sense by themselves, only their product does), we obtain that for* (59) *the symbolic notation has the form*

$$a^j a_\alpha a_\beta a_\gamma a^k a_\delta a_\mu a_\nu, \tag{60}$$

which can mean, for example, $a^j_{\alpha\delta\mu}a^k_{\beta\gamma\nu}$, as well as $a^j_{\alpha\beta\delta}a^k_{\gamma\mu\nu}$ and so on. To remove this polysemy, another ideal vector b is introduced for which analogously we have

$$a^j_{\alpha\beta\gamma} = b^j b_\alpha b_\beta b_\gamma. \tag{61}$$

*In this way, the **ideal vectors** a and b are different, but the products* (58) *and* (61) *are the same. Such ideal vectors a and b are called **parallel ideal vectors**, or they are also called **parallel symbols**. Then for the product* (59) *we obtain the exclusion of any misunderstanding and will write*

$$a^j_{\alpha\beta\gamma}a^k_{\delta\mu\nu} = a^j a_\alpha a_\beta a_\gamma b^k b_\delta b_\mu b_\nu,$$

or

$$a^j_{\alpha\beta\gamma}a^k_{\delta\mu\nu} = b^j b_\alpha b_\beta b_\gamma a^k a_\delta a_\mu a_\nu.$$

These two symbolic writing are equal in rights.

In the case of the cubic product (57) *of the coordinates of the tensor coefficient* $a^j_{\alpha\beta\gamma}$ *it is necessary to introduce three parallel symbols to each other and so on, when we have products of higher degree.*

We proceed analogously in the case of the symbolic writing of polynomial differential systems with different tensor coefficients.

Now we will examine the product (57) with three parallel symbols a, b, c by writing the first expression

$$j_{1,3} = (\widetilde{a}a)(\widetilde{b}b)(\widetilde{c}c)[abc]^2, \tag{62}$$

where

$$\begin{gathered}(\widetilde{a}a) = a^1a_1 + a^2a_2 + a^3a_3,\ (\widetilde{b}b) = b^1b_1 + b^2b_2 + b^3b_3,\\ (\widetilde{c}c) = c^1c_1 + c^2c_2 + c^3c_3,\\ [abc] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix},\end{gathered} \tag{63}$$

with the tensorial product

$$j_{1,3} = a^\alpha_{\alpha ps}a^\beta_{\beta qt}a^\gamma_{\gamma ru}\varepsilon^{pqr}\varepsilon^{stu}. \tag{64}$$

Examining all the possibilities of placing the parallel symbols a, b, c in the round brackets (in those squares only the sign can be changed, but since it is squared, this does not matter, but it is necessary that the same symbol does not occur in the square bracket, because a priori it is zero) we obtain the following expressions:

$$\begin{aligned} j_{2,3} &= (\widetilde{a}a)(\widetilde{b}c)(\widetilde{c}b)[abc]^2,\\ j_3 &= (\widetilde{a}b)(\widetilde{b}a)(\widetilde{c}c)[abc]^2,\\ j_{3,3} &= (\widetilde{a}b)(\widetilde{b}c)(\widetilde{c}a)[abc]^2,\\ j_4 &= (\widetilde{a}c)(\widetilde{b}a)(\widetilde{c}b)[abc]^2,\\ j_5 &= (\widetilde{a}c)(\widetilde{b}b)(\widetilde{c}a)[abc]^2. \end{aligned} \tag{65}$$

Interchanging the parallel symbols we obtain

$$\underset{a\leftrightarrow b}{j_{2,3} = j_5},\quad \underset{a\leftrightarrow c}{j_{2,3} = j_3},\quad \underset{a\leftrightarrow b}{j_{3,3} = j_4}. \tag{66}$$

From these 5 expressions form (68) only two are obtained, to which the tensor expressions correspond

$$j_{2,3} = a^\alpha_{\alpha ps}a^\beta_{\gamma qt}a^\gamma_{\beta ru}\varepsilon^{pqr}\varepsilon^{stu},\ j_{3,3} = a^\alpha_{\beta ps}a^\beta_{\gamma qt}a^\gamma_{\alpha ru}\varepsilon^{pqr}\varepsilon^{stu}. \tag{67}$$

Using the Jacobi matrix constructed on these elements we determine that they are functionally independent.

The next type that exists is $(0,6;0)$, using the procedure described above, 15 functionally independent invariants were determined

$$
\begin{aligned}
j_{4,3} &= a^{\alpha}_{\beta\alpha_1\gamma_1} a^{\beta}_{\alpha\beta_1\delta_1} a^{\gamma}_{\gamma\beta_2\gamma_2} a^{\delta}_{\delta\alpha_2\beta_3} a^{\mu}_{\mu\alpha_3\delta_2} a^{\nu}_{\nu\gamma_3\delta_3} \varepsilon^{\alpha_1\alpha_2\alpha_3} \varepsilon^{\beta_1\beta_2\beta_3} \varepsilon^{\gamma_1\gamma_2\gamma_3} \varepsilon^{\delta_1\delta_2\delta_3},\\
j_{5,3} &= a^{\alpha}_{\nu\alpha_1\gamma_1} a^{\beta}_{\gamma\alpha_2,\delta_1} a^{\gamma}_{\delta\alpha_3\gamma_2} a^{\delta}_{]\beta\beta_1,\delta_2} a^{\mu}_{\alpha\beta_2\gamma_3} a^{\nu}_{\nu\beta_3\delta_3} \varepsilon^{\alpha_1\alpha_2\alpha_3} \varepsilon^{\beta_1\beta_2\beta_3} \varepsilon^{\gamma_1\gamma_2\gamma_3} \varepsilon^{\delta_1\delta_2\delta_3},\\
j_{6,3} &= a^{\alpha}_{\beta\alpha_1\gamma_1} a^{\beta}_{\alpha\beta_1\delta_1} a^{\gamma}_{\delta\beta_2\gamma_2} a^{\delta}_{\gamma\alpha_2\beta_3} a^{\mu}_{\mu\alpha_3\delta_2} a^{\nu}_{\nu\gamma_3\delta_3} \varepsilon^{\alpha_1\alpha_2\alpha_3} \varepsilon^{\beta_1\beta_2\beta_3} \varepsilon^{\gamma_1\gamma_2\gamma_3} \varepsilon^{\delta_1\delta_2\delta_3},\\
j_{7,3} &= a^{\alpha}_{\gamma\alpha_1\gamma_1} a^{\beta}_{\delta\beta_1\delta_1} a^{\gamma}_{\beta\beta_2\gamma_2} a^{\delta}_{\alpha\alpha_2\beta_3} a^{\mu}_{\nu\alpha_3\delta_2} a^{\nu}_{\mu\gamma_3\delta_3} \varepsilon^{\alpha_1\alpha_2\alpha_3} \varepsilon^{\beta_1\beta_2\beta_3} \varepsilon^{\gamma_1\gamma_2\gamma_3} \varepsilon^{\delta_1\delta_2\delta_3},\\
j_{8,3} &= a^{\alpha}_{\mu\alpha_1\gamma_1} a^{\beta}_{\nu\beta_1\delta_1} a^{\gamma}_{\delta\beta_2\gamma_2} a^{\delta}_{\gamma\alpha_2\beta_3} a^{\mu}_{\alpha\alpha_3\delta_2} a^{\nu}_{\beta\gamma_3\delta_3} \varepsilon^{\alpha_1\alpha_2\alpha_3} \varepsilon^{\beta_1\beta_2\beta_3} \varepsilon^{\gamma_1\gamma_2\gamma_3} \varepsilon^{\delta_1\delta_2\delta_3},\\
j_{9,3} &= a^{\alpha}_{\nu\alpha_1\gamma_1} a^{\beta}_{\delta\beta_1\delta_1} a^{\gamma}_{\mu\beta_2\gamma_2} a^{\delta}_{\gamma\alpha_2\beta_3} a^{\mu}_{\alpha\alpha_3\delta_2} a^{\nu}_{\beta\gamma_3\delta_3} \varepsilon^{\alpha_1\alpha_2\alpha_3} \varepsilon^{\beta_1\beta_2\beta_3} \varepsilon^{\gamma_1\gamma_2\gamma_3} \varepsilon^{\delta_1\delta_2\delta_3},\\
j_{10,3} &= a^{\alpha}_{\beta\alpha_1\gamma_1} a^{\beta}_{\nu\beta_1\delta_1} a^{\gamma}_{\alpha\beta_2\gamma_2} a^{\delta}_{\mu\alpha_2\beta_3} a^{\mu}_{\gamma\alpha_3\delta_2} a^{\nu}_{\delta\gamma_3\delta_3} \varepsilon^{\alpha_1\alpha_2\alpha_3} \varepsilon^{\beta_1\beta_2\beta_3} \varepsilon^{\gamma_1\gamma_2\gamma_3} \varepsilon^{\delta_1\delta_2\delta_3},\\
j_{11,3} &= a^{\alpha}_{\beta\alpha_1\gamma_1} a^{\beta}_{\nu\beta_1\delta_1} a^{\gamma}_{\mu\beta_2\gamma_2} a^{\delta}_{\gamma\alpha_2\beta_3} a^{\mu}_{\delta\alpha_3\delta_2} a^{\nu}_{\alpha\gamma_3\delta_3} \varepsilon^{\alpha_1\alpha_2\alpha_3} \varepsilon^{\beta_1\beta_2\beta_3} \varepsilon^{\gamma_1\gamma_2\gamma_3} \varepsilon^{\delta_1\delta_2\delta_3},\\
j_{12,3} &= a^{\alpha}_{\beta\alpha_1\gamma_1} a^{\beta}_{\gamma\beta_1\delta_1} a^{\gamma}_{\nu\beta_2\gamma_2} a^{\delta}_{\mu\alpha_2\beta_3} a^{\mu}_{\alpha\alpha_3\delta_2} a^{\nu}_{\delta\gamma_3\delta_3} \varepsilon^{\alpha_1\alpha_2\alpha_3} \varepsilon^{\beta_1\beta_2\beta_3} \varepsilon^{\gamma_1\gamma_2\gamma_3} \varepsilon^{\delta_1\delta_2\delta_3},\\
j_{13,3} &= a^{\alpha}_{\beta\alpha_1\gamma_1} a^{\beta}_{\nu\beta_1\delta_1} a^{\gamma}_{\delta\beta_2\gamma_2} a^{\delta}_{\alpha\alpha_2\beta_3} a^{\mu}_{\mu\alpha_3\delta_2} a^{\nu}_{\gamma\gamma_3\delta_3} \varepsilon^{\alpha_1\alpha_2\alpha_3} \varepsilon^{\beta_1\beta_2\beta_3} \varepsilon^{\gamma_1\gamma_2\gamma_3} \varepsilon^{\delta_1\delta_2\delta_3},\\
j_{14,3} &= a^{\alpha}_{\beta\alpha_1\gamma_1} a^{\beta}_{\alpha\beta_1\delta_1} a^{\gamma}_{\gamma\beta_2\gamma_2} a^{\delta}_{\mu\alpha_2\beta_3} a^{\mu}_{\nu\alpha_3\delta_2} a^{\nu}_{\delta\gamma_3\delta_3} \varepsilon^{\alpha_1\alpha_2\alpha_3} \varepsilon^{\beta_1\beta_2\beta_3} \varepsilon^{\gamma_1\gamma_2\gamma_3} \varepsilon^{\delta_1\delta_2\delta_3},\\
j_{15,3} &= a^{\alpha}_{\beta\alpha_1\gamma_1} a^{\beta}_{\alpha\beta_1\delta_1} a^{\gamma}_{\delta\beta_2\gamma_2} a^{\delta}_{\mu\alpha_2\beta_3} a^{\mu}_{\nu\alpha_3\delta_2} a^{\nu}_{\gamma\gamma_3\delta_3} \varepsilon^{\alpha_1\alpha_2\alpha_3} \varepsilon^{\beta_1\beta_2\beta_3} \varepsilon^{\gamma_1\gamma_2\gamma_3} \varepsilon^{\delta_1\delta_2\delta_3},\\
j_{16,3} &= a^{\alpha}_{\beta\alpha_1\gamma_1} a^{\beta}_{\gamma\beta_1\delta_1} a^{\gamma}_{\delta\beta_2\gamma_2} a^{\delta}_{\mu\alpha_2\beta_3} a^{\mu}_{\alpha\alpha_3\delta_2} a^{\nu}_{\nu\gamma_3\delta_3} \varepsilon^{\alpha_1\alpha_2\alpha_3} \varepsilon^{\beta_1\beta_2\beta_3} \varepsilon^{\gamma_1\gamma_2\gamma_3} \varepsilon^{\delta_1\delta_2\delta_3},\\
j_{17,3} &= a^{\alpha}_{\beta\alpha_1\gamma_1} a^{\beta}_{\delta\beta_1\delta_1} a^{\gamma}_{\mu\beta_2\gamma_2} a^{\delta}_{\gamma\alpha_2\beta_3} a^{\mu}_{\alpha\alpha_3\delta_2} a^{\nu}_{\nu\gamma_3\delta_3} \varepsilon^{\alpha_1\alpha_2\alpha_3} \varepsilon^{\beta_1\beta_2\beta_3} \varepsilon^{\gamma_1\gamma_2\gamma_3} \varepsilon^{\delta_1\delta_2\delta_3},\\
j_{18,3} &= a^{\alpha}_{\gamma\alpha_1\gamma_1} a^{\beta}_{\beta\beta_1\delta_1} a^{\gamma}_{\alpha\beta_2\gamma_2} a^{\delta}_{\delta\alpha_2\beta_3} a^{\mu}_{\nu\alpha_3\delta_2} a^{\nu}_{\mu\gamma_3\delta_3} \varepsilon^{\alpha_1\alpha_2\alpha_3} \varepsilon^{\beta_1\beta_2\beta_3} \varepsilon^{\gamma_1\gamma_2\gamma_3} \varepsilon^{\delta_1\delta_2\delta_3}.
\end{aligned}
\tag{68}
$$

Remark 5.3. *In this way,* 18 *functionally independent irreducible invariants were constructed that are contained in a functional basis of the invariants of the ternary differential system* $s^3(3)$.

Analogously, we construct the centro-affine comitants of the system $s^3(3)$, which according to the form (51) will be written

$$
\underbrace{x^{-}x^{-}\cdots x^{-}}_{\delta}\;\underbrace{a^{-}_{---}a^{-}_{---}\cdots a^{-}_{---}}_{d}\;\underbrace{\varepsilon^{\alpha_1\alpha_2\alpha_3}\ldots\varepsilon^{\omega_1\omega_2\omega_3}}_{g_1}\;\underbrace{\varepsilon_{\mu_1\mu_2\mu_3}\cdots\varepsilon_{\upsilon_1\upsilon_2\upsilon_3}}_{g_2}, \tag{69}
$$

where each upper index meets once again lower. We obtain that $3d + 3g_2 = \delta + d + 3g_1$. From here we have

$$
2d = \delta + 3g_1 - 3g_2, \tag{70}
$$

where d, δ, g_1 and g_2 are nonegative integer numbers.

The first possible type is $(2,1;0)$, of this type we have only one comitant

$$k_{1,3} = a^{\alpha}_{\alpha\beta\gamma} x^{\beta} x^{\gamma}. \tag{71}$$

There are two comitants of the type $(4,2;0)$

$$k_{2,3} = a^{\alpha}_{\beta\gamma\mu} a^{\beta}_{\alpha\delta\nu} x^{\gamma} x^{\delta} x^{\mu} x^{\nu},\ k_{3,3} = a^{\alpha}_{\alpha\beta\gamma} a^{\beta}_{\delta\mu\theta} x^{\gamma} x^{\delta} x^{\mu} x^{\theta}. \tag{72}$$

There are four comitants of the type $(3,3;0)$

$$\begin{aligned} k_{4,3} &= a^{\alpha}_{\beta\delta\mu} a^{\beta}_{\gamma\nu\pi} a^{\gamma}_{\alpha\varrho\delta} x^{\delta} x^{\nu} x^{\varrho} \varepsilon^{\mu\pi\delta},\ k_{5,3} = a^{\alpha}_{\alpha\beta\delta} a^{\beta}_{\gamma\mu\pi} a^{\gamma}_{\nu\varrho\sigma} x^{\mu} x^{\nu} x^{\varrho} \varepsilon^{\delta\pi\sigma}, \\ k_{6,3} &= a^{\alpha}_{\beta\delta\pi} a^{\beta}_{\alpha\gamma\mu} a^{\gamma}_{\nu\varrho\sigma} x^{\delta} x^{\nu} x^{\varrho} \varepsilon^{\pi\mu\sigma},\ k_{7,3} = a^{\alpha}_{\alpha\gamma\delta} a^{\beta}_{\beta\mu\nu} a^{\gamma}_{\pi\varrho\sigma} x^{\mu} x^{\pi} x^{\varrho} \varepsilon^{\delta\nu\sigma}. \end{aligned} \tag{73}$$

There are two comitants of the type $(6,3;0)$

$$\begin{aligned} k_{8,3} &= a^{\alpha}_{\beta\nu\tau} a^{\beta}_{\alpha\gamma\phi} a^{\gamma}_{\delta\mu\psi} x^{\delta} x^{\mu} x^{\nu} x^{\tau} x^{\phi} x^{\psi}, \\ k_{9,3} &= a^{\alpha}_{\gamma\tau p} a^{\beta}_{\mu\phi q} a^{\gamma}_{\nu\psi r} x^{\delta} x^{\mu} x^{\nu} x^{\tau} x^{\phi} x^{\psi} \varepsilon^{pqr} \varepsilon_{\alpha\beta\gamma}. \end{aligned} \tag{74}$$

Theorem 5.1. *The invariants $j_{1,3} - j_{16,3}$ from* (68) *together with the comitants $k_{1,3} - k_{9,3}$ from* (71) – (74) *form a functional basis of the irreducible comitants of the ternary differential system $s^3(3)$.*

Analogously, we construct the centro-affine contravariants of the system $s^3(3)$ for which ($\delta = g_2 = 0$), and according to the form (51) there will be written

$$\underbrace{a^{-}_{---} a^{-}_{---} \cdots a^{-}_{---}}_{d}\ \underbrace{u_{-} u_{-} \cdots u_{-}}_{\omega}\ \underbrace{\varepsilon^{\alpha_1\alpha_2\alpha_3} \ldots \varepsilon^{\omega_1\omega_2\omega_3}}_{g_1}, \tag{75}$$

where each index from above meets again from below. We get $3d + \omega = d + 3g_1$. From here we have

$$2d = 3g_1 - \omega, \tag{76}$$

where d, ω and g_1 are natural numbers.

Two contravariants of the type $(0,2;2)$ were constructed

$$\begin{aligned} r_{1,3} &= a^{\alpha}_{\alpha\alpha_1\beta_1} a^{\beta}_{\beta\alpha_2\beta_2} u_{\alpha_3} u_{\beta_3} \varepsilon^{\alpha_1\alpha_2\alpha_3} \varepsilon^{\beta_1\beta_2\beta_3}, \\ r_{2,3} &= a^{\alpha}_{\beta\alpha_1\beta_1} a^{\beta}_{\alpha\alpha_2\beta_2} u_{\alpha_3} u_{\beta_3} \varepsilon^{\alpha_1\alpha_2\alpha_3} \varepsilon^{\beta_1\beta_2\beta_3}. \end{aligned} \tag{77}$$

Four contravariants of the type $(0,3;3)$ were constructed

$$\begin{aligned} r_{3,3} &= a^{\alpha}_{\alpha\beta\alpha_1} a^{\beta}_{\gamma\alpha_2\beta_1} a^{\gamma}_{\beta_2\alpha_3\beta_3} u_{\gamma_1} u_{\gamma_2} u_{\gamma_3} \varepsilon^{\alpha_1\alpha_3\gamma_1} \varepsilon^{\alpha_2\beta_2\gamma_2} \varepsilon^{\beta_1\beta_3\gamma_3}, \\ r_{4,3} &= a^{\alpha}_{\alpha\gamma\alpha_1} a^{\beta}_{\beta\alpha_2\beta_1} a^{\gamma}_{\beta_2\alpha_3\beta_3} u_{\gamma_1} u_{\gamma_2} u_{\gamma_3} \varepsilon^{\alpha_1\alpha_3\gamma_1} \varepsilon^{\alpha_2\beta_2\gamma_2} \varepsilon^{\beta_1\beta_3\gamma_3}, \\ r_{5,3} &= a^{\alpha}_{\beta\gamma\alpha_1} a^{\beta}_{\alpha\alpha_2\beta_1} a^{\gamma}_{\beta_2\alpha_3\beta_3} u_{\gamma_1} u_{\gamma_2} u_{\gamma_3} \varepsilon^{\alpha_1\alpha_3\gamma_1} \varepsilon^{\alpha_2\beta_2\gamma_2} \varepsilon^{\beta_1\beta_3\gamma_3}, \\ r_{6,3} &= a^{\alpha}_{\beta\alpha_1\beta_1} a^{\beta}_{\gamma\alpha_2\beta_2} a^{\gamma}_{\alpha\alpha_3\beta_3} u_{\gamma_1} u_{\gamma_2} u_{\gamma_3} \varepsilon^{\alpha_1\alpha_3\gamma_1} \varepsilon^{\beta_1\beta_2\gamma_2} \varepsilon^{\alpha_2\beta_3\gamma_3}. \end{aligned} \tag{78}$$

One contravariant of the type $(0,4;1)$ was constructed

$$r_{7,3} = a^{\alpha}_{\beta\alpha_1\beta_1} a^{\beta}_{\gamma\alpha_2\beta_2} a^{\gamma}_{\delta\alpha_3\beta_3} a^{\delta}_{\alpha\gamma_1\gamma_2} u_{\gamma_3} \varepsilon^{\alpha_1\alpha_2\alpha_3} \varepsilon^{\beta_1\beta_2\gamma_1} \varepsilon^{\beta_3\gamma_2\gamma_3}. \tag{79}$$

Theorem 5.2. *The invariants* $j_{1,3} - j_{18,3}$ *from* (68) *together with the contravariants* $r_{1,3} - r_{7,3}$ *from* (77)–(79) *form a functional basis of the irreducible contravariants of the ternary differential system* $s^3(3)$.

Analogously, we construct the centro-affine mixed comitants of the system $s^3(3)$ with the type (51), therefore we have

$$2d = \delta + 3g_1 - 3g_2 - \omega, \tag{80}$$

where d, δ, ω, g_1 and g_2 are nonnegative integer numbers.

One mixed comitant of the type $(3,1;1)$ was constructed

$$s_{1,3} = a^{\alpha}_{\beta\gamma\delta} x^{\beta} x^{\gamma} x^{\delta} u_{\alpha}. \tag{81}$$

Two mixed comitants of the type $(1,3;1)$ were constructed

$$\begin{aligned} s_{2,3} &= a^{\alpha}_{\alpha\alpha_1\beta_1} a^{\beta}_{\beta\alpha_2\beta_2} a^{\gamma}_{\gamma_1\gamma_2\gamma_3} x^{\gamma_1} u_{\gamma} \varepsilon^{\alpha_1\alpha_2\gamma_2} \varepsilon^{\beta_1\beta_2\gamma_3}, \\ s_{3,3} &= a^{\alpha}_{\beta\alpha_1\beta_1} a^{\beta}_{\gamma\alpha_2\beta_2} a^{\gamma}_{\gamma_1\gamma_2\gamma_3} x^{\gamma_1} u_{\alpha} \varepsilon^{\alpha_1\alpha_2\gamma_2} \varepsilon^{\beta_1\beta_2\gamma_3}. \end{aligned} \tag{82}$$

One mixed comitant of the type $(2,3;2)$ was constructed

$$s_{4,3} = a^{\alpha}_{\gamma\alpha_1\beta_1} a^{\beta}_{\alpha_2\beta_2\gamma_1} a^{\gamma}_{\alpha_3\beta_3\gamma_2} x^{\alpha_2} x^{\alpha_3} u_{\alpha} u_{\beta} \varepsilon^{\alpha_1\beta_2\beta_3} \varepsilon_{\beta_1\gamma_1\gamma_2}. \tag{83}$$

One mixed comitant of the type $(3,3;3)$ was constructed

$$s_{5,3} = a^{\alpha}_{\alpha_1\beta_1\gamma_1} a^{\beta}_{\alpha_2\beta_2\gamma_2} a^{\gamma}_{\alpha_3\beta_3\gamma_3} x^{\alpha_1} x^{\alpha_2} x^{\alpha_3} u_{\alpha} u_{\beta} u_{\gamma} \varepsilon^{\beta_1\beta_2\beta_3} \varepsilon^{\gamma_1\gamma_2\gamma_3}. \tag{84}$$

Remark 5.4. *In this way,* 5 *functionally independent irreducible mixed comitants were constructed that are contained in a functional basis of the mixed comitants of the ternary differential system* $s^3(3)$.

Remark 5.5. *To prove the functional independence of the invariant polynomials, the rank of the Jacobi matrix of these elements was calculated at the point*

$$x^1 = 1,\ x^2 = 1,\ x^3 = 1,\ u_1 = 1,\ u_2 = 1,\ u_3 = 1,\ a^1_{111} = 1,\ a^1_{112} = 0,\ a^1_{113} = 1,$$
$$a^1_{122} = 1,\ a^1_{123} = -1,\ a^1_{133} = -1,\ a^1_{222} = 0,\ a^1_{223} = 0,\ a^1_{233} = 1,\ a^1_{333} = 1,$$
$$a^2_{111} = 1,\ a^2_{112} = -1,\ a^2_{113} = 0,\ a^2_{122} = 1,\ a^2_{123} = 0,\ a^2_{133} = 1,\ a^2_{222} = 0,$$
$$a^2_{223} = 0,\ a^2_{233} = -1,\ a^2_{333} = 1,\ a^3_{111} = 1,\ a^3_{112} = -1,\ a^3_{113} = 0,\ a^3_{122} = -1,$$
$$a^3_{123} = 0,\ a^3_{133} = -1,\ a^3_{222} = 0,\ a^3_{223} = 0,\ a^3_{233} = 1,\ a^3_{333} = -1.$$

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An encryption algorithm using a generalization of the Markovski algorithm and T-quasigroups

NADEZHDA MALYUTINA AND VICTOR SHCHERBACOV

Abstract. Here is a more detailed description of the algorithm proposed in [1]. This algorithm simultaneously uses two cryptographic procedures: encryption using a generalization of the Markovski algorithm [2] and encryption using a system of orthogonal operations. In this paper, we present an implementation of this algorithm based on T-quasigroups, more precisely, based on medial quasigroups.

2020 Mathematics Subject Classification: 20N05, 20N15, 05B15, 94A60.
Keywords: cipher, ciphertext, plaintext, Markovski algorithm, quasigroup, n-ary groupoid, T-quasigroup.

Un algoritm de criptare bazat pe o generalizare a algoritmului Markovski și pe T-quasigrupuri

Rezumat. Se prezintă o descriere detaliată a algoritmului propus în [1], care utilizează simultan două procedee criptografice: criptarea bazată pe o generalizare a algoritmului Markovski [2] și criptarea printr-un sistem de operații ortogonale. În această lucrare este realizată o implementare a acestui algoritm utilizând T-quasigrupuri, mai precis quasigrupuri mediale.

Cuvinte-cheie: cifru, text criptat, text necriptat, algoritmul Markovski, quasigrup, grupoid n-ar, T-quasigrup.

1. Introduction

This paper generalizes a cryptographic algorithm based on the use of quasigroups of a special form and provides an example illustrating the operation of this algorithm.

Additional information is required for the modifications proposed in this work [3, 4].

2. Basic concepts and definitions

Definition 2.1. *A binary groupoid (Q, A) is a non-empty set Q with a binary operation A defined on it. An n-ary groupoid (Q, A) is a non-empty set Q with an Q-ary operation f defined on it.*

Definition 2.2. *n-ary groupoids $(Q, A_1), (Q, A_2), \dots, (Q, A_n)$ are called orthogonal if for any fixed n-tuple $a_1, a_2, \dots, a_n$ the system of the following equations*

$$\begin{cases} A_1(x_1, x_2, \dots, x_n) = a_1, \\ A_2(x_1, x_2, \dots, x_n) = a_2, \\ \qquad\qquad\qquad\qquad \dots \\ A_n(x_1, x_2, \dots, x_n) = a_n \end{cases}$$

has a unique solution.

If the set Q is finite, then any system of n orthogonal n-ary groupoids (Q, A_i), $i \in \overline{1, n}$, defines a permutation of the set Q^n and vice versa [5, 6, 7]. Therefore, if $|Q| = q$, then there are $(q^n)!$ systems of n-ary orthogonal groupoids defined on the set Q.

There are various generalizations of the definition of orthogonality of n-ary operations [8, 9].

Definition 2.3. *A binary groupoid (Q, A) is called a quasigroup if $\forall a, b \in Q$ two equations are uniquely solvable: $A(x, a) = b$ and $A(a, x) = b$.*

This definition of a quasigroup is called equational.

Definition 2.4. *Garret Birkhoff* [10, 11] *defined an equational quasigroup as an algebra with three binary operations $(Q, \cdot, \backslash, /)$ that satisfy the following six identities:*

$$\begin{aligned} x \cdot (x \backslash y) &= y, \\ (y / x) \cdot x &= y, \\ x \backslash (x \cdot y) &= y, \\ (y \cdot x) / x &= y, \\ x / (y \backslash x) &= y, \\ (x / y) \backslash x &= y. \end{aligned}$$

Definition 2.5. *With a given binary quasigroup (Q, A) there are $(3! - 1)$ other, so-called parastrophes of the quasigroup (Q, A):*

$$\begin{aligned} A(x_1, x_2) &= x_3, \\ A^{(12)}(x_2, x_1) &= x_3, \\ A^{(13)}(x_3, x_2) &= x_1, \\ A^{(23)}(x_1, x_3) &= x_2, \\ A^{(123)}(x_2, x_3) &= x_1 \\ A^{(132)}(x_3, x_1) &= x_2. \end{aligned}$$

Definition 2.6. *Let $(Q, \cdot)$ be a groupoid and a be a fixed element of Q, then we denote the left, right, and middle translations as L_a, R_a, and P_a, respectively, and define them as follows: $L_a x = a \cdot x$, $R_a x = x \cdot a$, $x \cdot P_a x = a$, $\forall x \in Q$.*

Translations play an important role in the theory of quasigroups.

In the following Table 1, for each type of translation, the equivalent parastrophe of the quasigroup $(Q, \cdot)$ is shown in each of the six cases [12].

Table 1. Translations of quasigroup parastrophes

	ϵ	(12)	(13)	(23)	(123)	(132)
R	R	L	R^{-1}	P	P^{-1}	L^{-1}
L	L	R	P^{-1}	L^{-1}	R^{-1}	P
P	P	P^{-1}	L^{-1}	R	L	R^{-1}
R^{-1}	R^{-1}	L^{-1}	R	P^{-1}	P	L
L^{-1}	L^{-1}	R^{-1}	P	L	R	P^{-1}
P^{-1}	P^{-1}	P	L	R^{-1}	L^{-1}	R

Definition 2.7. *A non-empty set Q with an n-ary operation f defined on it, such that in the equation $f(x_1, x_2, \dots, x_n) = x_{n+1}$ the knowledge of any n elements from $x_1, x_2, \dots, x_n, x_{n+1}$, makes it possible to uniquely determine the remaining $(n+1)$th element, is called an n-ary quasigroup.*

Definition 2.8. *An n-ary groupoid (Q, f) is said to be isotopic to an n-ary groupoid (Q, g) if there are permutations $\mu_1, \mu_2, \dots, \mu_n$ of the set Q such that: $f(x_1, x_2, \dots, x_n) = \mu^{-1} g(\mu_1 x_1, \mu_2 x_2, \dots, \mu_n x_n), \forall x_1, x_2, \dots, x_n \in Q$.*

It is written: $(Q, f) = (Q, g)T$, where $T = (\mu_1, \mu_2, \dots, \mu_n)-$ is an isotopy of an n-ary groupoid.

If $f = g$, then we get an autotopy of the n-ary groupoid (Q, f). The last component of an autotopy is called a quasiautomorphism.

If $\mu_1 = \mu_2 = \cdots = \mu_n$, then the groupoids (Q, f) and (Q, g) are said to be isomorphic.

And finally, if $\mu_1 = \mu_2 = \cdots = \mu_n = \mu$, and $f = g$, then we obtain an automorphism of the groupoid (Q, f).

Definition 2.9. *n-ary quasigroup (Q, f) of the form:*

$$f(x_1^n) = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n + a = \textstyle\sum_{i=1}^n \alpha_i x_i + a,$$

where $(Q,+)$ is a group, $\alpha_1, \alpha_2, \dots, \alpha_n$ are some automorphisms of the group $(Q,+)$, and an element a is some fixed element from the set Q, we will call a linear n-ary quasigroup (over the group $(Q,+)$) [12].

Definition 2.10. *An n-ary linear quasigroup (Q, f) over an Abelian group $(Q,+)$ is called an n-ary T-quasigroup* [13]. *If $n = 2$, then a quasigroup from this class of quasigroups is called a T- quasigroup* [14, 15].

The following identity for an n-ary quasigroup (Q, f)

$$f(f(x_{11}, x_{12}, \dots, x_{1n}), f(x_{21}, x_{22}, \dots, x_{2n}), \dots, f(x_{n1}, x_{n2}, \dots, x_{nn})) =$$
$$f(f(x_{11}, x_{21}, \dots, x_{n1}), f(x_{12}, x_{22}, \dots, x_{n2}), \dots, f(x_{1n}, x_{2n}, \dots, x_{nn}))$$

is called the medial identity [16]. *An n-ary quasigroup with a medial identity is called an n-ary medial quasigroup.*

In the binary case, we get the usual medial identity:

$$xy \cdot uv = xu \cdot yv.$$

Definition 2.11. *A quasigroup $(Q, \cdot)$ is a T-quasigroup if and only if there exists an Abelian group $(Q,+)$, its automorphisms ϕ and ψ, and a fixed element $c \in Q$ such that $x \cdot y = \phi x + \psi y + c$, for all $x, y \in Q$* [14, 15]. *A T-quasigroup with the additional condition $\phi\psi = \psi\phi$ is called medial.*

Medial quasigroups, like other classes of quasigroups isotopic to groups, allow one to construct quasigroups with predetermined properties. Often these properties can be expressed in terms of the properties of groups and isotopy components.

In the following theorem, the expression $A \perp {}^{(23)}A$ means that the quasigroups (Q, A) and $(Q, {}^{(23)}A)$ are orthogonal [12].

Theorem 2.1. *For a finite quasigroup (Q, A), the following equivalences hold:*
$(i) A \perp {}^{(12)}A \Leftrightarrow ((x\backslash z) \cdot x = (y\backslash z) \cdot y \Rightarrow x = y)$;
$(ii) A \perp {}^{(13)}A \Leftrightarrow (zx \cdot x = zy \cdot y \Rightarrow x = y)$;
$(iii) A \perp {}^{(23)}A \Leftrightarrow (x \cdot xz = y \cdot yz \Rightarrow x = y)$;
$(iv) A \perp {}^{(123)}A \Leftrightarrow (x \cdot zx = y \cdot zy \Rightarrow x = y)$;
$(v) A \perp {}^{(132)}A \Leftrightarrow (xz \cdot x = yz \cdot y \Rightarrow x = y)$, *for all $x, y, z \in Q$.*

To construct the quasigroups mentioned in Theorem 2.1, one can use GAP and Prover [17].

If $(Q, \cdot)$ is a T-quasigroup of the form $x \cdot y = \phi x + \psi y + c$, then its parastrophes have the following forms, respectively:

$$x \overset{(12)}{\cdot} y = \psi x + \phi y + c,$$

$$x \overset{(13)}{\cdot} y = \phi^{-1}x - \phi^{-1}\psi y - \phi^{-1}c,$$
$$x \overset{(23)}{\cdot} y = -\psi^{-1}\phi x + \psi^{-1}y - \psi^{-1}c,$$
$$x \overset{(123)}{\cdot} y = -\phi^{-1}\psi x + \phi^{-1}y - \phi^{-1}c,$$
$$x \overset{(132)}{\cdot} y = \psi^{-1}x - \psi^{-1}\phi y - \psi^{-1}c.$$

To construct a quasigroup (Q, A) orthogonal to its parastrophe in a more theoretical way, one can use the following theorem [4, 18].

Theorem 2.2. *For a T-quasigroup (Q, A) of the form: $A(x, y) = \phi x + \psi y + c$, the following equivalences hold over an abelian group $(Q, +)$:*
$(i) A \perp {}^{(12)}A \Leftrightarrow (\phi - \psi), (\phi + \psi))$ are permutations of the set Q;
$(ii) A \perp {}^{(13)}A \Leftrightarrow (\epsilon + \phi)$is a permutation of the set Q;
$(iii) A \perp {}^{(23)}A \Leftrightarrow (\epsilon + \psi)$is a permutation of the set Q;
$(iv) A \perp {}^{(123)}A \Leftrightarrow (\phi + \psi^2)$is a permutation of the set Q;
$(v) A \perp {}^{(132)}A \Leftrightarrow (\phi^2 + \psi)$is a permutation of the set Q.

Corollary 2.1. *T-quasigroup $(Z_p, \circ)$ of the form $x \circ y = k \cdot x + m \cdot y + c$, where $(Z_p, +)$ is a cyclic group of prime order p, $k, m, c \in Z_p$; $k, m, k+m, k-m, k+1, m+1, k^2+m, k+m^2 \not\equiv 0 \pmod p$, where the operation "$\cdot$" is multiplication modulo p, is orthogonal to any of its parastrophe.*

Quasigroups from Corollary 1 are suitable objects for constructing the above algorithms [4].

Belousov [19] has a proof of Toyoda's theorem, namely: if (Q, A) is a medial quasigroup, then there exists an Abelian group $(Q, +)$ such that the operation A has the form: $A(x, y) = \phi x + \psi y + c$, where ϕ, ψ are automorphisms of the group $(Q, +)$, moreover, $\phi\psi = \psi\phi$, and c is some fixed element of Q.

The converse Toyoda theorem also holds.

Theorem 2.3. *(Toyoda's inverse theorem) Let $(Q, +)$ be an arbitrary abelian group, ϕ, ψ be automorphisms of the group $(Q, +)$, where $\phi\psi = \psi\phi$, $c \in Q$ is a fixed arbitrary element. Then $(Q, \cdot)$ is a medial quasigroup, where $x \cdot y = \phi x + \psi y + c, \forall x, y \in Q$.*

3. Results

Below we denote the action of the left (right, middle) translation to the power of a of a binary quasigroup (Q, g_1) on an element u_1 by the symbol ${}_{g_1}T^{a}_{l_1}(u_1)$. And so on. Here l_1 means the leader element.

A description of Algorithm 1 and an example that implements the operation of this algorithm can be found in [3]. Now we give a generalization of Algorithm 1. Note that the algorithm works for texts with even length.

Algorithm 3.1. ***(Algorithm 1*).***

Encryption. Initially, we have the plaintext $u_1, u_2, \ldots, u_{2n}$.

Step 1.

$$ {}_{g_1}T_{l_1}^{a_{11}}(u_1) = v_1,\ {}_{g_2}T_{l_2}^{a_{12}}(u_2) = v_2,\ F_1^{a_{13}}(v_1, v_2) = (v_1^{'}, v_2^{'}); $$

Step 2.

$$ {}_{g_3}T_{v_1^{'}}^{a_{21}}(u_3) = v_3,\ {}_{g_4}T_{v_2^{'}}^{a_{22}}(u_4) = v_4,\ F_2^{a_{23}}(v_3, v_4) = (v_3^{'}, v_4^{'}); $$

Step 3.

$$ {}_{g_5}T_{v_3^{'}}^{a_{31}}(u_5) = v_5,\ {}_{g_6}T_{v_4^{'}}^{a_{32}}(u_6) = v_6,\ F_3^{a_{33}}(v_5, v_6) = (v_5^{'}, v_6^{'}); $$

$$ \ldots $$

Step N.

$$ {}_{g_{2n-1}}T_{v_{2n-1}^{'}}^{a_{n_1}}(u_{2n-1}) = v_{2n-1},\ {}_{g_{2n}}T_{v_{2n}^{'}}^{a_{n_2}}(u_{2n}) = v_{2n},\ F_n^{a_{n_3}}(v_{2n-1}, v_{2n}) = (v_{2n-1}^{'}, v_{2n}^{'}). $$

We get the ciphertext $v_1^{'}, v_2^{'}, \ldots, v_{2n}^{'}$.

Thus, to encrypt a plaintext of length $2n$, we need $2n$ different operations on the quasigroups used in Algorithm 1*, n different values of the function F, 2 leader elements, and $3n$ different powers used in the translation algorithm.

Decryption. Let us have a ciphertext of the form $v_1^{'}, v_2^{'}, \ldots, v_{2n}^{'}$.

Step 1.

$$ F_1^{-a_{13}}(v_1^{'}, v_2^{'}) = (v_1, v_2),\ {}_{g_1}T_{l_1}^{-a_{11}}(v_1) = u_1,\ {}_{g_2}T_{l_2}^{-a_{12}}(v_2) = u_2; $$

Step 2.

$$ F_2^{-a_{23}}(v_3', v_4') = (v_3, v_4);\ {}_{g_3}T_{v_1^{'}}^{-a_{21}}(v_3) = u_3,\ {}_{g_4}T_{v_2^{'}}^{-a_{22}}(v_4) = u_4, $$

Step 3.

$$ F_3^{-a_{33}}(v_5^{'}, v_6^{'}) = (v_5, v_6);\ {}_{g_5}T_{v_3^{'}}^{-a_{31}}(v_5) = u_5,\ {}_{g_6}T_{v_4^{'}}^{-a_{32}}(v_6) = u_6, $$

$$ \ldots $$

Step N.

$$ F_n^{-a_{n_3}}(v_{2n-1}^{'}, v_{2n}^{'}) = (v_{2n-1}, v_{2n}). $$

$$ {}_{g_{2n-1}}T_{v_{2n-1}^{'}}^{-a_{n_1}}(v_{2n-1}) = u_{2n-1},\ {}_{g_{2n}}T_{v_{2n}^{'}}^{-a_{n_2}}(v_{2n}) = u_{2n}, $$

We received a plaintext of the form $u_1, u_2, \ldots, u_{2n}$.

From Algorithm 1* we obtain the classical Markovski algorithm if we take only one quasigroup, one kind of quasigroup translation (left translations), each of which is taken to the first degree, and if the system of orthogonal operations (crypto procedures F) is not used. Some generalizations of Algorithm 1 are given in [4].

The Markovski algorithm is a special case of Algorithm 1*. As in the Markovski algorithm, in Algorithm 1*, the powers $a_{11}, a_{12}, \dots, a_{n3}$ must be different to protect this algorithm from chosen plaintext and ciphertext attacks [3].

Consider the operation of Algorithm 1* based on T-quasigroups using a specific example.

Example 3.1. *Take a cyclic group* $(Z_{313}, +) = (A, +)$.

Initially, we have a plaintext $u_1, u_2, u_3, u_4, u_5, u_6$ *of length* $k = 6$.

1) *First, we define all the* T*-quasigroups we need.*

1. *Define a* T*-quasigroup* $(A, g_1) = (A, *)$ *of the form:*

$$x * y = 25 \cdot x + 37 \cdot y + 11$$

with the leader l_1. *The mapping* $x \longrightarrow x * l_1$ *will be denoted by* R_{l_1},*i.e.,* $R_{l_1}(x) = x * l_1$ *for all* $x \in A$.

To find the mapping $R_{l_1}^{-1}$ *taking into account Table 1, we find the type of operation* $\overset{(13)}{*}$ *using the formula:* $x \overset{(13)}{*} y = \phi^{-1}x - \phi^{-1}\psi y - \phi^{-1}c$.

$$\phi = 25, \psi = 37, c = 11,$$
$$\phi^{-1} \equiv 25^{311}(\mod 313) \equiv 288, -\phi^{-1} \equiv 25,$$
$$-\phi^{-1}\psi \equiv 25 \cdot 37(\mod 313) \equiv 299,$$
$$-\phi^{-1}c \equiv 25 \cdot 11(\mod 313) \equiv 275.$$

We have:

$$x \overset{(13)}{*} y = 288 \cdot x + 299 \cdot y + 275,$$

$R_{l_1}^{-1} = x \overset{(13)}{*} l_1 = R_{l_1}^{(13)}x$. *In a sense, the quasigroup* $(A, \overset{(13)}{*})$ *is a "right inverse quasigroup" to the quasigroup* $(A, *)$. *Note that Corollary 2.1 implies that* $(A, *) \perp (A, \overset{(13)}{*})$.

2. *Define a* T*-quasigroup* $(A, g_2) = (A, \circ)$ *of the form:*

$$x \circ y = 75 \cdot x + 39 \cdot y + 100$$

with the leader l_2. *Mapping* $x \longrightarrow l_2 \circ x$ *is denoted by* L_{l_2},*i.e.,* $L_{l_2}(x) = l_2 \circ x$ *for all* $x \in A$.

To find the mapping $L_{l_2}^{-1}$, *we use Table 1 and find the type of operation* $\overset{(23)}{\circ}$ *by the formula:* $x \overset{(23)}{\circ} y = -\psi^{-1}\phi x + \psi^{-1}y - \psi^{-1}c$.

$$\phi = 75, \psi = 39, c = 100,$$
$$\psi^{-1} \equiv 39^{311}(\mod 313) \equiv 305, -\psi^{-1} \equiv 8,$$
$$-\psi^{-1}\phi \equiv 8 \cdot 75(\mod 313) \equiv 287,$$
$$-\psi^{-1}c \equiv 8 \cdot 100(\mod 313) \equiv 174.$$

We have:

$$x \overset{(23)}{\circ} y = 287 \cdot x + 305 \cdot y + 174.$$

3. *Let us define a T-quasigroup $(A, g_3) = (A, \star)$ of the form:*

$$x \star y = 127 \cdot x + 213 \cdot y + 19.$$

Mapping $x \longrightarrow y \star x$ is denoted by L_y,i.e., $L_y(x) = y \star x$ for all $x \in A$.

To find the mapping L_y^{-1}, use Table 1 and find the type of operation $\overset{(23)}{\star}$ by the formula: $x \overset{(23)}{\star} y = -\psi^{-1}\phi x + \psi^{-1}y - \psi^{-1}c$.

$$\phi = 127, \psi = 213, c = 19,$$
$$\psi^{-1} \equiv 213^{311}(\mod 313) \equiv 241, -\psi^{-1} \equiv 72,$$
$$-\psi^{-1}\phi \equiv 72 \cdot 127(\mod 313) \equiv 67,$$
$$-\psi^{-1}c \equiv 72 \cdot 19(\mod 313) \equiv 116.$$

We have:

$$x \overset{(23)}{\star} y = 67 \cdot x + 241 \cdot y + 116.$$

4. *Let us define a T-quasigroup $(A, g_4) = (A, \lozenge)$ of the form:*

$$x \lozenge y = 151 \cdot x + 301 \cdot y + 199.$$

Denote the mapping $x \longrightarrow x \lozenge y$ by R_y,i.e., $R_y(x) = x \lozenge y$ for all $x \in A$.

To find the mapping R_y^{-1}, taking into account Table 1, we find the type of operation $\overset{(13)}{\lozenge}$ using the formula: $x \overset{(13)}{\lozenge} y = \phi^{-1}x - \phi^{-1}\psi y - \phi^{-1}c$.

$$\phi = 151, \psi = 301, c = 199,$$
$$\phi^{-1} \equiv 151^{311}(\mod 313) \equiv 199, -\phi^{-1} \equiv 114,$$
$$-\phi^{-1}\psi \equiv 114 \cdot 301(\mod 313) \equiv 197,$$
$$-\phi^{-1}c \equiv 114 \cdot 199(\mod 313) \equiv 150.$$

We have:

$$x \overset{(23)}{\lozenge} y = 199 \cdot x + 197 \cdot y + 150.$$

5. *Let us define a T-quasigroup $(A, g_5) = (A, \odot)$ of the form:*

$$x \odot y = 213 \cdot x + 3 \cdot y + 9.$$

The mapping $x \longrightarrow x \odot y$ will be denoted by R_y,i.e., $R_y(x) = x \odot y$ for all $x \in A$.

To find the mapping R_y^{-1}, taking into account Table 1, we find the type of operation $\overset{(13)}{\odot}$ using the formula: $x \overset{(13)}{\odot} y = \phi^{-1}x - \phi^{-1}\psi y - \phi^{-1}c$.

$$\phi = 213, \psi = 3, c = 9,$$
$$\phi^{-1} \equiv 213^{311}(\mod 313) \equiv 241, -\phi^{-1} \equiv 72,$$
$$-\phi^{-1}\psi \equiv 72 \cdot 3(\mod 313) \equiv 216,$$
$$-\phi^{-1}c \equiv 72 \cdot 9(\mod 313) \equiv 22.$$

We have:

$$x \overset{(23)}{\odot} y = 241 \cdot x + 216 \cdot y + 22.$$

6. *Let us define a T-quasigroup* $(A, g_6) = (A, \circledast)$ *of the form:*

$$x \circledast y = 303 \cdot x + 200 \cdot y + 99.$$

Mapping $x \longrightarrow y \circledast x$ *is denoted by* L_y*,i.e.,* $L_y(x) = y \circledast x$ *for all* $x \in A$.

To find the mapping L_y^{-1}*, we use Table 1 and find the type of operation* $\overset{(23)}{\circledast}$ *by the formula:* $x \overset{(23)}{\circledast} y = -\psi^{-1}\phi x + \psi^{-1}y - \psi^{-1}c$.

$$\phi = 303, \psi = 200, c = 99,$$
$$\psi^{-1} \equiv 200^{311} (\mod 313) \equiv 36, -\psi^{-1} \equiv 277,$$
$$-\psi^{-1}\phi \equiv 277 \cdot 303 (\mod 313) \equiv 47,$$
$$-\psi^{-1}c \equiv 277 \cdot 99 (\mod 313) \equiv 192.$$

We have:

$$x \overset{(23)}{\circledast} y = 47 \cdot x + 36 \cdot y + 192.$$

2) *We define systems of two parastrophic orthogonal T-quasigroups. We need* 3 *systems of cryptofunctions* F.

1. *We define a system of two parastrophic orthogonal T-quasigroups* $(A, g_7) = (A, \oslash)$ *and* $(A, \overset{(23)}{\oslash})$ *as follows:*

$$\begin{cases} x \oslash y = 7 \cdot x + 12 \cdot y + 13 \\ x \overset{(23)}{\oslash} y = 182 \cdot x + 287 \cdot y + 25. \end{cases}$$

Denote the quasigroup system $(A, \oslash, \overset{(23)}{\oslash})$ *by* $F_1(x, y)$*, since this system is a function of two variables. To find the mapping* $F_1^{-1}(x, y)$*, we solve the system of linear equations:*

$$\begin{cases} 7 \cdot x + 12 \cdot y + 13 = a \\ 182 \cdot x + 287 \cdot y + 25 = b \end{cases} \Leftrightarrow \begin{cases} 175y = 182 \cdot a + 306 \cdot b \\ 138x = 287 \cdot a + 301 \cdot b + 12 \end{cases} \Rightarrow$$

$$F_1^{-1}(x, y) : \begin{cases} x = 86 \cdot a + 136 \cdot b + 177 \\ y = 289 \cdot a + 25 \cdot b. \end{cases}$$

Therefore, we have, if $F_1(x, y) = (a, b) = (7 \cdot x + 12 \cdot y + 13, 182 \cdot x + 287 \cdot y + 25)$*, then* $F_1^{-1}(a, b) = (86 \cdot a + 136 \cdot b + 177, 289 \cdot a + 25 \cdot b)$.

2. *We define a system of two parastrophic orthogonal T-quasigroups* $(A, g_8) = (A, \circledcirc)$ *and* $(A, \overset{(23)}{\circledcirc})$ *as follows:*

$$\begin{cases} x \circledcirc y = 79 \cdot x + 113 \cdot y + 23 \\ x \overset{(23)}{\circledcirc} y = 27 \cdot x + 277 \cdot y + 202. \end{cases}$$

Denote the quasigroup system $(A, \circledcirc, \overset{(23)}{\circledcirc})$ *by* $F_2(x, y)$*, since this system is a function of two variables. To find the mapping* $F_2^{-1}(x, y)$*, we solve the system of linear equations:*

$$\begin{cases} 79 \cdot x + 113 \cdot y + 23 = a \\ 27 \cdot x + 277 \cdot y + 202 = b \end{cases} \Leftrightarrow \begin{cases} 52y = -27 \cdot a + 79 \cdot b \\ 52x = 277 \cdot a - 113 \cdot b + 179 \end{cases} \Rightarrow$$

$$F_2^{-1}(x, y) : \begin{cases} x = 216 \cdot a + 52 \cdot b + 178 \\ y = 162 \cdot a + 152 \cdot b. \end{cases}$$

Therefore, we have, if $F_2(x, y) = (79 \cdot x + 113 \cdot y + 23, 27 \cdot x + 277 \cdot y + 202)$*, then* $F_2^{-1}(a, b) = (216 \cdot a + 52 \cdot b + 178, 162 \cdot a + 152 \cdot b)$.

3. *We define a system of two parastrophic orthogonal* T*-quasigroups* $(A, g_9) = (A, \otimes)$ *and* $(A, \overset{(13)}{\otimes})$ *as follows:*

$$\begin{cases} x \otimes y = 81 \cdot x + 101 \cdot y + 99 \\ x \overset{(13)}{\otimes} y = 228 \cdot x + 134 \cdot y + 277. \end{cases}$$

Denote the quasigroup system $(A, \otimes, \overset{(13)}{\otimes})$ *by* $F_3(x, y)$*, since this system is a function of two variables. To find the mapping* $F_3^{-1}(x, y)$*, we solve the system of linear equations:*

$$\begin{cases} 81 \cdot x + 101 \cdot y + 99 = a \\ 228 \cdot x + 134 \cdot y + 277 = b \end{cases} \Leftrightarrow \begin{cases} 280y = 228 \cdot a - 81 \cdot b - 135 \\ 280x = -134 \cdot a + 101 \cdot b \end{cases} \Rightarrow$$

$$F_3^{-1}(x, y) : \begin{cases} x = 42 \cdot a + 272 \cdot b \\ y = 50 \cdot a + 287 \cdot b + 61. \end{cases}$$

Therefore, we have, if $F_3(x, y) = (81 \cdot x + 101 \cdot y + 99, 228 \cdot x + 134 \cdot y + 277)$*, then* $F_3^{-1}(a, b) = (42 \cdot a + 272 \cdot b, 50 \cdot a + 287 \cdot b + 61)$.

Now we can use all the operations described above in Algorithm 1.*

Let the plaintext look like ***56; 43; 105; 59; 61; 19***.

In the algorithm, we will use the following values of the degrees:

$a_{11} = 3; a_{12} = 1; a_{13} = 2; a_{21} = 1; a_{22} = 3; a_{23} = 2; a_{31} = 3; a_{32} = 2; a_{33} = 1.$

Let us choose the elements as leaders: $l_1 = 110; l_2 = 210.$

Encryption.

Step 1.

$$_{g_1}T_{l_1}^{a_{11}}(u_1) = {}_{g_1}T_{110}^{3}(56) = v_1,$$
$$_{g_1}T_{110}^{1}(56) = u_1 * l_1 = 56 * 110 = 25 \cdot 56 + 37 \cdot 110 + 11 = 160,$$
$$_{g_1}T_{110}^{2}(56) = {}_{g_1}T_{110}^{1}(160) = 160 * 110 = 25 \cdot 160 + 37 \cdot 110 + 11 = 256,$$

$$_{g_1}T^3_{110}(56) = {_{g_1}T^1_{110}}(256) = 256 * 110 = 25 \cdot 256 + 37 \cdot 110 + 11 = 152 = v_1,$$
$$_{g_2}T^{a_{12}}_{l_2}(u_2) = {_{g_2}T^1_{210}}(43) = v_2,$$
$$_{g_2}T^1_{210}(43) = l_2 \circ 43 = 75 \cdot 210 + 39 \cdot 43 + 100 = 312 = v_2,$$
$$F_1^{a_{13}}(v_1, v_2) = F_1^2(152, 312) = (v_1^{'}, v_2^{'}),$$
$$F_1^1(152, 312) = (7 \cdot 152 + 12 \cdot 312 + 13, 182 \cdot 152 + 287 \cdot 312 + 25) = (126, 171),$$
$$F_1^2(152, 312) = F_1^1(126, 171) = (7 \cdot 126 + 12 \cdot 171 + 13, 182 \cdot 126 + 287 \cdot 171 + 25) = (130, 44) = (v_1^{'}, v_2^{'}),$$

Step 2.

$$_{g_3}T^{a_{21}}_{v_1^{'}}(u_3) = {_{g_3}T^1_{130}}(105) = v_3,$$
$$_{g_3}T^1_{130}(105) = 130 \star 105 = 127 \cdot 130 + 213 \cdot 105 + 19 = 82 = v_3,$$
$$_{g_4}T^{a_{22}}_{v_2^{'}}(u_4) = {_{g_4}T^3_{44}}(59) = v_4,$$
$$_{g_4}T^1_{44}(59) = 59 \Diamond 44 = 151 \cdot 59 + 301 \cdot 44 + 199 = 129,$$
$$_{g_4}T^2_{44}(59) = {_{g_4}T^1_{44}}(129) = 129 \Diamond 44 = 151 \cdot 129 + 301 \cdot 44 + 199 = 57,$$
$$_{g_4}T^3_{44}(59) = {_{g_4}T^1_{44}}(57) = 57 \Diamond 44 = 151 \cdot 57 + 301 \cdot 44 + 199 = 140 = v_4,$$
$$F_2^{a_{23}}(v_3, v_4) = F_2^2(82, 140) = (v_3^{'}, v_4^{'}),$$
$$F_2^1(82, 140) = (79 \cdot 82 + 113 \cdot 140 + 23, 27 \cdot 82 + 277 \cdot 140 + 202) = (98, 193),$$
$$F_2^2(82, 140) = F_2^1(98, 193) = (79 \cdot 98 + 113 \cdot 193 + 23, 27 \cdot 98 + 277 \cdot 193 + 202) = (152, 282) = (v_3^{'}, v_4^{'}),$$

Step 3.

$$_{g_5}T^{a_{31}}_{v_3^{'}}(u_5) = {_{g_5}T^3_{152}}(61) = v_5,$$
$$_{g_5}T^1_{152}(61) = 61 \odot 152 = 213 \cdot 61 + 3 \cdot 152 + 9 = 312,$$
$$_{g_5}T^2_{152}(61) = {_{g_5}T^1_{152}}(312) = 312 \odot 152 = 213 \cdot 312 + 3 \cdot 152 + 9 = 252,$$
$$_{g_5}T^3_{152}(61) = {_{g_5}T^1_{152}}(252) = 252 \odot 152 = 213 \cdot 252 + 3 \cdot 152 + 9 = 305 = v_5,$$
$$_{g_6}T^{a_{32}}_{v_4^{'}}(u_6) = {_{g_6}T^2_{282}}(19) = v_6,$$
$$_{g_6}T^1_{282}(19) = 282 \circledast 19 = 303 \cdot 282 + 200 \cdot 19 + 99 = 140,$$
$$_{g_6}T^2_{282}(19) = {_{g_6}T^1_{282}}(140) = 282 \circledast 140 = 303 \cdot 282 + 200 \cdot 140 + 99 = 239 = v_6,$$
$$F_3^{a_{33}}(v_5, v_6) = F_3^1(305, 239) = (v_5^{'}, v_6^{'}),$$
$$F_3^1(305, 239) = (81 \cdot 305 + 101 \cdot 239 + 99, 228 \cdot 305 + 134 \cdot 239 + 277) = (115, 118) = (v_5^{'}, v_6^{'}).$$

We get the following ciphertext: ***130; 44; 152; 282; 115; 118***.

Decryption.

Step 1.

$$F_1^{-a_{13}}(v_1^{'}, v_2^{'}) = F_1^{-2}(130, 44) = (v_1, v_2),$$
$$F_1^{-1}(130, 144) = (86 \cdot 130 + 136 \cdot 44 + 177, 289 \cdot 130 + 25 \cdot 44) = (126, 171),$$

$$F_1^{-2}(130,144) = F_1^{-1}(126,171) = (86\cdot 126 + 136\cdot 171 + 177, 289\cdot 126 + 25\cdot 171) = (152,312) = (v_1,v_2),$$

$$_{g_1}T_{l_1}^{-a_{11}}(v_1) = {}_{g_1}T_{110}^{-3}(152) = u_1,$$

$$_{g_1}T_{110}^{-1}(152) = 152 \overset{(13)}{*} 110 = 288\cdot 152 + 299\cdot 110 + 275 = 256,$$

$$_{g_1}T_{110}^{-2}(152) = {}_{g_1}T_{110}^{-1}(256) = 256 \overset{(13)}{*} 110 = 288\cdot 256 + 299\cdot 110 + 275 = 160,$$

$$_{g_1}T_{110}^{-3}(152) = {}_{g_1}T_{110}^{-1}(160) = 160 \overset{(13)}{*} 110 = 288\cdot 160 + 299\cdot 110 + 275 = 56 = u_1,$$

$$_{g_2}T_{l_2}^{-a_{12}}(v_2) = {}_{g_2}T_{210}^{-1}(312) = u_2,$$

$$_{g_2}T_{210}^{-1}(312) = 210 \overset{(23)}{\circ} 312 = 287\cdot 210 + 305\cdot 312 + 174 = 43 = u_2,$$

Step 2.

$$F_2^{-a_{23}}(v_3', v_4') = F_2^{-2}(152,282) = (v_3,v_4),$$

$$F_2^{-1}(152,282) = (216\cdot 152 + 52\cdot 282 + 178, 162\cdot 152 + 152\cdot 282) = (98,193),$$

$$F_2^{-2}(152,282) = F_2^{-1}(98,193) = (216\cdot 98 + 52\cdot 193 + 178, 162\cdot 98 + 152\cdot 193) = (82,140) = (v_3,v_4),$$

$$_{g_3}T_{v_1'}^{-a_{21}}(v_3) = {}_{g_3}T_{130}^{-1}(82) = u_3,$$

$$_{g_3}T_{130}^{-1}(82) = 130 \overset{(23)}{\star} 82 = 67\cdot 130 + 241\cdot 82 + 116 = 105 = u_3,$$

$$_{g_4}T_{v_2'}^{-a_{22}}(v_4) = {}_{g_4}T_{44}^{-3}(140) = u_4,$$

$$_{g_4}T_{44}^{-1}(140) = 140 \overset{(13)}{\diamond} 44 = 199\cdot 140 + 197\cdot 44 + 150 = 57,$$

$$_{g_4}T_{44}^{-2}(140) = {}_{g_4}T_{44}^{-1}(57) = 57 \overset{(13)}{\diamond} 44 = 199\cdot 57 + 197\cdot 44 + 150 = 129,$$

$$_{g_4}T_{44}^{-3}(140) = {}_{g_4}T_{44}^{-1}(129) = 129 \overset{(13)}{\diamond} 44 = 199\cdot 129 + 197\cdot 44 + 150 = 59 = u_4,$$

Step 3.

$$F_3^{-a_{33}}(v_5', v_6') = F_3^{-2}(115,118) = (v_5,v_6),$$

$$F_3^{-1}(115,118) = (42\cdot 115 + 272\cdot 118, 50\cdot 115 + 287\cdot 118 + 61) = (305,239) = (v_5,v_6),$$

$$_{g_5}T_{v_3'}^{-a_{31}}(v_5) = {}_{g_5}T_{152}^{-3}(305) = u_5,$$

$$_{g_5}T_{152}^{-1}(305) = 305 \overset{(13)}{\circledcirc} 152 = 241\cdot 305 + 216\cdot 152 + 22 = 252,$$

$$_{g_5}T_{152}^{-2}(305) = {}_{g_5}T_{152}^{-1}(252) = 252 \overset{(13)}{\circledcirc} 152 = 241\cdot 252 + 216\cdot 152 + 22 = 312,$$

$$_{g_5}T_{152}^{-3}(305) = {}_{g_5}T_{152}^{-1}(312) = 312 \overset{(13)}{\circledcirc} 152 = 241\cdot 312 + 216\cdot 152 + 22 = 61 = u_5,$$

$$_{g_6}T_{v_4'}^{-a_{32}}(v_6) = {}_{g_6}T_{282}^{-2}(239) = u_6,$$

$$_{g_6}T_{282}^{-1}(239) = 282 \overset{(23)}{\circledast} 239 = 47\cdot 282 + 36\cdot 239 + 192 = 140,$$

$$_{g_6}T_{282}^{-2}(239) = {}_{g_6}T_{282}^{-1}(140) = 282 \overset{(23)}{\circledast} 140 = 47\cdot 282 + 36\cdot 140 + 192 = 19 = u_6.$$

We get the following plaintext: ***56; 43; 105; 59; 61; 19***.

4. Conclusion

A program has been developed that uses a free version of the Pascal ABC programming language. The conducted experiments show that encoding/decoding is performed quite quickly. The program works for any values of leaders, powers, and any plaintext of length 6. It can be easily modified for a text of any length and any used quasigroups and functions.

Algorithm 1* makes it possible to obtain an almost "natural" stream cipher, i.e., a stream cipher that encodes a pair of plaintext elements at any step. It is easy to see that Algorithm 1* can be generalized to the n-ary case.

Remark 4.1. *Proper binary groupoids are preferable to linear quasigroups in terms of the construction of the map $F(x, y)$ for greater encryption security, but in this case, decryption may be slower than in the case of linear quasigroups, and the definition of these groupoids requires more memory. The same remark is valid for the choice of the function g. Perhaps the golden mean in this choice problem is the use of linear quasigroups over non-Abelian, especially simple, groups.*

Remark 4.2. *In this cipher, there is a possibility of protection against the standard statistical attack. For this area, more commonly used letters or pairs of letters can be denoted by more than one integer or more than one pair of integers* [3].

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The Lie algebra admitted by the ternary differential system with nonlinearities of degree six

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Abstract. In this paper, the ternary differential system with nonlinearities of degree six is investigated. For this system, nine Lie operators providing a linear representation of one-parameter elementary groups are determined in the space of phase variables and coefficients. The commutators of these operators, which form a Lie algebra L_9, are constructed. The structure constants are determined, and the type of the Lie algebra is analyzed. By means of the Killing form, it is proved that the Lie algebra L_9 is reductive.
2020 Mathematics Subject Classification: 37C05.
Keywords: differential system, Lie algebra, continuous one-parameter elementary group, structure equation, associated application matrix, Killing's form.

Algebra Lie admisă de sistemul diferenţial ternar cu neliniarităţi de gradul șase

Rezumat. În acest articol a fost studiat sistemul diferenţial ternar cu neliniarităţi de gradul şase. Pentru acest sistem au fost determinaţi nouă operatori Lie de reprezentare liniară a grupurilor elementare uniparametrice în spaţiul coeficienţilor şi variabilelor de fază. Au fost construiţi comutatorii acestor operatori, ce formează o algebră Lie L_9. Au fost determinate constantele de structură şi studiat tipul algebrei Lie. Cu ajutorul formei lui Killing, a fost demonstrat că algebra Lie L_9 este reductivă.
Cuvinte-cheie: sistem diferenţial, algebră Lie, grup elementar continuu uniparametric, ecuaţie de structură, matricea aplicaţiei asociate, forma Killing.

1. Introduction

An important role in the theory of ordinary differential equations is attributed to the class of autonomous systems of first-order differential equations whose right-hand sides are polynomials with real coefficients. The study of these systems is actual because they arise in many problems of a theoretical or applied nature.

In this paper, we examine the differential system $s^3(1,6)$ of the form

$$\frac{dx^j}{dt} = a^j_\alpha x^\alpha + a^j_{\alpha\beta\gamma\delta\mu\nu} x^\alpha x^\beta x^\gamma x^\delta x^\mu x^\nu \quad (j,\alpha,\beta,\gamma,\delta,\mu,\nu = \overline{1,3}), \tag{1}$$

where $a^j_{\alpha\beta\gamma\delta\mu\nu}$ is a symmetric tensor in lower indices in which the total convolution is done [1], [2], [3], [4]. The coefficients and variables of this system take values from the field of real numbers $\mathbb{R}$.

2. Lie operators of representation of elementary uniparametric transformation groups of the ternary differential system with nonlinearities of degree six

Remark 2.1. *In three-dimensional space, the number of elementary one-parameter continuous groups is determined by the following nine cases* [8]*:*

$$\bar{x}^1 = \lambda_1 x^1,\ \bar{x}^2 = x^2,\ \bar{x}^3 = x^3\ (\lambda_1 \in \mathbb{R}\setminus\{0\}), \tag{2}$$

$$\bar{x}^1 = x^1,\ \bar{x}^2 = \lambda_2 x^2,\ \bar{x}^3 = x^3\ (\lambda_2 \in \mathbb{R}\setminus\{0\}), \tag{3}$$

$$\bar{x}^1 = x^1,\ \bar{x}^2 = x^2,\ \bar{x}^3 = \lambda_3 x^3\ (\lambda_3 \in \mathbb{R}\setminus\{0\}), \tag{4}$$

$$\bar{x}^1 = x^1 + \mu_1 x^2,\ \bar{x}^2 = x^2,\ \bar{x}^3 = x^3\ (\mu_1 \in \mathbb{R}), \tag{5}$$

$$\bar{x}^1 = x^1 + \mu_2 x^3,\ \bar{x}^2 = x^2,\ \bar{x}^3 = x^3\ (\mu_2 \in \mathbb{R}), \tag{6}$$

$$\bar{x}^1 = x^1,\ \bar{x}^2 = \mu_3 x^1 + x^2,\ \bar{x}^3 = x^3\ (\mu_3 \in \mathbb{R}), \tag{7}$$

$$\bar{x}^1 = x^1,\ \bar{x}^2 = x^2 + \mu_4 x^3,\ \bar{x}^3 = x^3\ (\mu_4 \in \mathbb{R}), \tag{8}$$

$$\bar{x}^1 = x^1,\ \bar{x}^2 = x^2,\ \bar{x}^3 = \mu_5 x^1 + x^3\ (\mu_5 \in \mathbb{R}), \tag{9}$$

$$\bar{x}^1 = x^1,\ \bar{x}^2 = x^2,\ \bar{x}^3 = \mu_6 x^2 + x^3\ (\mu_6 \in \mathbb{R}), \tag{10}$$

the groups denoted respectively by H_i with parameters λ_i $(i = \overline{1,3})$ and H_{j+3} with parameters μ_j $(j = \overline{1,6})$, in the phase space $E^3(x)$ of system (1), where $x = (x^1, x^2, x^3)$.

It is known [2], [3], [7] that to each one-parameter transformation group acting in the space $E^{96}(x,a)$ of system (1), we can associate an operator of the form

$$X = \xi^j \frac{\partial}{\partial x^j} + \mathsf{d}\ \ (j = \overline{1,3}), \tag{11}$$

where

$$\mathsf{d} = \eta^j_\alpha \frac{\partial}{\partial a^j_\alpha} + \eta^j_{\alpha\beta\gamma\delta\mu\nu} \frac{\partial}{\partial a^j_{\alpha\beta\gamma\delta\mu\nu}}\ \ (j,\alpha,\beta,\gamma,\delta,\mu,\nu = \overline{1,3}), \tag{12}$$

and

$$\xi^j = \frac{\partial \bar{x}^j}{\partial \alpha}|_{\alpha=\alpha_0},\ \ \eta^j_\alpha = \frac{\partial \bar{a}^j_\alpha}{\partial \alpha}|_{\alpha=\alpha_0},\ \ \eta^j_{\alpha\beta\gamma\delta} = \frac{\partial \bar{a}^j_{\alpha\beta\gamma\delta\mu\nu}}{\partial \alpha}|_{\alpha=\alpha_0}, \tag{13}$$
$$(j,\alpha,\beta,\gamma,\delta,\mu,\nu = \overline{1,3})$$

when $\alpha_0 = 0$, and conversely.

The following theorem holds

Theorem 2.1. *The Lie operators corresponding to the representations of the groups H_i $(i=\overline{1,3})$ and H_{j+3} $(j=\overline{1,6})$, from (2) - (10), in the space of variables and coefficients $E^{96}(x,a)$ of system (1), are*

$$\begin{aligned} &X_1 = x^1\frac{\partial}{\partial x^1} + \mathsf{d}_1, \quad X_2 = x^2\frac{\partial}{\partial x^2} + \mathsf{d}_2, \quad X_3 = x^3\frac{\partial}{\partial x^3} + \mathsf{d}_3,\\ &X_4 = x^2\frac{\partial}{\partial x^1} + \mathsf{d}_4, \quad X_5 = x^3\frac{\partial}{\partial x^1} + \mathsf{d}_5, \quad X_6 = x^1\frac{\partial}{\partial x^2} + \mathsf{d}_6,\\ &X_7 = x^3\frac{\partial}{\partial x^2} + \mathsf{d}_7, \quad X_8 = x^1\frac{\partial}{\partial x^3} + \mathsf{d}_8, \quad X_9 = x^2\frac{\partial}{\partial x^3} + \mathsf{d}_9, \end{aligned} \tag{14}$$

where

$$\mathsf{d}_i = \mathsf{D}_i^{(1)} + \mathsf{D}_i^{(6)} \ \ (i=\overline{1,9}), \tag{15}$$

and

$$\begin{aligned} \mathsf{D}_1^{(1)} &= a_2^1\frac{\partial}{\partial a_2^1} + a_3^1\frac{\partial}{\partial a_3^1} - a_1^2\frac{\partial}{\partial a_1^2} - a_1^3\frac{\partial}{\partial a_1^3},\\ \mathsf{D}_2^{(1)} &= -a_2^1\frac{\partial}{\partial a_2^1} + a_1^2\frac{\partial}{\partial a_1^2} + a_3^2\frac{\partial}{\partial a_3^2} - a_2^3\frac{\partial}{\partial a_2^3},\\ \mathsf{D}_3^{(1)} &= -a_3^1\frac{\partial}{\partial a_3^1} - a_3^2\frac{\partial}{\partial a_3^2} + a_1^3\frac{\partial}{\partial a_1^3} + a_2^3\frac{\partial}{\partial a_2^3},\\ \mathsf{D}_4^{(1)} &= a_1^2\frac{\partial}{\partial a_1^1} + (a_2^2 - a_1^1)\frac{\partial}{\partial a_2^1} + a_3^2\frac{\partial}{\partial a_3^1} - a_1^2\frac{\partial}{\partial a_2^2} - a_1^3\frac{\partial}{\partial a_2^3},\\ \mathsf{D}_5^{(1)} &= a_1^3\frac{\partial}{\partial a_1^1} + a_2^3\frac{\partial}{\partial a_2^1} + (a_3^3 - a_1^1)\frac{\partial}{\partial a_3^1} - a_1^2\frac{\partial}{\partial a_3^2} - a_1^3\frac{\partial}{\partial a_3^3},\\ \mathsf{D}_6^{(1)} &= -a_2^1\frac{\partial}{\partial a_1^1} + (a_1^1 - a_2^2)\frac{\partial}{\partial a_1^2} + a_2^1\frac{\partial}{\partial a_2^2} + a_3^1\frac{\partial}{\partial a_3^2} - a_2^3\frac{\partial}{\partial a_1^3},\\ \mathsf{D}_7^{(1)} &= -a_2^1\frac{\partial}{\partial a_3^1} + a_1^3\frac{\partial}{\partial a_1^2} + a_2^3\frac{\partial}{\partial a_2^2} + (a_3^3 - a_2^2)\frac{\partial}{\partial a_3^2} - a_2^3\frac{\partial}{\partial a_3^3},\\ \mathsf{D}_8^{(1)} &= -a_3^1\frac{\partial}{\partial a_1^1} - a_3^2\frac{\partial}{\partial a_1^2} + (a_1^1 - a_3^3)\frac{\partial}{\partial a_1^3} + a_2^1\frac{\partial}{\partial a_2^3} + a_3^1\frac{\partial}{\partial a_3^3},\\ \mathsf{D}_9^{(1)} &= -a_3^1\frac{\partial}{\partial a_2^1} - a_3^2\frac{\partial}{\partial a_2^2} + a_1^2\frac{\partial}{\partial a_1^3} + (a_2^2 - a_3^3)\frac{\partial}{\partial a_2^3} + a_3^2\frac{\partial}{\partial a_3^3}, \end{aligned} \tag{16}$$

$$\begin{aligned} \mathsf{D}_1^{(6)} = &-5a^1_{111111}\frac{\partial}{\partial a^1_{111111}} - 4a^1_{111112}\frac{\partial}{\partial a^1_{111112}} - 4a^1_{111113}\frac{\partial}{\partial a^1_{111113}} -\\ &-3a^1_{111122}\frac{\partial}{\partial a^1_{111122}} - 3a^1_{111123}\frac{\partial}{\partial a^1_{111123}} - 3a^1_{111133}\frac{\partial}{\partial a^1_{111133}} - 2a^1_{111222}\frac{\partial}{\partial a^1_{111222}} -\\ &-2a^1_{111223}\frac{\partial}{\partial a^1_{111223}} - 2a^1_{111233}\frac{\partial}{\partial a^1_{111233}} - 2a^1_{111333}\frac{\partial}{\partial a^1_{111333}} - a^1_{112222}\frac{\partial}{\partial a^1_{112222}} - \end{aligned}$$

$$-a^1_{112223}\frac{\partial}{\partial a^1_{112223}} - a^1_{112233}\frac{\partial}{\partial a^1_{112233}} - a^1_{112333}\frac{\partial}{\partial a^1_{112333}} - a^1_{113333}\frac{\partial}{\partial a^1_{113333}} +$$
$$+a^1_{222222}\frac{\partial}{\partial a^1_{222222}} + a^1_{222223}\frac{\partial}{\partial a^1_{222223}} + a^1_{222233}\frac{\partial}{\partial a^1_{222233}} + a^1_{222333}\frac{\partial}{\partial a^1_{222333}} +$$
$$+a^1_{223333}\frac{\partial}{\partial a^1_{223333}} + a^1_{233333}\frac{\partial}{\partial a^1_{233333}} + a^1_{333333}\frac{\partial}{\partial a^1_{333333}} - 6a^2_{111111}\frac{\partial}{\partial a^2_{111111}} -$$
$$-5a^2_{111112}\frac{\partial}{\partial a^2_{111112}} - 5a^2_{111113}\frac{\partial}{\partial a^2_{111113}} - 4a^2_{111122}\frac{\partial}{\partial a^2_{111122}} - 4a^2_{111123}\frac{\partial}{\partial a^2_{111123}} -$$
$$-4a^2_{111133}\frac{\partial}{\partial a^2_{111133}} - 3a^2_{111222}\frac{\partial}{\partial a^2_{111222}} - 3a^2_{111223}\frac{\partial}{\partial a^2_{111223}} - 3a^2_{111233}\frac{\partial}{\partial a^2_{111233}} -$$
$$-3a^2_{111333}\frac{\partial}{\partial a^2_{111333}} - 2a^2_{112222}\frac{\partial}{\partial a^2_{112222}} - 2a^2_{112223}\frac{\partial}{\partial a^2_{112223}} - 2a^2_{112233}\frac{\partial}{\partial a^2_{112233}} -$$
$$-2a^2_{112333}\frac{\partial}{\partial a^2_{112333}} - 2a^2_{113333}\frac{\partial}{\partial a^2_{113333}} - a^2_{122222}\frac{\partial}{\partial a^2_{122222}} - a^2_{122223}\frac{\partial}{\partial a^2_{122223}} -$$
$$-a^2_{122233}\frac{\partial}{\partial a^2_{122233}} - a^2_{122333}\frac{\partial}{\partial a^2_{122333}} - a^2_{123333}\frac{\partial}{\partial a^2_{123333}} - a^2_{133333}\frac{\partial}{\partial a^2_{133333}} -$$
$$-6a^3_{111111}\frac{\partial}{\partial a^3_{111111}} - 5a^3_{111112}\frac{\partial}{\partial a^3_{111112}} - 5a^3_{111113}\frac{\partial}{\partial a^3_{111113}} - 4a^3_{111122}\frac{\partial}{\partial a^3_{111122}} -$$
$$-4a^3_{111123}\frac{\partial}{\partial a^3_{111123}} - 4a^3_{111133}\frac{\partial}{\partial a^3_{111133}} - 3a^3_{111222}\frac{\partial}{\partial a^3_{111222}} - 3a^3_{111223}\frac{\partial}{\partial a^3_{111223}} -$$
$$-3a^3_{111233}\frac{\partial}{\partial a^3_{111233}} - 3a^3_{111333}\frac{\partial}{\partial a^3_{111333}} - 2a^3_{112222}\frac{\partial}{\partial a^3_{112222}} - 2a^3_{112223}\frac{\partial}{\partial a^3_{112223}} -$$
$$-2a^3_{112233}\frac{\partial}{\partial a^3_{112233}} - 2a^3_{112333}\frac{\partial}{\partial a^3_{112333}} - 2a^3_{113333}\frac{\partial}{\partial a^3_{113333}} - a^3_{122222}\frac{\partial}{\partial a^3_{122222}} -$$
$$-a^3_{122223}\frac{\partial}{\partial a^3_{122223}} - a^3_{122233}\frac{\partial}{\partial a^3_{122233}} - a^3_{122333}\frac{\partial}{\partial a^3_{122333}} - a^3_{123333}\frac{\partial}{\partial a^3_{123333}} -$$
$$-a^3_{133333}\frac{\partial}{\partial a^3_{133333}},$$

$$\mathsf{D}_2^{(6)} = -a^1_{111112}\frac{\partial}{\partial a^1_{111112}} - 2a^1_{111122}\frac{\partial}{\partial a^1_{111122}} - a^1_{111123}\frac{\partial}{\partial a^1_{111123}} - 3a^1_{111222}\frac{\partial}{\partial a^1_{111222}} -$$
$$-2a^1_{111223}\frac{\partial}{\partial a^1_{111223}} - a^1_{111233}\frac{\partial}{\partial a^1_{111233}} - 4a^1_{112222}\frac{\partial}{\partial a^1_{112222}} - 3a^1_{112223}\frac{\partial}{\partial a^1_{112223}} -$$
$$-2a^1_{112233}\frac{\partial}{\partial a^1_{112233}} - a^1_{112333}\frac{\partial}{\partial a^1_{112333}} - 5a^1_{122222}\frac{\partial}{\partial a^1_{122222}} - 4a^1_{122223}\frac{\partial}{\partial a^1_{122223}} -$$
$$-3a^1_{122233}\frac{\partial}{\partial a^1_{122233}} - 2a^1_{122333}\frac{\partial}{\partial a^1_{122333}} - a^1_{123333}\frac{\partial}{\partial a^1_{123333}} - 6a^1_{222222}\frac{\partial}{\partial a^1_{222222}} -$$

$$-5a^1_{222223}\frac{\partial}{\partial a^1_{222223}}-4a^1_{222233}\frac{\partial}{\partial a^1_{222233}}-3a^1_{222333}\frac{\partial}{\partial a^1_{222333}}-2a^1_{223333}\frac{\partial}{\partial a^1_{223333}}-$$
$$-a^1_{233333}\frac{\partial}{\partial a^1_{233333}}+a^2_{111111}\frac{\partial}{\partial a^2_{111111}}+a^2_{111113}\frac{\partial}{\partial a^2_{111113}}-a^2_{111122}\frac{\partial}{\partial a^2_{111122}}+$$
$$+a^2_{111133}\frac{\partial}{\partial a^2_{111133}}-2a^2_{111222}\frac{\partial}{\partial a^2_{111222}}-a^2_{111223}\frac{\partial}{\partial a^2_{111223}}+a^2_{111333}\frac{\partial}{\partial a^2_{111333}}-$$
$$-3a^2_{112222}\frac{\partial}{\partial a^2_{112222}}-2a^2_{112223}\frac{\partial}{\partial a^2_{112223}}-a^2_{112233}\frac{\partial}{\partial a^2_{112233}}+a^2_{113333}\frac{\partial}{\partial a^2_{113333}}-$$
$$-4a^2_{122222}\frac{\partial}{\partial a^2_{122222}}-3a^2_{122223}\frac{\partial}{\partial a^2_{122223}}-2a^2_{122233}\frac{\partial}{\partial a^2_{122233}}-a^2_{122333}\frac{\partial}{\partial a^2_{122333}}+$$
$$+a^2_{133333}\frac{\partial}{\partial a^2_{133333}}-5a^2_{222222}\frac{\partial}{\partial a^2_{222222}}-4a^2_{222223}\frac{\partial}{\partial a^2_{222223}}-3a^2_{222233}\frac{\partial}{\partial a^2_{222233}}-$$
$$-2a^2_{222333}\frac{\partial}{\partial a^2_{222333}}-a^2_{223333}\frac{\partial}{\partial a^2_{223333}}+a^2_{333333}\frac{\partial}{\partial a^2_{333333}}-a^3_{111112}\frac{\partial}{\partial a^3_{111112}}-$$
$$-2a^3_{111122}\frac{\partial}{\partial a^3_{111122}}-a^3_{111123}\frac{\partial}{\partial a^3_{111123}}-3a^3_{111222}\frac{\partial}{\partial a^3_{111222}}-2a^3_{111223}\frac{\partial}{\partial a^3_{111223}}-$$
$$-a^3_{111233}\frac{\partial}{\partial a^3_{111233}}-4a^3_{112222}\frac{\partial}{\partial a^3_{112222}}-3a^3_{112223}\frac{\partial}{\partial a^3_{112223}}-2a^3_{112233}\frac{\partial}{\partial a^3_{112233}}-$$
$$-a^3_{112333}\frac{\partial}{\partial a^3_{112333}}-5a^3_{122222}\frac{\partial}{\partial a^3_{122222}}-4a^3_{122223}\frac{\partial}{\partial a^3_{122223}}-3a^3_{122233}\frac{\partial}{\partial a^3_{122233}}-$$
$$-2a^3_{122333}\frac{\partial}{\partial a^3_{122333}}-a^3_{123333}\frac{\partial}{\partial a^3_{123333}}-6a^3_{222222}\frac{\partial}{\partial a^3_{222222}}-5a^3_{222223}\frac{\partial}{\partial a^3_{222223}}-$$
$$-4a^3_{222233}\frac{\partial}{\partial a^3_{222233}}-3a^3_{222333}\frac{\partial}{\partial a^3_{222333}}-2a^3_{223333}\frac{\partial}{\partial a^3_{223333}}-a^3_{233333}\frac{\partial}{\partial a^3_{233333}},$$
$$\mathsf{D}_3^{(6)}=-a^1_{111113}\frac{\partial}{\partial a^1_{111113}}-a^1_{111123}\frac{\partial}{\partial a^1_{111123}}-2a^1_{111133}\frac{\partial}{\partial a^1_{111133}}-a^1_{111223}\frac{\partial}{\partial a^1_{111223}}-$$
$$-2a^1_{111233}\frac{\partial}{\partial a^1_{111233}}-3a^1_{111333}\frac{\partial}{\partial a^1_{111333}}-a^1_{112223}\frac{\partial}{\partial a^1_{112223}}-2a^1_{112233}\frac{\partial}{\partial a^1_{112233}}-$$
$$-3a^1_{112333}\frac{\partial}{\partial a^1_{112333}}-4a^1_{113333}\frac{\partial}{\partial a^1_{113333}}-a^1_{122223}\frac{\partial}{\partial a^1_{122223}}-2a^1_{122233}\frac{\partial}{\partial a^1_{122233}}-$$
$$-3a^1_{122333}\frac{\partial}{\partial a^1_{122333}}-4a^1_{123333}\frac{\partial}{\partial a^1_{123333}}-5a^1_{133333}\frac{\partial}{\partial a^1_{133333}}-a^1_{222223}\frac{\partial}{\partial a^1_{222223}}-$$
$$-2a^1_{222233}\frac{\partial}{\partial a^1_{222233}}-3a^1_{222333}\frac{\partial}{\partial a^1_{222333}}-4a^1_{223333}\frac{\partial}{\partial a^1_{223333}}-5a^1_{233333}\frac{\partial}{\partial a^1_{233333}}-$$
$$-6a^1_{333333}\frac{\partial}{\partial a^1_{333333}}-a^2_{111113}\frac{\partial}{\partial a^2_{111113}}-a^2_{111123}\frac{\partial}{\partial a^2_{111123}}-2a^2_{111133}\frac{\partial}{\partial a^2_{111133}}-$$

$$-a^2_{111223}\frac{\partial}{\partial a^2_{111223}} - 2a^2_{111233}\frac{\partial}{\partial a^2_{111233}} - 3a^2_{111333}\frac{\partial}{\partial a^2_{111333}} - a^2_{112223}\frac{\partial}{\partial a^2_{112223}} -$$
$$-2a^2_{112233}\frac{\partial}{\partial a^2_{112233}} - 3a^2_{112333}\frac{\partial}{\partial a^2_{112333}} - 4a^2_{113333}\frac{\partial}{\partial a^2_{113333}} - a^2_{122223}\frac{\partial}{\partial a^2_{122223}} -$$
$$-2a^2_{122233}\frac{\partial}{\partial a^2_{122233}} - 3a^2_{122333}\frac{\partial}{\partial a^2_{122333}} - 4a^2_{123333}\frac{\partial}{\partial a^2_{123333}} - 5a^2_{133333}\frac{\partial}{\partial a^2_{133333}} -$$
$$-a^2_{222223}\frac{\partial}{\partial a^2_{222223}} - 2a^2_{222233}\frac{\partial}{\partial a^2_{222233}} - 3a^2_{222333}\frac{\partial}{\partial a^2_{222333}} - 4a^2_{223333}\frac{\partial}{\partial a^2_{223333}} -$$
$$-5a^2_{233333}\frac{\partial}{\partial a^2_{233333}} - 6a^2_{333333}\frac{\partial}{\partial a^2_{333333}} + a^3_{111111}\frac{\partial}{\partial a^3_{111111}} + a^3_{111112}\frac{\partial}{\partial a^3_{111112}} +$$
$$+a^3_{111122}\frac{\partial}{\partial a^3_{111122}} - a^3_{111133}\frac{\partial}{\partial a^3_{111133}} + a^3_{111222}\frac{\partial}{\partial a^3_{111222}} - a^3_{111233}\frac{\partial}{\partial a^3_{111233}} -$$
$$-2a^3_{111333}\frac{\partial}{\partial a^3_{111333}} + a^3_{112222}\frac{\partial}{\partial a^3_{112222}} - a^3_{112233}\frac{\partial}{\partial a^3_{112233}} - 2a^3_{112333}\frac{\partial}{\partial a^3_{112333}} -$$
$$-3a^3_{113333}\frac{\partial}{\partial a^3_{113333}} + a^3_{122222}\frac{\partial}{\partial a^3_{122222}} - a^3_{122233}\frac{\partial}{\partial a^3_{122233}} - 2a^3_{122333}\frac{\partial}{\partial a^3_{122333}} -$$
$$-3a^3_{123333}\frac{\partial}{\partial a^3_{123333}} - 4a^3_{133333}\frac{\partial}{\partial a^3_{133333}} + a^3_{222222}\frac{\partial}{\partial a^3_{222222}} - a^3_{222233}\frac{\partial}{\partial a^3_{222233}} -$$
$$-2a^3_{222333}\frac{\partial}{\partial a^3_{222333}} - 3a^3_{223333}\frac{\partial}{\partial a^3_{223333}} - 4a^3_{233333}\frac{\partial}{\partial a^3_{233333}} - 5a^3_{333333}\frac{\partial}{\partial a^3_{333333}},$$
$$\mathsf{D}_4^{(6)} = a^2_{111111}\frac{\partial}{\partial a^1_{111111}} + (a^2_{111112} - a^1_{111111})\frac{\partial}{\partial a^1_{111112}} + a^2_{111113}\frac{\partial}{\partial a^1_{111113}} +$$
$$+(a^2_{111122} - 2a^1_{111112})\frac{\partial}{\partial a^1_{111122}} + (a^2_{111123} - a^1_{111113})\frac{\partial}{\partial a^1_{111123}} + a^2_{111133}\frac{\partial}{\partial a^1_{111133}} +$$
$$+(a^2_{111222} - 3a^1_{111122})\frac{\partial}{\partial a^1_{111222}} + (a^2_{111223} - 2a^1_{111123})\frac{\partial}{\partial a^1_{111223}} + (a^2_{111233} -$$
$$-a^1_{111133})\frac{\partial}{\partial a^1_{111233}} + a^2_{111333}\frac{\partial}{\partial a^1_{111333}} + (a^2_{112222} - 4a^1_{111222})\frac{\partial}{\partial a^1_{112222}} + (a^2_{112223} -$$
$$-3a^1_{111223})\frac{\partial}{\partial a^1_{112223}} + (a^2_{112233} - 2a^1_{111233})\frac{\partial}{\partial a^1_{112233}} + (a^2_{112333} - a^1_{111333})\frac{\partial}{\partial a^1_{112333}} +$$
$$+a^2_{113333}\frac{\partial}{\partial a^1_{113333}} + (a^2_{122222} - 5a^1_{112222})\frac{\partial}{\partial a^1_{122222}} + (a^2_{122223} - 4a^1_{112223})\frac{\partial}{\partial a^1_{122223}} +$$
$$+(a^2_{122233} - 3a^1_{112233})\frac{\partial}{\partial a^1_{122233}} + (a^2_{122333} - 2a^1_{112333})\frac{\partial}{\partial a^1_{122333}} + (a^2_{123333} -$$
$$-a^1_{113333})\frac{\partial}{\partial a^1_{123333}} + a^2_{133333}\frac{\partial}{\partial a^1_{133333}} + (a^2_{222222} - 6a^1_{122222})\frac{\partial}{\partial a^1_{222222}} + (a^2_{222223} -$$

$$-5a^1_{122223})\frac{\partial}{\partial a^1_{222223}}+(a^2_{222233}-4a^1_{122233})\frac{\partial}{\partial a^1_{222233}}+(a^2_{222333}-3a^1_{122333})\frac{\partial}{\partial a^1_{222333}}+$$

$$+(a^2_{223333}-2a^1_{123333})\frac{\partial}{\partial a^1_{223333}}+(a^2_{233333}-a^1_{133333})\frac{\partial}{\partial a^1_{233333}}+$$

$$+a^2_{333333}\frac{\partial}{\partial a^1_{333333}}-a^2_{111111}\frac{\partial}{\partial a^2_{111112}}-2a^2_{111112}\frac{\partial}{\partial a^2_{111122}}-a^2_{111113}\frac{\partial}{\partial a^2_{111123}}-$$

$$-3a^2_{111122}\frac{\partial}{\partial a^2_{111222}}-2a^2_{111123}\frac{\partial}{\partial a^2_{111223}}-a^2_{111133}\frac{\partial}{\partial a^2_{111233}}-4a^2_{111222}\frac{\partial}{\partial a^2_{112222}}-$$

$$-3a^2_{111223}\frac{\partial}{\partial a^2_{112223}}-2a^2_{111233}\frac{\partial}{\partial a^2_{112233}}-a^2_{111333}\frac{\partial}{\partial a^2_{112333}}-5a^2_{112222}\frac{\partial}{\partial a^2_{122222}}-$$

$$-4a^2_{112223}\frac{\partial}{\partial a^2_{122223}}-3a^2_{112233}\frac{\partial}{\partial a^2_{122233}}-2a^2_{112333}\frac{\partial}{\partial a^2_{122333}}-a^2_{113333}\frac{\partial}{\partial a^2_{123333}}-$$

$$-6a^2_{122222}\frac{\partial}{\partial a^2_{222222}}-5a^2_{122223}\frac{\partial}{\partial a^2_{222223}}-4a^2_{122233}\frac{\partial}{\partial a^2_{222233}}-3a^2_{122333}\frac{\partial}{\partial a^2_{222333}}-$$

$$-2a^2_{123333}\frac{\partial}{\partial a^2_{223333}}-a^2_{133333}\frac{\partial}{\partial a^2_{233333}}-a^3_{111111}\frac{\partial}{\partial a^3_{111112}}-2a^3_{111112}\frac{\partial}{\partial a^3_{111122}}-$$

$$-a^3_{111113}\frac{\partial}{\partial a^3_{111123}}-3a^3_{111122}\frac{\partial}{\partial a^3_{111222}}-2a^3_{111123}\frac{\partial}{\partial a^3_{111223}}-a^3_{111133}\frac{\partial}{\partial a^3_{111233}}-$$

$$-4a^3_{111222}\frac{\partial}{\partial a^3_{112222}}-3a^3_{111223}\frac{\partial}{\partial a^3_{112223}}-2a^3_{111233}\frac{\partial}{\partial a^3_{112233}}-a^3_{111333}\frac{\partial}{\partial a^3_{112333}}-$$

$$-5a^3_{112222}\frac{\partial}{\partial a^3_{122222}}-4a^3_{112223}\frac{\partial}{\partial a^3_{122223}}-3a^3_{112233}\frac{\partial}{\partial a^3_{122233}}-2a^3_{112333}\frac{\partial}{\partial a^3_{122333}}-$$

$$-a^3_{113333}\frac{\partial}{\partial a^3_{123333}}-6a^3_{122222}\frac{\partial}{\partial a^3_{222222}}-5a^3_{122223}\frac{\partial}{\partial a^3_{222223}}-4a^3_{122233}\frac{\partial}{\partial a^3_{222233}}-$$

$$-3a^3_{122333}\frac{\partial}{\partial a^3_{222333}}-2a^3_{123333}\frac{\partial}{\partial a^3_{223333}}-a^3_{133333}\frac{\partial}{\partial a^3_{233333}},$$

$$\mathsf{D}_5^{(6)}=a^3_{111111}\frac{\partial}{\partial a^1_{111111}}+a^3_{111112}\frac{\partial}{\partial a^1_{111112}}+(a^3_{111113}-a^1_{111111})\frac{\partial}{\partial a^1_{111113}}+$$

$$+a^3_{111122}\frac{\partial}{\partial a^1_{111122}}+(a^3_{111123}-a^1_{111112})\frac{\partial}{\partial a^1_{111123}}+(a^3_{111133}-2a^1_{111113})\frac{\partial}{\partial a^1_{111133}}+$$

$$+a^3_{111222}\frac{\partial}{\partial a^1_{111222}}+(a^3_{111223}-a^1_{111122})\frac{\partial}{\partial a^1_{111223}}+(a^3_{111233}-2a^1_{111123})\frac{\partial}{\partial a^1_{111233}}+$$

$$+(a^3_{111333}-3a^1_{111133})\frac{\partial}{\partial a^1_{111333}}+a^3_{112222}\frac{\partial}{\partial a^1_{112222}}+(a^3_{112223}-a^1_{111222})\frac{\partial}{\partial a^1_{112223}}+$$

$$+(a^3_{112233}-2a^1_{111223})\frac{\partial}{\partial a^1_{112233}}+(a^3_{112333}-3a^1_{111233})\frac{\partial}{\partial a^1_{112333}}+(a^3_{113333}-$$

$$-4a^1_{111333})\frac{\partial}{\partial a^1_{113333}}+a^3_{122222}\frac{\partial}{\partial a^1_{122222}}+(a^3_{122223}-a^1_{112222})\frac{\partial}{\partial a^1_{122223}}+(a^3_{122233}-$$

$$-2a^1_{112223})\frac{\partial}{\partial a^1_{122233}}+(a^3_{122333}-3a^1_{112233})\frac{\partial}{\partial a^1_{122333}}+(a^3_{123333}-4a^1_{112333})\frac{\partial}{\partial a^1_{123333}}+$$

$$+(a^3_{133333}-5a^1_{113333})\frac{\partial}{\partial a^1_{133333}}+a^3_{222222}\frac{\partial}{\partial a^1_{222222}}+(a^3_{222223}-a^1_{122222})\frac{\partial}{\partial a^1_{222223}}+$$

$$+(a^3_{222233}-2a^1_{122223})\frac{\partial}{\partial a^1_{222233}}+(a^3_{222333}-3a^1_{122233})\frac{\partial}{\partial a^1_{222333}}+(a^3_{223333}-$$

$$-4a^1_{122333})\frac{\partial}{\partial a^1_{223333}}+(a^3_{233333}-5a^1_{123333})\frac{\partial}{\partial a^1_{233333}}+(a^3_{333333}-6a^1_{133333})\frac{\partial}{\partial a^1_{333333}}-$$

$$-a^2_{111111}\frac{\partial}{\partial a^2_{111113}}-a^2_{111112}\frac{\partial}{\partial a^2_{111123}}-2a^2_{111113}\frac{\partial}{\partial a^2_{111133}}-a^2_{111122}\frac{\partial}{\partial a^2_{111223}}-$$

$$-2a^2_{111123}\frac{\partial}{\partial a^2_{111233}}-3a^2_{111133}\frac{\partial}{\partial a^2_{111333}}-a^2_{111222}\frac{\partial}{\partial a^2_{112223}}-2a^2_{111223}\frac{\partial}{\partial a^2_{112233}}-$$

$$-3a^2_{111233}\frac{\partial}{\partial a^2_{112333}}-4a^2_{111333}\frac{\partial}{\partial a^2_{113333}}-a^2_{112222}\frac{\partial}{\partial a^2_{122223}}-2a^2_{112223}\frac{\partial}{\partial a^2_{122233}}-$$

$$-3a^2_{112233}\frac{\partial}{\partial a^2_{122333}}-4a^2_{112333}\frac{\partial}{\partial a^2_{123333}}-5a^2_{113333}\frac{\partial}{\partial a^2_{133333}}-a^2_{122222}\frac{\partial}{\partial a^2_{222223}}-$$

$$-2a^2_{122223}\frac{\partial}{\partial a^2_{222233}}-3a^2_{122233}\frac{\partial}{\partial a^2_{222333}}-4a^2_{122333}\frac{\partial}{\partial a^2_{223333}}-5a^2_{123333}\frac{\partial}{\partial a^2_{233333}}-$$

$$-6a^2_{133333}\frac{\partial}{\partial a^2_{333333}}-a^3_{111111}\frac{\partial}{\partial a^3_{111113}}-a^3_{111112}\frac{\partial}{\partial a^3_{111123}}-2a^3_{111113}\frac{\partial}{\partial a^3_{111133}}-$$

$$-a^3_{111122}\frac{\partial}{\partial a^3_{111223}}-2a^3_{111123}\frac{\partial}{\partial a^3_{111233}}-3a^3_{111133}\frac{\partial}{\partial a^3_{111333}}-a^3_{111222}\frac{\partial}{\partial a^3_{112223}}-$$

$$-2a^3_{111223}\frac{\partial}{\partial a^3_{112233}}-3a^3_{111233}\frac{\partial}{\partial a^3_{112333}}-4a^3_{111333}\frac{\partial}{\partial a^3_{113333}}-a^3_{112222}\frac{\partial}{\partial a^3_{122223}}-$$

$$-2a^3_{112223}\frac{\partial}{\partial a^3_{122233}}-3a^3_{112233}\frac{\partial}{\partial a^3_{122333}}-4a^3_{112333}\frac{\partial}{\partial a^3_{123333}}-5a^3_{113333}\frac{\partial}{\partial a^3_{133333}}-$$

$$-a^3_{122222}\frac{\partial}{\partial a^3_{222223}}-2a^3_{122223}\frac{\partial}{\partial a^3_{222233}}-3a^3_{122233}\frac{\partial}{\partial a^3_{222333}}-4a^3_{122333}\frac{\partial}{\partial a^3_{223333}}-$$

$$-5a^3_{123333}\frac{\partial}{\partial a^3_{233333}}-6a^3_{133333}\frac{\partial}{\partial a^3_{333333}},$$

$$\mathsf{D}_6^{(6)}=-6a^1_{111112}\frac{\partial}{\partial a^1_{111111}}-5a^1_{111122}\frac{\partial}{\partial a^1_{111112}}-5a^1_{111123}\frac{\partial}{\partial a^1_{111113}}-$$

$$-4a^1_{111222}\frac{\partial}{\partial a^1_{111122}}-4a^1_{111223}\frac{\partial}{\partial a^1_{111123}}-4a^1_{111233}\frac{\partial}{\partial a^1_{111133}}-3a^1_{112222}\frac{\partial}{\partial a^1_{111222}}-$$

$$-3a^{1}_{112223}\frac{\partial}{\partial a^{1}_{111223}}-3a^{1}_{112233}\frac{\partial}{\partial a^{1}_{111233}}-3a^{1}_{112333}\frac{\partial}{\partial a^{1}_{111333}}-2a^{1}_{122222}\frac{\partial}{\partial a^{1}_{112222}}-$$

$$-2a^{1}_{122223}\frac{\partial}{\partial a^{1}_{112223}}-2a^{1}_{122233}\frac{\partial}{\partial a^{1}_{112233}}-2a^{1}_{122333}\frac{\partial}{\partial a^{1}_{112333}}-2a^{1}_{123333}\frac{\partial}{\partial a^{1}_{113333}}-$$

$$-a^{1}_{222222}\frac{\partial}{\partial a^{1}_{122222}}-a^{1}_{222223}\frac{\partial}{\partial a^{1}_{122223}}-a^{1}_{222233}\frac{\partial}{\partial a^{1}_{122233}}-a^{1}_{222333}\frac{\partial}{\partial a^{1}_{122333}}-$$

$$-a^{1}_{223333}\frac{\partial}{\partial a^{1}_{123333}}-a^{1}_{233333}\frac{\partial}{\partial a^{1}_{133333}}+(a^{1}_{111111}-6a^{2}_{111112})\frac{\partial}{\partial a^{2}_{111111}}+(a^{1}_{111112}-$$

$$-5a^{2}_{111122})\frac{\partial}{\partial a^{2}_{111112}}+(a^{1}_{111113}-5a^{2}_{111123})\frac{\partial}{\partial a^{2}_{111113}}+(a^{1}_{111122}-4a^{2}_{111222})\frac{\partial}{\partial a^{2}_{111122}}+$$

$$+(a^{1}_{111123}-4a^{2}_{111223})\frac{\partial}{\partial a^{2}_{111123}}+(a^{1}_{111133}-4a^{2}_{111233})\frac{\partial}{\partial a^{2}_{111133}}+(a^{1}_{111222}-$$

$$-3a^{2}_{112222})\frac{\partial}{\partial a^{2}_{111222}}+(a^{1}_{111223}-3a^{2}_{112223})\frac{\partial}{\partial a^{2}_{111223}}+(a^{1}_{111233}-3a^{2}_{112233})\frac{\partial}{\partial a^{2}_{111233}}+$$

$$+(a^{1}_{111333}-3a^{2}_{112333})\frac{\partial}{\partial a^{2}_{111333}}+(a^{1}_{112222}-2a^{2}_{122222})\frac{\partial}{\partial a^{2}_{112222}}+(a^{1}_{112223}-$$

$$-2a^{2}_{122223})\frac{\partial}{\partial a^{2}_{112223}}+(a^{1}_{112233}-2a^{2}_{122233})\frac{\partial}{\partial a^{2}_{112233}}+(a^{1}_{112333}-2a^{2}_{122333})\frac{\partial}{\partial a^{2}_{112333}}+$$

$$+(a^{1}_{113333}-2a^{2}_{123333})\frac{\partial}{\partial a^{2}_{113333}}+(a^{1}_{122222}-a^{2}_{222222})\frac{\partial}{\partial a^{2}_{122222}}+(a^{1}_{122223}-$$

$$-a^{2}_{222223})\frac{\partial}{\partial a^{2}_{122223}}+(a^{1}_{122233}-a^{2}_{222233})\frac{\partial}{\partial a^{2}_{122233}}+(a^{1}_{122333}-a^{2}_{222333})\frac{\partial}{\partial a^{2}_{122333}}+$$

$$+(a^{1}_{123333}-a^{2}_{223333})\frac{\partial}{\partial a^{2}_{123333}}+(a^{1}_{133333}-a^{2}_{233333})\frac{\partial}{\partial a^{2}_{133333}}+a^{1}_{222222}\frac{\partial}{\partial a^{2}_{222222}}+$$

$$+a^{1}_{222223}\frac{\partial}{\partial a^{2}_{222223}}+a^{1}_{222233}\frac{\partial}{\partial a^{2}_{222233}}+a^{1}_{222333}\frac{\partial}{\partial a^{2}_{222333}}+a^{1}_{223333}\frac{\partial}{\partial a^{2}_{223333}}+$$

$$+a^{1}_{233333}\frac{\partial}{\partial a^{2}_{233333}}+a^{1}_{333333}\frac{\partial}{\partial a^{2}_{333333}}-6a^{3}_{111112}\frac{\partial}{\partial a^{3}_{111111}}-5a^{3}_{111122}\frac{\partial}{\partial a^{3}_{111112}}-$$

$$-5a^{3}_{111123}\frac{\partial}{\partial a^{3}_{111113}}-4a^{3}_{111222}\frac{\partial}{\partial a^{3}_{111122}}-4a^{3}_{111223}\frac{\partial}{\partial a^{3}_{111123}}-4a^{3}_{111233}\frac{\partial}{\partial a^{3}_{111133}}-$$

$$-3a^{3}_{112222}\frac{\partial}{\partial a^{3}_{111222}}-3a^{3}_{112223}\frac{\partial}{\partial a^{3}_{111223}}-3a^{3}_{112233}\frac{\partial}{\partial a^{3}_{111233}}-3a^{3}_{112333}\frac{\partial}{\partial a^{3}_{111333}}-$$

$$-2a^{3}_{122222}\frac{\partial}{\partial a^{3}_{112222}}-2a^{3}_{122223}\frac{\partial}{\partial a^{3}_{112223}}-2a^{3}_{122233}\frac{\partial}{\partial a^{3}_{112233}}-2a^{3}_{122333}\frac{\partial}{\partial a^{3}_{112333}}-$$

$$-2a^{3}_{123333}\frac{\partial}{\partial a^{3}_{113333}}-a^{3}_{222222}\frac{\partial}{\partial a^{3}_{122222}}-a^{3}_{222223}\frac{\partial}{\partial a^{3}_{122223}}-a^{3}_{222233}\frac{\partial}{\partial a^{3}_{122233}}-$$

$$-a^3_{222333}\frac{\partial}{\partial a^3_{122333}} - a^3_{223333}\frac{\partial}{\partial a^3_{123333}} - a^3_{233333}\frac{\partial}{\partial a^3_{133333}},$$

$$\begin{aligned}
\mathsf{D}_7^{(6)} = {} & -a^1_{111112}\frac{\partial}{\partial a^1_{111113}} - a^1_{111122}\frac{\partial}{\partial a^1_{111123}} - 2a^1_{111123}\frac{\partial}{\partial a^1_{111133}} - a^1_{111222}\frac{\partial}{\partial a^1_{111223}} - \\
& -2a^1_{111223}\frac{\partial}{\partial a^1_{111233}} - 3a^1_{111233}\frac{\partial}{\partial a^1_{111333}} - a^1_{112222}\frac{\partial}{\partial a^1_{112223}} - 2a^1_{112223}\frac{\partial}{\partial a^1_{112233}} - \\
& -3a^1_{112233}\frac{\partial}{\partial a^1_{112333}} - 4a^1_{112333}\frac{\partial}{\partial a^1_{113333}} - a^1_{122222}\frac{\partial}{\partial a^1_{122223}} - 2a^1_{122223}\frac{\partial}{\partial a^1_{122233}} - \\
& -3a^1_{122233}\frac{\partial}{\partial a^1_{122333}} - 4a^1_{122333}\frac{\partial}{\partial a^1_{123333}} - 5a^1_{123333}\frac{\partial}{\partial a^1_{133333}} - a^1_{222222}\frac{\partial}{\partial a^1_{222223}} - \\
& -2a^1_{222223}\frac{\partial}{\partial a^1_{222233}} - 3a^1_{222233}\frac{\partial}{\partial a^1_{222333}} - 4a^1_{222333}\frac{\partial}{\partial a^1_{223333}} - 5a^1_{223333}\frac{\partial}{\partial a^1_{233333}} - \\
& -6a^1_{233333}\frac{\partial}{\partial a^1_{333333}} + a^3_{111111}\frac{\partial}{\partial a^2_{111111}} + a^3_{111112}\frac{\partial}{\partial a^2_{111112}} + (-a^2_{111112} + a^3_{111113})\frac{\partial}{\partial a^2_{111113}} + \\
& +a^3_{111122}\frac{\partial}{\partial a^2_{111122}} + (-a^2_{111122} + a^3_{111123})\frac{\partial}{\partial a^2_{111123}} + (-2a^2_{111123} + a^3_{111133})\frac{\partial}{\partial a^2_{111133}} + \\
& +a^3_{111222}\frac{\partial}{\partial a^2_{111222}} + (-a^2_{111222} + a^3_{111223})\frac{\partial}{\partial a^2_{111223}} + (-2a^2_{111223} + a^3_{111233})\frac{\partial}{\partial a^2_{111233}} + \\
& +(-3a^2_{111233} + a^3_{111333})\frac{\partial}{\partial a^2_{111333}} + a^3_{112222}\frac{\partial}{\partial a^2_{112222}} + (-a^2_{112222} + a^3_{112223})\frac{\partial}{\partial a^2_{112223}} + \\
& +(-2a^2_{112223} + a^3_{112233})\frac{\partial}{\partial a^2_{112233}} + (-3a^2_{112233} + a^3_{112333})\frac{\partial}{\partial a^2_{112333}} + (-4a^2_{112333} + \\
& +a^3_{113333})\frac{\partial}{\partial a^2_{113333}} + a^3_{122222}\frac{\partial}{\partial a^2_{122222}} + (-a^2_{122222} + a^3_{122223})\frac{\partial}{\partial a^2_{122223}} + (-2a^2_{122223} + \\
& +a^3_{122233})\frac{\partial}{\partial a^2_{122233}} + (-3a^2_{122233} + a^3_{122333})\frac{\partial}{\partial a^2_{122333}} + (-4a^2_{122333} + a^3_{123333})\frac{\partial}{\partial a^2_{123333}} + \\
& +(-5a^2_{123333} + a^3_{133333})\frac{\partial}{\partial a^2_{133333}} + a^3_{222222}\frac{\partial}{\partial a^2_{222222}} + (-a^2_{222222} + a^3_{222223})\frac{\partial}{\partial a^2_{222223}} + \\
& +(-2a^2_{222223} + a^3_{222233})\frac{\partial}{\partial a^2_{222233}} + (-3a^2_{222233} + a^3_{222333})\frac{\partial}{\partial a^2_{222333}} + (-4a^2_{222333} + \\
& +a^3_{223333})\frac{\partial}{\partial a^2_{223333}} + (-5a^2_{223333} + a^3_{233333})\frac{\partial}{\partial a^2_{233333}} + (-6a^2_{233333} + a^3_{333333})\frac{\partial}{\partial a^2_{333333}} - \\
& -a^3_{111112}\frac{\partial}{\partial a^3_{111113}} - a^3_{111122}\frac{\partial}{\partial a^3_{111123}} - 2a^3_{111123}\frac{\partial}{\partial a^3_{111133}} - a^3_{111222}\frac{\partial}{\partial a^3_{111223}} - \\
& -2a^3_{111223}\frac{\partial}{\partial a^3_{111233}} - 3a^3_{111233}\frac{\partial}{\partial a^3_{111333}} - a^3_{112222}\frac{\partial}{\partial a^3_{112223}} - 2a^3_{112223}\frac{\partial}{\partial a^3_{112233}} -
\end{aligned}$$

$$-3a^3_{112233}\frac{\partial}{\partial a^3_{112333}}-4a^3_{112333}\frac{\partial}{\partial a^3_{113333}}-a^3_{122222}\frac{\partial}{\partial a^3_{122223}}-2a^3_{122223}\frac{\partial}{\partial a^3_{122233}}-$$
$$-3a^3_{122233}\frac{\partial}{\partial a^3_{122333}}-4a^3_{122333}\frac{\partial}{\partial a^3_{123333}}-5a^3_{123333}\frac{\partial}{\partial a^3_{133333}}-a^3_{222222}\frac{\partial}{\partial a^3_{222223}}-$$
$$-2a^3_{222223}\frac{\partial}{\partial a^3_{222233}}-3a^3_{222233}\frac{\partial}{\partial a^3_{222333}}-4a^3_{222333}\frac{\partial}{\partial a^3_{223333}}-5a^3_{223333}\frac{\partial}{\partial a^3_{233333}}-$$
$$-6a^3_{233333}\frac{\partial}{\partial a^3_{333333}},$$

$$\mathsf{D}_8^{(6)}=-6a^1_{111113}\frac{\partial}{\partial a^1_{111111}}-5a^1_{111123}\frac{\partial}{\partial a^1_{111112}}-5a^1_{111133}\frac{\partial}{\partial a^1_{111113}}-4a^1_{111223}\frac{\partial}{\partial a^1_{111122}}-$$
$$-4a^1_{111233}\frac{\partial}{\partial a^1_{111123}}-4a^1_{111333}\frac{\partial}{\partial a^1_{111133}}-3a^1_{112223}\frac{\partial}{\partial a^1_{111222}}-3a^1_{112233}\frac{\partial}{\partial a^1_{111223}}-$$
$$-3a^1_{112333}\frac{\partial}{\partial a^1_{111233}}-3a^1_{113333}\frac{\partial}{\partial a^1_{111333}}-2a^1_{122223}\frac{\partial}{\partial a^1_{112222}}-2a^1_{122233}\frac{\partial}{\partial a^1_{112223}}-$$
$$-2a^1_{122333}\frac{\partial}{\partial a^1_{112233}}-2a^1_{123333}\frac{\partial}{\partial a^1_{112333}}-2a^1_{133333}\frac{\partial}{\partial a^1_{113333}}-a^1_{222223}\frac{\partial}{\partial a^1_{122222}}-$$
$$-a^1_{222233}\frac{\partial}{\partial a^1_{122223}}-a^1_{222333}\frac{\partial}{\partial a^1_{122233}}-a^1_{223333}\frac{\partial}{\partial a^1_{122333}}-a^1_{233333}\frac{\partial}{\partial a^1_{123333}}-$$
$$-a^1_{333333}\frac{\partial}{\partial a^1_{133333}}-6a^2_{111113}\frac{\partial}{\partial a^2_{111111}}-5a^2_{111123}\frac{\partial}{\partial a^2_{111112}}-5a^2_{111133}\frac{\partial}{\partial a^2_{111113}}-$$
$$-4a^2_{111223}\frac{\partial}{\partial a^2_{111122}}-4a^2_{111233}\frac{\partial}{\partial a^2_{111123}}-4a^2_{111333}\frac{\partial}{\partial a^2_{111133}}-3a^2_{112223}\frac{\partial}{\partial a^2_{111222}}-$$
$$-3a^2_{112233}\frac{\partial}{\partial a^2_{111223}}-3a^2_{112333}\frac{\partial}{\partial a^2_{111233}}-3a^2_{113333}\frac{\partial}{\partial a^2_{111333}}-2a^2_{122223}\frac{\partial}{\partial a^2_{112222}}-$$
$$-2a^2_{122233}\frac{\partial}{\partial a^2_{112223}}-2a^2_{122333}\frac{\partial}{\partial a^2_{112233}}-2a^2_{123333}\frac{\partial}{\partial a^2_{112333}}-2a^2_{133333}\frac{\partial}{\partial a^2_{113333}}-$$
$$-a^2_{222223}\frac{\partial}{\partial a^2_{122222}}-a^2_{222233}\frac{\partial}{\partial a^2_{122223}}-a^2_{222333}\frac{\partial}{\partial a^2_{122233}}-a^2_{223333}\frac{\partial}{\partial a^2_{122333}}-$$
$$-a^2_{233333}\frac{\partial}{\partial a^2_{123333}}-a^2_{333333}\frac{\partial}{\partial a^2_{133333}}+(a^1_{111111}-6a^3_{111113})\frac{\partial}{\partial a^3_{111111}}+(a^1_{111112}-$$
$$-5a^3_{111123})\frac{\partial}{\partial a^3_{111112}}+(a^1_{111113}-5a^3_{111133})\frac{\partial}{\partial a^3_{111113}}+(a^1_{111122}-4a^3_{111223})\frac{\partial}{\partial a^3_{111122}}+$$
$$+(a^1_{111123}-4a^3_{111233})\frac{\partial}{\partial a^3_{111123}}+(a^1_{111133}-4a^3_{111333})\frac{\partial}{\partial a^3_{111133}}+(a^1_{111222}-$$
$$-3a^3_{112223})\frac{\partial}{\partial a^3_{111222}}+(a^1_{111223}-3a^3_{112233})\frac{\partial}{\partial a^3_{111223}}+(a^1_{111233}-3a^3_{112333})\frac{\partial}{\partial a^3_{111233}}+$$

$$+(a^1_{111333}-3a^3_{113333})\frac{\partial}{\partial a^3_{111333}}+(a^1_{112222}-2a^3_{122223})\frac{\partial}{\partial a^3_{112222}}+(a^1_{112223}-$$
$$-2a^3_{122233})\frac{\partial}{\partial a^3_{112223}}+(a^1_{112233}-2a^3_{122333})\frac{\partial}{\partial a^3_{112233}}+(a^1_{112333}-2a^3_{123333})\frac{\partial}{\partial a^3_{112333}}+$$
$$+(a^1_{113333}-2a^3_{133333})\frac{\partial}{\partial a^3_{113333}}+(a^1_{122222}-a^3_{222223})\frac{\partial}{\partial a^3_{122222}}+(a^1_{122223}-$$
$$-a^3_{222233})\frac{\partial}{\partial a^3_{122223}}+(a^1_{122233}-a^3_{222333})\frac{\partial}{\partial a^3_{122233}}+(a^1_{122333}-a^3_{223333})\frac{\partial}{\partial a^3_{122333}}+$$
$$+(a^1_{123333}-a^3_{233333})\frac{\partial}{\partial a^3_{123333}}+(a^1_{133333}-a^3_{333333})\frac{\partial}{\partial a^3_{133333}}+a^1_{222222}\frac{\partial}{\partial a^3_{222222}}+$$
$$+a^1_{222223}\frac{\partial}{\partial a^3_{222223}}+a^1_{222233}\frac{\partial}{\partial a^3_{222233}}+a^1_{222333}\frac{\partial}{\partial a^3_{222333}}+a^1_{223333}\frac{\partial}{\partial a^3_{223333}}+$$
$$+a^1_{233333}\frac{\partial}{\partial a^3_{233333}}+a^1_{333333}\frac{\partial}{\partial a^3_{333333}},$$

$$\mathsf{D}^{(6)}_9=-a^1_{111113}\frac{\partial}{\partial a^1_{111112}}-2a^1_{111123}\frac{\partial}{\partial a^1_{111122}}-a^1_{111133}\frac{\partial}{\partial a^1_{111123}}-3a^1_{111223}\frac{\partial}{\partial a^1_{111222}}-$$
$$-2a^1_{111233}\frac{\partial}{\partial a^1_{111223}}-a^1_{111333}\frac{\partial}{\partial a^1_{111233}}-4a^1_{112223}\frac{\partial}{\partial a^1_{112222}}-3a^1_{112233}\frac{\partial}{\partial a^1_{112223}}-$$
$$-2a^1_{112333}\frac{\partial}{\partial a^1_{112233}}-a^1_{113333}\frac{\partial}{\partial a^1_{112333}}-5a^1_{122223}\frac{\partial}{\partial a^1_{122222}}-4a^1_{122233}\frac{\partial}{\partial a^1_{122223}}-$$
$$-3a^1_{122333}\frac{\partial}{\partial a^1_{122233}}-2a^1_{123333}\frac{\partial}{\partial a^1_{122333}}-a^1_{133333}\frac{\partial}{\partial a^1_{123333}}-6a^1_{222223}\frac{\partial}{\partial a^1_{222222}}-$$
$$-5a^1_{222233}\frac{\partial}{\partial a^1_{222223}}-4a^1_{222333}\frac{\partial}{\partial a^1_{222233}}-3a^1_{223333}\frac{\partial}{\partial a^1_{222333}}-2a^1_{233333}\frac{\partial}{\partial a^1_{223333}}-$$
$$-a^1_{333333}\frac{\partial}{\partial a^1_{233333}}-a^2_{111113}\frac{\partial}{\partial a^2_{111112}}-2a^2_{111123}\frac{\partial}{\partial a^2_{111122}}-a^2_{111133}\frac{\partial}{\partial a^2_{111123}}-$$
$$-3a^2_{111223}\frac{\partial}{\partial a^2_{111222}}-2a^2_{111233}\frac{\partial}{\partial a^2_{111223}}-a^2_{111333}\frac{\partial}{\partial a^2_{111233}}-4a^2_{112223}\frac{\partial}{\partial a^2_{112222}}-$$
$$-3a^2_{112233}\frac{\partial}{\partial a^2_{112223}}-2a^2_{112333}\frac{\partial}{\partial a^2_{112233}}-a^2_{113333}\frac{\partial}{\partial a^2_{112333}}-5a^2_{122223}\frac{\partial}{\partial a^2_{122222}}-$$
$$-4a^2_{122233}\frac{\partial}{\partial a^2_{122223}}-3a^2_{122333}\frac{\partial}{\partial a^2_{122233}}-2a^2_{123333}\frac{\partial}{\partial a^2_{122333}}-a^2_{133333}\frac{\partial}{\partial a^2_{123333}}-$$
$$-6a^2_{222223}\frac{\partial}{\partial a^2_{222222}}-5a^2_{222233}\frac{\partial}{\partial a^2_{222223}}-4a^2_{222333}\frac{\partial}{\partial a^2_{222233}}-3a^2_{223333}\frac{\partial}{\partial a^2_{222333}}-$$
$$-2a^2_{233333}\frac{\partial}{\partial a^2_{223333}}-a^2_{333333}\frac{\partial}{\partial a^2_{233333}}+a^2_{111111}\frac{\partial}{\partial a^3_{111111}}+(a^2_{111112}-$$

$$-a^3_{111113})\frac{\partial}{\partial a^3_{111112}}+a^2_{111113}\frac{\partial}{\partial a^3_{111113}}+(a^2_{111122}-2a^3_{111123})\frac{\partial}{\partial a^3_{111122}}+(a^2_{111123}-$$
$$-a^3_{111133})\frac{\partial}{\partial a^3_{111123}}+a^2_{111133}\frac{\partial}{\partial a^3_{111133}}+(a^2_{111222}-3a^3_{111223})\frac{\partial}{\partial a^3_{111222}}+(a^2_{111223}-$$
$$-2a^3_{111233})\frac{\partial}{\partial a^3_{111223}}+(a^2_{111233}-a^3_{111333})\frac{\partial}{\partial a^3_{111233}}+a^2_{111333}\frac{\partial}{\partial a^3_{111333}}+(a^2_{112222}-$$
$$-4a^3_{112223})\frac{\partial}{\partial a^3_{112222}}+(a^2_{112223}-3a^3_{112233})\frac{\partial}{\partial a^3_{112223}}+(a^2_{112233}-2a^3_{112333})\frac{\partial}{\partial a^3_{112233}}+$$
$$+(a^2_{112333}-a^3_{113333})\frac{\partial}{\partial a^3_{112333}}+a^2_{113333}\frac{\partial}{\partial a^3_{113333}}+(a^2_{122222}-5a^3_{122223})\frac{\partial}{\partial a^3_{122222}}+$$
$$+(a^2_{122223}-4a^3_{122233})\frac{\partial}{\partial a^3_{122223}}+(a^2_{122233}-3a^3_{122333})\frac{\partial}{\partial a^3_{122233}}+(a^2_{122333}-$$
$$-2a^3_{123333})\frac{\partial}{\partial a^3_{122333}}+(a^2_{123333}-a^3_{133333})\frac{\partial}{\partial a^3_{123333}}+a^2_{133333}\frac{\partial}{\partial a^3_{133333}}+(a^2_{222222}-$$
$$-6a^3_{222223})\frac{\partial}{\partial a^3_{222222}}+(a^2_{222223}-5a^3_{222233})\frac{\partial}{\partial a^3_{222223}}+(a^2_{222233}-4a^3_{222333})\frac{\partial}{\partial a^3_{222233}}+$$
$$+(a^2_{222333}-3a^3_{223333})\frac{\partial}{\partial a^3_{222333}}+(a^2_{223333}-2a^3_{233333})\frac{\partial}{\partial a^3_{223333}}+$$
$$+(a^2_{233333}-a^3_{333333})\frac{\partial}{\partial a^3_{233333}}+a^2_{333333}\frac{\partial}{\partial a^3_{333333}}. \tag{17}$$

Proof. We will examine the transformation group H_1, given by formulas (2), in the space of variables and coefficients of system (1). Then, this system will be written in the form

$$\frac{d\bar{x}^j}{dt}=\bar{a}^j_\alpha x^\alpha+\bar{a}^j_{\alpha\beta\gamma\delta\mu\nu}\bar{x}^\alpha\bar{x}^\beta\bar{x}^\gamma\bar{x}^\delta\bar{x}^\mu\bar{x}^\nu\quad(j,\alpha,\beta,\gamma,\delta,\mu,\nu=\overline{1,3}), \tag{18}$$

where

$$\bar{a}^1_1=a^1_1,\ \bar{a}^1_2=\lambda_1a^1_2,\ \bar{a}^1_3=\lambda_1a^1_3,\ \bar{a}^2_1=\frac{a^2_1}{\lambda_1},\ \bar{a}^2_2=a^2_2,\ \bar{a}^2_3=a^2_3,\ \bar{a}^3_1=\frac{a^3_1}{\lambda_1},\ \bar{a}^3_2=a^3_2,$$
$$\bar{a}^3_3=a^3_3,\ \bar{a}^1_{111111}=\frac{a^1_{111111}}{\lambda_1^5},\ \bar{a}^1_{111112}=\frac{a^1_{111112}}{\lambda_1^4},\ \bar{a}^1_{111122}=\frac{a^1_{111122}}{\lambda_1^3},\ \bar{a}^1_{111222}=\frac{a^1_{111222}}{\lambda_1^2},$$
$$\bar{a}^1_{112222}=\frac{a^1_{112222}}{\lambda_1},\ \bar{a}^1_{122222}=a^1_{122222},\ \bar{a}^1_{222222}=\lambda_1a^1_{222222},\ \bar{a}^1_{111113}=\frac{a^1_{111113}}{\lambda_1^4},$$
$$\bar{a}^1_{111123}=\frac{a^1_{111123}}{\lambda_1^3},\ \bar{a}^1_{111223}=\frac{a^1_{111223}}{\lambda_1^2},\ \bar{a}^1_{112223}=\frac{a^1_{112223}}{\lambda_1},\ \bar{a}^1_{122223}=a^1_{122223},$$
$$\bar{a}^1_{222223}=\lambda_1a^1_{222223},\ \bar{a}^1_{111133}=a^1_{111133}\frac{1}{\lambda_1^3},\ \bar{a}^1_{111233}=a^1_{111233}\frac{1}{\lambda_1^2},\ \bar{a}^1_{112233}=\frac{a^1_{112233}}{\lambda_1},$$
$$\bar{a}^1_{122233}=a^1_{122233},\ \bar{a}^1_{222233}=\lambda_1a^1_{222233},\ \bar{a}^1_{111333}=\frac{a^1_{111333}}{\lambda_1^2},\ \bar{a}^1_{112333}=\frac{a^1_{112333}}{\lambda_1},$$

$$\bar{a}^1_{122333} = a^1_{122333},\ \bar{a}^1_{222333} = \lambda_1 a^1_{222333},\ \bar{a}^1_{113333} = \frac{a^1_{113333}}{\lambda_1},\ \bar{a}^1_{123333} = a^1_{123333},$$

$$\bar{a}^1_{223333} = \lambda_1 a^1_{223333},\ \bar{a}^1_{133333} = a^1_{133333},\ \bar{a}^1_{233333} = \lambda_1 a^1_{233333},\ \bar{a}^1_{333333} = \lambda_1 a^1_{333333},$$

$$\bar{a}^2_{111111} = \frac{a^2_{111111}}{\lambda_1^6},\ \bar{a}^2_{111112} = \frac{a^2_{111112}}{\lambda_1^5},\ \bar{a}^2_{111122} = \frac{a^2_{111122}}{\lambda_1^4},\ \bar{a}^2_{111222} = \frac{a^2_{111222}}{\lambda_1^3},$$

$$\bar{a}^2_{112222} = \frac{a^2_{112222}}{\lambda_1^2},\ \bar{a}^2_{122222} = \frac{a^2_{122222}}{\lambda_1},\ \bar{a}^2_{222222} = a^2_{222222},\ \bar{a}^2_{111113} = \frac{a^2_{111113}}{\lambda_1^5},$$

$$\bar{a}^2_{111123} = \frac{a^2_{111123}}{\lambda_1^4},\ \bar{a}^2_{111223} = \frac{a^2_{111223}}{\lambda_1^3},\ \bar{a}^2_{112223} = \frac{a^2_{112223}}{\lambda_1^2},\ \bar{a}^2_{122223} = \frac{a^2_{122223}}{\lambda_1},$$

$$\bar{a}^2_{222223} = a^2_{222223},\ \bar{a}^2_{111133} = \frac{a^2_{111133}}{\lambda_1^4},\ \bar{a}^2_{111233} = \frac{a^2_{111233}}{\lambda_1^3},\ \bar{a}^2_{112233} = \frac{a^2_{112233}}{\lambda_1^2},$$

$$\bar{a}^2_{122233} = \frac{a^2_{122233}}{\lambda_1},\ \bar{a}^2_{222233} = a^2_{222233},\ \bar{a}^2_{111333} = \frac{a^2_{111333}}{\lambda_1^3},\ \bar{a}^2_{112333} = \frac{a^2_{112333}}{\lambda_1^2},$$

$$\bar{a}^2_{122333} = \frac{a^2_{122333}}{\lambda_1},\ \bar{a}^2_{222333} = a^2_{222333},\ \bar{a}^2_{113333} = \frac{a^2_{113333}}{\lambda_1^2},\ \bar{a}^2_{123333} = \frac{a^2_{123333}}{\lambda_1},$$

$$\bar{a}^2_{223333} = a^2_{223333},\ \bar{a}^2_{133333} = \frac{a^2_{133333}}{\lambda_1},\ \bar{a}^2_{233333} = a^2_{233333},\ \bar{a}^2_{333333} = a^2_{333333},$$

$$\bar{a}^3_{111111} = \frac{a^3_{111111}}{\lambda_1}^{6},\ \bar{a}^3_{111112} = \frac{a^3_{111112}}{\lambda_1^5},\ \bar{a}^3_{111122} = \frac{a^3_{111122}}{\lambda_1^4},\ \bar{a}^3_{111222} = \frac{a^3_{111222}}{\lambda_1^3},$$

$$\bar{a}^3_{112222} = \frac{a^3_{112222}}{\lambda_1^2},\ \bar{a}^3_{122222} = \frac{a^3_{122222}}{\lambda_1},\ \bar{a}^3_{222222} = a^3_{222222},\ \bar{a}^3_{111113} = \frac{a^3_{111113}}{\lambda_1^5},$$

$$\bar{a}^3_{111123} = \frac{a^3_{111123}}{\lambda_1^4},\ \bar{a}^3_{111223} = \frac{a^3_{111223}}{\lambda_1^3},\ \bar{a}^3_{112223} = \frac{a^3_{112223}}{\lambda_1^2},\ \bar{a}^3_{122223} = \frac{a^3_{122223}}{\lambda_1},$$

$$\bar{a}^3_{222223} = a^3_{222223},\ \bar{a}^3_{111133} = \frac{a^3_{111133}}{\lambda_1^4},\ \bar{a}^3_{111233} = \frac{a^3_{111233}}{\lambda_1^3},\ \bar{a}^3_{112233} = \frac{a^3_{112233}}{\lambda_1^2},$$

$$\bar{a}^3_{122233} = \frac{a^3_{122233}}{\lambda_1},\ \bar{a}^3_{222233} = a^3_{222233},\ \bar{a}^3_{111333} = \frac{a^3_{111333}}{\lambda_1^3},\ \bar{a}^3_{112333} = \frac{a^3_{112333}}{\lambda_1^2},$$

$$\bar{a}^3_{122333} = \frac{a^3_{122333}}{\lambda_1},\ \bar{a}^3_{222333} = a^3_{222333},\ \bar{a}^3_{113333} = \frac{a^3_{113333}}{\lambda_1^2},\ \bar{a}^3_{123333} = \frac{a^3_{123333}}{\lambda_1},$$

$$\bar{a}^3_{223333} = a^3_{223333},\ \bar{a}^3_{133333} = \frac{a^3_{133333}}{\lambda_1},\ \bar{a}^3_{233333} = a^3_{233333},\ \bar{a}^3_{333333} = a^3_{333333}.$$

Considering these equalities and introducing into (13) the notations $\alpha = \exp(\bar{\lambda}_1)$ [**?**], where $\exp(\bar{\lambda}_1) = |\lambda_1|$, we obtain

$$\xi^1 = x^1,\ \ \xi^2 = \xi^3 = 0,$$

$$\eta^1_1 = 0,\ \ \eta^1_2 = a^1_2,\ \ \eta^1_3 = a^1_3,\ \ \eta^2_1 = -a^2_1,\ \ \eta^2_2 = \eta^2_3 = 0,\ \ \eta^3_1 = -a^3_1,\ \ \eta^3_2 = \eta^3_3 = 0,$$

$$\eta^1_{111111} = -5a^1_{111111},\ \eta^1_{111112} = -4a^1_{111112},\ \eta^1_{111122} = -3a^1_{111122},\ \eta^1_{111222} = -2a^1_{111222},$$

$$\eta^1_{112222} = -a^1_{112222},\ \eta^1_{122222} = 0,\ \eta^1_{222222} = a^1_{222222},\ \eta^1_{111113} = -4a^1_{111113},$$

$$\eta^1_{111123} = -3a^1_{111123},\ \eta^1_{111223} = -2a^1_{111223},\ \eta^1_{112223} = -a^1_{112223},\ \eta^1_{122223} = 0,$$

$$\eta^1_{222223} = a^1_{222223},\ \eta^1_{111133} = -3a^1_{111133},\ \eta^1_{111233} = -2a^1_{111233},\ \eta^1_{112233} = -a^1_{112233},$$

$$\eta^1_{122233} = 0,\ \eta^1_{222233} = a^1_{222233},\ \eta^1_{111333} = -2a^1_{111333},\ \eta^1_{112333} = -a^1_{112333},$$

$$\eta^1_{122333} = 0,\ \eta^1_{222333} = a^1_{222333},\ \eta^1_{113333} = -a^1_{113333},\ \eta^1_{123333} = 0,$$

$$\eta^1_{223333} = a^1_{223333},\ \eta^1_{133333} = 0,\ \eta^1_{233333} = a^1_{233333},\ \eta^1_{333333} = a^1_{333333},$$

$$\eta^2_{111111} = -6a^2_{111111},\ \eta^2_{111112} = -5a^2_{111112},\ \eta^2_{111122} = -4a^2_{111122},\ \eta^2_{111222} = -3a^2_{111222},$$

$$\eta^2_{112222} = -2a^2_{112222},\ \eta^2_{122222} = -a^2_{122222},\ \eta^2_{222222} = 0,\ \eta^2_{111113} = -5a^2_{111113},$$

$$\eta^2_{111123} = -4a^2_{111123},\ \eta^2_{111223} = -3a^2_{111223},\ \eta^2_{112223} = -2a^2_{112223},\ \eta^2_{122223} = -a^2_{122223},$$

$$\eta^2_{222223} = 0,\ \eta^2_{111133} = -4a^2_{111133},\ \eta^2_{111233} = -3a^2_{111233},\ \eta^2_{112233} = -2a^2_{112233},$$

$$\eta^2_{122233} = -a^2_{122233},\ \eta^2_{222233} = 0,\ \eta^2_{111333} = -3a^2_{111333},\ \eta^2_{112333} = -2a^2_{112333},$$

$$\eta^2_{122333} = -a^2_{122333},\ \eta^2_{222333} = 0,\ \eta^2_{113333} = -2a^2_{113333},\ \eta^2_{123333} = -a^2_{123333},$$

$$\eta^2_{223333} = 0,\ \eta^2_{133333} = -a^2_{133333},\ \eta^2_{233333} = 0,\ \eta^2_{333333} = 0,$$

$$\eta^3_{111111} = -6a^3_{111111},\ \eta^3_{111112} = -5a^3_{111112},\ \eta^3_{111122} = -4a^3_{111122},\ \eta^3_{111222} = -3a^3_{111222},$$

$$\eta^3_{112222} = -2a^3_{112222},\ \eta^3_{122222} = -a^3_{122222},\ \eta^3_{222222} = 0,\ \eta^3_{111113} = -5a^3_{111113},$$

$$\eta^3_{111123} = -4a^3_{111123},\ \eta^3_{111223} = -3a^3_{111223},\ \eta^3_{112223} = -2a^3_{112223},\ \eta^3_{122223} = -a^3_{122223},$$

$$\eta^3_{222223} = 0,\ \eta^3_{111133} = -4a^3_{111133},\ \eta^3_{111233} = -3a^3_{111233},\ \eta^3_{112233} = -2a^3_{112233},$$

$$\eta^3_{122233} = -a^3_{122233},\ \eta^3_{222233} = 0,\ \eta^3_{111333} = -3a^3_{111333},\ \eta^3_{112333} = -2a^3_{112333},$$

$$\eta^3_{122333} = -a^3_{122333},\ \eta^3_{222333} = 0,\ \eta^3_{113333} = -2a^3_{113333},\ \eta^3_{123333} = -a^3_{123333},$$

$$\eta^3_{223333} = 0,\ \eta^3_{133333} = -a^3_{133333},\ \eta^3_{233333} = 0,\ \eta^3_{333333} = 0.$$

By substituting the values of these coordinates into (11) - (12), we obtain the operator X_1 and d_1, from (14) - (16).

Similarly, one can prove the determination of the operators X_k and d_k $(k = \overline{2,9})$, from (14) - (16), for the transformation groups H_i $(i = \overline{2,3})$ and H_{j+3} $(j = \overline{1,6})$, when $\mu_j = 0$ $(j = \overline{1,6})$, given by formulas (3) - (10), in the space of variables and coefficients $E^{96}(x,a)$ of system (1). Theorem 1 is proved. □

3. The Lie algebra L_9 of the admissible representation operators for the differential system with nonlinearities of degree six

For the operators X_i $(\overline{1,9})$ from (14) - (17), admitted by the system $s^3(1,6)$, the following holds

Lemma 3.1. *The operators X_i $(\overline{1,9})$, from (14) - (17), form a Lie algebra L_9 with the structure equations*

$$
\begin{gathered}
[X_1, X_4] = -X_4, \quad [X_1, X_5] = -X_5, \quad [X_1, X_6] = X_6, \quad [X_1, X_8] = X_8, \quad [X_2, X_4] = X_4, \\
[X_2, X_6] = -X_6, \quad [X_2, X_7] = -X_7, \quad [X_2, X_9] = X_9, \quad [X_3, X_5] = X_5, \quad [X_3, X_7] = X_7, \\
[X_3, X_8] = -X_8, \quad [X_3, X_9] = -X_9, \quad [X_4, X_6] = X_2 - X_1, \quad [X_4, X_7] = -X_5, \\
[X_4, X_9] = X_8, \quad [X_5, X_6] = X_7, \quad [X_5, X_8] = X_3 - X_1, \quad [X_5, X_9] = -X_4, \\
[X_6, X_9] = X_8, \quad [X_7, X_8] = -X_6, \quad [X_7, X_9] = X_3 - X_2,
\end{gathered} \tag{19}
$$

and for other commutators $[X_i, X_j] = 0$.

We introduce the notations

$$
\begin{gathered}
Y_1 = X_9, \quad Y_2 = X_1 - X_3, \quad Y_3 = X_1 - X_2, \quad Y_4 = X_4, \quad Y_5 = X_5, \quad Y_6 = X_6, \\
Y_7 = X_7, \quad Y_8 = X_8, \quad Y_9 = X_1 + X_2 + X_3.
\end{gathered} \tag{20}
$$

The following lemma holds.

Lemma 3.2. *The Lie algebra $L_9\langle X_1, \ldots, X_9\rangle$, with structure equations (19), is isomorphic to the Lie algebra $L_9\langle Y_1, \ldots, Y_9\rangle$ with the following commutator table:*

Table 1. The commutators of the Lie algebra $L_9\langle Y_1, \ldots, Y_9\rangle$

	Y_1	Y_2	Y_3	Y_4	Y_5	Y_6	Y_7	Y_8	Y_9
Y_1	0	$-Y_1$	Y_1	0	Y_4	$-Y_8$	$Y_2 - Y_3$	0	0
Y_2	Y_1	0	0	$-Y_4$	$-2Y_5$	Y_6	$-Y_7$	$2Y_8$	0
Y_3	$-Y_1$	0	0	$-2Y_4$	$-Y_5$	$2Y_6$	Y_7	$-Y_8$	0
Y_4	0	Y_4	$2Y_4$	0	0	$-Y_3$	$-Y_5$	Y_1	0
Y_5	$-Y_4$	$2Y_5$	Y_5	0	0	Y_7	0	$-Y_2$	0
Y_6	Y_8	$-Y_6$	$-2Y_6$	Y_3	$-Y_7$	0	0	0	0
Y_7	$-Y_2 + Y_3$	Y_7	$-Y_7$	Y_5	0	0	0	$-Y_6$	0
Y_8	0	$-2Y_8$	$-Y_8$	$-Y_1$	Y_2	0	Y_6	0	0
Y_9	0	0	0	0	0	0	0	0	0

where the non-zero structure constants, except for those determined by antisymmetry with respect to the lower indices, are

$$C^1_{12} = -1, \quad C^1_{13} = 1, \quad C^4_{15} = 1, \quad C^8_{16} = -1, \quad C^2_{17} = 1, \quad C^3_{17} = -1, \quad C^4_{24} = -1,$$
$$C^5_{25} = -2, \quad C^6_{26} = 1, \quad C^7_{27} = -1, \quad C^8_{28} = 2, \quad C^4_{34} = -2, \quad C^5_{35} = -1, \quad C^6_{36} = 2,$$
$$C^7_{37} = 1, \quad C^8_{38} = 1, \quad C^3_{46} = -1, \quad C^5_{47} = -1, \quad C^1_{48} = 1, \quad C^7_{56} = 1, \quad C^2_{58} = -1, \quad C^6_{78} = -1. \tag{21}$$

From Table 1 we observe that

$$L_9 = \langle Y_1, \ldots, Y_9 \rangle = L_8 \langle Y_1, \ldots, Y_8 \rangle \oplus \{Y_9\}, \tag{22}$$

where $Z = \{Y_9\}$ is the non-zero center of the Lie algebra $L_9 = \langle Y_1, \ldots, Y_9 \rangle$.

To determine the type of the Lie algebra $L_8\langle Y_1, \ldots, Y_8 \rangle$, we construct the associated application matrix [2], [3]:

$(ad(X)) =$

$$= \begin{pmatrix} x^3 - x^2 & x^1 & -x^1 & x^8 & 0 & 0 & 0 & -x^4 \\ x^7 & 0 & 0 & 0 & -x^8 & 0 & -x^1 & x^5 \\ -x^7 & 0 & 0 & -x^6 & 0 & x^4 & x^1 & 0 \\ x^5 & -x^4 & -2x^4 & x^2 + 2x^3 & -x^1 & 0 & 0 & 0 \\ 0 & -2x^5 & -x^5 & -x^7 & 2x^2 + x^3 & 0 & x^4 & 0 \\ 0 & x^6 & 2x^6 & 0 & 0 & -x^2 - 2x^3 & -x^8 & x^7 \\ 0 & -x^7 & x^7 & 0 & x^6 & -x^5 & x^2 - x^3 & 0 \\ -x^6 & 2x^8 & x^8 & 0 & 0 & x^1 & 0 & -2x^2 - x^3 \end{pmatrix} \tag{23}$$

In a similar way, we construct the associated application matrix for the element $Y \in L_8$:

$(ad(Y)) =$

$$= \begin{pmatrix} y^3 - y^2 & y^1 & -y^1 & y^8 & 0 & 0 & 0 & -y^4 \\ y^7 & 0 & 0 & 0 & -y^8 & 0 & -y^1 & y^5 \\ -y^7 & 0 & 0 & -y^6 & 0 & y^4 & y^1 & 0 \\ y^5 & -y^4 & -2y^4 & y^2 + 2y^3 & -y^1 & 0 & 0 & 0 \\ 0 & -2y^5 & -y^5 & -y^7 & 2y^2 + y^3 & 0 & y^4 & 0 \\ 0 & y^6 & 2y^6 & 0 & 0 & -y^2 - 2y^3 & -y^8 & y^7 \\ 0 & -y^7 & y^7 & 0 & y^6 & -y^5 & y^2 - y^3 & 0 \\ -y^6 & 2y^8 & y^8 & 0 & 0 & y^1 & 0 & -2y^2 - y^3 \end{pmatrix} \tag{24}$$

Using the structure constants $C_{\alpha\beta}$ from (21) and the matrices (23) - (24), we obtain the trace of the product matrix $tr((ad(X))(ad(Y)))$. By means of the trace of the matrix product $tr((ad(X))(ad(Y)))$, we determine the Killing form [2], [3]:

$$K < X, Y >= tr((ad(X))(ad(Y))). \tag{25}$$

Therefore

$$K < X, Y >= 6x^1y^7 + 12x^2y^2 + 6x^2y^3 + 6x^3y^2 + 12x^3y^3 + 6x^4y^6 + 6x^5y^8 + 6x^6y^4 + \\ +6x^7y^1 + 6x^8y^5. \tag{26}$$

Taking into account the Killing form (26), we obtain that

$$\begin{gathered} K_{17} = K_{23} = K_{32} = K_{46} = K_{58} = K_{64} = K_{71} = K_{85} = 6, \\ K_{22} = K_{33} = 12, \end{gathered} \tag{27}$$

and for other elements $K_{\alpha\beta} = 0$, $(\alpha, \beta = \overline{1, 8})$ for which we have

$$det(K_{\alpha\beta}) = det \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 6 & 0 \\ 0 & 12 & 6 & 0 & 0 & 0 & 0 & 0 \\ 0 & 6 & 12 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 \\ 6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 6 & 0 & 0 & 0 \end{pmatrix} \neq 0. \tag{28}$$

4. Conclusions

Since $det(K_{\alpha\beta}) \neq 0$, it follows that the Killing's form is non-degenerate, and in accordance with Cartan's criterion [3], we obtain that the Lie algebra L_8 is semisimple. Therefore, based on relation (22), we obtain that the Lie algebra L_9 is a reductive Lie algebra.

The Lie algebra L_9 obtained for the ternary differential system with nonlinearities of degree six (1) is isomorphic to the Lie algebra L_9 admitted by the ternary differential system with quadratic nonlinearities, from [8].

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Semantic non-compositional exploration of Bessarabian idioms by LLMs

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Abstract. The study explores the potential of LLMs in interpreting and translating Bessarabian idioms. The central problem addressed is the semantic non-compositionality of idiomatic expressions, which poses a significant challenge for Natural Language Processing since their figurative meaning cannot be derived from literal components. As part of the CI ARiA project, 1000 proverbs were digitized, using a corpus of 400 of them to evaluate the performance of 10 AI models (such as ChatGPT, Gemini, Grok). The methodology is multi-algorithmic, combining textual distance metrics (Levenshtein, Jaccard) with semantic similarity analysis via Sentence Transformers. The results indicate that while models demonstrate a solid capacity to grasp metaphorical meanings, significant differences exist regarding consistency and explanatory style.

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Keywords: Large Language Models (LLMs), idioms, semantic non-compositionality.

Explorarea non-compozițională semantică a idiomurilor basarabene de către LLM-uri

Rezumat. Lucrarea explorează potențialul LLM-urilor în interpretarea și traducerea idiomurilor basarabene. Problema centrală abordată este non-compoziționalitatea semantică a expresiilor idiomatice, care reprezintă o provocare majoră pentru procesarea limbajului natural deoarece sensul lor figurat nu poate fi dedus din componentele literale. În cadrul proiectului CI ARiA, au fost digitalizate 1000 de proverbe, utilizând un corpus de 400 dintre ele pentru a evalua performanța a 10 modele AI (precum ChatGPT, Gemini, Grok). Metodologia utilizată este multi-algoritmică, combinând metrici de distanță textuală (Levenshtein, Jaccard) cu analiza similitudinii semantice prin Sentence Transformers. Rezultatele indică faptul că, deși modelele demonstrează o capacitate solidă de a înțelege sensurile metaforice, există diferențe semnificative în ceea ce privește consistența și stilul explicativ.

Cuvinte-cheie: Modele de Limbaj Mari (LLM), idiomuri, non-compoziționalitate semantică.

1. Introduction

In recent years, the accelerated development of Generative Artificial Intelligence (GenAI) has fundamentally shifted the paradigm of how cultural, linguistic, and visual information is created, interpreted, and analyzed. Large Language Models (LLMs) and deep learning-based image generation systems have evolved from experimental tools into core technologies applied in education, research, and the creative industries. In this context, the intersection of traditional cultural heritage and modern AI technologies has emerged as a critical area of exploration.

The central problem addressed in this study is the complexity of idiomatic expressions within computational linguistics. Unlike standard sentence structures, idioms represent systematic deviations from the literal meaning of language. Most automated models, whether for machine translation, semantic analysis, or text generation, rely heavily on the principle of semantic compositionality, assuming that the meaning of a sentence can be derived from the meanings of its constituent words and their grammatical combination. However, this principle fails in the context of idiomatic expressions. As Multiword Expressions (MWEs), idioms possess figurative meanings that are stable and culturally recognized but cannot be deduced from the literal definitions of their components (e.g., the Bessarabian idiom ”a da ortul popii” literally means ”to give the coin to the priest,” but figuratively signifies ”to die”).

As noted by Iordache Golescu [1], “Vorba fără glume pare ca bucatele fără sare, iar pildele la vorbă, tocma ca nişte glume (Speech without humor is like food without salt, and idioms in conversation serve the very same purpose as jokes)”, idioms function like ”jokes that carry wisdom,” which are essential for the flavor and nuance of natural communication. Without them, language risks becoming ”unsalted” or flat. Consequently, identifying and accurately interpreting these expressions presents a significant challenge for Natural Language Processing (NLP). This challenge is particularly acute for under-resourced or regionally specific dialects, such as the Moldavian idioms characteristic of the Bessarabian region.

This research is part of the national project CI ARiA that aims to bridge the gap between cultural heritage and technological advancement. The project focuses on the digitization and augmentation of regional idiomatic expressions from the Republic of Moldova, intending to facilitate deep language learning and promote indigenous linguistic heritage. The project involves collecting and classifying approximately 10,000 tokens and creating interactive educational experiences using Artificial Intelligence and Augmented Reality (AR).

Specifically, this paper analyzes the "state of the art" of current LLMs, examining their capacity to comprehend, translate, and visualize the semantic depth of Bessarabian idioms. By evaluating how these models process the figurative meanings of idioms, we provide insights into the limitations and strengths of current AI systems in handling deep semantic and cultural concepts.

2. The nature and taxonomy of idioms

In computational linguistics and lexicography, idiomatic expressions are generally categorized under the broader umbrella of multiword expressions. According to the Cambridge Dictionary [2] an **idiom** is a group of words in a fixed order that has a particular meaning different from the meanings of each word on its own. An idiom is traditionally defined as a lexical unit consisting of two or more words, characterized by semantic non-compositionality. This means that the overarching sense of the expression cannot be derived simply by aggregating the literal meanings of its constituent parts.

From a structural and semantic perspective, idioms are defined by three primary characteristics. First, they possess **figurative meaning**, where the relationship between the literal components and the actual significance is metaphorical or metonymic. For example, the Bessarabian expression "*a sparge gheața*" (to break the ice) refers to initiating social interaction, without involving any physical act of breaking frozen water. Second, idioms exhibit **syntactic frozenness** (fixity), which means that they resist grammatical alteration or lexical substitution. One cannot replace components with synonyms without losing the idiomatic sense; for example, modifying "*Dintr-un bob răsar o mie*" (From a grain, a thousand sprouts) to "*Dintr-un bob, o grămadă*" (From a grain, a heap) destroys the established resonance of the phrase. Finally, **institutionalization** requires that the expression be recognized by the linguistic community as a single semantic unit. A clear example is "*a tăia frunză la câini*" (literally: cutting leaves for dogs), which is universally understood as "wasting time." In contrast, a constructed phrase like "*a tăia iarbă la pisici*" (to cut grass for cats)—though grammatically valid and semantically parallel—carries - does not have meaning because the community has not encoded it. Similarly, altering "*a fi de râsul găinilor*" (to be the laughingstock of hens) to "*gâștelor*" (geese) breaks the specific cultural code, rendering the phrase without its idiomatic impact.

Folklore researchers have proposed various taxonomies for idioms based on their degree of opacity and grammatical structure. A widely accepted classification distinguishes between four primary categories, ranging from completely opaque to statistically probable. At one end of the spectrum are **pure or opaque idioms**, expressions in which the link between the literal and figurative meaning is either severed or historically obscured.

In the Moldavian linguistic context, a prime example is “*la paștele cailor*” (literally: at the Easter of Horses); thus the constituent words offer no logical clue to the figurative meaning of ”*never*” or ”*an impossible time*.” Similarly, “*a spăla putina*” (to wash the barrel) is semantically opaque, as the literal action of cleaning a vessel has no intrinsic connection to the act of fleeing, except through a lost historical context.

Moving toward greater transparency, **semi-idioms** (or metaphorical idioms) allow the figurative meaning to be partially inferred through cognitive mapping. For example, the Bessarabian expression “*a pune paie pe foc*” (to put straw on the fire) operates on a clear metaphorical logic where the physical act of intensifying a flame maps directly onto the abstract concept of aggravating a conflict.

Distinct from these phrase-level structures are **proverbs and sayings**, which function as complete sentences conveying general truths, moral lessons, or advice. This category is particularly relevant to the Bessarabian corpus, as proverbs often encapsulate the collective experience and archaic wisdom of the community. An illustrative example is “*apa trece, pietrele rămân*” (water passes, stones remain), which conveys the permanence of fundamental values amidst transient events, or “*nu da vrabia din mână pe cioara de pe gard*” (do not trade the sparrow in your hand for the crow on the fence), which advises against risking a certainty for a larger but uncertain gain.

Finally, the taxonomy includes **collocations**, defined as words that statistically co-occur more often than chance. While these exist on the boundary of idiomaticity, they are crucial for natural fluency. In the Moldovan linguistic context, one observes specific pairings such as “*ploaie torențială*” (torrential rain) rather than “*ploaie puternică*” (strong rain), demonstrating a fixed lexical preference despite the semantic interchangeability of the adjectives.

A distinct and fascinating category of idiomatic expressions is one that encodes encodes empirical knowledge of the physical world. Before the formalization of the scientific method, communities relied on acute observation and pattern recognition to survive, eventually crystallizing these insights regarding gravity, thermodynamics, optics, and causality into idioms and proverbs to ensure their transmission across generations. In cognitive linguistics, this phenomenon is framed within Conceptual Metaphor Theory [5], where a source domain rooted in physical reality is mapped onto a target domain representing abstract social or psychological situations. Consequently, the source domain often reveals a surprisingly accurate intuitive understanding of natural laws, frequently referred to as ”**Naïve Physics**” or ”**Folk Physics**.”

This encoding of natural laws is evident across various scientific domains. Principles of thermodynamics and material science, for instance, are preserved in the proverb ”*bate* *fierul cât e cald*" (strike while the iron is hot) a direct instruction derived from metallurgy regarding the increased malleability of metals at high temperatures, applied metaphorically to the seizing of fleeting opportunities. Similarly, the Moldavian proverb "*aurul nu se unește nici cu fierul, nici cu oțelul* " (gold unites with neither iron nor steel) reflects an intuitive understanding of chemical properties and the distinct nature of noble metals; it uses the physical resistance of gold to alloy with base metals to metaphorically describe the fundamental incompatibility between moral nobility and common baseness.

Furthermore, the inescapability of gravitational force is captured in the idiom "*what goes up must come down*" which applies the laws of physics to the rise and fall of power or fortune, while "*Mărul nu cade departe de pom*" (The apple doesn't fall far from the tree) or "*Așchia nu sare departe de trunchi*" (The woodchip does not jump far from the trunk) relies on the observation of vertical trajectories to explain hereditary traits.

Concepts of causality and fluid dynamics are equally prevalent. The idiom "*Fără foc nu iese fum*" (There is no smoke without fire) inextricably links the observable effect of smoke to its necessary cause, combustion. Finally, the behavior of water provides a compelling case of cross-cultural divergence in physical observation. Although the English proverb "*Still waters run deep*" correctly contrasts laminar flow with turbulence to suggest hidden complexity (a state of being), its Moldavian equivalent, "*Apa lină sapă adânc*" (Smooth water digs deep), emphasizes the erosive power and persistence of the quiet force (an action), perfectly aligning with the geological principles of erosion.

Therefore, idioms are not merely linguistic ornaments, but serve as a repository of a community's empirical interaction with the physical universe, preserving observation-based truths that predate formal scientific classification. Despite substantial progress, LLMs continue to face notable challenges in processing idiomatic language. In particular, they often struggle not only to recognize the empirical natural laws and phenomena encoded in such expressions but, more importantly, to map these elements onto their intended figurative meanings. In light of these challenges, we aim to assess the proficiency of state-of-the-art LLMs in interpreting Bessarabian idioms, testing whether current model architectures can successfully navigate this semantic complexity.

3. Methodology

To assess the semantic alignment between linguistic comprehension and the visual representation of idioms, this study employed a cross-modal experimental design. The methodology combines quantitative textual analysis with qualitative visual evaluation to determine how AI systems navigate the non-compositional nature of Moldavian idioms.

3.1. Data collection and corpus

The primary source for this study was the volume of G. Botezatu et al[3]. From this volume, a dataset of 1000 representative Bessarabian idioms was manually digitized and curated [4]. Within the selected dataset, the corpus includes both idioms characteristic of the Bessarabia region and a significant number of common proverbs also used in Romania, thus this broader linguistic space provides a diverse framework for exploring the semantic non-compositionality of cultural heritage.

These idioms were selected based on their cultural relevance, linguistic complexity, and distinct figurative nature. From these 1000 proverbs, 400 were used to analyze 10 LLMs such as: ChatGPT, Grok, Gemini, DeepSeek, Mistral, Qwen, Alibaba Cloud, Copilot, Perplexity, Claude.

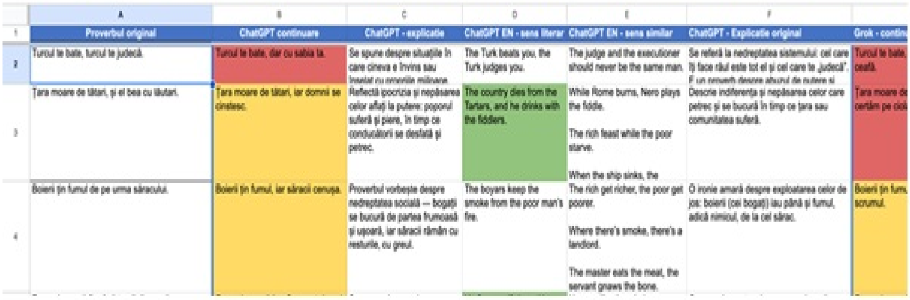
The research was structured into two distinct but interconnected experiments:

(1) **Textual analysis (LLM evaluation)**: Measuring the capacity of generative models to complete, explain, and translate idioms.
(2) **Multimodal visualization:** Investigating how text-to-image models interpret figurative prompts derived from the idioms.

To rigorously assess the performance of the selected models, the experimental workflow was divided into four distinct phases: generation, structuring, algorithmic evaluation, and aggregation.

(1) **Multi-model generation** Each proverb in the dataset was processed individually using ten distinct Large Language Models ChatGPT, Grok, Gemini, DeepSeek, Mistral, Qwen, Alibaba Cloud, Copilot, Perplexity, Claude to generate a comprehensive set of linguistic outputs. For every entry, the models were prompted to provide:
   - **Idiom completion**: Generating the missing segment of the proverb (either the beginning or the ending) to test knowledge recall.
   - **Generate explanation**: When LLMs generate the missing segment of a proverb, we also request an explanation in order to understand the rationale behind the generated continuation and how the figurative meaning is interpreted within the native cultural context.
   - **English interpretations**: Providing both the literal translation and a semantic equivalent (or similar English idiom).
   - **Generate explanation of original idiom**: Explaining the meaning of the original idiom allows us to verify whether the idiom has been generated in its correct form; a strong alignment between this explanation and the explanation

provided during the generation phase indicates deep comprehension. All steps are presented in Figure 1.



	A	B	C	D	E	F	G
1	Proverbul original	ChatGPT continuare	ChatGPT - explicație	ChatGPT EN - sens literar	ChatGPT EN - sens similar	ChatGPT - Explicație original	Grok - contin
2	Turcul te bate, turcul te judecă.	Turcul te bate, dar cu sabia ta.	Se spune despre situațiile în care cineva e învins sau	The Turk beats you, the Turk judges you.	The judge and the executioner should never be the same man.	Se referă la nedreptatea sistemului: cel care îți face răul este tot el și cel care te „judecă”.	Turcul te bate, ceafă.
3	Țara moare de tătari, și el bea cu lăutari.	Țara moare de tătari, iar domnii se cinstesc.	Reflectă ipocrizia și nepăsarea celor aflați la putere: poporul suferă și piere, în timp ce conducătorii se desfată și petrec.	The country dies from the Tartars, and he drinks with the fiddlers.	While Rome burns, Nero plays the fiddle. The rich feast while the poor starve. When the ship sinks, the	Descrie indiferența și nepăsarea celor care petrec și se bucură în timp ce țara sau comunitatea suferă.	Țara moare de certăm pe cio
4	Boierii țin fumul de pe urma săracului.	Boierii țin fumul, iar săracii cenușa.	Proverbul vorbește despre nedreptatea socială — bogații se bucură de partea frumoasă și ușoară, iar săracii rămân cu resturile, cu greul.	The boyars keep the smoke from the poor man's fire.	The rich get richer, the poor get poorer. Where there's smoke, there's a landlord. The master eats the meat, the servant gnaws the bone.	O ironie amară despre exploatarea celor de jos: boierii (cei bogați) iau până și fumul, adică nimicul, de la cel sărac.	Boierii țin fum, scrumul.

Figure 1. Sample of the Collected Data

(2) **Structured data organization** The raw outputs generated by each AI model were systematically organized into a structured CSV format. This step ensured precise alignment between the original proverb (ground truth) and the respective model responses, establishing a clean dataset for comparative analysis.

(3) **Automated comparative analysis (Python implementation)** To evaluate the accuracy and fidelity of the generated texts, a custom Python script was implemented. This script compared the original proverbs against the AI-generated content using an extensive suite of textual and semantic similarity algorithms, specifically:

- **Normalized Levenshtein distance**: To measure character-level edit distance.
- **Jaccard similarity**: To assess the overlap of token sets between texts.
- **SequenceMatcher**: To identify the longest contiguous matching sequences.
- **Jaro-Winkler similarity**: To evaluate string similarity with a bias toward matching prefixes
- **TF-IDF + Cosine similarity**: To measure the angle between vector representations of the texts based on term frequency.
- **Longest common subsequence (LCS)**: To verify the structural integrity of the recalled idioms.
- **Semantic similarity (Neural embeddings)**: Utilizing a multilingual **Sentence transformers** model to evaluate meaning preservation beyond mere lexical overlap.

(4) **Data aggregation and scoring** The resulting scores for each algorithm were collected and aggregated into a unified data structure (CSV/JSON). This consolidation facilitated a robust quantitative analysis, allowing for a statistical comparison of how each model navigates the complexities of Bessarabian idiomatic language.

4. Results analysis using text comparison algorithms

Following the collection and structuring of the Bessarabian idioms corpus, the next stage of the research focused on a comparative analysis of the results generated by various Artificial Intelligence models. The primary objective of this stage was to evaluate the degree of fidelity, consistency, and semantic understanding of the proverbs in relation to their traditional forms and established meanings in Bessarabian popular culture.

Given that idioms are short, semantically dense, and highly figurative linguistic constructions, a simple visual or subjective comparison would have been insufficient. Therefore, the analysis was conducted using text comparison algorithms capable of capturing both formal similarity (at the character and word level) and semantic proximity (meaning, intention, message).

The chosen approach is multi-algorithmic, as no single algorithm can capture all dimensions of a proverb: the exact form, syntactic structure, vocabulary used, and, especially, the figurative meaning. By combining several methods, the results obtained become more robust and their interpretation more credible.

4.1. Algorithms used

Several established algorithms in the field of Natural Language Processing (NLP) were used, each with a well-defined role in the comparative analysis:

String-based algorithms (such as Levenshtein, Jaro-Winkler, or SequenceMatcher) were used to evaluate similarity at the character level and textual order. These are particularly useful for identifying minor variations, spelling differences, or reformulations very close to the original.

Word-based algorithms (Jaccard, Word Overlap) allowed measuring vocabulary overlap while ignoring word order. This approach is relevant for proverbs, where meaning can be preserved even if the phrase structure is modified.

Vector and statistical methods, such as Cosine Similarity using TF-IDF, were used to evaluate the relative importance of terms in each idiom, reducing the influence of very frequent words, and highlighting key concepts.

Deep semantic similarity algorithms, based on pre-trained neural models (Sentence Transformers), completed the analysis. These allow for comparing the actual meaning of texts, even when there is no obvious lexical overlap, being essential for interpreting figurative meanings of idioms and paraphrases.

By using these algorithms in combination, the analysis is not limited to ”how similar the texts are,” but responds more deeply to the question: do they convey the same meaning?

4.2. Results comparison and metrics interpretation

To evaluate the performance of AI models in completing, explaining and translating proverbs, the results were compared using an extensive set of quantitative metrics, grouped into thematic categories:

Proverb completion accuracy (Figure 2): It analyzes how correctly the original proverb is reproduced. The exact match indicator represents cases where the model generated the established form of the proverb identically. The fact that ChatGPT, Grok, and Gemini obtain the same number of exact matches suggests a similar level of familiarity with the canonical forms.

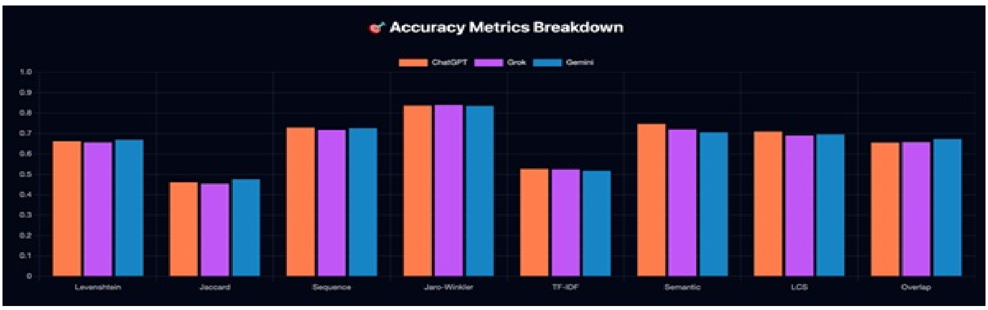


Figure 2. Accuracy Metrics

Textual similarity metrics: For non-identical reproductions, Normalized Levenshtein measures necessary modifications, Jaccard calculates word set overlap (reflecting synonym usage), and Sequence Matcher/LCS ratio analyzes the continuity of sequences to indicate how much of the original structure is preserved. Jaro-Winkler is more lenient with minor differences and favors prefix matches.

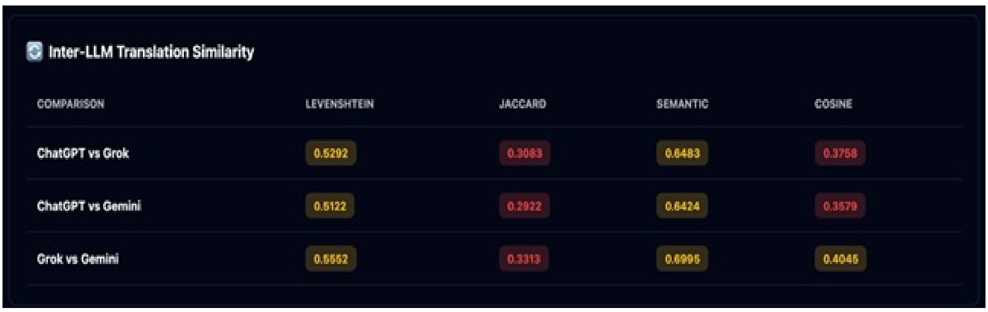
Semantic similarity: Essential for idioms, cosine similarity with TF-IDF evaluates the statistical proximity of important terms, while the semantic score (SentenceTransformer) measures the actual proximity of meaning. High values here indicate that models largely succeed in preserving the central idea of the proverb.

Length and style statistics: Metrics such as average word count and character count show the stylistic consistency of the completions.

Explanation quality (Figure 3): The consistency score reflects how well the explanation aligns with the idiom's meaning. Complexity metrics (length, number of sentences, readability) show the explanatory style: some models are concise, others more elaborate. A low word overlap score is a positive result here, as a good explanation interprets rather than repeats the idiom.

Figure 3. Explanation Quality

Inter-LLM and cross-comparisons (Figure 4): It analyzes convergence between models (e.g., ChatGPT vs. Gemini). High semantic scores versus lower lexical scores indicate agreement on meaning rather than wording



Inter-LLM Translation Similarity

COMPARISON	LEVENSHTEIN	JACCARD	SEMANTIC	COSINE
ChatGPT vs Grok	0.5292	0.3083	0.6483	0.3758
ChatGPT vs Gemini	0.5122	0.2922	0.6424	0.3579
Grok vs Gemini	0.5552	0.3313	0.6995	0.4045

Figure 4. Cross-Comparisons

The comparative analysis of the results generated by the ChatGPT, Grok and Gemini models highlights that all three tools demonstrate a solid capacity to work with Bessarabian idioms, both at the formal and semantic level.

However, the differences between them (Figure 5) become visible when the results are viewed in detail, through the prism of the used metrics.

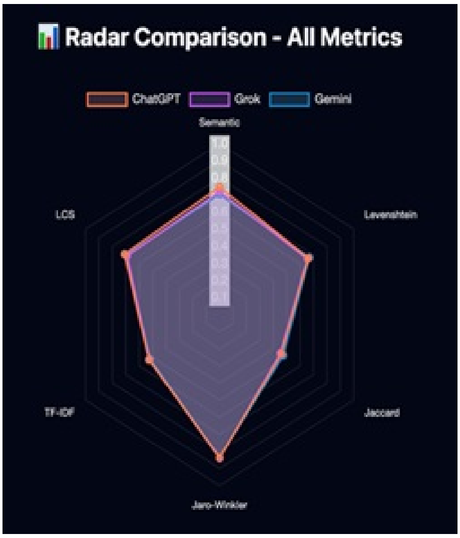


Figure 5. Radar Comparison

5. Conclusions

The exploration of Bessarabian idioms through Large Language Models (LLMs) confirms that current AI architectures have reached a threshold where they can effectively bridge the gap between traditional cultural heritage and digital innovation. This research demonstrates that while semantic non-compositionality remains a fundamental challenge in computational linguistics, modern models possess the necessary depth to navigate figurative meanings that cannot be derived from literal components.

Ultimately, the results indicate that LLMs are no longer merely experimental tools but have become core technologies capable of preserving and interpreting the ”unsalted” nuances of natural communication. This research lays the groundwork for future interactive educational experiences that leverage the synergy between Bessarabian folklore and Generative AI.

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On methods of constructing AD-quasigroups

Liubomir Chiriac, Natalia Josu, and Natalia Lupashco

Abstract. We research T- *quasigroups* with AD identity and Schweitzer identity. The necessary and sufficient condition has been determined that in the T- *quasigroup* G of the form $x \cdot y = \varphi(x) + \psi(y)$, AD identity and the Schweitzer identity are true. We have used the concept of a special direct product of an topological Abelian group G and proved that a binary operation can be defined on the set $G \times G$, such that the new algebraic structure is a non-associative topological quasigroups obeying medial, semimedial and AD laws.

2020 Mathematics Subject Classification: 34C14, 34C40.
Keywords: T-quasigroup, AG-quasigroup, AD-quasigroup, Schweitzer quasigroup, medial, semimedial quasigroup, topological quasigroup.

Asupra metodelor de construcție a quasigrupurilor AD

Rezumat. Lucrarea este dedicata studiului T- *quasigrupurilor* cu identitate AD şi identitate Schweitzer. A fost determinată condiţia necesară şi suficientă ca în T- *quasigrupul* G de forma $x \cdot y = \varphi(x) + \psi(y)$, identitatea AD si identitatea Schweitzer să fie adevarate. A fost utilizat conceptul de produs direct special al unui grup abelian topologic G şi s-a demonstrat ca o operaţie binară poate fi definită pe mulţimea $G \times G$, astfel încât noua structură algebrică să fie un quasigrup topologic neasociativ care este medial, semimedial şi AD.

Cuvinte-cheie: T-quasigrup, AG-quasigrup, AD-quasigrup, quasigrup Schweitzer, quasigrup medial, quasigrup semimedial, quasigrup topologic.

1. Introduction

In this paper we study T-*quasigroups* with AD identity (or Abel-Grassmann's right invertive groupoid identity) $a \cdot bc = c \cdot ba$ and Schweitzer identity $xy \cdot xz = zy$.

A T-*quasigroup* is a specific category of quasigroups $(G, \cdot)$ that can be expressed in terms of automorphisms of an abelian group $(G, +)$, using the relation $x \cdot y = \varphi(x) + \psi(y) + a$, where φ and ψ are automorphisms, and a is a fixed element.

We will say that the isotope $(G, \cdot)$ is generated by the automorphisms φ, ψ and a fixed element $a \in G$ of the Abelian group $(G, +)$, and, we write $(G, \cdot) = g(G, +, \varphi, \psi, a)$.

It is shown that if G is a T-*quasigroup*, then G is AD-*quasigroup* if and only if for any of its T-*forms* $(G, +, \varphi, \psi)$, $\varphi(x) = \psi^2(x)$.

At the same time it is proven that in T-*quasigroup* $(G, \cdot)$ of the form $(G, +, \varphi, \psi)$ Schweitzer identity is true if and only if $\varphi^2(y) + \psi\varphi(y) = 0$, $\varphi(\psi(x)) = \psi(x)$, $\psi^2(z) = \varphi(z)$.

Based on the demonstrated statements, examples of AD and Sweitzer quasigroups are constructed.

A new construction method of medial, semimedial and AD topological quasigroups has been proposed.

The authors have used the concept of a special direct product of an topological Abelian group G and proved that a binary operation can be defined on the set $G \times G$, such that the new algebraic structure is a non-associative topological quasigroups obeying medial, semimedial and AD laws.

The results established are related to the results of L. Chiriac and N. Josu in [1, 2] and to the research papers [3, 4, 7, 8, 9, 10].

2. Basic Notions

In this section we recall some fundamental definitions and notations [5, 6, 11].

A non-empty set G is said to be a *groupoid* with respect to a binary operation denoted by $\{\cdot\}$, if for every ordered pair (a, b) of elements of G there is a unique element $ab \in G$.

A quasigroup is a binary algebraic structure in which one-sided multiplication is a bijection in that all equations of the form $ax = b$ and $ya = b$ have unique solutions.

A groupoid G is called a primitive groupoid with divisions, if there exist two binary operation $l : G \times G \to G$, $r : G \times G \to G$ such that $l(a, b) \cdot a = b, a \cdot r(a, b) = b$ for all $a, b \in G$. Thus, a primitive groupoid with divisions is a universal algebra with three binary operations.

A primitive groupoid G with divisions is called a quasigroup if the equations $ax = b$ and $ya = b$ have unique solutions. In a quasigroup G the divisions l, r are unique. If the multiplication operation in a quasigroup $(G, \cdot)$ with a topology is continuous, then G is called a semitopoligical quasigroup. If in a semitopological quasigroup G the divisions l and r are continuous, then G is called a topological quasigroup.

An element $e \in G$ is called an *identity* if $ex = xe = x$ every $x \in G$. A quasigroup $(G, \cdot)$ with an identity element $e \in G$ is called a loop.

A groupoid $(G, \cdot)$ is called *medial* if it satisfies the law $xy \cdot zt = xz \cdot yt$ for all $x, y, z, t \in G$. A groupoid $(G, \cdot)$ is called *paramedial* if it satisfies the law $xy \cdot zt = ty \cdot zx$ for all $x, y, z, t \in G$.

A groupoid $(G, \cdot)$ is called *bicommutative* if it satisfies the law $xy \cdot zt = tz \cdot yx$ for all $x, y, z, t \in G$.

A groupoid $(G, \cdot)$ is called a *groupoid Abel-Grassmann* or AG*-groupoid* if it satisfies the left invertive law $(ab) \cdot c = (cb) \cdot a$ for all $a, b, c \in G$.

An Abel-Grassmann's right invertive groupoid (also known as a right almost semigroup or RA-semigroup) is a groupoid that satisfies the right invertive law $a(bc) = c(ba)$. This structure is the dual of the AG-groupoid (or left invertive law). We will use a right invertive groupoid as AD groupoid.

A groupoid $(G, \cdot)$ is called AD*-groupoid* if it satisfies the law $a \cdot bc = c \cdot ba$ for all $a, b, c \in G$.

A *Schweitzer* quasigroup $(G, \cdot)$ is a quasigroup that satisfies the Schweitzer identity $(xy) \cdot (xz) = zy$ for all elements $x, y, z \in G$.

A *Neumann* quasigroup $(G, \cdot)$ is a quasigroup that satisfies the Neumann identity $x(yz \cdot yx) = z$ for all elements $x, y, z \in G$.

Left semi-medial identity in a groupoid $(G, \cdot)$ has the following form: $xx \cdot zt = xy \cdot xz$ for all $x, y, z, t \in G$. R.H. Bruck [15] used this identity to define commutative Moufang loops in the class of loops.

3. T-quasigroups with AD and Sweitzer identities

In this section we study some aspects of charaterization of abelian groups isotopic to T-$quasigroups$.

Definition 3.1. *Quasigroup $(G, \cdot)$ is a T-$quasigroup$ if and only if there exists an abelian group $(G, +)$, its automorphisms φ and ψ, and a fixed element $a \in G$ such that $x \cdot y = \varphi(x) + \psi(y) + a$ for all $x, y \in G$.*

Under the conditions of Definition 3.1 we shall say that the isotope $(G, \cdot)$ is generated by the automorphisms φ, ψ and a fixed element $a \in G$ of the abelian group $(G, +)$, and, we write $(G, \cdot) = g(G, +, \varphi, \psi, a)$.

We study the problem formulated below.

Problem 3.1. *Let $(Q, \cdot)$ be a T-$quasigroup$. Under which conditions the Q is a quasigroup (of its T-$forms$ $(Q, +, \varphi, \psi, a)$) satisfying the identities P_i of some algebraic structure, where $i = 1, 2, ..., k$?*

In [12, 13], V. Shcherbacov and his students studied T-quasigroups with Schroder identity, Stein 2-nd and 3-rd identity.

We examine the T-$quasigroups$ with AD and Schweitzer identities.

Theorem 3.1. *Let $(G,\cdot)$ be a T-$quasigroup$. Then $(G,\cdot)$ is AD-$quasigroup$ if and only if for any of its T-$forms$ $(G,+,\varphi,\psi)$, $\varphi(x)=\psi(x)^2$.*

Proof. We rewrite the identity of the AD-$quasigroup$

$$x\cdot yz = z\cdot yx, \tag{1}$$

in the following form:

$$\varphi(x)+\psi(yz)=\varphi(z)+\psi(yx). \tag{2}$$

From (2) we have

$$\varphi(x)+\psi(\varphi(y)+\psi(z))=\varphi(z)+\psi(\varphi(y)+\psi(x)). \tag{3}$$

$$\varphi(x)+\psi\varphi(y)+\psi^2(z)=\varphi(z)+\psi\varphi(y)+\psi^2(x). \tag{4}$$

$$\varphi(x)+\psi^2(z)=\varphi(z)+\psi^2(x). \tag{5}$$

If we substitute $x=0$ in equality (5), then we obtain

$$\psi^2(z)=\varphi(z). \tag{6}$$

If in equality (5) $z=0$, then we obtain

$$\varphi(x)=\psi^2(x). \tag{7}$$

Converse. Substituting the expression $x\cdot y=\varphi(x)+\psi(y)$ in identity (1), then we get (4). Substituting equalities (6) and (7) in (4), we obtain that identity (4) is true. In this way, in this case, we have that identity (1) is true. The proof is complete. □

Example 3.1. *Examine the group $\mathbb{Z}_{17}$ of residues modulo 17. Define the quasigroup $(G,\cdot)$. We define the binary operation $x\cdot y=16x+4y(mod17)$ for all $x,y\in G$. Then $(G,\cdot)$ is an $AD-quasigroup$. Check. Let $x\cdot yz=z\cdot yx$. Then* $16x+4(16y+4z)=16z+4(16y+4x)(mod17)$, $16x+64y+16z=16z+64y+16x(mod17)$, $0=0(mod17)$.

Example 3.2. *Examine the group $\mathbb{Z}_{27}$ of residues modulo 27. Define the quasigroup $(G,\cdot)$, where binary operation $x\cdot y=25x+5y(mod27)$ for all $x,y\in G$. Then $(G,\cdot)$ is an $AD-quasigroup$. Check. Let $x\cdot yz=z\cdot yx$. Then* $25x+5(25y+5z)=25z+5(25y+5x)(mod27)$, $25x+125y+25z=25z+125y+25x(mod27)$, $0=0(mod27)$.

Using Theorem 3.1 from [7], we can construct the example of AG-quasigroups. A T-$quasigroup$ $(G,\cdot)$ is an AG-quasigroup if and only if the following condition holds for its automorphisms:$\varphi^2(x)=\psi(x)$ for all $x\in G$

Example 3.3. *Examine the group $\mathbb{Z}_{37}$ of residues modulo 37. Define the quasigroup $(G,\cdot)$. We define the binary operation $x \cdot y = 6x + 36y (mod37)$ for all $x, y \in G$. Then $(G,\cdot)$ is an AG-quasigroup. Check. Let $(x \cdot y) \cdot z = (z \cdot y) \cdot x$. Then $6(6x+36y)+36z = 6(6z+36y)+36x (mod37)$, $36x+108y+36z = 36z+108y+36x (mod37)$, $0 = 0 (mod37)$.*

Example 3.4. *Denote by $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z} = \{0, 1, ..., p-1\}$ the cyclic group of order p. Let $(G,+) = (\mathbb{Z}_5,+)$, $\varphi(x) = 4x$, $\psi(x) = 2x$. Then $x \cdot y = 4x + 2y$ and $\varphi(x) = \psi^2(x)$. Hence, $(G,\cdot) = g(G,+,\varphi,\psi)$ is an AD-quasigroup. Below we have constructed the Cayley table for AD-quasigroup $(G,\cdot)$*

$(\cdot)$	0	1	2	3	4
0	0	2	4	1	3
1	4	1	3	0	2
2	3	0	2	4	1
3	2	4	1	3	0
4	1	3	0	2	4

In this case, $(G,\cdot) = g(G,+,4x,2x)$ is an AD-quasigroup, but it is not AG-quasigroup. For example $(2 \cdot 4) \cdot 3 \neq (3 \cdot 4) \cdot 2$.

Example 3.5. *Let $(G,+) = (\mathbb{Z}_5,+)$, $\varphi(x) = 3x$, $\psi(x) = 2x$. Then $x \cdot y = 3x + 2y$ and $\varphi(x) \neq \psi^2(x)$. Hence, $(G,\cdot) = g(G,+,\varphi,\psi)$ is not an AD-quasigroup. Below we have constructed the Cayley table for quasigroup $(G,\cdot)$, where $a \cdot (bc) = c \cdot (ba)$ does not hold in $(G,\cdot)$. For example, $3 \cdot (4 \cdot 2) \neq 2 \cdot (4 \cdot 3)$. At the same time, $(G,\cdot) = g(G,+,3x,2x)$ is not AG-quasigroup, for example, $(3 \cdot 4) \cdot 2 \neq (2 \cdot 4) \cdot 3$.*

$(\cdot)$	0	1	2	3	4
0	0	2	4	1	3
1	3	0	2	4	1
2	1	3	0	2	4
3	4	1	3	0	2
4	2	4	1	3	0

Theorem 3.2. *Let $(G,\cdot)$ be a T-quasigroup. Then $(G,\cdot)$ is Schweitzer quasigroup if and only if for any of its T-forms $(G,+,\varphi,\psi)$, $\varphi^2(y) + \psi\varphi(y) = 0$, $\varphi(\psi(x)) = \psi(x)$, $\psi^2(z) = \varphi(z)$.*

Proof. The identity of the Schweitzer quasigroup

$$(yx) \cdot (yz) = zx, \tag{8}$$

we rewrite in the following form:

$$\varphi(yx) + \psi(yz) = \varphi(z) + \psi(x). \tag{9}$$

From (2) we have

$$\varphi(\varphi(y) + \psi(x)) + \psi(\varphi(y) + \psi(z)) = \varphi(z) + \psi(x), \tag{10}$$

$$\varphi^2(y) + \varphi\psi(x) + \psi\varphi(y) + \psi^2(z) = \varphi(z) + \psi(x). \tag{11}$$

If we substitute $x = z = 0$ in equality (11), then we obtain

$$\varphi^2(y) + \psi\varphi(y) = 0. \tag{12}$$

Similarly, if we substitute $z = y = 0$ in equality (11), then we obtain

$$\varphi(\psi(x)) = \psi(x). \tag{13}$$

If in equality (11) $x = y = 0$, then we get

$$\psi^2(z) = \varphi(z). \tag{14}$$

Converse. Substituting the expression $x \cdot y = \varphi(x) + \psi(y)$ in identity (8), we get (11). Substituting equalities (12), (13) and (14) in (11), we obtain that the identity (11) is true. In this way, in this case, identity (8) is true. The proof is complete. □

Remark 3.1. *We can rewrite Theorem 3.3 in the next form.*

Theorem 3.3. *Let G be a T-quasigroup. Then G is Schweitzer quasigroup if and only if for any of its T-forms $(G, +, \varphi, \psi)$, $\psi^2 = \varepsilon$, $\varphi = \varepsilon$, $\varepsilon + \psi(\varepsilon) = 0$.*

Example 3.6. *Denote by $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z} = \{0, 1, ..., p-1\}$ the cyclic group of order p. Let $(G, +) = (\mathbb{Z}_7, +)$, according to Theorem 3.3, $\varphi(x) = x$, $\psi(x) = -x$. Then $x \cdot y = x - y$ and $\varphi(x) = x = -\psi(x)$, $x^2 = x \cdot x = 0$. Hence, $(G, \cdot) = g(G, +, \varphi, \psi)$ is an Schweitzer* *quasigroup. Below we have constructed the Cayley table for $Schweitzer$ quasigroup $(G, \cdot)$. This quasigroup is also AD-quasigroup.*

$(\cdot)$	0	1	2	3	4	5	6
0	0	6	5	4	3	2	1
1	1	0	6	5	4	3	2
2	2	1	0	6	5	4	3
3	3	2	1	0	6	5	4
4	4	3	2	1	0	6	5
5	5	4	3	2	1	0	6
6	6	5	4	3	2	1	0

Remark 3.2. *In* [14], *N. Diduric and V. Shcherbacov proved that any Neumann quasigroup $(Q, \cdot)$ can be presented in the form $x \cdot y = x - y$, where $(Q, +)$ is an abelian group. Automorphism group of Neumann quasigroup coincides with the group $Aut(Q, +)$. Any Schweizer quasigroup is a Neumann quasigroup and vice versa.*

4. On a method of constructing medial, semimedial and AD topological quasigroups

In Section 4 we examined:

Problem 4.1. *Let $(G, +)$ be a commutative topological group. Under which conditions on the set $G \times G$ can one define a binary operation $(\circ)$ such that, $(G \times G, \circ)$ is a nonassociative topological quasigroups obeying certain laws?*

We prove a new method of constructing non-associative medial, semimedial, non-paramedial and AD topological quasigroup.

Theorem 4.1. *Let $(G, +, \tau)$ be a commutative topological group where G is not a singleton. For (x_1, y_1) and (x_2, y_2) in $G \times G$ define*

$$(x_1, y_1) \circ (x_2, y_2) = (x_2 + y_2 + y_1, -x_1 - y_1 - x_2)$$

Then $(G \times G, \circ, \tau_G)$, relative to the product topology τ_G, is a medial, Moufang and AD non-associative topological quasigroup. Moreover, if (G, τ) is T_i-$space$, then $(G \times G, \tau_G)$ is T_i-$space$ too, where $i = 1, 2, 3, 3.5$.

Proof. **1.** We will prove that $(G \times G, \circ)$ is a quasigroup. To this end, we will show that the equations $y \circ a = b$ and $a \circ x = b$ have unique solutions in $(G \times G, \circ)$. Let $y = (y_1, y_2)$,

$x = (x_1, x_2)$, $a = (a_1, a_2)$ and $b = (b_1, b_2)$. Since $y \circ a = b$ we have

$$(y_1, y_2) \circ (a_1, a_2) = (b_1, b_2). \tag{15}$$

According to the conditions of Theorem 4.1

$$(y_1, y_2) \circ (a_1, a_2) = (a_1 + a_2 + y_2, -y_1 - y_2 - a_1). \tag{16}$$

From (15) and (16) we get

$$a_1 + a_2 + y_2 = b_1 \tag{17}$$

and

$$-y_1 - a_1 - y_2 = b_2. \tag{18}$$

From (17) and (18) we obtain

$$y_2 = b_1 - a_1 - a_2. \tag{19}$$

and

$$y_1 = a_2 - b_1 - b_2. \tag{20}$$

Hence, $y_1 = a_2 - b_1 - b_2$ and $y_2 = b_1 - a_1 - a_2$ are solutions of the equation $y \circ a = b$. It is easy to show that any other solutions of that equation coincide with y_1 and y_2.

In this case

$$l((a_1, a_2), (b_1, b_2)) = (a_2 - b_1 - b_2, b_1 - a_1 - a_2)$$

and

$$l((a_1, a_2), (b_1, b_2)) \circ (a_1, a_2) = (b_1, b_2).$$

We will show that the equation $a \circ x = b$ have unique solutions in $(G \times G, \circ)$. Let $x = (x_1, x_2)$, $a = (a_1, a_2)$ and $b = (b_1, b_2)$. Since $a \circ x = b$ we have

$$(a_1, a_2) \circ (x_1, x_2) = (b_1, b_2). \tag{21}$$

According to the conditions of Theorem 4.1

$$(a_1, a_2) \circ (x_1, x_2) = (x_1 + x_2 + a_2, -a_1 - a_2 - x_1). \tag{22}$$

From (21) and (22) we get

$$x_1 + x_2 + a_2 = b_1 \tag{23}$$

and

$$-a_1 - a_2 - x_1 = b_2. \tag{24}$$

From (23) and (24) we obtain

$$x_2 = b_1 + b_2 + a_1. \tag{25}$$

and

$$x_1 = -a_1 - a_2 - b_2. \tag{26}$$

Hence, $x_1 = -a_1 - a_2 - b_2$ and $x_2 = b_1 + b_2 + a_1$ are solutions of the equation $a \circ x = b$. It is easy to show that any other solutions of that equation coincide with x_1 and x_2. Then $x_1 = -a_1 - b_2 - a_2$ and $x_2 = b_1 + b_2 + a_1$ are unique solutions.

In this case

$$r((a_1, a_2), (b_1, b_2)) = (-a_1 - b_2 - a_2, b_1 + b_2 + a_1)$$

and $(a_1, a_2) \circ r((a_1, a_2), (b_1, b_2)) = (b_1, b_2)$. Thus $(G \times G, \circ)$ is a quasigroup.

2. We will prove that associativity

$$((x_1, y_1) \circ (x_2, y_2)) \circ (x_3, y_3) = (x_1, y_1) \circ ((x_2, y_2) \circ (x_3, y_3)) \tag{27}$$

does not hold in $(G \times G, \circ)$.

Indeed, for the first side of the law (27), we obtain

$$((x_1, y_1) \circ (x_2, y_2)) \circ (x_3, y_3) = (x_2 + y_2 + y_1, -x_1 - y_1 - x_2) \circ (x_3, y_3) =$$

$$= (x_3 + y_3 - x_1 - y_1 - x_2, -x_2 - y_2 - y_1 + x_1 + y_1 + x_2 - x_3). \tag{28}$$

$$= (x_3 + y_3 - x_1 - y_1 - x_2, -y_2 + x_1 - x_3). \tag{29}$$

Similarly, for the second side of the law (27), we have

$$(x_1, y_1) \circ ((x_2, y_2) \circ (x_3, y_3)) = (x_1, y_1) \circ (x_3 + y_3 + y_2, -x_2 - y_2 - x_3) =$$

$$= (y_3 + y_1 - x_2, -x_1 - y_1 - y_2 - y_3 - x_3). \tag{30}$$

From (29) and (30), it is clear that associativity does not hold in $(G \times G, \circ)$.

3. We will show that $(G \times G, \circ)$ is medial quasigroup, that is, the property $xy \cdot zt = xz \cdot yt$ holds.

Let $x = (x_1, y_1), y = (x_2, y_2), z = (x_3, y_3), t = (x_4, y_4)$, then

$$((x_1, y_1) \circ (x_2, y_2)) \circ ((x_3, y_3) \circ (x_4, y_4)) = ((x_1, y_1) \circ (x_3, y_3)) \circ ((x_2, y_2) \circ (x_4, y_7)). \tag{31}$$

According to the Theorem, for the first side of the law (31), we have

$$((x_1, y_1) \circ (x_2, y_2)) \circ ((x_3, y_3) \circ (x_4, y_4)) =$$

$$= ((x_2 + y_2 + y_1, -x_1 - y_1 - x_2)) \circ ((x_4 + y_4 + y_3, -x_3 - y_3 - x_4)) =$$

$$= (y_4 - x_3 - x_2 - x_1 - y_1, -y_2 + x_1 - x_4 - y_4 - y_3). \tag{32}$$

Similarly, for the other side of the law (31), we get

$$((x_1, y_1) \circ (x_3, y_3)) \circ ((x_2, y_2) \circ (x_4, y_4)) =$$

$$= (x_3 + y_3 + y_1, -x_1 - y_1 - x_3) \circ (x_4 + y_4 + y_2, -x_2 - y_2 - x_4) =$$

$$= (y_4 - x_3 - x_2 - x_1 - y_1, -y_2 + x_1 - x_4 - y_4 - y_3). \qquad (33)$$

From (32) and (33), we obtain that both sides are equal and $(G \times G, \circ)$ is a medial quasigroup.

Similarly, it is shown that $(G \times G, \circ)$ is a semimedial(commutative Moufang), non-paramedial quasigroup.

4. Similarly, we will show that AD identity $a \cdot bc = c \cdot ba$ is fulfilled in $(G \times G, \circ)$.

Let $x = (x_1, y_1), y = (x_2, y_2), z = (x_3, y_3)$. Then

$$(x_1, y_1) \circ ((x_2, y_2) \circ (x_3, y_3)) = (x_3, y_3) \circ ((x_2, y_2)) \circ (x_1, y_1)). \qquad (34)$$

Indeed, according to the Theorem for the first side of the law (34), we have

$$(x_1, y_1) \circ ((x_2, y_2) \circ (x_3, y_3)) =$$

$$(x_1, y_1) \circ ((x_3 + y_3 + y_2, -x_2 - y_2 - x_3)) =$$

$$(y_3 - x_2 + y_1, -x_1 - y_1 - x_3 - y_3 - y_2). \qquad (35)$$

Similarly, for the other side of the law (34) we get

$$(x_3, y_3) \circ ((x_2, y_2) \circ (x_1, y_1)) =$$

$$(x_3, y_3) \circ (x_1 + y_1 + y_2, -x_2 - y_2 - x_1) =$$

$$(y_3 - x_2 + y_1, -x_1 - y_1 - x_3 - y_3 - y_2). \qquad (36)$$

From (35) and (36), we obtain that both sides are equal and $(G \times G, \circ)$ is an AD quasigroup.

Multiplication $(\circ)$ and divisions $l(a, b)$ and $r(a, b)$ are jointly continuous relative to the product topology.

Consequently, $(G \times G, \circ, \tau_G)$ is a non-associative, medial, semimedial and AD topological quasigroup.

If (G, τ) is T_i-space, then according to Theorem 2.3.11 in [6], a product of T_i-spaces is a T_i-spaces, where $i = 1, 2, 3, 3.5$. The proof is complete. □

In [7] was proved the following Theorem.

Theorem 4.2. *Let $(G, +, \tau)$ be a commutative topological group where G is not a singleton. For (x_1, y_1) and (x_2, y_2) in $G \times G$ define*

$$(x_1, y_1) \circ (x_2, y_2) = (-x_1 - x_2, y_1 + y_2)$$

Then $(G \times G, \circ, \tau_G)$, relative to the product topology τ_G, is a medial, semimedial, paramedial, bicommutative, Manin, Cote and GA non-associative topological quasigroup. Moreover, if (G, τ) is $T_i - space$, then $(G \times G, \tau_G)$ is $T_i - space$ too, where $i = 1, 2, 3, 3.5$.

The following Theorem was proved in [2].

Theorem 4.3. *Let $(G, +, \tau)$ be a commutative topological group where G is not a singleton. For (x_1, y_1) and (x_2, y_2) in $G \times G$ define*

$$(x_1, y_1) \circ (x_2, y_2) = (-x_1 - y_1 + y_2, -x_2 - y_2 + x_1).$$

Then $(G \times G, \circ, \tau_G)$, relative to the product topology τ_G, is a non-associative, medial, AG and AD-topological quasigroup. Moreover, if (G, τ) is $T_i - space$, then $(G \times G, \tau_G)$ is $T_i - space$ too, where $i = 1, 2, 3, 3.5$.

Example 4.1. *Let $G = \{0, 1, 2\}$. We define the binary operation " + ".*

(+)	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

Then $(G, +)$ is a commutative group. Define a binary operation $(\circ)$ on the set $G \times G$ by $(x_1, y_1) \circ (x_2, y_2)$=$(x_2 + y_2 + y_1, -x_1 - y_1 - x_2)$ for all $x_1, y_1, x_2, y_2 \in G \times G$. If we label the elements as follows $(0, 0) \leftrightarrow 0$, $(0, 1) \leftrightarrow 1$, $(0, 2) \leftrightarrow 2$, $(1, 0) \leftrightarrow 3$, $(1, 1) \leftrightarrow 4$, $(1, 2) \leftrightarrow 5$, $(2, 0) \leftrightarrow 6$, $(2, 1) \leftrightarrow 7$, $(2, 2) \leftrightarrow 8$, then, we obtain:

(∘)	0	1	2	3	4	5	6	7	8
0	0	3	6	5	8	2	7	1	4
1	5	8	2	7	1	4	0	3	6
2	7	1	4	0	3	6	5	8	2
3	2	5	8	4	7	1	6	0	3
4	4	7	1	6	0	3	2	5	8
5	6	0	3	2	5	8	4	7	1
6	1	4	7	3	6	0	8	2	5
7	3	6	0	8	2	5	1	4	7
8	8	2	5	1	4	7	3	6	0

Then $(G \times G, \circ)$ is a medial, semimedial, non-paramedial, AD non-associative quasigroup.

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Maximal multiplicity of the line at infinity in a sextic polynomial differential system

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Abstract. We investigate planar polynomial differential systems of degree six having an invariant straight line at infinity with high algebraic multiplicity in the sense of the Poincaré compactification. By analyzing the determinant associated with the invariant line at infinity, we show that the maximal multiplicity attainable in this class is 16. The computations involved are extremely large, and only representative cases are presented. The obtained results support the conjecture that the bound $M_\infty(n) = 3n - 2$ is sharp for polynomial differential systems of degree $n \geq 2$.
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Keywords: planar polynomial differential systems, invariant straight line at infinity, algebraic multiplicity, Poincaré transformation.

Multiplicitatea maximală a dreptei de la infinit într-un sistem diferențial polinomial de gradul şase

Rezumat. În această lucrare investigăm sisteme diferențiale polinomiale plane de gradul șase care admit o dreaptă invariantă la infinit cu multiplicitate algebrică mare, în sensul compactificării lui Poincaré. Prin analiza determinantului asociat dreptei invariante de la infinit, arătăm că multiplicitatea maximă ce poate fi atinsă în această clasă este 16. Calculele implicate sunt extrem de voluminoase, fiind prezentate doar cazuri reprezentative. Rezultatele obținute susțin conjectura conform căreia limita $M_\infty(n) = 3n - 2$ este optimă pentru sisteme diferențiale polinomiale de grad $n \geq 2$.
Cuvinte-cheie: sisteme diferențiale polinomiale, dreaptă invariantă la infinit, multiplicitate algebrică, transformare Poincaré.

1. Introduction

We consider the real planar polynomial differential system

$$\begin{cases} \dfrac{dx}{dt} = P(x, y), \\ \dfrac{dy}{dt} = Q(x, y), \end{cases} \tag{1}$$

where $P, Q \in \mathbb{R}[x, y]$, $\gcd(P, Q) = 1$, and let

$$\mathbb{X} = P(x, y)\frac{\partial}{\partial x} + Q(x, y)\frac{\partial}{\partial y}$$

be the associated polynomial vector field. The degree of system (1) is defined as $n = \max\{\deg P, \deg Q\}$.

Definition 1.1. *An algebraic curve $f(x, y) = 0$, $f \in \mathbb{C}[x, y]$, is called an invariant algebraic curve of system* (1) *if there exists a polynomial $K(x, y) \in \mathbb{C}[x, y]$ such that*

$$\mathbb{X}(f) = f(x, y)K(x, y). \tag{2}$$

Invariant algebraic curves play an important role in the qualitative theory of polynomial differential systems [1], [2], [3], [4]. The problem of determining the maximal number of invariant straight lines of a polynomial differential system has been studied in [5].

Polynomial differential systems having an invariant straight line of maximal multiplicity were studied in [6], [7], [8], and polynomial differential systems with a center-focus critical point having an invariant straight line of maximal multiplicity were investigated in [9], [10], [11], [12]. Invariant straight lines are also used in the construction of Darboux first integrals; see, for example, [13], [14], [15].

In this paper, we use the notion of algebraic multiplicity introduced in [16].

Definition 1.2. *Let $\mathbb{C}_m[x]$ be the $\mathbb{C}$-vector space of polynomials in $\mathbb{C}[x]$ of degree at most m. Then it has dimension $R = \binom{n+m}{n}$. Let $v_1, v_2, \ldots, v_R$ be a base of $\mathbb{C}_m[x]$. If k is the greatest positive integer such that the k-th power of f divides* $\det M_R$*, where*

$$M_R = \begin{pmatrix} v_1 & v_2 & \ldots & v_R \\ \mathbb{X}(v_1) & \mathbb{X}(v_2) & \ldots & \mathbb{X}(v_R) \\ \ldots & \ldots & \ldots & \ldots \\ \mathbb{X}^{R-1}(v_1) & \mathbb{X}^{R-1}(v_2) & \ldots & \mathbb{X}^{R-1}(v_R) \end{pmatrix},$$

then the invariant algebraic curve f of degree m of the vector field $\mathbb{X}$ has algebraic multiplicity k.

In [7], it was proved that for planar polynomial differential systems of degree $n \geq 2$, the maximal algebraic multiplicity $M_a(n)$ of an affine invariant straight line satisfies

$$3n - 2 \leq M_a(n) \leq 3n - 1.$$

Moreover, for cubic systems the maximal multiplicity of a real straight line, including the line at infinity, is equal to seven. Following the notation introduced in [7], we denote by $M_\infty(n)$ the maximal multiplicity of the invariant straight line at infinity.

The aim of this paper is to study planar polynomial differential systems of degree six of the form

$$\begin{cases} \dot{x} = \sum_{n=0}^{6} \sum_{i=0}^{n} a_{i,n-i} x^i y^{n-i}, \\ \dot{y} = \sum_{n=0}^{6} \sum_{i=0}^{n} b_{i,n-i} x^i y^{n-i}, \end{cases} \tag{3}$$

where $a_{i,j}, b_{i,j} \in \mathbb{R}$.

The main contribution of this paper is the identification and classification, up to orbital equivalence, of planar polynomial differential systems of degree six whose invariant straight line at infinity has maximal algebraic multiplicity.

2. Obtaining systems with maximal multiplicity

For system (3), we apply the Poincaré transformation $x \mapsto \frac{1}{z}$, $y \mapsto \frac{y}{z}$, followed by the substitution $y \mapsto x$ and $z \mapsto y$. In this way, we obtain the system

$$\begin{cases} \dot{x} = \sum_{n=0}^{6} \sum_{i=0}^{n} \left(a_{i,n-i}\, x^{n-i+1} y^{6-n} + b_{i,n-i}\, x^{n-i} y^{6-n} \right), \\ \dot{y} = y \sum_{n=0}^{6} \sum_{i=0}^{n} a_{i,n-i}\, x^{n-i} y^{6-n}. \end{cases} \tag{4}$$

where the coefficients coincide with those of system (3). The invariant straight line $y = 0$ of system (4) represents the invariant straight line located at infinity of system (3).

Therefore, in order to obtain an invariant line at infinity of maximal multiplicity, it is necessary to impose that the polynomial $\det M_R$ admits y as a factor of maximal possible power. In this case, taking $v_1 = 1$, $v_2 = x$, and $v_3 = y$, we obtain the matrix M_R from Definition 1.2:

$$M_R = \begin{pmatrix} 1 & x & y \\ \mathbb{X}(1) & \mathbb{X}(x) & \mathbb{X}(y) \\ \mathbb{X}^2(1) & \mathbb{X}^2(x) & \mathbb{X}^2(y) \end{pmatrix} = \begin{pmatrix} 1 & x & y \\ 0 & P & Q \\ 0 & P\frac{\partial P}{\partial x} + Q\frac{\partial P}{\partial y} & P\frac{\partial Q}{\partial x} + Q\frac{\partial Q}{\partial y} \end{pmatrix}.$$

Computing the determinant of this matrix, we obtain

$$\det M_R = P\left(P\frac{\partial Q}{\partial x} + Q\frac{\partial Q}{\partial y}\right) - Q\left(P\frac{\partial P}{\partial x} + Q\frac{\partial P}{\partial y}\right) = y\sum_{i=1}^{17} A_i(x) y^i. \tag{5}$$

Thus, y is a factor of the polynomial (5), which shows that the line at infinity is invariant. The highest power of y appearing in (5) is 18, and therefore we can conclude that

$$M_\infty(6) \leq 18.$$

In order for the line at infinity of system (3) to have maximal multiplicity, we require that all the polynomials $A_i(x)$ vanish, that is, all coefficients of these polynomials with respect to x are equal to zero. From the condition $A_1(x) = 0$, we obtain that the multiplicity of the line at infinity is equal to 2, and so on.

Despite extensive computational efforts, some cases could not be fully resolved due to their extreme algebraic complexity. In particular, the corresponding polynomial systems require up to 4.5 GB of memory, and even on a workstation equipped with an Intel Core i7-14700K processor and 32 GB of RAM, the computations could not be completed after more than 100 hours of continuous runtime. Consequently, several cases have been omitted.

Despite these limitations, three polynomial differential systems attaining maximal multiplicity were identified. The first of these systems is presented below.

$$
\begin{cases}
\dot{x} = \dfrac{a_{6,0}^{11}\left(a_{6,0}b_{1,0}-a_{1,0}b_{6,0}\right)^4}{432b_{6,0}^5}\Big(432x^6a_{6,0}^4b_{6,0}^6 - 2592x^5ya_{6,0}^5b_{6,0}^5 + 432x^5a_{5,0}a_{6,0}^3b_{6,0}^6 + 6480x^4y^2a_{6,0}^6b_{6,0}^4 - \\
-2160x^4ya_{5,0}a_{6,0}^4b_{6,0}^5 + 180x^4a_{5,0}^2a_{6,0}^2b_{6,0}^6 - 8640x^3y^3a_{6,0}^7b_{6,0}^3 + 4320x^3y^2a_{5,0}a_{6,0}^5b_{6,0}^4 - 720x^3ya_{5,0}^2a_{6,0}^3b_{6,0}^5 + \\
+40x^3a_{5,0}^3a_{6,0}b_{6,0}^6 + 6480x^2y^4a_{6,0}^8b_{6,0}^2 - 4320x^2y^3a_{5,0}a_{6,0}^6b_{6,0}^3 + 1080x^2y^2a_{5,0}^2a_{6,0}^4b_{6,0}^4 - 120x^2ya_{5,0}^3a_{6,0}^2b_{6,0}^5 + \\
+5x^2a_{5,0}^4b_{6,0}^6 - 2592xy^5a_{6,0}^9b_{6,0} + 2160xy^4a_{5,0}a_{6,0}^7b_{6,0}^2 - 720xy^3a_{5,0}^2a_{6,0}^5b_{6,0}^3 + 120xy^2a_{5,0}^3a_{6,0}^3b_{6,0}^4 - \\
-10xya_{5,0}^4a_{6,0}b_{6,0}^5 + 432xa_{1,0}a_{6,0}^3b_{6,0}^6 - 432y^5a_{5,0}a_{6,0}^8b_{6,0} + 180y^4a_{5,0}^2a_{6,0}^6b_{6,0}^2 - 40y^3a_{5,0}^3a_{6,0}^4b_{6,0}^3 + \\
+5y^2a_{5,0}^4a_{6,0}^2b_{6,0}^4 + 2160ya_{6,0}^5b_{1,0}b_{6,0}^4 - 2592ya_{1,0}a_{6,0}^4b_{6,0}^5 + 432a_{6,0}^4b_{0,0}b_{6,0}^5 - 72a_{5,0}a_{6,0}^3b_{1,0}b_{6,0}^5 + \\
+72a_{1,0}a_{5,0}a_{6,0}^2b_{6,0}^6 + 432y^6a_{6,0}^{10}\Big), \\
\dot{y} = \dfrac{a_{6,0}^{10}\left(a_{6,0}b_{1,0}-a_{1,0}b_{6,0}\right)^4}{432b_{6,0}^4}\Big(432x^6a_{6,0}^4b_{6,0}^6 - 2592x^5ya_{6,0}^5b_{6,0}^5 + 432x^5a_{5,0}a_{6,0}^3b_{6,0}^6 + 6480x^4y^2a_{6,0}^6b_{6,0}^4 - \\
-2160x^4ya_{5,0}a_{6,0}^4b_{6,0}^5 + 180x^4a_{5,0}^2a_{6,0}^2b_{6,0}^6 - 8640x^3y^3a_{6,0}^7b_{6,0}^3 + 4320x^3y^2a_{5,0}a_{6,0}^5b_{6,0}^4 - 720x^3ya_{5,0}^2a_{6,0}^3b_{6,0}^5 + \\
+40x^3a_{5,0}^3a_{6,0}b_{6,0}^6 + 6480x^2y^4a_{6,0}^8b_{6,0}^2 - 4320x^2y^3a_{5,0}a_{6,0}^6b_{6,0}^3 + 1080x^2y^2a_{5,0}^2a_{6,0}^4b_{6,0}^4 - 120x^2ya_{5,0}^3a_{6,0}^2b_{6,0}^5 + \\
5x^2a_{5,0}^4b_{6,0}^6 - 2592xy^5a_{6,0}^9b_{6,0} + 2160xy^4a_{5,0}a_{6,0}^7b_{6,0}^2 - 720xy^3a_{5,0}^2a_{6,0}^5b_{6,0}^3 + 120xy^2a_{5,0}^3a_{6,0}^3b_{6,0}^4 - \\
-10xya_{5,0}^4a_{6,0}b_{6,0}^5 + 432xa_{6,0}^4b_{1,0}b_{6,0}^5 - 432y^5a_{5,0}a_{6,0}^8b_{6,0} + 180y^4a_{5,0}^2a_{6,0}^6b_{6,0}^2 - 40y^3a_{5,0}^3a_{6,0}^4b_{6,0}^3 + \\
+5y^2a_{5,0}^4a_{6,0}^2b_{6,0}^4 + 1728ya_{6,0}^5b_{1,0}b_{6,0}^4 - 2160ya_{1,0}a_{6,0}^4b_{6,0}^5 + 432a_{6,0}^4b_{0,0}b_{6,0}^5 + 432y^6a_{6,0}^{10}\Big),
\end{cases}
$$

where the nondegeneracy conditions $a_{6,0}b_{6,0}\left(a_{6,0}b_{1,0}-a_{1,0}b_{6,0}\right) \neq 0$ arise naturally in the process of solving the corresponding system of algebraic equations. After performing the time rescaling

$$
t = \frac{432b_{6,0}^5\tau}{a_{6,0}^{10}\left(a_{6,0}b_{1,0}-a_{1,0}b_{6,0}\right)^4}
$$

and the change of variables

$$\begin{cases} X = -6a_{6,0}b_{6,0}x + 6a_{6,0}^2 y + \dfrac{a_{5,0}^6 b_{6,0}^7}{108a_{6,0}^2} + \dfrac{72}{5}a_{6,0}^3 b_{6,0}^6 \left(6a_{6,0}b_{0,0} - a_{5,0}b_{1,0}\right) + a_{5,0}b_{6,0}, \\ Y = -\dfrac{b_{6,0}^6\left(1296a_{6,0}^5 b_{1,0} - a_{5,0}^5 b_{6,0}\right)}{18a_{6,0}}x - \dfrac{1}{18}b_{6,0}^5\left(a_{5,0}^5 b_{6,0} + 6480a_{6,0}^5 b_{1,0} - \right. \\ \left. -7776a_{1,0}a_{6,0}^4 b_{6,0}\right) y + 12a_{5,0}a_{6,0}^3 b_{1,0}b_{6,0}^6 - \dfrac{a_{5,0}^6 b_{6,0}^7}{108a_{6,0}^2}, \end{cases}$$

the system is transformed into

$$\begin{cases} \dot{X} = X, \\ \dot{Y} = -5Y + \frac{b_{6,0}}{108a_{6,0}^2}X^6. \end{cases}$$

By rescaling the OY-axis using the transformation $X = x$, $Y = \dfrac{b_{6,0}}{108a_{6,0}^2}y$, we obtain the system

$$\begin{cases} \dot{x} = x, \\ \dot{y} = -5y + x^6. \end{cases} \tag{6}$$

The other two systems are omitted, as they are equivalent, up to affine transformations and time rescaling, to system (6). Consequently, up to this point, we have obtained a single system whose invariant straight line at infinity has multiplicity 16. Since some cases could not be fully resolved, we formulate the following conjecture:

Conjecture 2.1. *Let $\mathbb{X}$ be a real planar polynomial differential system of degree six. Then the maximal algebraic multiplicity of the invariant straight line at infinity is*

$$M_\infty(6) = 16.$$

Moreover, up to affine changes of coordinates and time rescaling, there exists a unique system attaining this maximal multiplicity, which can be reduced to the normal form (6).

3. The general case

From system (6) and from the polynomial systems with maximal multiplicity at infinity studied in [17], we observe that some of these systems have the form

$$\begin{cases} \dot{x} = x, \\ \dot{y} = (1-n)y + x^n, \end{cases} \tag{7}$$

where n is the degree of the system.

Applying the Poincaré transformation $x \mapsto \frac{1}{z}$, $y \mapsto \frac{y}{z}$, followed by the substitution $y \mapsto x$ and $z \mapsto y$, system (7) is transformed into

$$\begin{cases} \dot{x} = nxy^{n-1} - 1, \\ \dot{y} = y^n. \end{cases} \tag{8}$$

The polynomial $\det M_R$ associated with system (8), which determines the multiplicity of the invariant straight line at infinity for system (7), is given by

$$\det M_R = n(n-1)xy^{3n-2}.$$

This implies that any planar polynomial differential system may admit the invariant line at infinity with multiplicity at least equal to $3n - 2$. Therefore, in general, we obtain that

$$M_\infty(n) \geq 3n - 2.$$

As shown in [17] for the cases $n \in \{2, 3, 4, 5\}$, this value indeed represents the maximal multiplicity, that is,

$$M_\infty(n) = 3n - 2, \quad n \in \{2, 3, 4, 5\}.$$

This suggests that the bound $M_\infty(n) = 3n - 2$ is sharp for all $n \geq 2$, although a complete proof for arbitrary degree remains open.

Conjecture 3.1. *For any planar polynomial differential system of degree $n \geq 2$, the maximal algebraic multiplicity of the invariant straight line at infinity, in the sense of the Poincaré compactification, is given by*

$$M_\infty(n) = 3n - 2.$$

4. Conclusions

In this paper, we studied planar polynomial differential systems of degree six admitting an invariant straight line at infinity of maximal algebraic multiplicity. Using the Poincaré compactification and the associated determinant criterion, we provide strong evidence that the maximal multiplicity of the invariant straight line at infinity for sextic systems is equal to 16 and formulate a conjecture asserting the uniqueness of the normal form attaining this value. Moreover, by considering a general family of polynomial systems, we showed that the multiplicity $3n - 2$ of the invariant straight line at infinity can be attained for any degree $n \geq 2$. This result confirms and extends previous classifications for quadratic, cubic, quartic, and quintic systems, and places the sextic case into a unified theoretical framework.

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About some commutative operations with classes of subcategories

Dumitru Botnaru and Alina Turcanu

Abstract. Two operations on the class of subcategories of some categories are defined and conditions are indicated when these operations are commutative. Examples of factorization structures are given, whose subcategory classes satisfy the necessary conditions for these operations applied to them to be commutative.
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Keywords: the reflective, coreflective subcategories, the pair of conjugate subcategories, the factorization structures, category of locally convex topological vector spaces.

Despre unele operații comutative cu clase de subcategorii

Rezumat. Sunt definite două operații asupra claselor de subcategorii ale unor categorii și sunt indicate condițiile în care aceste operații sunt comutative. Sunt date exemple de structuri de factorizare, ale căror clase de subcategorii îndeplinesc condițiile necesare pentru ca aceste operații aplicate asupra lor să fie comutative.
Cuvinte-cheie: subcategorii reflective, coreflective, perechi de subcategorii conjugate, structuri de factorizare, categoria spațiilor local convexe topologice vectoriale.

1. Introduction

In this section introduces the notation and basic concepts that will be used throughout the paper. We recall standard definitions of subobjects and factor objects determined by classes of morphisms, as well as the associated classes of reflective and coreflective subcategories. Furthermore, we introduce pairs of conjugate subcategories and several notions of stability with respect to subobjects and factor objects, together with their corresponding dual notions.

Notations: Let C be a subcategory and $\mathcal{I}$ a class of monomorphisms. Then $\mathcal{S}_{\mathcal{I}}(C)$ is the subcategory of all $\mathcal{I}$-subobjects of objects in C. Dual: $Q_{\mathcal{E}}(C)$.

Let $\mathbb{R}$ be the class of non-zero reflective subcategories, $\mathbb{K}$ - the class of non-zero coreflective subcategories of category $C_2\mathcal{V}$ of locally convex topological vector spaces, $\mathcal{R} \in \mathbb{R}$, $\mathcal{K} \in \mathbb{K}$ and the corresponding functors $r : C_2\mathcal{V} \to \mathcal{R}$ and $k : C_2\mathcal{V} \to \mathcal{K}$.

Let be $\varepsilon\mathcal{R} = \{e \in \mathcal{E}pi | r(e) \in \mathcal{I}so\}$, $\mu\mathcal{K} = \{m \in \mathcal{M}ono | k(m) \in \mathcal{I}so\}$. Both $\mu\mathcal{K}$ and $\varepsilon\mathcal{R}$ are classes of bimorphisms (see [1,2]).

$e : X \to Y \in \varepsilon\mathcal{R} \Leftrightarrow r^X = f \cdot e$ for a morphism f;

$m : X \to Y \in \mu\mathcal{K} \Leftrightarrow k^Y = m \cdot f$ for a morphism f.

Definition 1.1. *([1]) Let $\mathcal{R} \in \mathbb{R}$ and $\mathcal{K} \in \mathbb{K}$ be. The pair $(\mathcal{K}, \mathcal{R})$ is called a pair of conjugate subcategories if $\varepsilon\mathcal{R} = \mu\mathcal{K}$. These classes $\varepsilon\mathcal{R}$ (or $\mu\mathcal{K}$) form the class Bic of bicomplete classes.*

Let $\mathcal{E} \in \mathcal{E}pi$. Then $\mathbb{V}(\mathcal{E})$ is the class of reflective subcategories closed with respect to $\mathcal{E}$-factorobjects. Dual notion: $\mathbb{V}^c(\mathcal{M})$.

$\mathbb{V}^s(\mathcal{B})$ is the class of reflective subcategories of $\mathbb{V}(\mathcal{B})$ closed with respect to $\mathcal{B}$-subobjects. Dual notion: $\mathbb{V}_f(\mathcal{B})$. $\mathbb{V}^s_f(\mathcal{B}) = \mathbb{V}^s(\mathcal{B}) \cap \mathbb{V}_f(\mathcal{B})$.

$\mathbb{R}^s(\mathcal{B})$ is the class of reflective subcategories closed with respect to $\mathcal{B}$-subobjects. Dual notion: $\mathbb{R}_f(\mathcal{B})$. $\mathbb{R}^s_f(\mathcal{B}) = \mathbb{R}^s(\mathcal{B}) \cap \mathbb{R}_f(\mathcal{B})$.

$\mathbb{V}(\mathcal{B}, \mathcal{E}) = \mathbb{R}(\mathcal{B}) \bigcap \mathbb{V}(\mathcal{E})$.

$\mathbb{V}^s(\mathcal{B}, \mathcal{E}) = \mathbb{R}^s(\mathcal{B}) \bigcap \mathbb{V}(\mathcal{E})$.

$\mathbb{V}_f(\mathcal{B}, \mathcal{E}) = \mathbb{R}_f(\mathcal{B}) \bigcap \mathbb{V}(\mathcal{E})$.

$\mathbb{V}^s_f(\mathcal{B}, \mathcal{E}) = \mathbb{V}^s(\mathcal{B}, \mathcal{E}) \bigcap \mathbb{V}_f(\mathcal{B}, \mathcal{E})$.

Dual notions: $\mathbb{K}(\mathcal{B})$, $\mathbb{K}^s(\mathcal{B})$, $\mathbb{K}_f(\mathcal{B})$, $\mathbb{K}^s_f(\mathcal{B})$, $\overline{\mathbb{V}}^c(\mathcal{M})$, $\overline{\mathbb{V}}^s(\mathcal{B}, \mathcal{M})$, $\overline{\mathbb{V}}_f(\mathcal{B}, \mathcal{M})$, $\overline{\mathbb{V}}^s_f(\mathcal{B}, \mathcal{M})$.

2. Commutative operations with subcategory classes

In this section we investigate the commutativity of the operators $(\mathcal{S}_\mathcal{I})$ and $(Q_\mathcal{E})$ acting on classes of subcategories. Under appropriate stability assumptions on the injection class $(\mathcal{I})$ and the projection class $(\mathcal{E})$ arising from left and right factorization structures, we show that the formation of subobjects and factor objects commutes, that is, ($Q_\mathcal{E}\mathcal{S}_\mathcal{I}(C) = \mathcal{S}_\mathcal{I}Q_\mathcal{E}(C)$). Several consequences for reflective and coreflective subcategories are then derived.

Theorem 2.1. *Let $(\mathcal{P}, \mathcal{I})$ a left factorization structure with injection class $\mathcal{I}$ stable on the right, $(\mathcal{E}, \mathcal{M})$ a right factorization structure with projection class $\mathcal{E}$ stable on the left, and $C \subset C_2\mathcal{V}$. Then*

$$Q_\mathcal{E}\mathcal{S}_\mathcal{I}(C) = \mathcal{S}_\mathcal{I}Q_\mathcal{E}(C).$$

Proof. $\Rightarrow Q_\mathcal{E}\mathcal{S}_\mathcal{I}(C) \subset \mathcal{S}_\mathcal{I}Q_\mathcal{E}(C)$. Let $Z \in |Q_\mathcal{E}\mathcal{S}_\mathcal{I}(C)|$ be. There is then an object $A \in \|C\|$ and the morfisms $i : X \to A \in \mathcal{I}$, $e : X \to Z \in \mathcal{E}$.

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "e"'] & A \arrow[d, dashed, "e^1"] \\ Z \arrow[r, dashed, "i^1"] & T \end{tikzcd}$$

Figure 1. The puschout square built on the morphisms i and e.

Let be

$$i' \cdot e = e' \cdot i \tag{1}$$

the puschout square built on the morphisms i and e. Then $e' \in \mathcal{E}$ and $i' \in \mathcal{I}$ and $Z \in |\mathcal{S}_{\mathcal{I}}\mathcal{Q}_{\mathcal{E}}(\mathcal{C})|$.

$\Leftarrow$ $\mathcal{S}_{\mathcal{I}}\mathcal{Q}_{\mathcal{E}}(\mathcal{C}) \subset \mathcal{Q}_{\mathcal{E}}\mathcal{S}_{\mathcal{I}}(\mathcal{C})$. Dual demonstration. □

Corollary 2.1. *1. Let $(\mathcal{P}, \mathcal{I})$ a left factorization structure with stable injection class on the right, $(\mathcal{E}, \mathcal{M})$ a right factorization structure with stable projection class on the left, $\mathcal{R} \in \mathbb{R}$ and $\mathcal{K} \in \mathbb{K}$. Then*

$$a)\ \mathcal{Q}_{\mathcal{E}}\mathcal{S}_{\mathcal{I}}(\mathcal{R}) = \mathcal{S}_{\mathcal{I}}\mathcal{Q}_{\mathcal{E}}(\mathcal{R}),\quad b)\ \mathcal{Q}_{\mathcal{E}}\mathcal{S}_{\mathcal{I}}(\mathcal{K}) = \mathcal{S}_{\mathcal{I}}\mathcal{Q}_{\mathcal{E}}(\mathcal{K}).$$

2. Let $(\mathcal{E}_u, \mathcal{M}_p)$ = (the class of universal epimorphisms, the class of exact monomorphisms) = (the class of surjective morphisms, the class of topological inclusions);

$(\mathcal{E}_p, \mathcal{M}_u)$ = (the class of exact epimorphisms, the class of universal monomorphisms) (see [3,4]). Then:

$$a)\ \mathcal{Q}_{\mathcal{E}_u}\mathcal{I}_{\mathcal{M}_u}(\mathcal{R}) = \mathcal{I}_{\mathcal{M}_u}\mathcal{Q}_{\mathcal{E}_u}(\mathcal{R}),\quad b)\ \mathcal{Q}_{\mathcal{E}_u}\mathcal{I}_{\mathcal{M}_p}(\mathcal{R}) = \mathcal{I}_{\mathcal{M}_p}\mathcal{Q}_{\mathcal{E}_u}(\mathcal{R}).$$

3. Some properties of operations with subcategory classes

In this section we analyze structural properties of the operations defined on classes of subcategories. In particular, we examine the behavior of the operators $(\mathcal{Q}_{\mathcal{B}})$ and $(\mathcal{S}_{\mathcal{B}})$, if $\mathcal{B}$ is a bicomplete class of morphisms, with respect to products, subobjects, and factorizations associated with classes of morphisms. Theorem 3.1 establishes the commutativity of certain canonical diagrams associated with the operators $(\mathcal{Q}_{\mathcal{B}})$ and $(\mathcal{S}_{\mathcal{B}})$, as well as stability conditions with respect to products and subobjects. As applications, the subsequent corollaries highlight consequences for reflective and coreflective subcategories associated with left and right factorization structures.

Theorem 3.1. *Let $\mathcal{B} \in \mathbb{B}ic$ be. Then the following diagrams are commutative:*

a)

$$\begin{tikzcd}
& \mathbb{R}^{S}(\mathcal{B}) \arrow[dr, "Q_{\mathcal{B}}"] & \\
\mathbb{R} \arrow[ur, "S_{\mathcal{B}}"] \arrow[rr, "S_{\mathcal{B}}Q_{\mathcal{B}} = Q_{\mathcal{B}}S_{\mathcal{B}}"] \arrow[dr, "Q_{\mathcal{B}}"'] & & \mathbb{R}^{S}_{f}(\mathcal{B}) \\
& \mathbb{R}_{f}(\mathcal{B}) \arrow[ur, "S_{\mathcal{B}}"'] &
\end{tikzcd}$$

Figure 2. Commutativity of operations $(Q_{\mathcal{B}})$ and $(S_{\mathcal{B}})$ on class $\mathbb{R}$.

b)

$$\begin{tikzcd}
& \mathbb{K}^{S}(\mathcal{B}) \arrow[dr, "Q_{\mathcal{B}}"] & \\
\mathbb{K} \arrow[ur, "S_{\mathcal{B}}"] \arrow[rr, "S_{\mathcal{B}}Q_{\mathcal{B}} = Q_{\mathcal{B}}S_{\mathcal{B}}"] \arrow[dr, "Q_{\mathcal{B}}"'] & & \mathbb{K}^{S}_{f}(\mathcal{B}) \\
& \mathbb{K}_{f}(\mathcal{B}) \arrow[ur, "S_{\mathcal{B}}"'] &
\end{tikzcd}$$

Figure 3. Commutativity of operations $(Q_{\mathcal{B}})$ and $(S_{\mathcal{B}})$ on class $\mathbb{K}$.

Proof. a) $Q_{\mathcal{B}}(\mathcal{R})$ is closed in relation to the products. Indeed, let $\{A_i | i \in \mathcal{I}\}$ be a family of objects in $Q_{\mathcal{B}}(\mathcal{R})$. Then there exist the objects $T_i \in |\mathcal{R}|$ and the morphisms $b_i : T_i \longrightarrow A_i \in \mathcal{B}, i \in \mathcal{I}$. The morphism

$$b = \prod b_i : \prod T_i \longrightarrow \prod A_i$$

belongs to class $\mathcal{B}$ and $\prod T_i \in |\mathcal{R}|$. So $\prod A_i \in |Q_{\mathcal{B}}(\mathcal{R})|$.

Let's verify that $Q_{\mathcal{B}}(\mathcal{R})$ is closed with respect to $(\varepsilon\mathcal{R})^{\perp}$-subobjects. Let $A \in |Q_{\mathcal{B}}(\mathcal{R})|$ and $m : X \longrightarrow (\varepsilon\mathcal{R})^{\perp}$ be. There are $T \in |\mathcal{R}|$ and the morphism $b : T \longrightarrow A \in \mathcal{B}$. We examine the pullback square:

$$\begin{tikzcd}
P \arrow[r, dashed, "b^{1}"] \arrow[d, dashed, "m^{1}"'] & X \arrow[d, "m"] \\
T \arrow[r, "b"] & A
\end{tikzcd}$$

Figure 4. The pullback square built on the morphisms b and m

$$b \cdot m' = m \cdot b' \tag{1}$$

Then $b' \in \mathcal{B}$ and $m' \in (\varepsilon\mathcal{R})^{\perp}$. So $P \in \mathcal{R}$, and $X \in |Q_{\mathcal{B}}(\mathcal{R})|$. Thus we have proven that $Q_{\mathcal{B}}(\mathcal{R})$ is a $(\varepsilon\mathcal{R})$-reflective subcategory.

Let's verify that $\mathcal{S}_{\mathcal{B}}(\mathcal{R})$ is a $(\varepsilon\mathcal{R})$-reflective subcategory. Let be $X \in |C_2\mathcal{V}|$, $r^X : X \to rX$, his $\mathcal{R}$ - replica,

$$r^X = b^X \cdot r_1^X \tag{2}$$

$(\mathcal{B}^{\top}, \mathcal{B})$-factorization of morphism r^X and we will demonstrate that $r_1^X : X \to r_1 X$ is $\mathcal{S}_{\mathcal{B}}(\mathcal{R})$-replica of object X. Let $Z \in |\mathcal{S}_{\mathcal{B}}(\mathcal{R})|$ and $f : X \to Z$. It exists $T \in |\mathcal{R}|$ and $b : Z \to T \in \mathcal{B}$. Then

$$\begin{tikzcd}
X \arrow[r, "r_1^X"] \arrow[dr, "f"'] & r_1 X \arrow[r, "b^X"] \arrow[d, dashed, "h"] & rX \arrow[d, dashed, "g"] \\
 & Z \arrow[r, "b"] & T
\end{tikzcd}$$

Figure 5. r_1^X is $\mathcal{S}_{\mathcal{B}}(\mathcal{R})$-replica of object X.

$$b \cdot f = g \cdot r^X \tag{3}$$

or

$$b \cdot f = g \cdot b^X \cdot r_1^X \tag{4}$$

for un morphism g. Because $r_1^X \perp b$, we deduce that

$$f = h \cdot r_1^X \tag{5}$$

Thus f is extended by r_1^X.

b) Dual demonstration. □

Corollary 3.1. *Let be $\mathcal{B} \in \mathbb{B}ic$, $(\mathcal{E}, \mathcal{M})$ a right factorization structure with projection class $\mathcal{E}$ stable on the left and $\mathcal{R} \in \mathbb{R}$. Then the following diagram is commutative.*

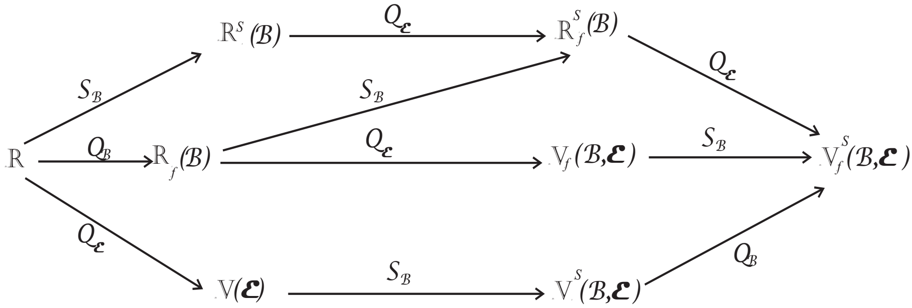

Figure 6. Commutativity of operations $(Q_\mathcal{B})$, $(\mathcal{S}_\mathcal{B})$ and $(\mathcal{S}_\mathcal{E})$ on class $\mathbb{R}$.

Corollary 3.2. *Let be $\mathcal{B} \in \mathbb{B}ic$, $(\mathcal{E}, \mathcal{M})$ a left factorization structure with bijection class $\mathcal{E}$ stable on the right and $\mathcal{K} \in \mathbb{K}$. Then the following diagram is commutative.*

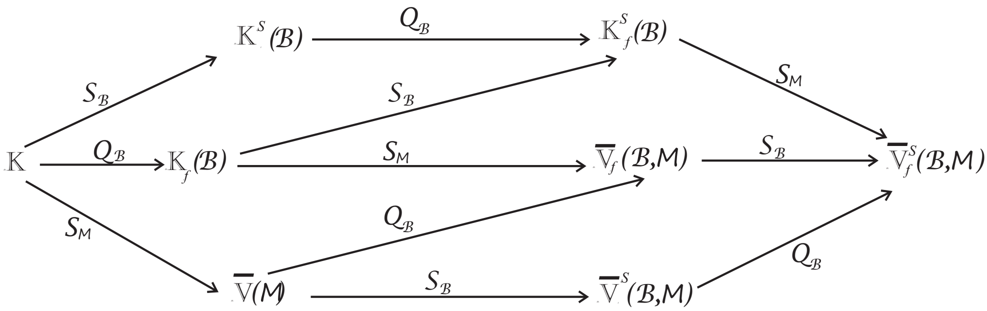

Figure 7. Commutativity of operations $(Q_\mathcal{B})$, $(\mathcal{S}_\mathcal{B})$ and $(\mathcal{S}_\mathcal{M})$ on class $\mathbb{K}$.

4. Conclusions

In this paper we have investigated the behavior of operations with classes of subcategories generated by subobjects and factor objects in the category $C_2\mathcal{V}$ of locally convex topological vector spaces. The main emphasis was placed on the interaction between the operators $\mathcal{S}_\mathcal{I}$ and $Q_\mathcal{E}$ associated with left and right factorization structures, as well as on their impact on reflective and coreflective subcategories.

First, we established sufficient stability conditions on the classes of monomorphisms and epimorphisms under which the formation of $\mathcal{I}$-subobjects and $\mathcal{E}$-factor objects is commutative. Theorem 2.1 shows that, in the presence of $\mathcal{I}$-right-stable injection classes and $\mathcal{E}$-left-stable projection classes, the equality

$$\mathcal{Q}_{\mathcal{E}}\mathcal{S}_{\mathcal{I}}(\mathcal{C}) = \mathcal{S}_{\mathcal{I}}\mathcal{Q}_{\mathcal{E}}(\mathcal{C}),$$

holds for arbitrary subcategories $\mathcal{C}$. This result provides a categorical framework in which subobject and factor object constructions can be interchanged without loss of generality. As a consequence, several commutativity properties were derived for reflective and coreflective subcategories, including important cases arising from classical factorization structures in $\mathcal{C}_2\mathcal{V}$, such as those determined by universal morphisms ($\mathcal{M}_u$) and exact morphisms ($\mathcal{M}_p$). These results clarify the role of bicomplete classes and conjugate pairs of subcategories in the study of stability with respect to subobjects and factor objects.

In the second part of the paper, we analyzed structural properties of the operators $\mathcal{S}_{\mathcal{B}}$ and $\mathcal{Q}_{\mathcal{B}}$ for bicomplete classes $\mathcal{B}$. Theorem 3.1 shows that these operators give rise to commutative canonical diagrams and preserve essential categorical constructions such as products and suitable classes of subobjects. In particular, we proved that $\mathcal{Q}_{\mathcal{B}}(\mathcal{R})$ and $\mathcal{S}_{\mathcal{B}}(\mathcal{R})$ inherit reflectivity properties from $\mathcal{R}$, while their dual counterparts preserve coreflectivity. The obtained results unify and extend several known facts concerning reflective and coreflective subcategories in categories with factorization structures. They also provide a systematic approach to studying stability properties via bicomplete classes and conjugate subcategories. Further research may focus on applications of these constructions to more general topological or functional-analytic categories, as well as on the interaction with other categorical closure operators.

Open problems and perspectives. The results obtained in this paper naturally lead to the following open problems:

Problem 4.1 (Generality of commutativity). To what extent does the commutativity $\mathcal{Q}_{\mathcal{E}}\mathcal{S}_{\mathcal{I}}(\mathcal{C}) = \mathcal{S}_{\mathcal{I}}\mathcal{Q}_{\mathcal{E}}(\mathcal{C})$, remain valid in categories beyond $\mathcal{C}_2\mathcal{V}$? In particular, can analogous results be established in quasi-abelian, exact, or locally presentable categories equipped with suitable factorization structures?

Problem 4.2 (Minimal stability assumptions). Are the stability conditions imposed on the injection and projection classes in Theorem 2.1 necessary? Determine minimal or alternative sets of assumptions on $\mathcal{I}$ and $\mathcal{E}$ ensuring the commutativity of the operators $\mathcal{S}_{\mathcal{I}}$ and $\mathcal{Q}_{\mathcal{E}}$.

Problem 4.3 (Structure of bicomplete classes). Provide an intrinsic characterization and, if possible, a classification of bicomplete classes of morphisms in categories of topological or functional-analytic nature. How do such classes determine and reflect conjugate pairs of reflective and coreflective subcategories?

Problem 4.4 (Interaction with other categorical closures). Investigate the relationship between the operators $\mathcal{S}_{\mathcal{B}}$, $\mathcal{Q}_{\mathcal{B}}$ and other categorical closure operators, such as torsion theories, localizations, or reflective hulls. Under what conditions do these constructions coincide or interact in a nontrivial way?
Problem 4.5 (Functorial behavior and invariance). Study the behavior of the constructions introduced here under functorial changes of categories. In particular, determine which properties of bicomplete classes and conjugate subcategories are preserved under adjoint functors or equivalences of categories.
Problem 4.6 (Homological and topological applications). Explore potential connections between the presented framework and homological methods in topological vector spaces. For instance, can the stability results be used to characterize classes of spaces closed under exact sequences, limits, or colimits of a prescribed type?
Problem 4.7 (Concrete realizations in functional analysis). Identify concrete classes of locally convex spaces for which the abstract results of this paper yield new closure or stability properties with respect to subspaces, quotients, and products. In particular, investigate applications to classical categories such as barreled, bornological, or nuclear spaces.

These open problems indicate that the theory developed here opens several avenues for further research, both at the abstract categorical level and in concrete functional-analytic contexts.

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